TRIANGULAR INTUITIONISTIC FUZZY TRIPLE BONFERRONI HARMONIC MEAN OPERATORS AND APPLICATION TO MULTI-ATTRIBUTE GROUP DECISION MAKING

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Abstract. As an special intuitionistic fuzzy set defined on the real number set, triangular intuitionistic fuzzy number (TIFN) is a fundamental tool for quantifying an ill-known quantity. In order to model the decision maker's overall preference with mandatory requirements, it is necessary to develop some Bonferroni harmonic mean operators for TIFNs which can be used to effectively intergrate the information of attribute values for multi-attribute group decision making (MAGDM) with TIFNs. The purpose of this paper is to develop some Bonferroni harmonic operators of TIFNs and apply to the MAGDM problems with TIFNs. The weighted possibility means of TIFN are firstly defined. Hereby, a new lexicographic approach is presented to rank TIFNs sufficiently considering the risk preference of decision maker. The sensitivity analysis on the risk preference parameter is made. Then, three kinds of triangular intuitionistic fuzzy Bonferroni harmonic aggregation operators are defined, including a triangular intuitionistic fuzzy triple weighted Bonferroni harmonic mean operator (TIFTWBHM) operator, a triangular intuitionistic fuzzy triple ordered weighted Bonferroni harmonic mean (TIFTOWBHM) operator and a triangular intuitionistic fuzzy triple hybrid Bonferroni harmonic mean (TIFTHBHM) operator. Some desirable properties for these operators are discussed in detail. By using the TIFTWBHM operator, we can obtain the individual overall attribute values of alternatives, which are further integrated into the collective ones by the TIFTHBHM operator. The ranking order of alternatives is generated according to the collective overall attribute values of alternatives. A real investment selection case study verifies the validity and applicability of the proposed method.

1. Introduction

In order to describe and depict uncertainty of real world, Zedeh [37] initiated the concept of fuzzy sets (FSs) in 1965, which can be used to represent the fuzziness nature. The fuzzy decision making analysis appears. However, the decision making problems often involve many incomplete information and relate to many complex factors, such as economy, politics, psychological behavior, ideology and so on [18-20]. Therefore, there often exist some hesitation degrees in the judgments of

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decision maker (DM) [1,21-24]. For example, for a real investment selection problem, because of incompleteness and uncertainty of information in the evaluation of the candidate's prospects, the evaluation value can be expressed by triangular intuitionistic fuzzy number (TIFN) [10,11-14,16,21-23] ((3.33, 3.53, 3.75); 0.7, 0.2), which means that the minimum value is 3.33, the maximum value is 3.75, and the most possible value is 3.53. Meanwhile, the maximum membership degree for the most possible value 3.53 is 70%, the minimum non-membership degree for the most possible value 3.53 is 20%, and the indeterminacy is 10%. That is to say, the DM has a hesitation degree for the estimation on this judgement, this hesitation influences the decision making on the investment selection.

As a generalization of fuzzy numbers, TIFN is an special intuitionistic fuzzy set (IFS) [1] defined on the real number set, which seems to suitably describe an ill-known quantity [10-22]. Shu et al. [16] defined a TIFN in a similar way to that of the fuzzy number and introduced an algorithm for intuitionistic fuzzy fault tree analysis. Li [11] corrected some errors in the definition of the four arithmetic operations for the TIFNs in [16]. There exist several investigations on the ranking of TIFNs and application to MADM and MAGDM. Based on the concept of a ratio of the value index to the ambiguity index, Li [12] discussed the concept of the TIFN, ranking method and applied them to MADM problems in depth. Li et al. [13] defined the values and ambiguities for TIFN as well as the value-index and ambiguity-index, developed a value and ambiguity based ranking method and applied to MADM with TIFNs. Nan et al. [14] defined the ranking order relations of TIFNs, which were applied to matrix games with payoffs of TIFNs. Wan [25] introduced the possibility mean, variance and variance coefficient of TIFNs and proposed a method based on possibility variance coefficient for MADM with TIFNs. Wan et al. [26] defined the crisp weighted possibility mean of TIFNs and the Hamming distance for TIFNs, and extended the classic VlseKriterijumska Optimizacija I Kompromisno Resenje (VIKOR) method to solve MAGDM with TIFNs. Wang et al. [30] introduced new arithmetic operations and logic operators for TIFNs and applied them to fault analysis of a printed circuit board assembly system. Wan and Dong [27] proposed the possibility method for MAGDM with TIFNs and incomplete weight information. Wan et al. [19] developed some new generalized aggregation operators for TIFNs and applied to MAGDM. Dong and Wan [6] studied the MAGDM problems in which the attribute values are the TIFNs, the attribute weights are completely unknown and the weights of DMs are given by linguistic variables. Dong and Wan [7] investigated the prioritized multi-criteria group decision making with TIFNs. Wan et al. [29] developed two triangular intuitionistic fuzzy generalized aggregation operators: TAIFGOWA operator and TAIFGHWA operator and applied them to MAGDM.

The aforementioned research about IFNs mainly deals with the operation laws [11, 16, 30], aggregation operators [7, 30, 33, 34], ranking methods [6, 12-14], decision making methods [6, 30]. Moreover, the study on aggregation operators of TIFNs concentrates the weighted arithemetic average operator and logic operators. Most of existing aggregation operators for IFNs consider the input arguments as independent, i.e., the inter-relationship of the individual arguments has not been captured by those operators. On the contrary, the Boneferoni mean (BM) originally introduced by Bonferroni [3] has the capability to capture the correlations between the input arguments to be aggregated. The BM was recently generalized through replacing the simple averaging by other mean type operators as well as associating differing importance with the arguments [36]. Xu and Yager [34] first studied BM under intuitionistic fuzzy environments and applied the weighted intuitionistic fuzzy BM to multicriteria decision making. Beliakov et al. [2] proposed the generalized Bonferroni mean (GBM) operator, which models the average of the conjunctive expressions and the average of remaining. Both the BM and GBM operators ignore some aggregation information and the weight vector of the aggregated arguments. To overcome this drawback, Xia et al. [32] developed a generalized weighted Bonferroni mean (GWBM) operator as the weighted version of the GBM, they also developed the generalized Bonferroni geometric mean (GWBGM) operator and extended to intuitionistic fuzzy environment. Park and Park [15] also defined a generalized fuzzy weighted Bonferroni harmonic mean (GFWBHM) operator, a generalized fuzzy ordered weighted Bonferroni harmonic mean (GFOWBHM) operator, generalized uncertain weighted Bonferroni harmonic mean (GUWBHM) operator, and a fuzzy weighted generalized harmonic mean (FWGHM) operator.

In mathematics, the harmonic mean (HM) is a special case of the power mean. As it tends strongly toward the least elements of the list, it may alleviate the influence of large outliers and increase the influence of small values. Recently, harmonic mean has been extended to the case of triangular fuzzy numbers (TFNs). Xu [35] developed the fuzzy harmonic mean (FHM) operators such as fuzzy weighted harmonic mean (FWHM) operator, fuzzy ordered weighted harmonic mean (FOWHM) operator and fuzzy hybrid harmonic mean (FHHM). Sun and Sun [17] further extended the BM operator to fuzzy environment. They introduced fuzzy Bonferroni harmonic mean (FBHM) operator and fuzzy ordered Bonferroni harmonic mean (FOBHM) operator, and applied to MADM.

TIFNs are special subset of IFSs. Compared with the definition of IFS, TIFN makes the membership and non-membership no longer just relative to a fuzzy concept of "excellent" or "good" by adding the triangular fuzzy number. Hence, TIFNs have better capability to model ill-known quantities than IFSs. TIFNs can be used to describe the supply chain, environment pollution, outsouring service provider selection, water resource management, environment assesset, military weapon evaluation, etc. [27-29]. Sun and Sun [17], Park and Park [15] studied the BM operator of triangular fuzzy number. But triangular fuzzy number does not consider the hesitant degree. The BM operator of triangular fuzzy number can not be directly applied to TIFNs. However, there is no investigation on the extension of the BM and HM operators for TIFNs. As far as we know, only Dutta and Guha [8] researched the BM for trapezoidal intuitionistic fuzzy numbers (TrIFNs). They pointed out that the arithmetic mean does not allow us to model any kind of mandatory requirements while the geometric mean employs the partial satisfaction of all the criteria even those criteria are not mandatory. The key characteristic of the BM is the behavior along some of the edges of the unit hypercube. In particular, the BM can model mandatory requirements. In order to model the DM's overall preference with the mandatory requirements, it is necessary to develop some Bonferroni harmonic mean operators for TIFNs which can be used to effectively intergrate the information of attribute values for MAGDM with TIFNs. Nevertheless, there exist some difficulties and challenges, such as how to reasonably define the triple Bonferroni harmonic mean operators for TIFNs and how to discuss the desirable properties of these operators. Consequently, the aim of this paper is to extend the Boneferoni harmonic means under triangular intuitionistic fuzzy environment. Some triangular intuitionistic fuzzy Bonferroni harmonic mean aggregation operators are developed, including a triangular intuitionistic fuzzy triple weighted Bonferroni harmonic mean (TIFTWBHM) operator, a triangular intuitionistic fuzzy triple ordered weighted Bonferroni harmonic mean (TIFTOWBHM) operator and a triangular intuitionistic fuzzy triple hybrid Bonferroni harmonic mean (TIFTHBHM) operator. Then, a new decision method based on TIFTWBHM and TIFTHBHM operators is proposed for solving the MAGDM problems with TIFNs. The main works and contributions of this paper are summarized as follows:

(i) The weighted possibility means of TIFN are defined. Hereby, a new lexicographic approach is presented to rank the TIFNs sufficiently considering the risk preference of DM. The sensitivity analyses on the risk preference parameter are made.

(ii) The TIFTWBHM, TIFTOWBHM and TIFTHBHM operators developed in this paper extend most of the existing BM and HM operators. For example, the GFWBHM, GUWBHM, and FWGHM operators defined in [15] are a special case of the TIFTWBHM operator and the triangular fuzzy weighted Bonferroni harmonic mean (TFWBHM) operator defined in this paper respectively. The FBHM operator defined in [17] is a specal case of the triangular intuitionistic fuzzy weighted Bonferroni harmonic mean (TIFWBHM) operator defined in this paper. The FHM operator and the uncertain weighted harmonic mean (UWHM) operator defined in [35] are a specal case of the triangular intuitionistic fuzzy weight harmonic mean (TIFWHM) operator defined in this paper. The BM for TrIFNs developed by Dutta and Guha [8] only considered the BM and ignored HM, whereas the TIFTWBHM, TIFTOWBHM and TIFTHBHM operators developed in this paper take the BM and HM into consideration simultaneously.

(iii) Existing BM, HM operators and their extensions are mainly focused on the weighted average and the ordered weighted average operators. There were few studies about the hybrid aggregation operators. The TIFTHBHM operator can reflect the important degrees of both the given arguments and the ordered positions of the arguments. It is usually applied to integrate the individual comprehensive attribute values of alternatives into the collective ones, which can sufficiently reflect the importance degrees of different experts.

The rest of this paper is structured as follows. In Section 2, we define the weighted possibility means of TIFNs. Hereby, a new lexicographic approach to ranking TIFNs is presented. In Section 3, the TIFTWBHM, TIFTOWBHM and TIFTHBHM operators are defined and their desirable properties are discussed in detail. A new method for MAGDM with TIFNs is developed in Section 4. In section 5, a numerical example is illustrated. Short conclusions are made in section 6.

2. Weighted Possibility Means and New Ranking Approach for TIFNs

In this section, the weighted possibility means of TIFNs are defined. Thereby, a new lexicographic approach is developed to rank the TIFNs.

2.1. Definition and Operation Laws of TIFNs.

Definition 2.1. [13] A TIFN $\tilde{a} = ((\underline{a}, a, \bar{a}); \omega_{\tilde{a}}, u_{\tilde{a}})$ is an special IFS on a real number set R, whose membership function and non-membership function are defined as:

$$\mu_{\tilde{a}}(x) = \begin{cases} \frac{x-\underline{a}}{a-\underline{a}}\omega_{\tilde{a}}, & \text{if} \quad \underline{a} \leq x < a \\ \omega_{\tilde{a}}, & \text{if} \quad x = a \\ \frac{\bar{a}-x}{\bar{a}-a}\omega_{\tilde{a}}, & \text{if} \quad a < x \leq \bar{a} \\ 0, & \text{if} \quad x < \underline{a} \quad \text{or} \quad x > \bar{a} \end{cases}$$

and

$$\nu_{\tilde{a}}(x) = \begin{cases} \frac{a-x+(x-\underline{a})u_{\tilde{a}}}{a-\underline{a}}, & \text{if} \quad \underline{a} \leq x < a \\ u_{\tilde{a}}, & \text{if} \quad x = a \\ \frac{x-a+(\bar{a}-x)u_{\tilde{a}}}{\bar{a}-a}, & \text{if} \quad a < x \leq \bar{a} \\ 1, & \text{if} \quad x < \underline{a} \quad \text{or} \quad x > \bar{a} \end{cases}$$

respectively, depicted as in Figure 1 below. The values $\omega_{\tilde{a}}$ and $u_{\tilde{a}}$ represent the maximum degree of membership and the minimum degree of non-membership, respectively, such that they satisfy the conditions: $0 \leq \omega_{\tilde{a}} \leq 1$, $0 \leq u_{\tilde{a}} \leq 1$ and $\omega_{\tilde{a}} + u_{\tilde{a}} \leq 1$. Let $\pi_{\tilde{a}}(x) = 1 - \mu_{\tilde{a}}(x) - \nu_{\tilde{a}}(x)$, which is called an intuitionistic fuzzy index of an element x in $\tilde{a}$. If $\underline{a} \geq 0$ and one of the three values $\underline{a}$, a and $\bar{a}$ is not equal to 0, then the TIFN $\tilde{a} = (\underline{a}, a, \bar{a})$ is called a positive TIFN, denoted by $\tilde{a} > 0$.The TIFNs discussed in this paper are all positive TIFNs.

Definition 2.2. [12] Let $\tilde{a}_i = ((\underline{a}_i, a_i, \bar{a}_i); \omega_{\tilde{a}_i}, u_{\tilde{a}_i})$ $(i = 1, 2)$ be two TIFNs and $\lambda \geq 0$. Then the operation laws for TIFNs are defined as follows:

1) $\tilde{a}_1 + \tilde{a}_2 = ((\underline{a}_1 + \underline{a}_2, a_1 + a_2, \bar{a}_1 + \bar{a}_2); \omega_{\tilde{a}_1} \wedge \omega_{\tilde{a}_2}, u_{\tilde{a}_1} \vee u_{\tilde{a}_2})$;
2) $\lambda\tilde{a}_1 = ((\lambda\underline{s}_1, \lambda a_1, \lambda\bar{a}_1); \omega_{\tilde{a}_1}, u_{\tilde{a}_1})$;
3) $\tilde{a}_1\tilde{a}_2 = ((\underline{a}_1\underline{a}_2, a_1a_2, \bar{a}_1\bar{a}_2); \omega_{\tilde{a}_1} \wedge \omega_{\tilde{a}_2}, u_{\tilde{a}_1} \vee u_{\tilde{a}_2})$;
4) $\tilde{a}_1/\tilde{a}_2 = ((\underline{a}_1/\bar{a}_2, a_1/a_2, \bar{a}_1/\underline{a}_2); \omega_{\tilde{a}_1} \wedge \omega_{\tilde{a}_2}, u_{\tilde{a}_1} \vee u_{\tilde{a}_2})$;
5) $\tilde{a}_1^{\lambda} = ((\underline{a}_1^{\lambda}, a_1^{\lambda}, \bar{a}_1^{\lambda}); \omega_{\tilde{a}_1}, u_{\tilde{a}_1})$;
6) $\tilde{a}_1^{-1} = ((\bar{a}_1^{-1}, a_1^{-1}, \underline{a}_1^{-1}); \omega_{\tilde{a}_1}, u_{\tilde{a}_1})$;

where the symbols "$\wedge$" and "$\vee$" mean min and max operators, respectively.

2.2. Weighted Possibility Means of TIFNs.

Definition 2.3. [12] Let $0 \leq \alpha \leq \omega_{\tilde{a}}$, $u_{\tilde{a}} \leq \beta \leq 1$ and $0 \leq \alpha + \beta \leq 1$. The α-cut set and β- cut set of a TIFN $\tilde{a} = ((\underline{a}, a, \bar{a}); \omega_{\tilde{a}}, u_{\tilde{a}})$ are respectively defined as $\tilde{a}_{\alpha} = \{x \,|\mu_{\tilde{a}}(x) \geq \alpha\}$ and $\tilde{a}_{\beta} = \{x \,|\upsilon_{\tilde{a}}(x) \leq \beta\}$ which can be calculated as follows:

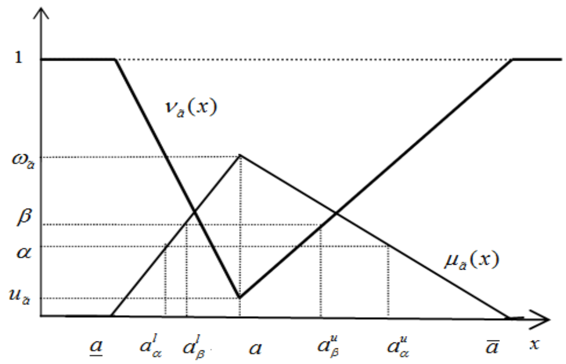

FIGURE 1. α-cut Set and β-cut Set TIFN

$$\tilde{a}_\alpha = [a^l_\alpha, a^u_\alpha] = [\underline{a} + \frac{(a-\underline{a})\alpha}{\omega_{\tilde{a}}}, \bar{a} - \frac{(\bar{a}-a)\alpha}{\omega_{\tilde{a}}}] \tag{1}$$

and

$$\tilde{a}_\beta = [a^l_\beta, a^u_\beta] = [\frac{(1-\beta)a + (\beta - u_{\tilde{a}})\underline{a}}{1-u_{\tilde{a}}}, \frac{(1-\beta)a + (\beta - u_{\tilde{a}})\bar{a}}{1-u_{\tilde{a}}}] \tag{2}$$

The α-cut set and β-cut set are also depicted in Figure 1.

Motivated by [10], we give the definitions of the weighted possibility means of TIFNs as follows.

Definition 2.4. [26] The f weighted lower and upper possibility means of membership function for a TIFN $\tilde{a}$ are respectively defined as:

$$\underline{m}_\mu(\tilde{a}) = \int_0^{\omega_{\tilde{a}}} f(\text{Pos}[\tilde{a} \le a^l_\alpha])a^l_\alpha \text{d}\alpha = \int_0^{\omega_{\tilde{a}}} f(\alpha)a^l_\alpha \text{d}\alpha, \tag{3}$$

$$\bar{m}_\mu(\tilde{a}) = \int_0^{\omega_{\tilde{a}}} f(\text{Pos}[\tilde{a} \ge a^u_\alpha])a^u_\alpha \text{d}\alpha = \int_0^{\omega_{\tilde{a}}} f(\alpha)a^u_\alpha \text{d}\alpha, \tag{4}$$

where Pos means possibility [10] and the weighting function $f : [0, \omega_{\tilde{a}}] \to R$ is non-negative, monotone increasing and satisfies the conditions: $\int_0^{\omega_{\tilde{a}}} f(\alpha)\text{d}\alpha = \omega_{\tilde{a}}$ and $f(0) = 0$.

Definition 2.5. [26] The g weighted lower and upper possibility means of membership function for a TIFN $\tilde{a}$ are respectively defined as:

$$\underline{m}_\nu(\tilde{a}) = \int_{u_{\tilde{a}}}^1 g(\text{Pos}[\tilde{a} \le a^l_\beta])a^l_\beta \text{d}\beta = \int_{u_{\tilde{a}}}^1 g(\beta)a^l_\beta \text{d}\beta, \tag{5}$$

$$\bar{m}_\nu(\tilde{a}) = \int_{u_{\tilde{a}}}^1 g(\text{Pos}[\tilde{a} \ge a^u_\beta])a^u_\beta \text{d}\beta = \int_{u_{\tilde{a}}}^1 g(\beta)a^u_\beta \text{d}\beta, \tag{6}$$

where the weighting function $g : [u_{\tilde{a}}, 1] \to R$ is non-negative, monotone decreasing and satisfies the conditions: $\int_{u_{\tilde{a}}}^1 g(\beta)\text{d}\beta = 1 - u_{\tilde{a}}$ and $g(1) = 0$.

Definition 2.6. For a TIFN $\tilde{a} = ((\underline{a}, a, \bar{a}); \omega_{\tilde{a}}, u_{\tilde{a}})$ and the f weighted possibility mean of membership function and g weighted possibility mean of non-membership function are respectively defined as:

$$m_\mu(\tilde{a}, \theta) = (1-\theta)\underline{m}_\mu(\tilde{a}) + \theta\bar{m}_\mu(\tilde{a}) \tag{7}$$

and

$$m_\nu(\tilde{a}, \theta) = (1-\theta)\underline{m}_\nu(\tilde{a}) + \theta\bar{m}_\nu(\tilde{a}) \tag{8}$$

where $\theta \in [0,1]$ is the risk preference parameter of DM and can reflect different importance to the weighted lower and upper possibility means. Different DMs have various preferences for the lower and upper possibility means. $\theta \in (0.5, 1]$ implies that DM prefers the weighted upper possibility mean, namely DM is pessimistic; $\theta \in [0, 0.5)$ shows that DM prefers the weighted lower possibility mean, namely DM is optimistic; $\theta = 0.5$ indicates that DM is indifference between the weighted lower and upper possibility means, namely DM is preference neutral.

IF $\theta = 0$, then $m_\mu(\tilde{a}, 0) = \underline{m}_\mu(\tilde{a})$; if $\theta = 1$, then $m_\mu(\tilde{a}, 1) = \bar{m}_\mu(\tilde{a})$ and $m_\nu(\tilde{a}, 1) = \bar{m}_\nu(\tilde{a})$; if $\theta = 0.5$, then $m_\mu(\tilde{a}, 0.5) = \frac{1}{2}[(\underline{m}_\mu\tilde{a}) + \bar{m}_\mu(\tilde{a})]$ and

$$m_\nu(\tilde{a}, 0.5) = \tfrac{1}{2}[\underline{m}_\nu(\tilde{a}) + \bar{m}_\nu(\tilde{a})].$$

Thus, if the TrIFN $\tilde{a} = ((\underline{a}, a, \bar{a}); \omega_{\tilde{a}}, u_{\tilde{a}})$ degenerates to TFN $\tilde{a} = (\underline{a}, a, \bar{a})$, i.e., $\omega_{\tilde{a}} = 1$ and $u_{\tilde{a}} = 0$, then, $m_\mu(\tilde{a}, 0.5) = \frac{1}{2}[\underline{m}_\mu(\tilde{a}) + \bar{m}_\mu(\tilde{a})]$ is just the f weighted possibility mean of fuzzy number defined in Definition 2 of [10] (see pp. 365). Therefore, the f weighted possibility mean of fuzzy number defined in [10] is just a special case of that defined in this paper.

Obviously, $m_\mu(\tilde{a}, \theta)$ synthetically reflects the information on every membership degree, and $m_\mu(\tilde{a}, 0.5)$ may be regarded as a central value that represents from the membership function point of view. Likewise, $m_\nu(\tilde{a}, \theta)$ synthetically reflects the information on every non-membership degree, and $m_\nu(\tilde{a}, 0.5)$ may be regarded as a central value that represents from the non-membership function point of view.

Example 2.7. If f and g are chosen as follows:

$$f(\alpha) = 2\alpha/\omega_{\tilde{a}},\ (\alpha \in [0, \omega_{\tilde{a}}]), \tag{9}$$

and

$$g(\beta) = 2(1-\beta)/(1-u_{\tilde{a}}),\ (\beta \in [u_{\tilde{a}}, 1]), \tag{10}$$

respectively, then, according to the equations (3)-(6), we have

$$\underline{m}_\mu(\tilde{a}) = \tfrac{1}{3}(\underline{a} + 2a)\omega_{\tilde{a}}, \tag{11}$$

$$\bar{m}_\mu(\tilde{a}) = \tfrac{1}{3}(\bar{a} + 2a)\omega_{\tilde{a}}, \tag{12}$$

$$\underline{m}_\nu(\tilde{a}) = \tfrac{1}{3}(2a + \underline{a})(1 - u_{\tilde{a}}), \tag{13}$$

$$\bar{m}_\nu(\tilde{a}) = \tfrac{1}{3}(2a + \bar{a})(1 - u_{\tilde{a}}). \tag{14}$$

Further, by the equations (7) and (8), we have

$$m_\mu(\tilde{a}, \theta) = \tfrac{1}{3}[(1-\theta)(\underline{a} + 2a) + \theta(2a + \bar{a})]\omega_{\tilde{a}}, \tag{15}$$

$$m_\nu(\tilde{a}, \theta) = \tfrac{1}{3}[(1-\theta)(\underline{a} + 2a) + \theta(2a + \bar{a})](1 - u_{\tilde{a}}). \tag{16}$$

Remark 2.8. If a TrIFN $\tilde{a} = ((\underline{a}, a, \bar{a}); \omega_{\tilde{a}}, u_{\tilde{a}})$ degenerates to a TFN $\tilde{a} = (\underline{a}, a, \bar{a})$, it follows from the equations (11), (12) (or (13), (14)) and (15) (or (16) with $\theta = 0.5$ that the weighted lower possibility mean, weighted upper possibility mean and weighted possibility mean of the TFN $\tilde{a} = (\underline{a}, a, \bar{a})$ are obtained as follows: $M_*(\tilde{a}) = (\underline{a}+2a)/3$, $M^*(\tilde{a}) = (2a+\bar{a})/3$ and $\bar{M}(\tilde{a}) = (\underline{a}+4a+\bar{a})/6$ respectively.These results of TFN are the same as those of TFN in Examples 2.1 of [4].

The weighted possibility means have some useful properties outlined in Theorem 2.9.

Theorem 2.9. *Let* $\tilde{a}_i = ((\underline{a}_i, a_i, \bar{a}_i); \omega_{\tilde{a}_i}, u_{\tilde{a}_i})$ $(i = 1, 2)$ *be two TIFNs with* $\omega_{\tilde{a}_1} = \omega_{\tilde{a}_2}$ *and* $u_{\tilde{a}_1} = u_{\tilde{a}_2}$*. Then for* $\gamma > 0$ *any* $\tau > 0$*, the following equalities are valid:*

$$m_\mu(\gamma\tilde{a}_1 + \tau\tilde{a}_2, \theta) = \gamma m_\mu(\tilde{a}_1, \theta) + \tau m_\mu(\tilde{a}_2, \theta), \tag{17}$$

$$m_\nu(\gamma\tilde{a}_1 + \tau\tilde{a}_2, \theta) = \gamma m_\nu(\tilde{a}_1, \theta) + \tau m_\nu(\tilde{a}_2, \theta). \tag{18}$$

Proof. Since $\gamma > 0$ and $\tau > 0$., then by Definition 2.3, we get the α -cut set of TrIFN $\gamma\tilde{a} + \tau\tilde{b}$ as

$$(\gamma\tilde{a}_1 + \tau\tilde{a}_2)_\alpha = [\gamma a^l_{1\alpha} + \tau a^l_{2\alpha}, \gamma a^u_{1\alpha} + \tau a^u_{2\alpha}].$$

By the equation (7), $\omega_{\tilde{a}_1} = \omega_{\tilde{a}_2}$ and $u_{\tilde{a}_1} = u_{\tilde{a}_2}$, we obtain

$$\begin{aligned} m_\mu(\gamma\tilde{a}_1 + \tau\tilde{a}_2, \theta) &= (1-\theta)\underline{m}_\mu(\gamma\tilde{a}_1 + \tau\tilde{a}_2) + \theta\bar{m}_\mu(\gamma\tilde{a}_1 + \tau\tilde{a}_2) \\ &= (1-\theta)\int_0^{\omega_{\tilde{a}_1}\wedge\omega_{\tilde{a}_2}} f(\alpha)(\gamma a^l_{1\alpha} + \tau a^l_{2\alpha})\mathrm{d}\alpha \\ &+ \theta\int_0^{\omega_{\tilde{a}_1}\wedge\omega_{\tilde{a}_2}} f(\alpha)(\gamma a^u_{1\alpha} + \tau a^u_{2\alpha})\mathrm{d}\alpha \\ &= \gamma[(1-\theta)\underline{m}_\mu(\tilde{a}_1) + \theta\bar{m}_\mu(\tilde{a}_1)] + \tau[(1-\theta)\underline{m}_\mu(\tilde{a}_2) + \theta\bar{m}_\mu(\tilde{a}_2)] \\ &= \gamma m_\mu(\tilde{a}_1, \theta) + \tau m_\mu(\tilde{a}_2, \theta). \end{aligned}$$

Thus, the equation (17) holds. By the same way, the equation (18) can be proven. Namely, Theorem 2.9 is proven. □

Especially, if $\gamma = \tau = 1$, then by Theorem 2.9 the following equalities are valid:

$$m_\mu(\tilde{a}_1 + \tilde{a}_2, \theta) = m_\mu(\tilde{a}_1, \theta) + m_\mu(\tilde{a}_2, \theta), m_\nu(\tilde{a}_1 + \tilde{a}_2, \theta) = m_\nu(\tilde{a}_1, \theta) + m_\nu(\tilde{a}_2, \theta).$$

Remark 2.10. The weighting functions f and g can be chosen as several forms, for example,

$$f(\alpha) = (n+1)\alpha^n/(\omega_{\tilde{a}})^n, (\alpha \in [0, \omega_{\tilde{a}}]),$$

$$g(\beta) = (n+1)(1-\beta)^n/(1-u_{\tilde{a}})^n, (\beta \in [u_{\tilde{a}}, 1]),$$

where the power n is any positive integer, such as $n = 1, 2$, etc. These power forms of weighting functions are motivated by [10] (see Examples 1-3 in [10]). Hence, by introducing different f and g, we can give different (case-dependent) importance to α -cut set and β -cut set of TIFNs $\tilde{a}$.

In real-life application, the functions f and g can be selected according to the real need of the decision problems and the preferences of DMs. Hence, the weighted possibility mean of membership (non-membership) function not only reflects the information on every membership (non-membership) degree and represents a mean value of membership (non-membership) function, but also exhibits great flexibility and facility for different risk preferences of DMs. In what follows, the weighting functions f and g are respectively chosen as the equations (9) and (10) for computation convenience.

2.3. **A New Lexicographic Ranking Approach of TIFNs Based on Weighted Possibility Means.**

The possibility means of fuzzy numbers are similar to the mean of random variables. They can be used to quantitatively characterize the values of fuzzy numbers. Obviously, the greater the possibility means, the bigger the corresponding fuzzy number.

Let $m_\mu(\tilde{a}_i,\theta)$ and $m_\nu(\tilde{a}_i,\theta)$ be the weighted possibility means of the membership and non-membership functions for TIFNs $\tilde{a}_i = ((\underline{a}_i, a_i, \bar{a}_i); \omega_{\tilde{a}_i}, u_{\tilde{a}_i})$ $(i = 1,2)$, respectively. Thereby, a lexicographic approach to ranking two TIFNs $\tilde{a}_1$ and $\tilde{a}_2$ can be summarized as follows:

(1) If $m_\mu(\tilde{a}_1,\theta) < m_\mu(\tilde{a}_2,\theta)$, then $\tilde{a}_1$ is smaller than $\tilde{a}_2$, denoted by $\tilde{a}_1 < \tilde{a}_2$;
(2) If $m_\mu(\tilde{a}_1,\theta) > m_\mu(\tilde{a}_2,\theta)$, then $\tilde{a}_1$ is bigger than $\tilde{a}_2$, denoted by $\tilde{a}_1 > \tilde{a}_2$;
(3) If $m_\mu(\tilde{a}_1,\theta) = m_\mu(\tilde{a}_2,\theta)$, then
   (a) if $m_\nu(\tilde{a}_1,\theta) < m_\nu(\tilde{a}_2,\theta)$, then $\tilde{a}_1 < \tilde{a}_2$;
   (b) if $m_\nu(\tilde{a}_1,\theta) > m_\nu(\tilde{a}_2,\theta)$, then $\tilde{a}_1 > \tilde{a}_2$;
   (c) if $m_\nu(\tilde{a}_1,\theta) = m_\nu(\tilde{a}_2,\theta)$, then $\tilde{a}_1$ and $\tilde{a}_2$ represent the same information, denoted by $\tilde{a}_1 = \tilde{a}_2$.

Remark 2.11. Nan et al. [14] utilized the membership function average index $S_\mu(\tilde{a}) = \omega_{\tilde{a}}(\underline{a} + 2a + \bar{a})/4$ and the non-membership function average index $S_v(\tilde{a}) = (1 - u_{\tilde{a}})(\underline{a} + 2a + \bar{a})/4$ of TIFN $\tilde{a} = ((\underline{a}, a, \bar{a}); \omega_{\tilde{a}}, u_{\tilde{a}})$ to rank TIFNs. It is easily seen from the equations (15) and (16) that $S_\mu(\tilde{a})$ and $S_v(\tilde{a})$ are respectively equal to $m_\mu(\tilde{a},\theta)$ and $m_v(\tilde{a},\theta)$ if $\theta = (3\bar{a} - 2a - \underline{a})/[4(\bar{a} - \underline{a})]$. Therefore, the ranking approach [14] may be regarded as a special case of the proposed approach in this paper. In other words, this paper generalizes the ranking approach of [14].

Example 2.12. Consider two TIFNs $\tilde{a}_1 = ((3,6,8); 0.62, 0.22)$ and

$$\tilde{a}_2 = ((5,7,9); 0.52, 0.26).$$

By the equations (15) and (16), we obtain $\tilde{a}_1 > \tilde{a}_2$ if $\theta > 0.5686$; $\tilde{a}_1 < \tilde{a}_2$ if $\theta < 0.5686$. However, using the ranking approach of [14], we can calculate the average indexes as follows:

$S_\mu(\tilde{a}_1) = 3.617$, $S_v(\tilde{a}_1) = 4.550$, $S_\mu(\tilde{a}_2) = 3.640$, $S_v(\tilde{a}_2) = 5.180$.

Because $S_\mu(\tilde{a}_1) < S_\mu(\tilde{a}_2)$, the ranking order obtained by [14] is $\tilde{a}_1 < \tilde{a}_2$, which is the same as the ranking order of the case $\theta < 0.5686$. Namely, the ranking result obtained by [14] is just a special case of that obtained by this paper.

Example 2.12 implies that when chosen different values of risk preference parameter θ we can get different ranking orders. The new ranking approach of TIFNs proposed in this paper sufficiently takes into the risk preference of DM consideration, which can make the ranking result more reasonable. In contrast, the ranking approach [14] failed to consider the risk preference of DM.

To analyze the effect of risk preference parameter θ on the raking of TIFNs, we make sensitivity analyses for $m_\mu(\tilde{a},\theta)$ and $m_v(\tilde{a},\theta)$ with respect to θ, respectively.

Theorem 2.13. *Let $\Delta\theta$ be a perturbation of the risk preference parameter θ with $0 \le \theta+\Delta\theta \le 1$. If $m_\mu(\tilde{a}_1,\theta) \le m_\mu(\tilde{a}_2,\theta)$, then $m_\mu(\tilde{a}_1,\theta+\Delta\theta) \le m_\mu(\tilde{a}_2,\theta+\Delta\theta)$ if and only if*

$$\begin{cases} \max\{(m_\mu(\tilde{a}_1,\theta)-m_\mu(\tilde{a}_2,\theta))/(\eta_2-\eta_1),-\theta\} \le \Delta\theta \le 1-\theta, & \eta_2>\eta_1 \\ -\theta \le \Delta\theta \le 1-\theta, & \eta_2=\eta_1 \\ -\theta \le \Delta\theta \le \min\{(m_\mu(\tilde{a}_1,\theta)-m_\mu(\tilde{a}_2,\theta))/(\eta_2-\eta_1),1-\theta\}, & \eta_2<\eta_1 \end{cases}$$

where $\eta_i = \frac{1}{3}(\bar{a}_i - \underline{a}_i)\omega_{\tilde{a}_i} \quad (i=1,2)$.

Proof. Let $m_\mu(\tilde{a}_1,\theta+\Delta\theta) \le m_\mu(\tilde{a}_2,\theta+\Delta\theta)$, then by the equation (15) we get

$$\begin{aligned} &\tfrac{1}{3}[(1-\theta-\Delta\theta)(\underline{a}_1+2a_1)+(\theta+\Delta\theta)(2a_1+\bar{a}_1)]\omega_{\tilde{a}_1} \\ &\le \tfrac{1}{3}[(1-\theta-\Delta\theta)(\underline{a}_2+2a_2)+(\theta+\Delta\theta)(2a_2+\bar{a}_2)]\omega_{\tilde{a}_2}. \end{aligned}$$

Namely, $m_\mu(\tilde{a}_1,\theta)-m_\mu(\tilde{a}_2,\theta) \le (\eta_2-\eta_1)\Delta\theta$. Since $0\le\theta\le 1$ and $0\le\theta+\Delta\theta\le 1$, we obtain $-\theta \le \Delta\theta \le 1-\theta$. Then, if $\eta_2>\eta_1$, we have $\Delta\theta \ge (m_v(\tilde{a}_1,\theta)-m_v(\tilde{a}_2,\theta))/(\eta_2-\eta_1)$.
Thus,

$$\max\{(m_v(\tilde{a}_1,\theta)-m_v(\tilde{a}_2,\theta))/(\eta_2-\eta_1),-\theta\} \le \Delta\theta \le 1-\theta;$$

if $\eta_2<\eta_1$, we have$\Delta\theta \le (m_v(\tilde{a}_1,\theta)-m_v(\tilde{a}_2,\theta))/(\eta_2-\eta_1)$.
Thus,

$$-\theta \le \Delta\theta \le \min\{(m_v(\tilde{a}_1,\theta)-m_v(\tilde{a}_2,\theta))/(\eta_2-\eta_1),1-\theta\};$$

if $\eta_2=\eta_1$, we have $-\theta \le \Delta\theta \le 1-\theta$.
This completes the proof of Theorem 2.13. □

By the same way, we have the following Theorem 2.14.

Theorem 2.14. *Let $\Delta\theta$ be a perturbation of the risk preference parameter θ with $0 \le \theta+\Delta\theta \le 1$. If $m_v(\tilde{a}_1,\theta) \le m_v(\tilde{a}_2,\theta)$, then $m_v(\tilde{a}_1,\theta+\Delta\theta) \le m_v(\tilde{a}_2,\theta+\Delta\theta)$ if and only if*

$$\begin{cases} \max\{(m_v(\tilde{a}_1,\theta)-m_v(\tilde{a}_2,\theta))/\quad(\eta_2-\eta_1),-\theta\} \le \Delta\theta \le 1-\theta, & \xi_2>\xi_1 \\ -\theta \le \Delta\theta \le 1-\theta, & \xi_2=\xi_1 \\ -\theta \le \Delta\theta \le \min\{(m_v(\tilde{a}_1,\theta)-m_v(\tilde{a}_2,\theta))/(\eta_2-\eta_1),1-\theta\}, & \xi_2<\xi_1 \end{cases}$$

where $\xi_i = \frac{1}{3}(\bar{a}_i-\underline{a}_i)(1-u_{\tilde{a}_i})\ (i=1,2)$.

3. Triangular Intuitionistic Fuzzy Triple Bonferroni Harmonic Mean Operators

HM is the reciprocal of arithmetic mean of reciprocal, which is a conservative average to be used to provide for aggregation lying between the max and min operators, and is widely used as a tool to aggregate central tendency data [35]. In this section, we develop three kinds of triangular intuitionistic fuzzy triple Bonferroni harmonic mean operators.

3.1. Related Bonferroni Mean Operators and Bonferroni Harmonic Mean Operators.

The BM was originally introduced by Bonferroni [3], which was defined as follows:

Definition 3.1. [3] Let $p, q \geq 0$ and a_i $(i = 1, 2, \cdots, n)$ be a set of nonnegative real numbers. If

$$BM^{p,q}(a_1, a_2, ..., a_n) = \frac{1}{n(n-1)} \sum_{i,j=1, i \neq j}^{n} a_i^p a_j^q {}^{\frac{1}{p+q}}, \tag{19}$$

then $\mathrm{BM}^{p,q}$ is called the Bonferroni mean (BM) operator.

Definition 3.2. [3, 2] Let $p, q, r \geq 0$ and a_i $(i = 1, 2, \cdots, n)$ be a set of nonnegative numbers. If

$$\mathrm{TBM}^{p,q,r}(a_1, a_2, \cdots, a_n) = \frac{1}{n(n-1)(n-2)} \sum_{i,j,k=1, i \neq j \neq k}^{n} a_i^p a_j^q a_k^r {}^{\frac{1}{p+q+r}}, \tag{20}$$

then $\mathrm{TBM}^{p,q,r}$ is called the Triple Bonferroni mean (TBM) operator.

It is noted that both BM and TBM operators do not consider the weight vector of the aggregated arguments. To overcome this drawback, Xia et al. [32] defined a generalized weighted BM operator.

Definition 3.3. [32] Let $p, q, r \geq 0$ and a_i $(i = 1, 2, \cdots, n)$ be a set of nonnegative numbers with the weight vector $\mathbf{w} = (w_1, w_2, \cdots, w_n)^{\mathrm{T}}$ such that $w_i > 0$ $(i = 1, 2, \cdots, n)$ and $\sum_{i=1}^{n} w_i = 1$. If

$$\mathrm{GWBM}^{p,q,r}(a_1, a_2, \cdots, a_n) = \Big(\sum_{i,j,k=1, i \neq j \neq k}^{n} w_i w_j w_k a_i^p a_j^q a_k^r \Big)^{\frac{1}{p+q+r}}, \tag{21}$$

then $\mathrm{GWBM}^{p,q,r}$ is called the generalized weighted Bonferroni mean (GWBM) operator.

Xia et al. [32] also extended the GWBM operator for real numbers to suit the case for the intuitionistic fuzzy sets. To aggregate the triangular fuzzy correlated information, based on the BM and weighted HM operators, Sun and Sun [17] developed the Bonferroni harmonic mean operator for TFNs which is called fuzzy weighted Bonferroni harmonic mean (FWBHM) operator. This operator considers the weight vector of the aggregated arguments.

Definition 3.4. [17] Let a_i $(i = 1, 2, \cdots, n)$ be a collection of TFNs with the weight vector $\mathbf{w} = (w_1, w_2, \cdots, w_n)^{\mathrm{T}}$ such that $w_i > 0$ $(i = 1, 2, \cdots, n)$ and $\sum_{i=1}^{n} w_i = 1$. If

$$\mathrm{FWBHM}^{p,q}(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \frac{1}{\left(\sum_{i,j=1}^{n} \frac{w_i w_j}{\tilde{a}_i^p \tilde{a}_j^q}\right)^{\frac{1}{p+q}}} = \left(\frac{1}{\left(\sum_{i,j=1}^{n} \frac{w_i w_j}{\underline{a}_i^p \underline{a}_j^q}\right)^{\frac{1}{p+q}}}, \frac{1}{\left(\sum_{i,j=1}^{n} \frac{w_i w_j}{a_i^p a_j^q}\right)^{\frac{1}{p+q}}}, \frac{1}{\left(\sum_{i,j=1}^{n} \frac{w_i w_j}{\bar{a}_i^p \bar{a}_j^q}\right)^{\frac{1}{p+q}}}\right), \quad (22)$$

where $p, q > 0$, then $\mathrm{FWBHM}^{p,q}$ is called the triangular fuzzy weight Bonferroni harmonic mean (TFWBHM) operator.

3.2. **Triangular Intuitionistic Fuzzy Triple Weighted Bonferroni Harmonic Mean Operator.**

Based on the GBM and TFWBHM operators, we give the definition of the TIFTWBHM operator.

Definition 3.5. Let $\tilde{a}_i = ((\underline{a}_i, a_i, \bar{a}_i); \omega_{\tilde{a}_i}, v_{\tilde{a}_i})$ $(i = 1, 2, \cdots, n)$ be a collection of TIFNs with the weight vector $\mathbf{w} = (w_1, w_2, \cdots, w_n)^{\mathrm{T}}$ such that $w_i > 0$ $(i = 1, 2, \cdots, n)$ and $\sum_{i=1}^{n} w_i = 1$. If

$$\mathrm{TIFTWBHM}^{p,q,r}(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \frac{1}{\left(\sum_{i,j,k=1, i\neq j\neq k}^{n} \frac{w_i w_j w_k}{\tilde{a}_i^p \tilde{a}_j^q \tilde{a}_k^r}\right)^{\frac{1}{p+q+r}}}$$

where $p, q > 0$, then $\mathrm{TIFTWBHM}^{p,q,r}$ is called the TIFTWBHM operator.

Remark 3.6. The term "generalized" used by Xia et al. [32] is different from that used by Beliakov et al. [2], we think that the term "triple" originated by Bonferroni [3] is more suitable than the term "generalized". Thus, we use the term "triple" in this paper.

Theorem 3.7. *Let $\tilde{a}_i = ((\underline{a}_i, a_i, \bar{a}_i); \omega_{\tilde{a}_i}, v_{\tilde{a}_i})$ $(i = 1, 2, \cdots, n)$ be a collection of TIFNs. Their aggregation result by TIFTWBHM operator is also a TIFN and*

$$\mathrm{TIFTWBHM}^{p,q,r}(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n)$$
$$= \left(\left(\frac{1}{\left(\sum_{i,j,k=1, i\neq j\neq k}^{n} \frac{w_i w_j w_k}{\underline{a}_i^p \underline{a}_j^q \underline{a}_k^r}\right)^{\frac{1}{p+q+r}}}, \frac{1}{\left(\sum_{i,j,k=1, i\neq j\neq k}^{n} \frac{w_i w_j w_k}{a_i^p a_j^q a_k^r}\right)^{\frac{1}{p+q+r}}}, \frac{1}{\left(\sum_{i,j,k=1, i\neq j\neq k}^{n} \frac{w_i w_j w_k}{\bar{a}_i^p \bar{a}_j^q \bar{a}_k^r}\right)^{\frac{1}{p+q+r}}}\right); \min_{1\leq i\leq n}\{\omega_{\tilde{a}_i}\}, \max_{1\leq i\leq n}\{u_{\tilde{a}_i}\}\right). \quad (23)$$

Proof. By Definition 2.2, we have

$$\frac{w_i w_j w_k}{\tilde{a}_i^p \tilde{a}_j^q \tilde{a}_k^r} = \left(\left(\frac{w_i w_j w_k}{\bar{a}_i^p \bar{a}_j^q \bar{a}_k^r}, \frac{w_i w_j w_k}{a_i^p a_j^q a_k^r}, \frac{w_i w_j w_k}{\underline{a}_i^p \underline{a}_j^q \underline{a}_k^r}\right); \min_{t=i,j,k}\{\omega_{\tilde{a}_t}\}, \max_{t=i,j,k}\{u_{\tilde{a}_t}\}\right),$$

$$\sum_{i,j,k=1, i\neq j\neq k}^{n} \frac{w_i w_j w_k}{\tilde{a}_i^p \tilde{a}_j^q \tilde{a}_k^r} = \left(\left(\sum_{i,j,k=1, i\neq j\neq k}^{n} \frac{w_i w_j w_k}{\bar{a}_i^p \bar{a}_j^q \bar{a}_k^r}, \sum_{i,j,k=1, i\neq j\neq k}^{n} \frac{w_i w_j w_k}{a_i^p a_j^q a_k^r}, \sum_{i,j,k=1, i\neq j\neq k}^{n} \frac{w_i w_j w_k}{\underline{a}_i^p \underline{a}_j^q \underline{a}_k^r}\right); \min_{1\leq i\leq n}\{\omega_{\tilde{a}_i}\}, \max_{1\leq i\leq n}\{u_{\tilde{a}_i}\}\right),$$

$$(\sum_{i,j,k=1,i\neq j\neq k}^{n}\frac{w_iw_jw_k}{\tilde{a}_i^p\tilde{a}_j^q\tilde{a}_k^r})^{\frac{1}{p+q+r}}=((\sum_{i,j,k=1,i\neq j\neq k}^{n}\frac{w_iw_jw_k}{\bar{a}_i^p\bar{a}_j^q\bar{a}_k^r})^{\frac{1}{p+q+r}},(\sum_{i,j,k=1,i\neq j\neq k}^{n}\frac{w_iw_jw_k}{a_i^pa_j^qa_k^r})^{\frac{1}{p+q+r}},$$
$$(\sum_{i,j,k=1,i\neq j\neq k}^{n}\frac{w_iw_jw_k}{\underline{a}_i^p\underline{a}_j^q\underline{a}_k^r})^{\frac{1}{p+q+r}});\min_{1\leq i\leq n}\{\omega_{\tilde{a}_i}\},\max_{1\leq i\leq n}\{u_{\tilde{a}_i}\})$$

Thus,
$$\frac{1}{(\sum_{i,j,k=1,i\neq j\neq k}^{n}\frac{w_iw_jw_k}{\tilde{a}_i^p\tilde{a}_j^q\tilde{a}_k^r})^{\frac{1}{p+q+r}}}$$
$$=((\frac{1}{(\sum_{i,j,k=1,i\neq j\neq k}^{n}\frac{w_iw_jw_k}{\underline{a}_i^p\underline{a}_j^q\underline{a}_k^r})^{\frac{1}{p+q+r}}},\frac{1}{(\sum_{i,j,k=1,i\neq j\neq k}^{n}\frac{w_iw_jw_k}{a_i^pa_j^qa_k^r})^{\frac{1}{p+q+r}}},$$
$$\frac{1}{(\sum_{i,j,k=1,i\neq j\neq k}^{n}\frac{w_iw_jw_k}{\bar{a}_i^p\bar{a}_j^q\bar{a}_k^r})^{\frac{1}{p+q+r}}});\min_{1\leq i\leq n}\{\omega_{\tilde{a}_i}\},\max_{1\leq i\leq n}\{u_{\tilde{a}_i}\}),$$

which completes the proof of Theorem 3.7. □

The TIFTWBHM operator has some desirable properties, such as idempotency, boundedness, commutativity, monotonicity, etc.

Theorem 3.8. *1) Idempotency: If all* $\tilde{a}_i$ $(j=1,2,\cdots,n)$ *are equal, i.e.,* $\tilde{a}_j=\tilde{a}$, *then*

$$\text{TIFTWBHM}^{p,q,r}(\tilde{a}_1,\tilde{a}_1\cdots,\tilde{a}_n)=\tilde{a}.$$

2) Boundedness: Let

$$\tilde{a}^+=((\max_{1\leq i\leq n}\{\underline{a}_i\},\max_{1\leq i\leq n}\{a_i\},\max_{1\leq i\leq n}\{\bar{a}_i\});\max_{1\leq i\leq n}\{\omega_{\tilde{a}_i}\},\min_{1\leq i\leq n}\{u_{\tilde{a}_i}\}),$$

$$\tilde{a}^-=((\min_{1\leq i\leq n}\{\underline{a}_i\},\min_{1\leq i\leq n}\{a_i\},\min_{1\leq i\leq n}\{\bar{a}_i\});\min_{1\leq i\leq n}\{\omega_{\tilde{a}_i}\},\max_{1\leq i\leq n}\{u_{\tilde{a}_i}\}),$$

then

$$a^-\leq\text{TIFTWBHM}^{p,q,r}(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)\leq a^+. \tag{24}$$

3) Commutativity: Let $(\dot{\tilde{a}}_1,\dot{\tilde{a}}_2,\cdots,\dot{\tilde{a}}_n)$ *be any permutation of* $(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)$, *then*

$$\text{TIFTWBHM}^{p,q,r}(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)=\text{TIFTWBHM}^{p,q,r}(\dot{\tilde{a}}_1,\dot{\tilde{a}}_2,\cdots,\dot{\tilde{a}}_n). \tag{25}$$

Theorem 3.9. *(Monotonicity)Let* $\tilde{a}_i=((\underline{a}_i,a_i,\bar{a}_i);\omega_{\tilde{a}_i},u_{\tilde{a}_i})$*and*$\tilde{b}_i=((\underline{b}_i\ b_i,\bar{b}_i);\omega_{\tilde{b}_i},u_{\tilde{b}_i})$ $(i=1,2,\cdots,n)$ *be two collections of* $TIFNs$, *if*

$$\underline{a}_i\leq\underline{b}_i,\quad a_i\leq b_i,\quad \bar{a}_i\leq\bar{b}_i,\quad \omega_{\tilde{a}_i}\leq\omega_{\tilde{b}_i},\quad u_{\tilde{a}_i}\geq u_{\tilde{b}_i}, \tag{26}$$

then,

$$\text{TIFTWBHM}^{p,q,r}(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)\leq\text{TIFTWBHM}^{p,q,r}(\tilde{b}_1,\tilde{b}_2,\cdots,\tilde{b}_n). \tag{27}$$

Proof. By the equation (26), for all i, j, k, we have

$$\underline{a}_i^p\underline{a}_j^q\underline{a}_k^r \leq \underline{b}_i^p\underline{b}_j^q\underline{b}_k^r, a_i^p a_j^q a_k^r \leq b_i^p b_j^q b_k^r, \bar{a}_i^p\bar{a}_j^q\bar{a}_k^r \leq \bar{b}_i^p\bar{b}_j^q\bar{b}_k^r,$$

$$(\sum_{i,j,k=1,i\neq j\neq k}^{n} \frac{w_i w_j w_k}{\underline{a}_i^p\underline{a}_j^q\underline{a}_k^r})^{-\frac{1}{p+q+r}} \leq (\sum_{i,j,k=1,i\neq j\neq k}^{n} \frac{w_i w_j w_k}{\underline{b}_i^p\underline{b}_j^q\underline{b}_k^r})^{-\frac{1}{p+q+r}},$$

$$(\sum_{i,j,k=1,i\neq j\neq k}^{n} \frac{w_i w_j w_k}{a_i^p a_j^q a_k^r})^{-\frac{1}{p+q+r}} \leq (\sum_{i,j,k=1}^{n} \frac{w_i w_j w_k}{b_i^p b_j^q b_k^r})^{-\frac{1}{p+q+r}},$$

$$(\sum_{i,j,k=1,i\neq j\neq k}^{n} \frac{w_i w_j w_k}{\bar{a}_i^p\bar{a}_j^q\bar{a}_k^r})^{-\frac{1}{p+q+r}} \leq (\sum_{i,j,k=1,i\neq j\neq k}^{n} \frac{w_i w_j w_k}{\bar{b}_i^p\bar{b}_j^q\bar{b}_k^r})^{-\frac{1}{p+q+r}},$$

$$\min_{1\leq i\leq n}\{\omega_{\tilde{a}_i}\} \leq \min_{1\leq i\leq n}\{\omega_{\tilde{b}_i}\},$$

$$\max_{1\leq i\leq n}\{u_{\tilde{a}_i}\} \geq \max_{1\leq i\leq n}\{u_{\tilde{b}_i}\},$$

$$1 - \max_{1\leq i\leq n}\{u_{\tilde{a}_i}\} \leq 1 - \max_{1\leq i\leq n}\{u_{\tilde{b}_i}\}.$$

Let

$$\text{TIFTWBHM}^{p,q,r}(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \tilde{a} = ((\underline{a}, a, \bar{a}); \omega_{\tilde{a}}, u_{\tilde{a}}),$$

$$\text{TIFTWBHM}^{p,q,r}(\tilde{b}_1, \tilde{b}_2, \cdots, \tilde{b}_n) = \tilde{b} = ((\underline{b}, b, \bar{b}); \omega_{\tilde{b}}, u_{\tilde{b}}),$$

$$\underline{a} = (\sum_{i,j,k=1,i\neq j\neq k}^{n} \frac{w_i w_j w_k}{\underline{a}_i^p\underline{a}_j^q\underline{a}_k^r})^{-\frac{1}{p+q+r}}, a = (\sum_{i,j,k=1,i\neq j\neq k}^{n} \frac{w_i w_j w_k}{a_i^p a_j^q a_k^r})^{-\frac{1}{p+q+r}},$$

$$\bar{a} = (\sum_{i,j,k=1,i\neq j\neq k}^{n} \frac{w_i w_j w_k}{\bar{a}_i^p\bar{a}_j^q\bar{a}_k^r})^{-\frac{1}{p+q+r}},$$

$$\omega_{\tilde{a}} = \min_{1\leq i\leq n}\{\omega_{\tilde{a}_i}\}, u_{\tilde{a}} = \max_{1\leq i\leq n}\{u_{\tilde{a}_i}\},$$

$$\underline{b} = (\sum_{i,j,k=1,i\neq j\neq k}^{n} \frac{w_i w_j w_k}{\underline{b}_i^p\underline{b}_j^q\underline{b}_k^r})^{-\frac{1}{p+q+r}}, b = (\sum_{i,j,k=1,i\neq j\neq k}^{n} \frac{w_i w_j w_k}{b_i^p b_j^q b_k^r})^{-\frac{1}{p+q+r}},$$

$$\bar{b} = (\sum_{i,j,k=1,i\neq j\neq k}^{n} \frac{w_i w_j w_k}{\bar{b}_i^p\bar{b}_j^q\bar{b}_k^r})^{-\frac{1}{p+q+r}},$$

$$\omega_{\tilde{b}} = \min_{1\leq i\leq n}\{\omega_{\tilde{b}_i}\}, u_{\tilde{b}} = \max_{1\leq i\leq n}\{u_{\tilde{b}_i}\}.$$

Thus,

$$\tfrac{1}{3}[(1-\theta)(\underline{a}+2a)+\theta(2a+\bar{a})]\omega_{\tilde{a}} \leq \tfrac{1}{3}[(1-\theta)(\underline{b}+2b+\theta(2b+\bar{b})]\omega_{\tilde{b}},$$

$$\tfrac{1}{3}[(1-\theta)(\underline{a}+2a)+\theta(2a+\bar{a})](1-u_{\tilde{a}}) \leq \tfrac{1}{3}[(1-\theta)(\underline{b}+2b+\theta(2b+\bar{b})](1-u_{\tilde{b}}).$$

In terms of the equations (15) and (16), we get

$$m_\mu(\text{TIFTWBHM}^{p,q,r}(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n), \theta) \leq m_\mu(\text{TIFTWBHM}^{p,q,r}(\tilde{b}_1, \tilde{b}_2, \cdots, \tilde{b}_n), \theta),$$

$$m_v(\text{TIFTWBHM}^{p,q,r}(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n), \theta) \leq m_v(\text{TIFTWBHM}^{p,q,r}(\tilde{b}_1, \tilde{b}_2, \cdots, \tilde{b}_n), \theta).$$

According to the lexicographic ranking approach of TIFNs in subsection 3.2, the equation (27) holds. □

Theorem 3.10. *Let $\tilde{a}_i$ $(i = 1,2,\cdots,n)$ be a collection of TIFNs, $k > 0$, then*

$$\text{TIFTWBHM}^{p,q,r}(k\tilde{a}_1, k\tilde{a}_2, \cdots, k\tilde{a}_n) = k\text{TIFTWBHM}^{p,q,r}(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n).$$

Theorem 3.11. *Let $\tilde{b}$ be a TIFN and $\tilde{a}_i$ $(i = 1,2,\cdots,n)$ be a collection of TIFNs, then*

$$\text{TIFTWBHM}^{p,q,r}(\tilde{a}_1\tilde{b}, \tilde{a}_2\tilde{b}, \cdots, \tilde{a}_n\tilde{b}) = \text{TIFTWBHM}^{p,q,r}(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n)\tilde{b}.$$

Proof. By Definition 3.5, we have

$$\text{TIFTWBHM}^{p,q,r}(\tilde{a}_1\tilde{b}, \tilde{a}_2\tilde{b}, \cdots, \tilde{a}_n\tilde{b}) = ((\frac{1}{(\sum\limits_{i,j,k=1, i\neq j\neq k}^{n} \frac{w_i w_j w_k}{\underline{a}_i^p \underline{b}^p \underline{a}_j^q \underline{b}^q \underline{a}_k^r \underline{b}^r})^{\frac{1}{p+q+r}}},$$

$$\frac{1}{(\sum\limits_{i,j,k=1, i\neq j\neq k}^{n} \frac{w_i w_j w_k}{a_i^p b^p a_j^q b^q a_k^r b^r})^{\frac{1}{p+q+r}}}, \frac{1}{(\sum\limits_{i,j,k=1, i\neq j\neq k}^{n} \frac{w_i w_j w_k}{\bar{a}_i^p \bar{b}^p \bar{a}_j^q \bar{b}^q \bar{a}_k^r \bar{b}^r})^{\frac{1}{p+q+r}}});$$

$$\min_{1\leq i\leq n}\{\omega_{\tilde{a}_i}\} \wedge \omega_{\tilde{b}}, \max_{1\leq i\leq n}\{u_{\tilde{a}_i}\} \vee u_{\tilde{b}})$$

$$= ((\frac{\underline{b}}{(\sum\limits_{i,j,k=1, i\neq j\neq k}^{n} \frac{w_i w_j w_k}{\underline{a}_i^p \underline{a}_j^q \underline{a}_k^r})^{\frac{1}{p+q+r}}}, \frac{b}{(\sum\limits_{i,j,k=1, i\neq j\neq k}^{n} \frac{w_i w_j w_k}{a_i^p a_j^q a_k^r})^{\frac{1}{p+q+r}}}, \frac{\bar{b}}{(\sum\limits_{i,j,k=1, i\neq j\neq k}^{n} \frac{w_i w_j w_k}{\bar{a}_i^p \bar{a}_j^q \bar{a}_k^r})^{\frac{1}{p+q+r}}});$$
$$\min_{1\leq i\leq n}\{\omega_{\tilde{a}_i}\} \wedge \omega_{\tilde{b}}, \max_{1\leq i\leq n}\{u_{\tilde{a}_i}\} \vee u_{\tilde{b}})$$
$$= TIFTWBHM^{p,q,r}(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n)\tilde{b}.$$

Hence, the proof of Theorem 3.11 is completed. □

3.3. **Triangular Intuitionistic Fuzzy Triple Ordered Weighted Bonferroni Harmonic Mean.**

Chen et al. [5] developed an ordered weighted HM (OWHM) operator for real numbers.

Definition 3.12. [5] An OWHM operator of dimension n is a mapping OWHM : $R^n \to R$, and $\text{OWHM}_w(a_1, a_2, \cdots, a_n) = (\sum\limits_{i=1}^{n} \frac{w_i}{a_{\sigma(i)}})^{-1}$, where $\mathbf{w} = (w_1, w_2, \cdots, w_n)^{\text{T}}$ is the weight vector associated with OWHM satisfying that $w_i > 0$ $(i = 1, 2, \cdots, n)$ and $\sum\limits_{i=1}^{n} w_i = 1$, $(\sigma(1), \sigma(2), \cdots, \sigma(n))$ is a permutation of $(1, 2, \cdots, n)$ such that $a_{\sigma(i-1)} \geq a_{\sigma(i)}$ for all $i = 2, 3, \cdots, n$.

Xu [35] defined a fuzzy ordered Bonferroni harmonic mean operator for TFNs. To suit the case of TIFNs, we define the TIFTOWBHM operator as follows:

Definition 3.13. Let $\tilde{a}_i = ((\underline{a}_i, a_i, \bar{a}_i); \omega_{\tilde{a}_i}, u_{\tilde{a}_i})$ $(i = 1, 2, \cdots, n)$ The triple ordered weighted Bonferroni harmonic mean operator of dimension n is defined as

$$\text{TIFTOWBHM}^{p,q,r}(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \frac{1}{(\sum\limits_{i,j,k=1, i\neq j\neq k}^{n} \frac{w_i w_j w_k}{\tilde{a}_{\sigma(i)}^p \tilde{a}_{\sigma(j)}^q \tilde{a}_{\sigma(k)}^r})^{\frac{1}{p+q+r}}}$$

$$=((\frac{1}{(\sum\limits_{i,j,k=1,i\neq j\neq k}^{n}\frac{w_iw_jw_k}{\underline{a}_{\sigma(i)}^{p}\underline{a}_{\sigma(j)}^{q}\underline{a}_{\sigma(k)}^{r}})^{\frac{1}{p+q+r}}},\frac{1}{(\sum\limits_{i,j,k=1,i\neq j\neq k}^{n}\frac{w_iw_jw_k}{a_{\sigma(i)}^{p}a_{\sigma(j)}^{q}a_{\sigma(k)}^{r}})^{\frac{1}{p+q+r}}},$$
$$\frac{1}{(\sum\limits_{i,j,k=1,i\neq j\neq k}^{n}\frac{w_iw_jw_k}{\bar{a}_{\sigma(i)}^{p}\bar{a}_{\sigma(j)}^{q}\bar{a}_{\sigma(k)}^{r}})^{\frac{1}{p+q+r}}});$$
$$\min_{1\leq i\leq n}\{\omega_{\tilde{a}_i}\},\max_{1\leq i\leq n}\{u_{\tilde{a}_i}\}),\tag{28}$$

where $\mathbf{w}=(w_1,w_2,\cdots,w_n)^{\mathrm{T}}$ is the weight vector associated with, satisfying that $w_i>0$ $(i=1,2,\cdots,n)$ and $\sum\limits_{i=1}^{n}w_i=1$, $p,q,r\geq 0$, $(\sigma(1),\sigma(2),\cdots,\sigma(n))$ is a permutation of $(1,2,\cdots,n)$ such that $\tilde{a}_{\sigma(i-1)}\geq\tilde{a}_{\sigma(i)}$ for all $i=2,3,\cdots,n$.

Especially, if $r=0$, then the TIFTOWBHM operator is reduced to

$$\mathrm{TIFTOWBHM}^{p,q,0}(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)=((\frac{1}{(\sum\limits_{i,j=1}^{n}\frac{w_iw_j}{\underline{a}_{\sigma(i)}^{p}\underline{a}_{\sigma(j)}^{q}})^{\frac{1}{p+q}}},$$
$$\frac{1}{(\sum\limits_{i,j=1}^{n}\frac{w_iw_j}{a_{\sigma(i)}^{p}a_{\sigma(j)}^{q}})^{\frac{1}{p+q}}},\frac{1}{(\sum\limits_{i,j=1}^{n}\frac{w_iw_j}{\bar{a}_{\sigma(i)}^{p}\bar{a}_{\sigma(j)}^{q}})^{\frac{1}{p+q}}});$$
$$\min_{1\leq i\leq n}\{\omega_{\tilde{a}_i}\},\max_{1\leq i\leq n}\{u_{\tilde{a}_i}\}),\tag{29}$$

which is called the triangular intuitionistic fuzzy ordered weighted Bonferroni harmonic mean (TIFOWBHM) operator. If $r=q=0$ and $p=1$, then the TIFTOWBHM operator is reduced to

$$\mathrm{TIFTOWBHM}^{p,0,0}(\tilde{a}_1,\tilde{a}_2,\cdots,\tilde{a}_n)=\frac{1}{(\sum\limits_{i,j,k=1}^{n}\frac{w_iw_jw_k}{\tilde{a}_{\sigma(i)}^{p}\tilde{a}_{\sigma(j)}^{0}\tilde{a}_{\sigma(k)}^{0}})^{\frac{1}{p}}}$$
$$=((\frac{1}{(\sum\limits_{i=1}^{n}\frac{w_i}{\underline{a}_{\sigma(i)}^{p}})^{\frac{1}{p}}},\frac{1}{(\sum\limits_{i=1}^{n}\frac{w_i}{a_{\sigma(i)}^{p}})^{\frac{1}{p}}},\frac{1}{(\sum\limits_{i=1}^{n}\frac{w_i}{\bar{a}_{\sigma(i)}^{p}})^{\frac{1}{p}}});\min_{1\leq i\leq n}\{\omega_{\tilde{a}_i}\},\max_{1\leq i\leq n}\{u_{\tilde{a}_i}\}),\tag{30}$$

which is called the triangular intuitionistic fuzzy ordered weighted harmonic (TIFOWH) operator.

If TIFNs $\tilde{a}_i=((\underline{a}_i,a_i,\bar{a}_i);\omega_{\tilde{a}_i},u_{\tilde{a}_i})$ are reduced to TFNs $\widehat{a}_i=(\underline{a}_i,a_i,\bar{a}_i)$ $(i=1,2,\cdots,n)$, then:

(i) The TIFTOWBHM (i.e., the equation (28)) becomes

$$\mathrm{TIFTOWBHM}^{p,q,r}(\widehat{a}_1,\widehat{a}_2,\cdots,\widehat{a}_n)=(\frac{1}{(\sum\limits_{i,j,k=1,i\neq j\neq k}^{n}\frac{w_iw_jw_k}{\underline{a}_{\sigma(i)}^{p}\underline{a}_{\sigma(j)}^{q}\underline{a}_{\sigma(k)}^{r}})^{\frac{1}{p+q+r}}},$$
$$\frac{1}{(\sum\limits_{i,j,k=1,i\neq j\neq k}^{n}\frac{w_iw_jw_k}{a_{\sigma(i)}^{p}a_{\sigma(j)}^{q}a_{\sigma(k)}^{r}})^{\frac{1}{p+q+r}}},\frac{1}{(\sum\limits_{i,j,k=1,i\neq j\neq k}^{n}\frac{w_iw_jw_k}{\bar{a}_{\sigma(i)}^{p}\bar{a}_{\sigma(j)}^{q}\bar{a}_{\sigma(k)}^{r}})^{\frac{1}{p+q+r}}}).$$

Here we call it the TFTOWBHM operator which is just the GFOWBHM operator defined in [15].

(ii) The TIFOWBHM operator (i.e., the equation (29)) becomes

$$\mathrm{TIFOWBHM}^{p,q,0}(\widehat{a}_1,\widehat{a}_2,\cdots,\widehat{a}_n)=\frac{1}{(\sum\limits_{i,j=1}^{n}\frac{w_iw_j}{\widehat{a}_i^{p}\widehat{a}_j^{q}})^{\frac{1}{p+q}}}$$

$$= (\frac{1}{(\sum_{i,j=1}^{n} \frac{w_i w_j}{\underline{a}^p_{\sigma(i)} \underline{a}^q_{\sigma(j)}})^{\frac{1}{p+q}}}, \frac{1}{(\sum_{i,j=1}^{n} \frac{w_i w_j}{a^p_{\sigma(i)} a^q_{\sigma(j)}})^{\frac{1}{p+q}}}, \frac{1}{(\sum_{i,j=1}^{n} \frac{w_i w_j}{\bar{a}^p_{\sigma(i)} \bar{a}^q_{\sigma(j)}})^{\frac{1}{p+q}}}),$$

which is just the FOBHM operator defined in [17].

(iii) The TIFOWH operator (i.e., the equation (30)) becomes

$$\text{TIFOWH}(\widehat{a}_1, \widehat{a}_2, \cdots, \widehat{a}_n) = (\frac{1}{(\sum_{i=1}^{n} \frac{w_i}{\underline{a}^p_{\sigma(i)}})^{\frac{1}{p}}}, \frac{1}{(\sum_{i=1}^{n} \frac{w_i}{a^p_{\sigma(i)}})^{\frac{1}{p}}}, \frac{1}{(\sum_{i=1}^{n} \frac{w_i}{\bar{a}^p_{\sigma(i)}})^{\frac{1}{p}}}),$$

which is just the FOWHM operator defined in [35].

(iv) If the TFNs $\widehat{a}_i = (\underline{a}_i, a_i, \bar{a}_i)$ $(i = 1, 2, \cdots, n)$ are reduced to the interval numbers $\breve{a}_i = [\underline{a}_i, \bar{a}_i]$, then the TIFTOWBHM operator (i.e., the equation (28)) is reduced to the GUOWBHM operator defined in [15] as follows:

$$\text{GUOWBHM}^{p,q,r}(\breve{a}_1, \breve{a}_2, \cdots, \breve{a}_n) = [\frac{1}{(\sum_{i,j,k=1}^{n} \frac{w_i w_j w_k}{\underline{a}^p_i \underline{a}^q_j \underline{a}^r_k})^{\frac{1}{p+q+r}}}, \frac{1}{(\sum_{i,j,k=1}^{n} \frac{w_i w_j w_k}{\bar{a}^p_i \bar{a}^q_j \bar{a}^r_k})^{\frac{1}{p+q+r}}}]. \quad (31)$$

Furthermore, the TIFOWH operator (i.e., the equation (30) is reduced to the UOWHM operator defined in [35]:

$$\text{UOWHM}^p(\breve{a}_1, \breve{a}_2, \cdots, \breve{a}_n) = (\frac{1}{\sum_{i,j=1}^{n} \frac{w_i}{\underline{a}_i}}, \frac{1}{\sum_{i,j=1}^{n} \frac{w_i}{\bar{a}_i}}).$$

(v) If $\underline{a}_i = \bar{a}_i = a_i$, for all $(i = 1, 2, \cdots, n)$, i.e., the TFNs $\widehat{a}_i = (\underline{a}_i, a_i, \bar{a}_i)$ are reduced to real numbers, then the TIFTOWBHM operator (i.e., the equation (30)) is reduced to the triple ordered weighted Bonferroni harmonic mean (TOWBHM) operator:

$$\text{TOWBHM}^{p,q,r}(a_1, a_2, \cdots, a_n) = \frac{1}{(\sum_{i,j,k=1}^{n} \frac{w_i w_j w_k}{a^p_{\sigma(i)} a^q_{\sigma(j)} a^r_{\sigma(k)}})^{\frac{1}{p+q+r}}},$$

whic is just the GOWBHM defined in [15].

The weight vector $w = (w_1, w_2, \cdots, w_n)^{\mathrm{T}}$ associated with the TIFTOWBHM operator can be determined according to actual needs. Moreover, there are also many methods to obtain the associated weight vector (see [9, 33] for details). Similar to the TIFTWBHM operator, the TIFTOWBHM operator has some desirable properties, such as idempotency, boundedness, commutativity, monotonicity (omitted).

Theorem 3.14. *Let* $\tilde{a}_i$ $(i = 1, 2, \cdots, n)$ *be a collection of TIFNs,* $k > 0$*, then*

$$\text{TIFTOWBHM}^{p,q,r}(k\tilde{a}_1, k\tilde{a}_2, \cdots, k\tilde{a}_n) = k\text{TIFTOWBHM}^{p,q,r}(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n).$$

Theorem 3.15. *Let* $\tilde{b}$ *be a TIFNs and* $\tilde{a}_i$ $(i = 1, 2, \cdots, n)$ *be a collection of TIFNs, then*

$$\text{TIFTOWBHM}^{p,q,r}(\tilde{a}_1\tilde{b}, \tilde{a}_2\tilde{b}, \cdots, \tilde{a}_n\tilde{b}) = \text{TIFTOWBHM}^{p,q,r}(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n)\tilde{b}.$$

Proof. By Definition 3.13, we have

$$\text{TIFTOWBHM}^{p,q,r}(\tilde{a}_1\tilde{b}, \tilde{a}_2\tilde{b}, \cdots, \tilde{a}_n\tilde{b}) = ((\frac{1}{(\sum\limits_{i,j,k=1,i\neq j\neq k}^{n} \frac{w_i w_j w_k}{\underline{a}_i^p \underline{b}^p \underline{a}_j^q \underline{b}^q \underline{a}_k^r \underline{b}^r})^{\frac{1}{p+q+r}}},$$

$$\frac{1}{(\sum\limits_{i,j,k=1,i\neq j\neq k}^{n} \frac{w_i w_j w_k}{a_i^p b^p a_j^q b^q a_k^r b^r})^{\frac{1}{p+q+r}}}, \frac{1}{(\sum\limits_{i,j,k=1,i\neq j\neq k}^{n} \frac{w_i w_j w_k}{\bar{a}_i^p \bar{b}^p \bar{a}_j^q \bar{b}^q \bar{a}_k^r \bar{b}^r})^{\frac{1}{p+q+r}}});$$

$$\min_{1\leq i\leq n}\{\omega_{\tilde{a}_i}\} \wedge \omega_{\tilde{b}}, \max_{1\leq i\leq n}\{u_{\tilde{a}_i}\} u_{\tilde{b}})$$

$$= ((\frac{\underline{b}}{(\sum\limits_{i,j,k=1,i\neq j\neq k}^{n} \frac{w_i w_j w_k}{\underline{a}_i^p \underline{a}_j^q \underline{a}_k^r})^{\frac{1}{p+q+r}}}, \frac{b}{(\sum\limits_{i,j,k=1,i\neq j\neq k}^{n} \frac{w_i w_j w_k}{a_i^p a_j^q a_k^r})^{\frac{1}{p+q+r}}}, \frac{\bar{b}}{(\sum\limits_{i,j,k=1,i\neq j\neq k}^{n} \frac{w_i w_j w_k}{\bar{a}_i^p \bar{a}_j^q \bar{a}_k^r})^{\frac{1}{p+q+r}}});$$
$$\min_{1\leq i\leq n}\{\omega_{\tilde{a}_i}\} \wedge \omega_{\tilde{b}}, \max_{1\leq i\leq n}\{u_{\tilde{a}_i}\} \vee u_{\tilde{b}}).$$

which completes the proof of Theorem 3.15. □

The TIFTOWBHM operator has some special cases as follows:

(1) If $W^* = (1, 0, \cdots, 0)^{\mathrm{T}}$, then $\text{TIFTOWBHM}_{W^*}^{p,q,r}(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \max\limits_{1\leq i\leq n}\{\tilde{a}_i\}$;

(2) If $W_* = (0, 0, \cdots, 0, 1)^{\mathrm{T}}$, then $\text{TIFTOWBHM}_{W_*}^{p,q,r}(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \min\limits_{1\leq i\leq n}\{\tilde{a}_i\}$;

(3) If $W = (\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n})^{\mathrm{T}}$, then

$$\text{TIFTOWBHM}_{W}^{p,q,r}(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \frac{n^{\frac{1}{p+q+r}}}{(\sum\limits_{i,j,k=1,i\neq j\neq k}^{n} \frac{1}{\tilde{a}_i^p \tilde{a}_j^q \tilde{a}_k^r})^{\frac{1}{p+q+r}}}$$

$$= ((\frac{n^{\frac{1}{p+q+r}}}{(\sum\limits_{i,j,k=1,i\neq j\neq k}^{n} \frac{1}{\underline{a}_i^p \underline{a}_j^q \underline{a}_k^r})^{\frac{1}{p+q+r}}}, \frac{n^{\frac{1}{p+q+r}}}{(\sum\limits_{i,j,k=1,i\neq j\neq k}^{n} \frac{1}{a_i^p a_j^q a_k^r})^{\frac{1}{p+q+r}}}, \frac{n^{\frac{1}{p+q+r}}}{(\sum\limits_{i,j,k=1,i\neq j\neq k}^{n} \frac{1}{\bar{a}_i^p \bar{a}_j^q \bar{a}_k^r})^{\frac{1}{p+q+r}}});$$
$$\min_{1\leq i\leq n}\{\omega_{\tilde{a}_i}\}, \max_{1\leq i\leq n}\{u_{\tilde{a}_i}\}).$$

3.4. **Triangular Intuitionistic Fuzzy Triple Hybrid Bonferroni Harmonic Mean Operator.**

By combining the advantages of the weighted harmonic mean operator and the ordered weighted harmonic mean operator, Xu [35] developed the fuzzy hybrid harmonic mean (FHHM) operator for the triangular fuzzy numbers, Sun and Sun [17] proposed the notation of the fuzzy hybrid Bonferroni harmonic mean (FHBHM) operator for TFNs as follows:

Definition 3.16. [17] Let $\hat{a}_i = (\underline{a}_i, a_i, \bar{a}_i)$ $(i = 1, 2, \cdots, n)$ be a collection of TFNs. A FHBHM operator of dimension n is defined as $\text{FHBHM}^{p,q}(\hat{a}_1, \hat{a}_2, \cdots, \hat{a}_n) =$ $((\sum\limits_{i,j=1}^{n} \frac{w_i w_j}{(\dot{\underline{a}}_{\sigma(i)})^p (\dot{\underline{a}}_{\sigma(j)})^q})^{-\frac{1}{p+q}}, (\sum\limits_{i,j=1}^{n} \frac{w_i w_j}{(\dot{a}_{\sigma(i)})^p (\dot{a}_{\sigma(j)})^q})^{-\frac{1}{p+q}}, (\sum\limits_{i,j=1}^{n} \frac{w_i w_j}{(\dot{\bar{a}}_{\sigma(i)})^p (\dot{\bar{a}}_{\sigma(j)})^q})^{-\frac{1}{p+q}}),$

where $\dot{\hat{a}}_{\sigma(i)} = (\dot{\underline{a}}_{\sigma(i)}, \dot{a}_{\sigma(i)}, \dot{\bar{a}}_{\sigma(i)})$ is the i th largest of the weighted TFNs $\dot{\hat{a}}_i (\dot{\hat{a}}_i = nw_i \hat{a}_i,\ i = 1, 2, \cdots, n)$, $\mathbf{w} = (w_1, w_2, \cdots, w_n)^{\mathrm{T}}$ is the weight vector of $(\hat{a}_1, \hat{a}_2, \cdots, \hat{a}_n)$ satisfying $w_i > 0$ and $\sum_{i=1}^{n} w_i = 1$, and n is the balancing coefficient.

It should be pointed out that, Definition 3.16 is not right since it did not consider the associated weight vector with the FHBHM operator. Generally, the hybrid aggregation operator contains two kinds of weight vector: the associated weight vector (or position weight vector) and the weight vector of the fused arguments. However, there is only the weighted vector of the fused arguments in Definition 3.16. Namely, the associated weight vector is confused as the weighted vector of the fused arguments. Analogously, the FHHM operator defined by Xu [35] has the same error.

Definition 3.17. Let $\tilde{a}_i = ((\underline{a}_i, a_i, \bar{a}_i); \omega_{\tilde{a}_i}, u_{\tilde{a}_i})$ $(i = 1, 2, \cdots, n)$ be a collection of TIFNs. The triangular intuitionistic fuzzy triple hybrid Bonferroni harmonic mean operator of dimension n is defined as

$$\mathrm{TIFTHBHM}_{w,v}^{p,q,r}(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n) = \frac{1}{\left(\sum\limits_{i,j,k=1, i\neq j\neq k}^{n} \frac{w_i w_j w_k}{\dot{\tilde{a}}_{\sigma(i)}^{p} \dot{\tilde{a}}_{\sigma(j)}^{q} \dot{\tilde{a}}_{\sigma(k)}^{r}}\right)^{\frac{1}{p+q+r}}} =$$

$$\Bigg(\Bigg(\Bigg(\sum_{i,j,k=1, i\neq j\neq k}^{n} \frac{w_i w_j w_k}{(\dot{\underline{a}}_{\sigma(i)})^p (\dot{\underline{a}}_{\sigma(j)})^q (\dot{\underline{a}}_{\sigma(k)})^r}\Bigg)^{-\frac{1}{p+q+r}},$$

$$\Bigg(\sum_{i,j,k=1, i\neq j\neq k}^{n} \frac{w_i w_j w_k}{(\dot{a}_{\sigma(i)})^p (\dot{a}_{\sigma(j)})^q (\dot{a}_{\sigma(k)})^r}\Bigg)^{-\frac{1}{p+q+r}},$$

$$\Bigg(\sum_{i,j,k=1, i\neq j\neq k}^{n} \frac{w_i w_j w_k}{(\dot{\bar{a}}_{\sigma(i)})^p (\dot{\bar{a}}_{\sigma(j)})^q (\dot{\bar{a}}_{\sigma(k)})^r}\Bigg)^{-\frac{1}{p+q+r}}\Bigg); \min_{1\leq i\leq n}\{\omega_{\dot{\tilde{a}}_{\sigma(i)}}\}, \max_{1\leq i\leq n}\{u_{\dot{\tilde{a}}_{\sigma(i)}}\}\Bigg), \quad (32)$$

where $\mathbf{w} = (w_1, w_2, \cdots, w_n)^{\mathrm{T}}$ is the weight vector associated with TIFTHBHM, satisfying that $w_i > 0$ $(i = 1, 2, \cdots, n)$ and $\sum_{i=1}^{n} w_i = 1$, $\dot{\tilde{a}}_{\sigma(i)} = ((\dot{\underline{a}}_{\sigma(i)}, \dot{a}_{\sigma(i)}, \dot{\bar{a}}_{\sigma(i)}); \omega_{\dot{\tilde{a}}_{\sigma(i)}}, u_{\dot{\tilde{a}}_{\sigma(i)}})$ is the i th largest of the weighted TIFNs $\dot{\tilde{a}}_i$ $(\dot{\tilde{a}}_i = nv_i \tilde{a}_i, i = 1, 2, \cdots, n)$, and $v = (v_1, v_2, \cdots, v_n)^{\mathrm{T}}$ is the weight vector of $(\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n)$satisfying that $v_i > 0$ $(i = 1, 2, \cdots, n)$ and $\sum_{i=1}^{n} v_i = 1$, and n is the balancing coefficient.

Especially, if $v_i = 1/n$ $(i = 1, 2, \cdots, n)$, then the TIFTHBHM operator is reduced to the TIFTOWBHM operator. That is to say, the TIFTOWBHM operator is a special case of the TIFTHBHM operator. If the TIFNs reduce to TFNs and $r = 0$, then the TIFTHBHM operator is reduced to the corrected FHBHM operator defined by Sun and Sun [17], i.e., the FHBHM operator is also a special case of the TIFTHBHM operator.

The above three kinds of triangular intuitionistic fuzzy triple hybrid Bonferroni harmonic mean operators can take the given arguments and their relationships into consideration. The TIFTWBHM operator emphasizes the importance of each argument, the TIFTOWBHM operator stresses the importance of the ordered position of each argument, while the TIFTHBHM operator reflects the important degrees of both the given arguments and the ordered position of the arguments. Furthermore, the prominent characteristic of these operators is that they can take the correlationships between the TIFN arguments into account.

In MAGDM with TIFNs, since the attribute values of alternatives are represented as TIFNs, the TIFTWBHM operator can be directly used to integrate the attribute values of alternatives into the individual comprehensive attribute values of alternatives. Meanwhile, there are multiple experts participating decision making together, different experts should be allocated various weights. Thus, the TIFTHBHM operator can be used to integrate the individual comprehensive attribute values of alternatives into the collective ones by sufficiently considering the expert weights.

4. **A New Method for MAGDM with TIFNs**

In this section, employed the TIFTWBHM and TIFTHBHM operators, a new decision making method is developed to solve the MAGDM problems with TIFNs.

4.1. **Presentation of MAGDM Problems with TIFNs.**

Let $A = \{A_1, A_2, \cdots, A_m\}$ be an alternative set and $C = \{c_1, c_2, \cdots, c_n\}$ be the set of attributes. The attribute weight vector is $\mathbf{w} = (w_1, w_2, \cdots, w_n)^{\mathrm{T}}$, satisfying that $0 \le w_j \le 1$ $(j = 1, 2, \cdots, n)$ and $\sum_{j=1}^{n} w_j = 1$. There are s DMs (or experts) participating in decision making, denote the set of DMs by $E = \{e_1, e_2, \cdots, e_s\}$. The weight vector of the DMs is $v = (v_1, v_2, \cdots, v_s)^{\mathrm{T}}$, satisfying that $0 \le v_k \le 1$ $(k = 1, 2, \cdots, s)$ and $\sum_{k=1}^{s} v_k = 1$. The rating of an alternative A_i on an attribute c_j given by the DM e_k is a TIFN $\tilde{a}_{ij}^k = ((\underline{a}_{ij}^k, a_{ij}^k, \bar{a}_{ij}^k); \omega_{\tilde{a}_{ij}^k}, u_{\tilde{a}_{ij}^k})$, where $\omega_{\tilde{a}_{ij}^k}$ and $u_{\tilde{a}_{ij}^k}$ denote respectively the maximum membership degree and the minimum non-membership degree of alternative A_i on attribute c_j given by the DM e_k, satisfying $0 \le \omega_{\tilde{a}_{ij}^k} \le 1$, $0 \le u_{\tilde{a}_{ij}^k} \le 1$ and $0 \le \omega_{\tilde{a}_{ij}^k} + u_{\tilde{a}_{ij}^k} \le 1$. Hence, a MAGDM problem can be concisely expressed in matrix format as $\tilde{A}^k = (\tilde{a}_{ij}^k)_{m \times n}$ $(k = 1, 2, \cdots, s)$, which are referred to as TIFN decision matrices.

Since the attributes are generally incommensurate, the decision matrix needs to be normalized so as to transform the various attribute values into comparable values. The matrix $\tilde{A}^k = (\tilde{a}_{ij}^k)_{m \times n}$ is normalized into $\tilde{R}^k = (\tilde{r}_{ij}^k)_{m \times n}$, where

$$\tilde{r}_{ij}^k = ((\underline{r}_{ij}^k, r_{ij}^k, \bar{r}_{ij}^k); \omega_{\tilde{r}_{ij}^k}, u_{\tilde{r}_{ij}^k}), \omega_{\tilde{r}_{ij}^k} = \omega_{\tilde{a}_{ij}^k}, u_{\tilde{r}_{ij}^k} = u_{\tilde{a}_{ij}^k}$$

and

$$\tilde{r}_{ij}^k = ((\frac{\underline{a}_{ij}^k}{\bar{a}_j^+}, \frac{a_{ij}^k}{\bar{a}_j^+}, \frac{\bar{a}_{ij}^k}{\bar{a}_j^+}); \omega_{\tilde{r}_{ij}^k}, u_{\tilde{r}_{ij}^k}) \quad \text{for benefit attributes,} \tag{33}$$

$$\tilde{r}_{ij}^k = ((\frac{\underline{a}_j^-}{\bar{a}_{ij}^k}, \frac{\underline{a}_j^-}{a_{ij}^k}, \frac{\underline{a}_j^-}{{}_{ij}^{k}}); \omega_{\tilde{r}_{ij}^k}, u_{\tilde{r}_{ij}^k}) \quad \text{for cost attributes,} \tag{34}$$

where $\bar{a}_j^+ = \max\{\bar{a}_{ij}^k | i = 1, 2, \cdots .m; k = 1, 2, \cdots , s\}$ and $\underline{a}_j^- = \min\{_{ij}^k | i = 1, 2, \cdots .m; k = 1, 2, \cdots , s\}$ $(j = 1, 2, \cdots , n)$.

4.2. **Decision Method Based on TIFTWBHM and TIFTHBHM Operators.**

In sum, an algorithm and process for solving the MAGDM problems with TIFNs may be summarized as follows.

Step 1: Normalize the decision matrix $\tilde{A}^k$ into$\tilde{R}^k$ by the equations (33) and (33);

Step 2: Combined the TIFTWBHM operator with the attribute weight vector $\mathbf{w} = (w_1, w_2, \cdots , w_n)^{\mathrm{T}}$, the individual overall attribute value of alternative A_i given by DMe_k can be obtained as follows:

$$\begin{aligned}\tilde{r}_i^k = & ((\underline{r}_i^k, r_i^k, \bar{r}_i^k); \omega_{\tilde{r}_i^k}, u_{\tilde{r}_i^k}) = \mathrm{TIFTWBHM}(\tilde{r}_{i1}^k, \tilde{r}_{i2}^k, \cdots, \tilde{r}_{in}^k) \\ = & (((\sum_{t,j,u=1}^{n} \frac{w_t w_j w_u}{(\underline{r}_{it}^k)^p (\underline{r}_{ij}^k)^q (\underline{r}_{iu}^k)^r})^{-\frac{1}{p+q+r}}, (\sum_{t,j,u=1}^{n} \frac{w_t w_j w_u}{(r_{it}^k)^p (r_{ij}^k)^q (r_{iu}^k)^r})^{-\frac{1}{p+q+r}}, \\ & (\sum_{t,j,u=1}^{n} \frac{w_t w_j w_u}{(\bar{r}_{it}^k)^p (\bar{r}_{ij}^k)^q (\bar{r}_{iu}^k)^r})^{-\frac{1}{p+q+r}})(i = 1, 2, \cdots , m;\ k = 1, 2, \cdots , s). \end{aligned} \tag{35}$$

Step 3: Utilize the TIFTHBHM operator to aggregate the individual overall attribute values $\tilde{r}_i^k$, $(k = 1, 2, \cdots , s)$ and get the collective overall attribute value $\tilde{r}_i$ of alternative A_i as follows:

$$\begin{aligned}\tilde{r}_i = & \mathrm{TIFTHBHM}_{\omega,v}^{p,q,r}(\tilde{r}_i^1, \tilde{r}_i^2, \cdots , \tilde{r}_i^s) = (((\sum_{t,j,k=1}^{s} \frac{\omega_t \omega_j \omega_k}{(\underline{\dot{a}}_i^{\sigma(i)})^p (\underline{\dot{a}}_i^{\sigma(j)})^q (\underline{\dot{a}}_i^{\sigma(k)})^r})^{-\frac{1}{p+q+r}}, \\ & (\sum_{t,j,k=1}^{s} \frac{\omega_t \omega_j \omega_k}{(\dot{r}_i^{\sigma(i)})^p (\dot{r}_i^{\sigma(j)})^q (\dot{r}_i^{\sigma(k)})^r})^{-\frac{1}{p+q+r}}, (\sum_{t,j,k=1}^{s} \frac{\omega_t \omega_j \omega_k}{(\dot{\bar{r}}_i^{\sigma(i)})^p (\dot{\bar{r}}_i^{\sigma(j)})^q (\dot{\bar{r}}_i^{\sigma(k)})^r})^{-\frac{1}{p+q+r}}); \\ & \min_{1 \le k \le s}\{\omega_{\dot{r}_i^{\sigma(k)}}\}, \max_{1 \le k \le s}\{u_{\dot{r}_i^{\sigma(k)}}\}), \end{aligned} \tag{36}$$

where $\omega = (\omega_1, \omega_2, \cdots , \omega_s)^{\mathrm{T}}$ is the weight vector correlated with TIFTHBHM, satisfying that $\omega_i > 0$ $(i = 1, 2, \cdots , s)$ and $\sum_{i=1}^{s} \omega_i = 1$. $\dot{\tilde{r}}_i^{\sigma(i)} = ((\underline{\dot{r}}_i^{\sigma(i)}, \dot{r}_i^{\sigma(i)}, \dot{\bar{r}}_i^{\sigma(i)});$ $\omega_{\dot{r}_i^{\sigma(i)}}, u_{\dot{r}_i^{\sigma(i)}})$ is the ith largest of the weighted TIFNs $\dot{\tilde{r}}_i^k = sv_i\tilde{r}_i^k \quad (k = 1, 2, \cdots , s)$, and $v = (v_1, v_2, \cdots , v_s)^{\mathrm{T}}$ is the weight vector of the DMs.

Step 4: Use the equations (15) and (16) to calculate the weighted possibility means $m_\mu(\tilde{r}_i, \theta)$ and $m_v(\tilde{r}_i, \theta)$ of $\tilde{r}_i = ((\underline{r}_i, r_i, \bar{r}_i; \omega_{\tilde{r}_i}, u_{\tilde{r}_i})$. Then, the ranking order of alternatives can be obtained according to the lexicographic ranking approach in Subsection 2.3.

5. A Real Investment Selection Case Study and Comparative Analysis

In this section, a real investment selection case study is illustrated to demonstrate the applicability and implementation process of the MAGDM method proposed in this paper. The comparison analyses of computational results are also conducted to show the superiority of the proposed method.

5.1. **A Real Investment Selection Case Study.**

The proposed method is applied to a real investment selection example. An investment company, Shenzhen Capital Group (SCGC for short), was born in a hotbed of innovation, the city of Shenzhen (in China). The birth of SCGC itself is an innovation of the city government of Shenzhen. The city wants to create a shoulder to support high-tech industries and to promote business development. The scale of established government-backed funds is RMB 6.841 billion and the new funds being raised is RMB 1.77 billion. SCGC desires to invest a sum of money in the best option. After preliminary screening, five possible candidates (i.e., alternatives) remain for further evaluation. Candidate A_1 is a car company (Zhejiang geely automobile co., ltd), A_2 is a food company (Oreo company), A_3 is a computer company (Lenovo Group ltd), A_4 is an arms company (Xi'an days and defense technology Limited by Share Ltd) and A_5 is a TV company (Changhong company). The decision making committee consists of three DMs: president e_1, vice-president e_2 and department manager e_3. They assess the five candidates on the basis of four attributes, including the risk analysis c_1, the growth analysis c_2, the social-political impact analysis c_3, the environmental impact analysis c_4, the first attribute is cost attribute, and the other three attributes are benefit attributes. The weight vector of the attributes is $W = (0.2, 0.1, 0.3, 0.4)^{\mathrm{T}}$, the weight vector of the DMs is $\mathbf{v} = (0.2, 0.5, 0.3)^{\mathrm{T}}$. The ratings of the candidates with respect to attributes can be represented by TIFNs as in Tables 1-3. For example, TIFN ((3.33, 3.53, 3.75); 0.7, 0.2) in Table 1 indicates that the mark of the candidate with respect to the attribute is about 3.53 with the maximum satisfaction degree 0.7, while the minimum dissatisfaction degree 0.2. In other words, the hesitation degree is 0.1. This assessment value ((3.33, 3.53, 3.75); 0.7, 0.2) can be obtained by the following pre-processing phase:

1) After negotiation and discussion, the group of DMs all agree that the assessments on attribute c_1 should use triangular fuzzy numbers in 10-point scale (grades from 0 up to 10 are used in endpoints of mode, lower and upper limits, with 0 being worst and 10 being best) to score.

2) Each DM respectively gives the lower mark a, the most possible mark b, and the upper mark c for alternative A_1 on attribute c_1.

3) Meanwhile, for the most possible mark, each DM provides the maximum membership degree $w_{\tilde{a}}$ and the minimum non-membership degree $u_{\tilde{a}}$ according to his/her knowledge and experience or by statistical methods. Thus, TrIFN $\tilde{a} = ((a, b, c); \omega_{\tilde{a}}, u_{\tilde{a}})$ is produced.

The other TrIFNs in Tables 1-3 can be similarly explained and obtained.

	c_1	c_2	c_3	c_4
A_1	((3.33,3.53,3.75);0.7,0.2)	((3.60,3.80,4.00);0.4,0.5)	((5.46,5.58,5.76);0.5,0.4)	((4.34,4.55,4.76);0.8,0.1)
A_2	((3.22,3.33,3.40);0.6,0.3)	((3.35,3.85,4.15);0.6,0.3)	((3.60,4.02,4.20);0.7,0.2)	((4.83,5.04,5.25);0.6,0.3)
A_3	((3.00,3.09,3.15);0.8,0.1)	((4.50,4.65,4.75);0.7,0.2)	((4.62,4.74,4.92);0.5,0.2)	((6.51,6.65,6.72);0.8,0.1)
A_4	((3.40,3.52,3.65);0.6,0.4)	((4.85,4.90,5.00);0.6,0.3)	((5.88,5.94,6.00);0.6,0.2)	((6.79,6.93,7.00);0.7,0.1)
A_5	((3.70,3.79,3.84);0.6,0.3)	((3.90,3.95,4.00);0.6,0.2)	((4.98,5.10,5.28);0.7,0.2)	((6.58,6.79,6.93);0.6,0.2)

TABLE 1. TIFN Decision Matrix Given by DM e_1

Step 1: According to the equations (33) and (34), the normalized TIFN decision matrices are obtained and listed in Tables 4-6.

	c_1	c_2	c_3	c_4
A_1	((4.00,4.28,4.61);0.7,0.2)	((2.85,3.05,3.25);0.4,0.5)	((4.56,4.68,4.86);0.5,0.4)	((3.29,3.50,3.71);0.8,0.1)
A_2	((3.84,4.00,4.10);0.6,0.3)	((2.60,3.10,3.40);0.6,0.3)	((2.70,3.00,3.12);0.7,0.2)	((3.78,3.99,4.20);0.6,0.3)
A_3	((3.00,3.65,3.75);0.8,0.1)	((3.75,3.90,4.00);0.7,0.2)	((3.72,3.84,4.02);0.5,0.2)	((5.46,5.60,5.67);0.8,0.1)
A_4	((4.11,4.28,4.47);0.6,0.4)	((4.10,4.15,5.00);0.6,0.3)	((4.98,5.04,6.00);0.6,0.2)	((5.74,5.88,7.00);0.7,0.1)
A_5	((4.54,4.68,4.76);0.6,0.3)	((3.00,3.15,3.20);0.6,0.2)	((4.08,4.20,4.56);0.7,0.2)	((5.53,5.60,4.74);0.6,0.2)

TABLE 2. TIFN Decision Matrix Given by DM e_2

	c_1	c_2	c_3	c_4
A_1	((5.76,6.38,7.14);0.7,0.2)	((1.70,1.90,2.10);0.4,0.5)	((2.65,2.75,2.90);0.5,0.4)	((1.68,1.89,2.10);0.8,0.1)
A_2	((5.45,5.76,6.00);0.6,0.3)	((1.45,1.95,2.25);0.6,0.3)	((1.32,1.74,1.92);0.7,0.2)	((2.17,2.38,2.59);0.6,0.3)
A_3	((3.00,5.08,5.26);0.8,0.1)	((2.60,2.75,2.85);0.7,0.2)	((2.34,2.46,2.64);0.5,0.2)	((3.85,3.99,4.06);0.8,0.1)
A_4	((6.00,6.38,6.82);0.6,0.4)	((2.95,3.00,5.00);0.6,0.3)	((3.60,3.66,6.00);0.6,0.2)	((4.13,4.27,7.00);0.7,0.1)
A_5	((6.97,7.31,7.50);0.6,0.3)	((2.00,2.05,2.15);0.6,0.2)	((2.70,2.82,3.00);0.7,0.2)	((3.92,4.13,4.27);0.6,0.2)

TABLE 3. TIFN Decision Matrix Given by DM e_3

	c_1	c_2	c_3	c_4
A_1	((0.80,0.85,0.90);0.7,0.2)	((0.72,0.76,0.80);0.4,0.5)	((0.91,0.93,0.96);0.5,0.4)	((0.62,0.65,0.68);0.8,0.1)
A_2	((0.88,0.90,0.93);0.6,0.3)	((0.67,0.77,0.83);0.6,0.3)	((0.60,0.67,0.70);0.7,0.2)	((0.69,0.72,0.75);0.6,0.3)
A_3	((0.95,0.97,1.00);0.8,0.1)	((0.90,0.93,0.95);0.7,0.2)	((0.77,0.79,0.82);0.5,0.2)	((0.93,0.95,0.96);0.8,0.1)
A_4	((0.82,0.85,0.88);0.6,0.4)	((0.97,0.98,1.00);0.6,0.3)	((0.98,0.99,1.00);0.6,0.2)	((0.97,0.99,1.00);0.7,0.1)
A_5	((0.78,0.79,0.81);0.6,0.3)	((0.78,0.79,0.81);0.6,0.2)	((0.83,0.85,0.88);0.7,0.2)	((0.94,0.97,0.99);0.6,0.2)

TABLE 4. Normalized TIFN Decision Matrix Given by DM e_1

	c_1	c_2	c_3	c_4
A_1	((0.65,0.70,0.75);0.7,0.2)	((0.57,0.61,0.65);0.4,0.5)	((0.76,0.78,0.81);0.5,0.4)	((0.47,0.50,0.53);0.8,0.1)
A_2	((0.73,0.75,0.78);0.6,0.3)	((0.52,0.62,0.68);0.6,0.3)	((0.45,0.50,0.52);0.7,0.2)	((0.54,0.57,0.60);0.6,0.3)
A_3	((0.80,0.82,1.00);0.8,0.1)	((0.75,0.78,0.80);0.7,0.2)	((0.62,0.64,0.67);0.5,0.2)	((0.78,0.80,0.81);0.8,0.1)
A_4	((0.67,0.70,0.73);0.6,0.4)	((0.82,0.83,1.00);0.6,0.3)	((0.83,0.84,1.00);0.6,0.2)	((0.82,0.84,1.00);0.7,0.1)
A_5	((0.63,0.64,0.66);0.6,0.3)	((0.60,0.63,0.64);0.6,0.2)	((0.68,0.70,0.76);0.7,0.2)	((0.79,0.80,0.82);0.6,0.2)

TABLE 5. Normalized TIFN Decision Matrix Given by DM e_2

Step 2: Use TIFTWBHM operator to integrate the elements in the i-th row of the normalized matrix $\tilde{R}^k$ to obtain the individual overall TIFNs of the alternative $\tilde{r}_i^k$ $(i = 1, 2, 3, 4, 5; k = 1, 2, 3)$.

For example, let $p = \mathrm{q} = \mathrm{r} = 2$, then $\tilde{r}_j^k$ can be calculated by the equation (35) as follows:

$$\tilde{r}_1^1 = ((0.7239, 0.7573, 0.7587); 0.4, 0.5),\ \tilde{r}_2^1 = ((0.6812, 0.7341, 0.7670); 0.6, 0.3);$$
$$\tilde{r}_3^1 = ((0.8725, 0.8936, 0.9163); 0.5, 0.2),\ \tilde{r}_4^1 = ((0.9360, 0.9555, 0.9720); 0.6, 0.4);$$
$$\tilde{r}_5^1 = ((0.8508, 0.8700, 0.8920); 0.6, 0.3),\ \tilde{r}_1^2 = ((0.5669, 0.6008, 0.6358); 0.4, 0.5);$$
$$\tilde{r}_2^2 = ((0.5271, 0.5728, 0.6013); 0.6, 0.3),\ \tilde{r}_3^2 = ((0.7201, 0.7414, 0.7822); 0.5, 0.2);$$
$$\tilde{r}_4^2 = ((0.7845, 0.8044, 0.9224); 0.6, 0.3),\ \tilde{r}_5^2 = ((0.6948, 0.7111, 0.7417); 0.6, 0.3);$$
$$\tilde{r}_1^3 = ((0.3160, 0.3518, 0.3879); 0.2, 0.7),\ \tilde{r}_2^3 = ((0.2845, 0.3445, 0.3777); 0.6, 0.3);$$
$$\tilde{r}_3^3 = ((0.4833, 0.5050, 0.5550); 0.5, 0.2),\ \tilde{r}_4^3 = ((0.5502, 0.5712, 0.7905); 0.6, 0.4);$$

$$\tilde{r}_5^3 = ((0.4646, 0.4827, 0.5065); 0.6, 0.3)$$

Step 3: Combined the TIFTHBHM operator with the DMs' weight vector $v = (0.2, 0.5, 0.3)^{\mathrm{T}}$, the collective overall attribute values of alternatives A_i $(i = 1, 2, \cdots, 5)$ can be obtained.

For example, using the method of Xu [33] to obtain the associated weight vector $\omega = (0.06, 0.66, 0.28)^{\mathrm{T}}$ and taking $p = \mathrm{q} = \mathrm{r} = 2$, by the equation (36) we have

$$\tilde{r}_1 = ((\underline{r}_1, r_1, \bar{r}_1); \omega_{\tilde{r}_1}, u_{\tilde{r}_1}) = ((0.4338, 0.4740, 0.5132); 0.4, 0.5);$$
$$\tilde{r}_2 = ((\underline{r}_2, r_2, \bar{r}_2); \omega_{\tilde{r}_2}, u_{\tilde{r}_2}) = ((0.4009, 0.4668, 0.5012); 0.6, 0.3);$$
$$\tilde{r}_3 = ((\underline{r}_3, r_3, \bar{r}_3); \omega_{\tilde{r}_3}, u_{\tilde{r}_3}) = ((0.6198, 0.6429, 0.6858); 0.5, 0.2);$$
$$\tilde{r}_4 = ((\underline{r}_4, r_4, \bar{r}_4); \omega_{\tilde{r}_4}, u_{\tilde{r}_4}) = ((0.6913, 0.7139, 0.8583); 0.6, 0.4);$$
$$\tilde{r}_5 = ((\underline{r}_5, r_5, \bar{r}_5); \omega_{\tilde{r}_5}, u_{\tilde{r}_5}) = ((0.5990, 0.6171, 0.6453); 0.6, 0.3).$$

	c_1	c_2	c_3	c_4
A_1	((0.42,0.47,0.52);0.7,0.2)	((0.34,0.38,0.42);0.4,0.5)	((0.53,0.55,0.58);0.5,0.4)	((0.24,0.27,0.30);0.8,0.1)
A_2	((0.50,0.52,0.55);0.6,0.3)	((0.29,0.39,0.45);0.6,0.3)	((0.22,0.29,0.32);0.7,0.2)	((0.31,0.34,0.37);0.6,0.3)
A_3	((0.57,0.59,1.00);0.8,0.1)	((0.52,0.55,0.57);0.7,0.2)	((0.39,0.41,0.44);0.5,0.2)	((0.55,0.57,0.58);0.8,0.1)
A_4	((0.44,0.47,0.50);0.6,0.4)	((0.59,0.60,1.00);0.6,0.3)	((0.60,0.61,1.00);0.6,0.2)	((0.59,0.61,1.00);0.7,0.1)
A_5	((0.40,0.41,0.43);0.6,0.3)	((0.40,0.41,0.43);0.6,0.2)	((0.45,0.47,0.50);0.7,0.2)	((0.56,0.59,0.61);0.6,0.2)

TABLE 6. Normalized TIFN Decision Matrix Given by DM e_3

Step 4: Rank A_i $(i = 1, 2, 3, 4, 5)$ according to the lexicographic ranking approach.

For example, if $\theta = 0.5$, we get the weighted possibility means of the collective overall attribute values of the alternatives by the equations (15) and (16) as follows: $m_\mu(\tilde{r}_1, \theta) = 0.1895$, $m_\mu(\tilde{r}_2, \theta) = 0.2784$,$m_\mu(\tilde{r}_3, \theta) = 0.2949$,$m_\mu(\tilde{r}_4, \theta) = 0.3925$, $m_\mu(\tilde{r}_5, \theta) = 0.3713$,$m_v(\tilde{r}_1, \theta) = 0.2369$,$m_v(\tilde{r}_2, \theta) = 0.3248$, $m_v(\tilde{r}_3, \theta) = 0.4719$, $m_v(\tilde{r}_4, \theta) = 0.3925$,$m_v(\tilde{r}_5, \theta) = 0.4331$.

Thus, the ranking of candidates is generated as $A_4 \succ A_5 \succ A_3 \succ A_2 \succ A_1$.The best candidate is A_4.

For any other risk preference parameter values θ in the same way, we can obtain the ranking orders of candidates listed in Table 7.

θ	Ranking orders	Best candidates
0	$A_3 \succ A_5 \succ A_4 \succ A_2 \succ A_1$	A_3
0.1	$A_3 \succ A_5 \succ A_4 \succ A_2 \succ A_1$	A_3
0.2	$A_3 \succ A_5 \succ A_4 \succ A_2 \succ A_1$	A_3
0.3	$A_3 \succ A_5 \succ A_4 \succ A_2 \succ A_1$	A_3
0.4	$A_4 \succ A_5 \succ A_3 \succ A_2 \succ A_1$	A_4
0.5	$A_4 \succ A_5 \succ A_3 \succ A_2 \succ A_1$	A_4
0.6	$A_3 \succ A_4 \succ A_5 \succ A_2 \succ A_1$	A_3
0.7	$A_4 \succ A_5 \succ A_3 \succ A_2 \succ A_1$	A_3
0.8	$A_4 \succ A_5 \succ A_3 \succ A_2 \succ A_1$	A_3
0.9	$A_4 \succ A_5 \succ A_3 \succ A_2 \succ A_1$	A_3
1.0	$A_4 \succ A_5 \succ A_2 \succ A_3 \succ A_1$	A_4

TABLE 7. Ranking Orders For Different θ Values with $p = q = r = 2$

It can be seen from Table 7 that, for different risk preference parameter values, the ranking orders of candidates are also not completely the same. For instance, if $\theta \in [0, 0.3]$, then the ranking is$A_3 \succ A_5 \succ A_4 \succ A_2 \succ A_1$, the best is A_3 ; if $\theta \in [0.4, 0.6]$, then the ranking is $A_4 \succ A_5 \succ A_3 \succ A_2 \succ A_1$, the best is A_4 ; if $\theta \in [0.7, 1.0)$, then the ranking is $A_4 \succ A_5 \succ A_2 \succ A_3 \succ A_1$, the best is A_4.

In the same way, we can obtain the collective overall attribute values of candidates for any other preference parameter values p,q,r,θ. The computation results and ranking are listed in Tables 8 and 9.

It can be seen from Tables 8 and 9 that, for different parameter values p,q, r, the ranking orders of candidates are also not completely the same. For instance, if $\theta \in [0, 0.4)$ and $p = q = r = 1$, then the ranking is $A_3 \succ A_5 \succ A_4 \succ A_2 \succ A_1$, the best is A_3 ; if $\theta \in [0, 0.6)$ and $p = q = r = 3$, then the ranking is $A_3 \succ A_5 \succ A_2 \succ A_4 \succ A_1$, the best candidate is A_3.

The above analysis suggests that the risk preference of DM for the weighted lower and upper possibility means indeed plays an important role in the decision making. Since TIFN is a special kind of intuitionistic fuzzy set, involving DM's risk preference to rank the TIFNs is very reasonable and necessary. When the risk preference parameter values θ are different, the corresponding decision results

θ_1	Ranking orders	Best candidates
0	$A_3 \succ A_5 \succ A_4 \succ A_2 \succ A_1$	A_3
0.1	$A_3 \succ A_5 \succ A_4 \succ A_2 \succ A_1$	A_3
0.2	$A_3 \succ A_5 \succ A_4 \succ A_2 \succ A_1$	A_3
0.3	$A_3 \succ A_5 \succ A_4 \succ A_2 \succ A_1$	A_3
0.4	$A_3 \succ A_5 \succ A_4 \succ A_2 \succ A_1$	A_3
0.5	$A_3 \succ A_4 \succ A_5 \succ A_2 \succ A_1$	A_3
0.6	$A_4 \succ A_3 \succ A_5 \succ A_2 \succ A_1$	A_4
0.7	$A_4 \succ A_5 \succ A_3 \succ A_2 \succ A_1$	A_4
0.8	$A_4 \succ A_5 \succ A_3 \succ A_2 \succ A_1$	A_4
0.9	$A_4 \succ A_5 \succ A_3 \succ A_2 \succ A_1$	A_4
1.0	$A_4 \succ A_3 \succ A_5 \succ A_2 \succ A_1$	A_4

TABLE 8. Ranking Orders for Different θ Values with $p = q = r = 1$

θ	Ranking orders	Best candidates
0	$A_3 \succ A_5 \succ A_2 \succ A_4 \succ A_1$	A_3
0.1	$A_3 \succ A_5 \succ A_2 \succ A_4 \succ A_1$	A_3
0.2	$A_3 \succ A_5 \succ A_2 \succ A_4 \succ A_1$	A_3
0.3	$A_3 \succ A_5 \succ A_2 \succ A_4 \succ A_1$	A_3
0.4	$A_3 \succ A_5 \succ A_2 \succ A_4 \succ A_1$	A_3
0.5	$A_3 \succ A_5 \succ A_2 \succ A_4 \succ A_1$	A_3
0.6	$A_4 \succ A_5 \succ A_4 \succ A_2 \succ A_1$	A_5
0.7	$A_5 \succ A_4 \succ A_3 \succ A_2 \succ A_1$	A_5
0.8	$A_5 \succ A_4 \succ A_2 \succ A_3 \succ A_1$	A_5
0.9	$A_4 \succ A_5 \succ A_2 \succ A_3 \succ A_1$	A_4
1.0	$A_4 \succ A_5 \succ A_2 \succ A_3 \succ A_1$	A_4

TABLE 9. The Ranking Orders for Different θ Values with $p = q = r = 3$

may be different. In addition, for different parameter values p, q and r, the ranking orders of candidates are also not completely the same, which shows that the ranking results may depend on these parameters.

5.2. **Comparison Analysis with Triangular Fuzzy MAGDM.**

Wei [31] introduced the fuzzy induced ordered weighted harmonic mean (FIO-WHM) operator and applied to MAGDM with TFNs. If $\omega_{\tilde{a}_{ij}} = 1$ and $u_{\tilde{a}_{ij}} = 0$, then all the TIFNs $\tilde{a}_{ij} = ((\underline{a}_{ij}, a_{ij}, \bar{a}_{ij}); \omega_{\tilde{a}_{ij}}, u_{\tilde{a}_{ij}})$ in Tables 1-3 are changed into TFNs $(\underline{a}_{ij}, a_{ij}, \bar{a}_{ij})$. Thus, the above investment selection example is reduced to the MAGDM problem with TFNs. We use method [31] to solve this MAGDM problem with TFNs. The ranking order of alternatives by method [31] is $A_5 \succ A_1 \succ A_3 \succ A_2 \succ A_4$ and the best alternative is A_5.

The proposed method in this paper can also be used to solve this MAGDM problem with TFNs, the ranking order of alternatives is obtained as $A_4 \succ A_3 \succ A_5 \succ A_1 \succ A_2$, which is remarkably different from that obtained by method [31]. The main reasons are as follows:

(1) The proposed method in this paper can be used to solve not only the MAGDM problems with TIFNs but also the MAGDM with TFNs, while method [31] can only deal with the MAGDM problems with TFNs.

(2) This paper sufficiently considers the different preferences for the DMs, which makes the decision results more consistent with the actual situation, while Wei [31] did not consider the DM's preference (namely it assumes that all DMs are preference neutral).

In sum, the research problems and principles of decision making by method [31] and the proposed method in this paper are remarkably different. The former studied MAGDM with TFNs, while the latter researches MAGDM with TIFNs. The former defined the FIOWHM operator whereas the latter developed three kinds of triangular intuitionistic fuzzy triple Bonferroni harmonic mean operators. Furthermore, the latter is more flexible than the former since the later can derive different decision results by adjusting the parameter values θ, p, q and r, which can give more choices for different risk preference types of DMs.

5.3. Comparison with Extended VIKOR Method for MAGDM with TIFNs.

Wan et al. [26] put forward an extended VIKOR method for MAGDM with TIFNs. In this subsection, we utilized method [26] to solve the above investment selection example. Taking the coefficients of decision mechanism $\lambda = 0.5$, the coefficients $Q(A_1) = 0.1241$, $Q(A_2) = 0.4459$, $Q(A_3) = 0.6533$, $Q(A_4) = 0.2386$, $Q(A_5) = 0.6101$. Thus, the ranking order of alternatives obtained by method [26] is $A_3 \succ A_5 \succ A_2 \succ A_4 \succ A_1$, which is accodance with the result of case $\theta \in [0, 0.4)$ and $p = q = r = 3$ in Table 9.

Compared with method [26], the method in this paper has the following advantages:

(1) The former only employed the TAIFWA operator to obtain the group decision matrix, whereas the latter uses the TIFTWBHM and TIFTHBHM operators to derive the collective comprehensive values of alternatives. The TIFTHBHM operator can reflect not only the important degrees of both the given arguments and the ordered positions of the arguments but also the correlations between the input arguments. Therefore, the latter is more comprehensive than the former.

(2) Choosing different parameter values of θ, p, p and r in TIFTHBHM operator, the latter is able to obtain diversified ranking orders of alternatives, which greatly enhances the flexibility and agility of the proposed method.

6. Conclusions

This paper defined the weighted possibility means of membership and non-membership functions for TIFNs. Thereby, a new lexicographic ranking method of TIFNs was presented sufficiently considering the risk preference of DM. The sensitivity analyses with respect to the risk preference parameter were made. Then, three kinds of triangular intuitionistic fuzzy Bonferroni harmonic aggregation operators, including the TIFTWBHM, TIFTOWBHM and TIFTHBHM operators, were developed. Some of their desirable properties were also investigated in detail. A new decision method based on the TIFTWBHM and TIFTHBHM operators was proposed for solving the MAGDM problems with TIFNs. In this method, by the TIFTWBHM operator, the attribute values of alternatives were aggregated into the individual comprehensive attribute values of alternatives, which were further integrated into the collective ones by the TIFTHBHM operator. The ranking order of alternatives was generated according to the collective comprehensive attribute values of alternatives. The proposed MAGDM method sufficiently considers the different risk preferences of DMs, which can make the decision results more reasonable and consistent with the reality. We also compare different BM operators, the proposed BM operators in this paper can be reduced to existing BM operators.

Although the developed method in this paper was illustrated with an investment selection problem, it is expected to be applicable to the decision making problems in many areas, such as the supplier management, water environment assessment, threat evaluation and missile weapon system selection, warship combat plan evaluation, pattern recognition, approximation reasoning, just name a few. The TIFTWBHM, TIFTOWBHM and TIFTHBHM operators not only extend most of the existing Bonferroni mean operators and harmonic mean operators, but also provide the new tools for solving decision making problems. The triangular intuitionistic fuzzy triple geometric Bonferroni mean operators will be introduced and employed to MAGDM with TIFNs in the near future.

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