A HYERS-ULAM-RASSIAS STABILITY RESULT FOR FUNCTIONAL EQUATIONS IN INTUITIONISTIC FUZZY BANACH SPACES

N. C. KAYAL, T. K. SAMANTA, P. SAHA AND B. S. CHOUDHURY

Abstract. Hyers-Ulam-Rassias stability have been studied in the contexts of several areas of mathematics. The concept of fuzziness and its extensions have been introduced to almost all branches of mathematics in recent times. Here we define the cubic functional equation in 2-variables and establish that Hyers-Ulam-Rassias stability holds for such equations in intuitionistic fuzzy Banach spaces.

1. Introduction

The kind of stability we consider here was first formulated by Ulam [27] for linear equations which was solved by Hyers [8] for Cauchy functional equations in Banach spaces and was further generalized by Rassias [21]. The idea of such stability (which subsequently came to be known as Hyers-Ulam-Rassias stability) was generalized and extended to several areas of mathematics over the years.

For instances, such stabilities were considered for differential equations [10], functional equations [7], isometries [6] etc.

Fuzzy concepts, after its introduction by Zadeh [28], made quick inroads in many branches of mathematics including functional analysis. Particularly three versions of Hyers-Ulam-Rassias stability were studied by Mirmostafaee et al. [16] on fuzzy normed linear spaces.

Intuitionistic fuzzy sets [1] are further extensions of fuzzy sets in which an additional degree of non-belongingness is introduced. Intuitionistic fuzzy linear spaces have been defined in the work of Shakeri [26]. Stability studies of the above kind also appeared in the same work. Several important results of intuitionistic fuzzy linear spaces have been established in the papers [20, 22, 23]. Hyers-Ulam-Rassias stability of different functional equations have been discussed in various spaces like fuzzy Banach spaces, intuitionistic fuzzy Banach spaces, non-Archimedean Banach spaces etc. in many papers [3, 11, 12, 13, 14, 15, 16, 17, 19, 24, 25].

In this paper we deal with a Hyers-Ulam-Rassias stability problem in the intuitionistic fuzzy linear spaces as defined in [26]. For that purpose we consider the class of cubic functional equations in 2-variables. Cubic functional equations have appeared in such stability studies in works by Jun et al. [9], Najati [18] in linear

---

Received: December 2015; Revised: March 2016; Accepted: May 2016

Key words and phrases: Cubic functional equations, t-norm, t-conorm, Intuitionistic fuzzy Banach space, Hyers-Ulam-Rassias stability.

spaces. Stabilities of quadratic functional equations in two variables were studied by Bae et al [2].

Specifically, putting the above concepts together, we define cubic functional equations in 2-variables and establish the Hyers-Ulam-Rassias stability of these equations in intuitionistic fuzzy Banach spaces.

2. Preliminaries

We now define cubic functional equations in 2-variables as follows :
A mapping $g : X^2 \to \mathbb{R}$ is said to be a cubic form if $g(x, y) = ax^3 + bx^2y + cxy^2 + dy^3$ for all $x, y \in X$, where $a, b, c, d \in \mathbb{R}$ and X is a real vector space.

If X and Y are assumed to be real vector space and a Banach space respectively then for a mapping $f : X^2 \to Y$, consider the functional equation

$$f(mx + y, mz + w) + f(mx - y, mz - w) = mf(x + y, z + w) + mf(x - y, z - w) + 2(m^3 - m)f(x, z) \quad (1)$$

where $m \in \mathbb{N}$ and $m \geq 2$.

If $X = Y = \mathbb{R}$, for all $x, y \in \mathbb{R}$, the cubic form $g(x, y) = ax^3 + bx^2y + cxy^2 + dy^3$ is a solution of (1). Any solution of (1) is termed as cubic mapping.

Our aim is to determine Hyers-Ulam-Rassias stability results concerning the cubic functional equations (1) in two variables in intuitionistic fuzzy Banach spaces.

In the following we state some lemmas, definitions and examples.

Lemma 2.1. [5] *Consider the set* L^* *and the order relation* $\leq_{L^*}$ *defined by*

$$L^* = \{(x_1, x_2) : (x_1, x_2) \in [0, 1]^2 \text{ and } x_1 + x_2 \leq 1\},$$

$$(x_1, x_2) \leq_{L^*} (y_1, y_2) \Leftrightarrow x_1 \leq y_1, x_2 \geq y_2, \forall (x_1, x_2), (y_1, y_2) \in L^*.$$

Then $(L^*, \leq_{L^*})$ *is a complete lattice. We denote its units by* $0_{L^*} = (0, 1)$ *and* $1_{L^*} = (1, 0)$.

Definition 2.2. [1] An intuitionistic fuzzy set $A_{\zeta,\eta}$ in a universal set U is an object $A_{\zeta,\eta} = \{(\zeta_A(u), \eta_A(u)) : u \in U\}$, where $\zeta_A(u) \in [0, 1]$ and $\eta_A(u) \in [0, 1]$ for all $u \in U$ are called the membership degree and the non-membership degree respectively, of u in $A_{\zeta,\eta}$ and furthermore, satisfy $\zeta_A(u) + \eta_A(u) \leq 1$.

Definition 2.3. [4] A triangular norm (t-norm) on L^* is a mapping $\tau : (L^*)^2 \to L^*$ satisfying the following conditions :
(a) $(\forall x \in L^*)(\tau(x, 1_{L^*}) = x)$ (boundary condition);
(b) $(\forall (x, y) \in (L^*)^2)(\tau(x, y) = \tau(y, x))$ (commutativity);
(c) $(\forall (x, y, z) \in (L^*)^3)(\tau(x, \tau(y, z)) = \tau(\tau(x, y), z))$ (associativity);
(d) $(\forall (x, x', y, y') \in (L^*)^4)(x \leq_{L^*} x'$ and $y \leq_{L^*} y' \Rightarrow \tau(x, y) \leq_{L^*} \tau(x', y'))$ (monotonicity).

A t-norm τ on L^* is said to be continuous if for any $x, y \in L^*$ and any sequences $\{x_n\}$ and $\{y_n\}$ which converge to x and y respectively,

$$\lim_{n \to \infty} \tau(x_n, y_n) = \tau(x, y).$$

For example, let $a = (a_1, a_2), b = (b_1, b_2) \in L^*$, consider $\tau(a, b) = (a_1 b_1, \min\{a_2 + b_2, 1\})$ and $M(a, b) = (\min\{a_1, b_1\}, \max\{a_2, b_2\})$. Then $\tau(a, b)$ and $M(a, b)$ are continuous t-norm.

Now, we define a sequence τ^n recursively by $\tau^1 = \tau$ and

$$\tau^n\left(x^{(1)}, \cdots, x^{(n+1)}\right) = \tau\left(\tau^{n-1}\left(x^{(1)}, \cdots, x^{(n)}\right), x^{(n+1)}\right)$$

for all $n \geq 2$ and $x^i \in L^*$.

Definition 2.4. [4] A continuous t-norm τ on L^* is said to be continuous t-representable if there exists a continuous t-norm $*$ and a continuous t-conorm $\diamond$ on $[0, 1]$ such that, for all $x = (x_1, x_2), y = (y_1, y_2) \in L^*$,

$$\tau(x, y) = (x_1 * y_1, x_2 \diamond y_2).$$

Definition 2.5. [4] A negator on L^* is any decreasing mapping $N : L^* \to L^*$ satisfying $N(0_{L^*}) = 1_{L^*}$ and $N(1_{L^*}) = 0_{L^*}$. If $N(N(x)) = x$ for all $x \in L^*$, then N is called an involutive negator. A negator on $[0, 1]$ is a decreasing mapping $N : [0, 1] \to [0, 1]$ satisfying $N(0) = 1$ and $N(1) = 0$. N_s denotes the standard negator on $[0, 1]$ defined by $N_s(x) = 1 - x$ for all $x \in [0, 1]$.

Definition 2.6. [26] (1) Let $L = (L^*, \leq_{L^*})$. The triple (X, P, τ) is said to be an L-fuzzy normed space if X is a vector space, τ is a continuous t-norm on L^* and P is an L-fuzzy set on $X \times (0, +\infty)$ satisfying the following conditions for all $x, y \in X$ and $t, s > 0$,

(a) $P(x, t) > 0_{L^*}$;
(b) $P(x, t) = 1_{L^*}$ if and only if $x = 0$;
(c) $P(\alpha x, t) = P\left(x, \frac{t}{|\alpha|}\right)$ for all $\alpha \neq 0$;
(d) $P(x + y, t + s) \geq_{L^*} \tau(P(x, t), P(y, s))$;
(e) $P(x, .) : (0, \infty) \to L^*$ is continuous;
(f) $\lim_{t \to 0} P(x, t) = 0_{L^*}$ and $\lim_{t \to \infty} P(x, t) = 1_{L^*}$.

In this case P is called an L-fuzzy norm (briefly, L^*-fuzzy norm).

(2) If $P = P_{\mu, \nu}$ is an intuitionistic fuzzy set, then the triple $(X, P_{\mu, \nu}, \tau)$ is said to be an intuitionistic fuzzy normed space (briefly, IFN-space). In this case $P = P_{\mu, \nu}$ is called an intuitionistic fuzzy norm on X.

Note that, if P is an L^*-fuzzy norm on X, then the following are satisfied :
(i) $P(x, t)$ is nondecreasing with respect to t for all $x \in X$.
(ii) $P(x - y, t) = P(y - x, t)$ for all $x, y \in X$ and $t > 0$.

Example 2.7. Let $(X, \|.\|)$ be a normed space.
Let $\tau(a, b) = (a_1 b_1, \min\{a_2 + b_2, 1\})$ for all $a = (a_1, a_2), b = (b_1, b_2) \in L^*$ and μ, ν be membership and non-membership degree of an intuitionistic fuzzy set defined by

$$P_{\mu, \nu}(x, t) = (\mu_x(t), \nu_x(t)) = \left(\frac{t}{t + m\|x\|}, \frac{\|x\|}{t + \|x\|}\right)$$

for all $t \in \mathbb{R}^+$ in which $m > 1$. Then $(X, P_{\mu, \nu}, \tau)$ is an IFN-space. Here, $\mu(x, t) + \nu(x, t) = 1$ for $x = 0$ and $\mu(x, t) + \nu(x, t) < 1$ for $x \neq 0$.

Let $M(a, b) = (\min\{a_1, b_1\}, \max\{a_2, b_2\})$ for all $a = (a_1, a_2), b = (b_1, b_2) \in L^*$ and μ, ν be membership and non-membership degree of an intuitionistic fuzzy set defined by

$$P_{\mu,\nu}(x, t) = (\mu_x(t), \nu_x(t)) = \left(e^{-\frac{\|x\|}{t}}, e^{-\frac{\|x\|}{t}}\left(e^{\frac{\|x\|}{t}} - 1\right)\right)$$

for all $t \in \mathbb{R}^+$. Then $(X, P_{\mu,\nu}, M)$ is an IFN-space.

Definition 2.8. (1) A sequence $\{x_n\}$ in an IFN-space $(X, P_{\mu,\nu}, \tau)$ is said to be convergent to a point $x \in X$ (denoted by $x_n \to x$) if $P_{\mu,\nu}(x_n - x, t) \to 1_{L^*}$ as $n \to \infty$ for every $t > 0$.
(2) A sequence $\{x_n\}$ in an IFN-space $(X, P_{\mu,\nu}, \tau)$ is said to be a Cauchy sequence if, for any $0 < \epsilon < 1$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$P_{\mu,\nu}(x_n - x_m, t) >_{L^*} (N_s(\epsilon), \epsilon)$$

for all $n, m \geq n_0$, where N_s is the standard negator.
(3) An IFN-space $(X, P_{\mu,\nu}, \tau)$ is said to be complete if every Cauchy sequence in $(X, P_{\mu,\nu}, \tau)$ is convergent in $(X, P_{\mu,\nu}, \tau)$. A complete intuitionistic fuzzy normed space is called an intuitionistic fuzzy Banach space.

By Hyers-Ulam-Rassias stability here mean that for any mapping which deviates from the cubic mapping to a certain degree, there exists a cubic mapping which approximates the function to a degree which depends on the deviation. The precise technical meaning of it is given in the statement of theorem 3.1.

3. Hyers-Ulam-Rassias Stability of (1)

Throughout this section it is assumed that X is a real vector space, $(Y, P_{\mu,\nu}, \tau)$ is a complete IFN-space and $(Z, P'_{\mu,\nu}, \tau)$ is an IFN-space.

Theorem 3.1. *Let $\phi : X^4 \to Z$ be a mapping such that*

$$P'_{\mu,\nu}(\phi(mx, my, mz, mw), t) \geq_{L^*} P'_{\mu,\nu}(\alpha\,\phi(x, y, z, w), t) \tag{2}$$

for all $x, y, z, w \in X, t > 0$ and for some α satisfying $0 < \alpha < m^3$ where $m \in \mathbb{N}$ and $m \geq 2$. Let $f : X^2 \to Y$ be a mapping such that

$$P_{\mu,\nu}(f(mx + y, mz + w) + f(mx - y, mz - w) - mf(x + y, z + w)$$
$$- mf(x - y, z - w) - 2(m^3 - m)f(x, z), t) \geq_{L^*} P'_{\mu,\nu}(\phi(x, y, z, w), t) \tag{3}$$

for all $x, y, z, w \in X, t > 0$. Then there exists a unique cubic mapping $T : X^2 \to Y$ such that

$$P_{\mu,\nu}(f(x, y) - T(x, y), t) \geq_{L^*} P'_{\mu,\nu}\left(\phi(x, 0, y, 0), 2(m^3 - \alpha)t\right) \tag{4}$$

and

$$m^{-3n} f(m^n x, m^n y) \to T(x, y) \quad \text{as } n \to \infty \tag{5}$$

for all $x, y \in X, t > 0$.

Proof. Putting $y = 0, w = 0$ in (3), we have for all $x, z \in X$, and $t > 0$,

$$P_{\mu,\nu}(m^{-3} f(mx, mz) - f(x, z), t) \geq_{L^*} P'_{\mu,\nu}(\phi(x, 0, z, 0), 2m^3 t). \tag{6}$$

Replacing x, z by mx, mz respectively in (6) and using (2), we get for all $x, z \in X, t > 0$,

$$P_{\mu,\nu}(m^{-6} f(m^2 x, m^2 z) - m^{-3} f(mx, mz), t) \geq_{L^*} P'_{\mu,\nu}\left(\phi(x, 0, z, 0), \frac{2m^6 t}{\alpha}\right).$$

By the principle of mathematical induction, it can be shown that for all $x, y \in X, t > 0$ and $n \in \mathbb{N}$,

$$P_{\mu,\nu}(m^{-3n} f(m^n x, m^n y) - m^{-3(n-1)} f(m^{n-1} x, m^{n-1} y), t) \geq_{L^*} P'_{\mu,\nu}\left(\phi(x, 0, y, 0), \frac{2m^{3n} t}{\alpha^{n-1}}\right). \quad (7)$$

Now, for all $x, y \in X, t > 0, n \in \mathbb{N}$ and all integer $p \geq 0$ we get

$$\begin{aligned} &P_{\mu,\,,\nu}\left(m^{-3(n+p)} f(m^{n+p} x, m^{n+p} y) - m^{-3p} f(m^p x, m^p y), t\sum_{r=0}^{n-1} \frac{\alpha^{p+r}}{2m^{3(p+r+1)}}\right) \\ &\geq_{L^*} \tau^{n-1}\Bigg(P_{\mu,\nu}\left(m^{-3(n+p)} f(m^{n+p} x, m^{n+p} y) - m^{-3(n+p-1)} f(m^{n+p-1} x, m^{n+p-1} y), \frac{\alpha^{n+p-1} t}{2m^{3(p+n)}}\right), \\ &\quad P_{\mu,\nu}\left(m^{-3(n+p-1)} f(m^{n+p-1} x, m^{n+p-1} y) - m^{-3(n+p-2)} f(m^{n+p-2} x, m^{n+p-2} y), \frac{\alpha^{n+p-2} t}{2m^{3(p+n-1)}}\right), \ldots, \\ &\quad P_{\mu,\nu}\left(m^{-3(p+1)} f(m^{p+1} x, m^{p+1} y) - m^{-3p} f(m^p x, m^p y), \frac{\alpha^p t}{2m^{3(p+1)}}\right)\Bigg) \\ &= P'_{\mu,\nu}(\phi(x, 0, y, 0), t) \quad [\text{by } (7)]. \end{aligned}$$

It implies that, for all $x, y \in X, t > 0, n \in \mathbb{N}$ and all integer $p \geq 0$

$$P_{\mu,\nu}(m^{-3(n+p)} f(m^{n+p} x, m^{n+p} y) - m^{-3p} f(m^p x, m^p y), t) \geq_{L^*} P'_{\mu,\nu}\left(\phi(x, 0, y, 0), \frac{t}{\frac{\alpha^p}{2m^{3p}} \sum_{r=0}^{n-1} \frac{\alpha^r}{m^{3(r+1)}}}\right). \quad (8)$$

Putting $p = 0$ in (8), we get for all $x, y \in X, t > 0$ and $n \in \mathbb{N}$,

$$P_{\mu,\nu}(m^{-3n} f(m^n x, m^n y) - f(x, y), t) \geq_{L^*} P'_{\mu,\nu}\left(\phi(x, 0, y, 0), \frac{t}{\sum_{r=0}^{n-1} \frac{\alpha^r}{2m^{3(r+1)}}}\right). \quad (9)$$

Since $\left(\frac{\alpha}{m^3}\right)^p \to 0$ as $p \to \infty$, (8) implies that

$$P_{\mu,\nu}(m^{-3(n+p)} f(m^{n+p} x, m^{n+p} y) - m^{-3p} f(m^p x, m^p y), t) \to 1_{L^*}$$

as $p \to \infty$. Therefore the sequence $\{m^{-3n} f(m^n x, m^n y)\}$ is a Cauchy sequence in $(Y, P_{\mu,\nu}, \tau)$. Since $(Y, P_{\mu,\nu}, \tau)$ is a Banach space, there exists some function $T : X^2 \to Y$ such that (5) holds. Let $\delta > 0$. Now, using (9), we get for all $x, y \in X, t > 0$ and $n \in \mathbb{N}$,

$$P_{\mu,\nu}(f(x, y) - T(x, y), t + \delta)$$

$$\geq_{L^*} \tau\left(P'_{\mu,\nu}\left(\phi(x, 0, y, 0), \frac{t}{\sum_{r=0}^{n-1} \frac{\alpha^r}{2m^{3(r+1)}}}\right),\right.$$

$$\left.P_{\mu,\nu}(m^{-3n} f(m^n x, m^n y) - T(x, y), \delta)\right).$$

Taking limit as $n \to \infty$, we get for all $x, y \in X, t > 0$,

$$P_{\mu,\nu}(f(x, y) - T(x, y), t + \delta)$$

$$\geq_{L^*} \tau\left(P'_{\mu,\nu}\left(\phi(x, 0, y, 0), \frac{t}{\sum_{r=0}^{\infty} \frac{\alpha^r}{2m^{3(r+1)}}}\right), 1_{L^*}\right) \quad [\text{by } (5)].$$

Taking limit as $\delta \to 0$, we get (4). Replacing x, y, z, w by $m^n x, m^n y, m^n z, m^n w$ respectively in (3) and using (2), we get for all $x, y, z, w \in X, t > 0$ and $n \in \mathbb{N}$,

$$P_{\mu,\nu}(m^{-3n} f(m^n(mx+y), m^n(mz+w)) + m^{-3n} f(m^n(mx-y), m^n(mz-w))$$

$$-m^{-3n+1} f(m^n(x+y), m^n(z+w)) - m^{-3n+1} f(m^n(x-y), m^n(z-w))$$

$$-2(m^3 - m)m^{-3n} f(m^n x, m^n z), t) \geq_{L^*} P'_{\mu,\nu}\left(\phi(x, y, z, w), \frac{m^{3n}t}{\alpha^n}\right).$$

Taking limit as $n \to \infty$ and using (5), we get for all $x, y, z, w \in X, t > 0$,

$$P_{\mu,\nu}(T(mx+y, mz+w) + T(mx-y, mz-w) - mT(x+y, z+w)$$

$$-mT(x-y, z-w) - 2(m^3 - m)T(x, z), t) = 1_{L^*}.$$

It implies that for all $x, y, z, w \in X$,

$$T(mx+y, mz+w) + T(mx-y, mz-w) - mT(x+y, z+w)$$

$$-mT(x-y, z-w) - 2(m^3 - m)T(x, z) = 0.$$

This shows that T satisfies (1) and so T is cubic. From definition of T it is clear that for all $x, y \in X$ and $n \in \mathbb{N}$,

$$T(x, y) = m^{-3n} T(m^n x, m^n y). \tag{10}$$

To prove the uniqueness of T, assume that $T' : X^2 \to Y$ is another cubic mapping satisfying (4) and (5). Now, for all $x, y \in X, t > 0$ and $n \in \mathbb{N}$,

$$\begin{aligned} & P_{\mu,\nu}(T(x, y) - T'(x, y), t) \\ = \ & P_{\mu,\nu}(m^{-3n} T(m^n x, m^n y) - m^{-3n} T'(m^n x, m^n y), t) \quad [\text{by } (10)] \\ \geq_{L^*} \ & \tau \left(P_{\mu,\nu}\left(T(m^n x, m^n y) - f(m^n x, m^n y), \frac{m^{3n} t}{2} \right), \right. \\ & \left. P_{\mu,\nu}\left(f(m^n x, m^n y) - T'(m^n x, m^n y), \frac{m^{3n} t}{2} \right) \right) \\ \geq_{L^*} \ & P'_{\mu,\nu}\left(\phi(x, 0, y, 0), \frac{(m^3 - \alpha) m^{3n} t}{\alpha^n} \right). \end{aligned}$$

Taking limit as $n \to \infty$, we get for all $x, y \in X, t > 0$,

$$P_{\mu,\nu}(T(x, y) - T'(x, y), t) = 1_{L^*}.$$

That is, $T(x, y) = T'(x, y)$ for all $x, y \in X$. This shows that T is unique. Hence the proof. □

Corollary 3.2. *Let a, b, c, d, p, q, r, s be real numbers such that $a, b, c, d \geq 0$ and $p, q, r, s \in (0, 3)$ and $z_0 \in Z$. Let $f : X^2 \to Y$ be a mapping such that*

$$\begin{aligned} & P_{\mu,\nu}(f(mx + y, mz + w) + f(mx - y, mz - w) - mf(x + y, z + w) \\ & \qquad - mf(x - y, z - w) - 2(m^3 - m)f(x, z), t) \\ \geq_{L^*} \ & P'_{\mu,\nu}((a\|x\|^p + b\|y\|^q + c\|z\|^r + d\|w\|^s) z_0, t) \end{aligned}$$

for all $x, y \in X, t > 0$ and $m \in \mathbb{N}, m \geq 2$. Then there exists a unique cubic mapping $T : X^2 \to Y$ such that

$$P_{\mu,\nu}(f(x, y) - T(x, y), t) \geq_{L^*} P'_{\mu,\nu}\left((a\|x\|^p + c\|y\|^r) z_0, 2(m^3 - m^l) t\right)$$

where $l = \max\{p, q, r, s\}$ and $m^{-3n} f(m^n x, m^n y) \to T(x, y)$ as $n \to \infty$ for all $x, y \in X, t > 0$.

Proof. Define $\phi(x, y, z, w) = (a\|x\|^p + b\|y\|^q + c\|z\|^r + d\|w\|^s) z_0$ and take $\alpha = m^l$ where $l = \max\{p, q, r, s\}$. Then clearly, (2) is satisfied and $0 < \alpha < m^3$. By an application of the theorem 3.1 the proof is completed. □

Theorem 3.3. *Let $\phi : X^4 \to Z$ be a mapping such that*

$$P'_{\mu,\nu}\left(\phi\left(\frac{x}{m}, \frac{y}{m}, \frac{z}{m}, \frac{w}{m}\right), t\right) \geq_{L^*} P'_{\mu,\nu}\left(\frac{1}{\alpha} \phi(x, y, z, w), t\right) \tag{11}$$

for all $x, y, z, w \in X, t > 0$ and for some α satisfying $\alpha > m^3$ where $m \in \mathbb{N}$ and $m \geq 2$. Let $f : X^2 \to Y$ be a mapping such that

$$\begin{aligned} & P_{\mu,\nu}(f(mx + y, mz + w) + f(mx - y, mz - w) - mf(x + y, z + w) \\ & - mf(x - y, z - w) - 2(m^3 - m)f(x, z), t) \geq_{L^*} P'_{\mu,\nu}(\phi(x, y, z, w), t) \end{aligned}$$

for all $x, y \in X, t > 0$. *Then there exists a unique cubic mapping* $T : X^2 \to Y$ *such that* $P_{\mu,\nu}(f(x, y) - T(x, y), t) \geq_{L^*} P'_{\mu,\nu}\left(\phi(x, 0, y, 0), 2(\alpha - m^3)t\right)$ *and* $m^{3n} f\left(\frac{x}{m^n}, \frac{y}{m^n}\right) \to T(x, y)$ *as* $n \to \infty$ *for all* $x, y \in X, t > 0$.

Corollary 3.4. *Let* a, b, c, d, p, q, r, s *be real numbers such that* $a, b, c, d \geq 0$ *and* $p, q, r, s > 3$ *and* $z_0 \in Z$. *Let* $f : X^2 \to Y$ *be a mapping such that*

$$P_{\mu,\nu}(f(mx + y, mz + w) + f(mx - y, mz - w) - mf(x + y, z + w)$$
$$- mf(x - y, z - w) - 2(m^3 - m)f(x, z), t)$$
$$\geq_{L^*} P'_{\mu,\nu}((a\|x\|^p + b\|y\|^q + c\|z\|^r + d\|w\|^s) z_0, t)$$

for all $x, y \in X, t > 0$ *and* $m \in \mathbb{N}, m \geq 2$. *Then there exists a unique cubic mapping* $T : X^2 \to Y$ *such that*

$$P_{\mu,\nu}(f(x, y) - T(x, y), t) \geq_{L^*} P'_{\mu,\nu}\left((a\|x\|^p + c\|y\|^r)z_0, 2(m^l - m^3)t\right)$$

where $l = \min\{p, q, r, s\}$ *and* $m^{3n} f\left(\frac{x}{m^n}, \frac{y}{m^n}\right) \to T(x, y)$ *as* $n \to \infty$ *for all* $x, y \in X, t > 0$.

Proof. Define $\phi(x, y, z, w) = (a\|x\|^p + b\|y\|^q + c\|z\|^r + d\|w\|^s) z_0$ and take $\alpha = m^l$ where $l = \min\{p, q, r, s\}$. Then clearly, (11) is satisfied and $\alpha > m^3$. By an application of the theorem 3.3 the proof is completed. □

4. Conclusions

In this paper we have defined 2-variables cubic functional equations. The stability studies we have made for these equations are in intuitionistic fuzzy Banach spaces. It can be investigated whether the result holds for equations of arbitrary degree with any number of variables. This is the most general situation of the kind of problem described here. This can be treated as an open problem.

References

[1] K. T. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems, **20** (1986), 87–96.
[2] J. H. Bae and W. G. Park, *A fixed-point approach to the stability of a functional equation on quadratic forms*, Journal of Inequalities and Applications, **82** (2011), 1–7.
[3] J. H. Bae, W. G. Park,*On the Ulam stability of the Cauchy-Jensen equation and the additive-quadratic equation*, J. Nonlinear Sci. Appl. **8(5)** (2015), 710–718.
[4] G. Deschrijver, C. Cornelis and E. E. Kerre, *On the representation of intuitionistic fuzzy t-norms and t-conorms*, IEEE Transaction on Fuzzy Systems, **12** (2004), 45–61.
[5] G. Deschrijver and E. E. Kerre, *On the relationship between some extensions of fuzzy set theory*, Fuzzy Sets and Systems, **23** (2003), 227–235.
[6] Y. Dong, *On approximate isometries and application to stability of a function*, J. Math. Anal. Appl., **426(2)** (2015), 125–137.
[7] A. Grabiec, *The generalized Hyers-Ulam stability of a class of functional equations*, Publ. Math. Debrecen, **48** (1996), 217–235.
[8] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A., **27** (1941), 222–224.
[9] K. W. Jun and H. M. Kim *The generalized Hyers-Ulam-Rassias stability of a cubic functional equation*, J. Math. Anal. Appl., **274** (2002), 867–878.

[10] S. M. Jung, *Hyers-Ulam stability of linear differential equations of first order, II*, App. Math. Lett., **19** (2006), 854–858.

[11] N. C. Kayal, P. Mondal and T. K. Samanta, *The generalized Hyers - Ulam - Rassias stability of a quadratic functional equation in fuzzy banach spaces*, Journal of New Results in Science, **1(5)** (2014), 83–95.

[12] N. C. Kayal, P. Mondal and T. K. Samanta, *The fuzzy stability of a pexiderized functional equation*, Mathematica Moravica, **18(2)** (2014), 1–14.

[13] N. C. Kayal, P. Mondal and T. K. Samanta, *Intuitionistic fuzzy stability of a quadratic functional equation*, Tbilisi Mathematical Journal, **8(2)** (2015), 139–147.

[14] S. O. Kim, A. Bodaghi and C. Park, *Stability of functional inequalities associated with the Cauchy-Jensen additive functional equalities in non-Archimedean Banach spaces*, J. Nonlinear Sci. Appl., **8(5)** (2015), 776–786.

[15] Y. Lan and Y. Shen, *The general solution of a quadratic functional equation and Ulam stability*, J. Nonlinear Sci. Appl. **8(5)** (2015), 640–649.

[16] A. K. Mirmostafaee and M. S. Moslehian, *Fuzzy versions of Hyers-Ulam-Rassias theorem*, Fuzzy Sets and Systems, **159** (2008), 720–729.

[17] P. Mondal, N. C. Kayal and T. K. Samanta, *The stability of pexider type functional equation in intuitionistic fuzzy Banach spaces via fixed point technique*, Journal of Hyperstructures, **4(1)** (2015), 37–49.

[18] A. Najati, *The generalized Hyers-Ulam-Rassias stability of a cubic functional equation*, Turk J Math, **31** (2007), 395–408.

[19] C. Park, *Fuzzy stability of a functional equation associated with inner product space*, Fuzzy Sets and Systems, **160** (2009), 1632–1642.

[20] J. H. Park, *Intuitionistic fuzzy metric spaces*, Chaos, Solitons and Fractals, **22** (2004), 1039–1046.

[21] Th. M. Rassias, *On the stability of the linear mapping in Banach space*, Proc. Amer. Mathematical Society, **72(2)** (1978), 297–300.

[22] R. Saadati and J. H. Park, *On the intuitionistic fuzzy topological spaces*, Chaos, Solitons and Fractals, **27** (2006), 331–344.

[23] T. K. Samanta and Iqbal H. Jebril, *Finite dimentional intuitionistic fuzzy normed linear space*, Int. J. Open Problems Compt. Math., **2(4)** (2009), 574–591.

[24] T. K. Samanta, N. C.Kayal and P. Mondal, *The Stability of a General Quadratic Functional Equation in Fuzzy Banach Space*, Journal of Hyperstructures, **1(2)** (2012), 71–87.

[25] T. K. Samanta, P. Mondal and N. C. Kayal, *The generalized Hyers-Ulam-Rassias stability of a quadratic functional equation in fuzzy Banach spaces*, Annals of Fuzzy Mathematics and Informatics, **6(2)** (2013), 285–294.

[26] S. Shakeri, I*ntutionistic fuzzy stability of Jenson type mapping*, J. Non linear Sc. Appl., **2(2)** (2009), 105–112.

[27] S. M. Ulam, *Problems in Modern Mathematics*, Science Editions, Wiley, New York, 1964(Chapter VI, Some Questions in Analysis: §1, Stability).

[28] L. A. Zadeh, *Fuzzy sets*, Information and control, **8** (1965), 338–353.

Nabin Chandra Kayal*, Department of Mathematics, Indian Institute Of Engineering Science and Technology, Shibpur, Howrah - 711103, West Bengal, India
E-mail address: kayalnabin82@gmail.com

Tapas Kumar Samanta, Department of Mathematics, Uluberia College, Uluberia, Howrah - 711315, West Bengal, India
E-mail address: mumpu_tapas5@yahoo.co.in

Parbati Saha, Department of Mathematics, Indian Institute Of Engineering Science and Technology, Shibpur, Howrah - 711103, West Bengal, India
E-mail address: parbati_saha@yahoo.co.in

Binayak S. Choudhury, Department of Mathematics, Indian Institute Of Engineering Science and Technology, Shibpur, Howrah - 711103, West Bengal, India
E-mail address: binayak12@yahoo.co.in

∗Corresponding author