CARTESIAN-CLOSEDNESS OF THE CATEGORY OF L-FUZZY Q-CONVERGENCE SPACES

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Abstract. The definition of L-fuzzy Q-convergence spaces is presented by Pang and Fang in 2011. However, Cartesian-closedness of the category of L-fuzzy Q-convergence spaces is not investigated. This paper focuses on Cartesian-closedness of the category of L-fuzzy Q-convergence spaces, and it is shown that the category L-**QFCS** of L-fuzzy Q-convergence spaces is Cartesian-closed.

1. Introduction

In 1968, Chang [2] introduced fuzzy set theory to topology. Since then many researchers have successfully generalized the theory of general topology to the fuzzy setting in different directions. For example, Chang's I-topology is a crisp subset of I-powerset I^X. Differently Höhle [6] presented the notion of a fuzzy topology as an I-subset of a powerset 2^X, which is called fuzzifying topology in [19]. Then Kubiak [8] and Šostak [16] independently extended Höhle′s notion to an L-subset of L^X respectively, and the resulting topology is called L-fuzzy topology.

The category **Top** of topological spaces and continuous maps is not Cartesian closed, and then there is no natural function space structure. The category of convergence spaces, which is not only Cartesian closed, but also can embed **Top** as a reflective subcategory, can overcome this deficiency [15, 4]. Cartesian closedness, as a desired structural property in the category of convergence spaces, is the main reason for its prominent role in topological spaces. Convergence theory of filters provides a good tool for interpreting topological structures.

With the development of fuzzy topology, many researchers have extended convergence structures to the lattice-valued setting [7, 12, 14, 13, 17, 18]. The resulting categories have the desired structural property of Cartesian-closedness and contain the categories of suitable fuzzy topological spaces as reflective subcategories, respectively.

In 2011, Pang and Fang [10] introduced L-fuzzy Q-convergence structures and proved that the category of topological L-fuzzy Q-convergence spaces is isomorphic to that of L-fuzzy topological spaces. However, Cartesian-closedness of the category of L-fuzzy Q-convergence spaces is not investigated. A natural question is whether

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the category of L-fuzzy Q-convergence spaces is Cartesian-closed. In this paper, we give a positive answer to this problem.

2. Preliminaries

Throughout this paper, L denotes a completely distributive lattice and $'$ is an order-reversing involution on L. The smallest element and the largest element in L are denoted by 0 and 1, respectively. For $a, b \in L$, we say that a is wedge below b in L, in symbols $a \prec b$, if for every subset $D \subseteq L$, $b \leq \bigvee D$ implies $a \leq d$ for some $d \in D$. A complete lattice L is completely distributive if and only if $b = \bigvee\{a \in L | a \prec b\}$ for each $b \in L$. An element a in L is called co-prime if $a \leq b \vee c$ implies $a \leq c$ or $a \leq b$. The set of nonzero co-prime elements in L is denoted by $J(L)$.

For a nonempty set X, L^X denotes the set of all L-subsets on X. The smallest element and the largest element in L^X are denoted by 0_X and 1_X, respectively. L^X is also a completely distributive De Morgan algebra when it inherits the structure of the lattice L in a natural way. For each $x \in X$ and $a \in L$, we define the L-subset x_a by $x_a(y) = a$ if $y = x$, and $x_a(y) = 0$ otherwise. The L-subset x_a is called a fuzzy point of L^X. It is easy to see that $J(L^X) = \{x_\lambda | \lambda \in J(L), x \in X\}$. We say a fuzzy point x_λ quasi-coincides with A, denoted by $x_\lambda \hat{q} A$ [9], if $\lambda \not\leq A'(x)$ or equivalently $x_\lambda \not\leq A'$. Otherwise, we say x_λ does not quasi-coincide with A, denoted by $x_\lambda \neg\hat{q} A$.

In this paper, we assume that 0 is prime.

Definition 2.1. [5, 7] A mapping $\mathcal{F} : L^X \to L$ is called an L-fuzzy filter on X if it satisfies:

(LF1) $\mathcal{F}(0_X) = 0$, $\mathcal{F}(1_X) = 1$;
(LF2) $\mathcal{F}(A \wedge B) = \mathcal{F}(A) \wedge \mathcal{F}(B)$.

The family of all L-fuzzy filters on X is denoted by $\mathcal{F}_L(X)$. On the set $\mathcal{F}_L(X)$, we define an order by $\mathcal{F} \leq \mathcal{G}$ if $\mathcal{F}(A) \leq \mathcal{G}(A)$ for all $A \in L^X$. Every nonempty family $\{\mathcal{F}_i\}_{i \in I}$ of L-fuzzy filters has an infimum $\bigwedge_{i \in I} \mathcal{F}_i$, which can be calculated as $\forall A \in L^X$,

$$(\bigwedge_{i \in I} \mathcal{F}_i)(A) = \bigwedge_{i \in I} \mathcal{F}_i(A).$$

Example 2.2. [10] For $x_\lambda \in J(L^X)$, we define $\hat{q}(x_\lambda) : L^X \to L$ as follows:

$$\forall A \in L^X, \hat{q}(x_\lambda)(A) = \begin{cases} 1, & x_\lambda \hat{q} A; \\ 0, & \text{otherwise.} \end{cases}$$

Then $\hat{q}(x_\lambda)$ is an L-fuzzy filter on X, called a fuzzy point L-fuzzy filter on X.

Definition 2.3. [5, 7] Let $\mathcal{F} \in \mathcal{F}_L(X)$ and $f : X \to Y$ be a mapping. Then $f^{\Rightarrow}(\mathcal{F}) : L^Y \to L$, $A \mapsto \mathcal{F}(f^{\leftarrow}(A))$ is an L-fuzzy filter on Y and is called the image of $\mathcal{F}$ under f.

Lemma 2.4. [5, 7] (1) *Suppose* $\{\mathcal{F}_i\}_{i \in I} \subseteq \mathcal{F}_L(X)$ *and* $f : X \to Y$. *Then* $f^{\Rightarrow}(\bigwedge_{i \in I} \mathcal{F}_i) = \bigwedge_{i \in I} f^{\Rightarrow}(\mathcal{F}_i)$.

(2) *Let* $\mathcal{F} \in \mathcal{F}_L(X)$, $f : X \to Y$, $g : Y \to Z$. *Then* $(g \circ f)^{\Rightarrow}(\mathcal{F}) = g^{\Rightarrow} \circ f^{\Rightarrow}(\mathcal{F})$.

Lemma 2.5. [5, 7] *For a nonempty family $\{\mathcal{F}_i\}_{i\in I}$ of L-fuzzy filters, the following are equivalent:*

(1) *There exists an L-fuzzy filter $\mathcal{F}$ such that $\mathcal{F} \geq \mathcal{F}_i$ for all $i \in I$.*

(2) $\mathcal{F}_{i_1}(A_1) \wedge ... \wedge \mathcal{F}_{i_n}(A_n) = 0$ *if* $A_1 \wedge ... \wedge A_n = 0$ $(n \in \mathbf{N},\ A_1, ..., A_n \in L^X,\ i_1, ..., i_n \in I)$

In the case of existence, for a nonempty family $\{\mathcal{F}_i\}_{i\in I}$, the supremum in $(\mathcal{F}_L(X), \leq)$ is given by

$$\bigvee_{i\in I} \mathcal{F}_i(A) = \bigvee_{n\in \mathbf{N}} \{\mathcal{F}_{i_1}(A_1) \wedge ... \wedge \mathcal{F}_{i_n}(A_n) \mid A_1 \wedge ... \wedge A_n \leq A\}.$$

Let $\mathcal{F} \in \mathcal{F}_L(Y)$ and $f : X \to Y$ be a mapping. Then $f^{\Leftarrow}(\mathcal{F}) : L^X \to L$ defined by $f^{\Leftarrow}(\mathcal{F})(A) = \bigvee_{f^{\leftarrow}(B)\leq A} \mathcal{F}(B)$ is an L-fuzzy filter if and only if $\forall B \in L^Y$, $f^{\leftarrow}(B) = 0_X$ implies $\mathcal{F}(B) = 0$.

Lemma 2.6. [5, 7] *Let $\{X_i\}_{i\in I}$ be a family of nonempty sets, $p_j : \prod_{i\in I} X_i \to X_j$ the projection mapping and $\mathcal{F}_i \in \mathcal{F}_L(X_i)(\forall i \in I)$. Then $\bigvee_{i\in I} p_i^{\Leftarrow}(\mathcal{F}_i) \in \mathcal{F}_L(\prod_{i\in I} X_i)$, which is called the product L-fuzzy filter and denoted by $\prod_{i\in I} \mathcal{F}_i$.*

For two L-fuzzy filters $\mathcal{F}$ and $\mathcal{G}$, we write $\mathcal{F} \times \mathcal{G}$.

Lemma 2.7. [5, 7] *Let $\{X_i\}_{i\in I}$ be a family of nonempty sets, $p_j : \prod_{i\in I} X_i \to X_j$ the projection mapping, $\mathcal{F}_i \in \mathcal{F}_L(X_i)(\forall i \in I)$ and $\mathcal{F} \in \mathcal{F}_L(\prod_{i\in I} X_i)$. Then the following statements hold:*

(1) $\prod_{i\in I} p_i^{\Rightarrow}(\mathcal{F}) \leq \mathcal{F}$;

(2) $p_j^{\Rightarrow}(\prod_{i\in I} \mathcal{F}_i) \geq \mathcal{F}_j, \forall j \in I$;

(3) $p_j^{\Rightarrow}(\prod_{i\in I} p_i^{\Rightarrow}(\mathcal{F})) = \mathcal{F}_j, \forall j \in I$.

Lemma 2.8. [5, 7] *Let $\mathcal{F} \in \mathcal{F}_L(X_1)$, $\mathcal{G} \in \mathcal{F}_L(X_2)$, $f : X_1 \to Y_1$ and $g : X_2 \to Y_2$ be mappings. Then $(f \times g)^{\Rightarrow}(\mathcal{F} \times \mathcal{G}) = f^{\Rightarrow}(\mathcal{F}) \times g^{\Rightarrow}(\mathcal{G})$.*

Definition 2.9. [1] A category $\mathbf{C}$ is called a topological category over $\mathbf{Set}$ provided that for any set X, any class J, any family $((X_j, \xi_j))_{j\in J}$ of $\mathbf{C}$-objects and any family $(f_j : X \to X_j)_{j\in J}$ of mappings, there exists a unique $\mathbf{C}$-structure ξ on X which is initial with respect to the source $(f_j : X \to (X_j, \xi_j))_{j\in J}$. This means that for a $\mathbf{C}$-object (Y, η), a mapping $g : (Y, \eta) \to (X, \xi)$ is a $\mathbf{C}$-morphism if and only if for all $j \in J$, $f_j \circ g : (Y, \eta) \to (X_j, \xi_j)$ is a $\mathbf{C}$-morphism.

For more notions related to category theory, we refer to [1].

Definition 2.10. [10] An L-fuzzy Q-convergence structure on X is defined to be a mapping $c : \mathcal{F}_L(X) \to L^X$ such that $\forall x_\lambda \in J(L^X)$, $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L(X)$,

(LFQC1) $x_\lambda \leq c(\hat{q}(x_\lambda))$;

(LFQC2) $\mathcal{F} \leq \mathcal{G} \Rightarrow c(\mathcal{F}) \leq c(\mathcal{G})$.

The pair (X, c) is called an L-fuzzy Q-convergence space, and it will be called pretopological if it satisfies

(LFQC3) $x_\lambda \leq c(\mathcal{Q}^c_{x_\lambda})$, where $\mathcal{Q}^c_{x_\lambda} = \bigwedge_{x_\lambda \leq c(\mathcal{F})} \mathcal{F}$.

The pair (X, c) is called a topological L-fuzzy Q-convergence space if it satisfies moreover, (LFQC4) $\mathcal{Q}^c_{x_\lambda}(A) = \bigvee_{x_\lambda \hat{q} B \le A} \bigwedge_{y_\mu \hat{q} B} Q^c_{y_\mu}$.

A continuous mapping between two L-fuzzy Q-convergence spaces (X, c) and (Y, d) is a mapping $f : X \to Y$ such that for all $\mathcal{F} \in \mathcal{F}_L(X)$, $c(\mathcal{F}) \le f^{\leftarrow}(d(f^{\Rightarrow}(\mathcal{F})))$.

The category of L-fuzzy Q-convergence spaces and continuous mappings is denoted by L-**QFCS**. L-**QFTCS** denotes the full subcategory of L-**QFCS** consisting of topological L-fuzzy Q-convergence spaces.

Let (X, c) be an L-fuzzy Q-convergence space. Then for each $x_\lambda \in J(L^X)$, $\mathcal{Q}^c_{x_\lambda} = \bigwedge_{x_\lambda \le c(\mathcal{F})} \mathcal{F}$ is an L-fuzzy filter on X satisfying $\mathcal{Q}^c_{x_\lambda} \le \hat{q}(x_\lambda)$, and $\mathcal{Q}^c_{x_\lambda}(A) = 0$ whenever $x_\lambda \neg \hat{q} A$.

Theorem 2.11. [11] L-**QFCS** *is topological over* **Set***.*

Definition 2.12. [11] Let $\{(X_i, c_i)\}_{i \in I}$ be a family of L-fuzzy Q-convergence spaces and $\{p_j : \prod_{i \in I} X_i \to (X_j, c_j)\}_{j \in I}$ be the source formed by the family of the projection mappings $\{p_j : \prod_{i \in I} X_i \to X_j\}_{j \in I}$. The L-fuzzy Q-convergence structure on $X = \prod_{i \in I} X_i$, denoted by $\prod_{i \in I} c_i$, which is initial with respect to $\{p_j : X = \prod_{i \in I} X_i \to (X_j, c_j)\}_{j \in I}$, is called the product L-fuzzy Q-convergence structure and the pair $(X, \prod_{i \in I} c_i)$ is called the product space briefly. Thus for each $\mathcal{F} \in \mathcal{F}_L(X)$ and $x \in X$, we have

$$(\prod_{i \in I} c_i)(\mathcal{F})(x) = \bigwedge_{i \in I} c_i(p_i^{\Rightarrow}(\mathcal{F}))(p_i(x)).$$

For the product space of L-fuzzy Q-convergence spaces (X, c_X) and (Y, c_Y), we write $(X \times Y, c_X \times c_Y)$ for $(X, c_X) \times (Y, c_Y)$.

Example 2.13. Let X be a nonempty set.

We define $c^*(\mathcal{F}) = 1_X$ for all $\mathcal{F} \in \mathcal{F}_L(X)$. Then c^* is an L-fuzzy Q-convergence structure on X, called the indiscrete convergence structure on X.

We define $c_*(\mathcal{F}) = 1_X$ if $\mathcal{F} \ge \hat{q}(x_\lambda)$, and $c_*(\mathcal{F}) = 0_X$ otherwise. Then it is an L-fuzzy Q-convergence structure on X.

3. L-QFCS is a Cartesian Closed Category

Recall a category **C** is called Cartesian-closed [1] provided that the following conditions are satisfied:
(1) For each pair(X, Y) of **C**-objects there exists a product $X \times Y$ in **C**,
(2) For each pair of **C**-objects X and Y, there exists a **C**-object Y^X (called power object) and a **C**-morphism $evX, Y : Y^X \times Y \longrightarrow Y$ (called evaluation morphism) such that for each C-object Z and each C-morphism $f : Z \times X \longrightarrow Y$, there exists a unique C-morphism $g : Z \longrightarrow Y^X$ such that $evX, Y \circ (g \times id_X) = f$.

In the sequel, we will show that L-**QFCS** is Cartesian-closed.

Definition 3.1. [15] Let **C** be a category. A family $\{f_i : B_i \to B\}_{i \in I}$ of **C**-morphisms is called an epi-sink in **C** provided that for any pair $\alpha, \beta : B \to D$ of **C**-morphisms such that $\alpha \circ f_i = \beta \circ f_i$ for each $i \in I$, it follows that $\alpha = \beta$.

Lemma 3.2. [15] *Let* $\mathbf{C}$ *be a topological category. Then the following are equivalent:*

(1) $\mathbf{C}$ *is Cartesian closed.*

(2) *For each* $\mathbf{C}$*-object* A *holds: for any final epi-sink* $\{f_i : B_i \to B\}_{i\in I}$ *in* $\mathbf{C}$, $\{id_A \times f_i : A \times B_i \to A \times B\}_{i\in I}$ *is a final epi-sink in* $\mathbf{C}$.

Lemma 3.3. *Let* (X, c_X) *be an* L*-fuzzy Q-convergence space and* $\{(X_i, c_i)\}_{i\in I}$ *be a family of* L*-fuzzy Q-convergence spaces. Then* $\{f_i : (X_i, c_i) \to (X, c_X)\}_{i\in I}$ *is an epi-sink if and only if* $X = \bigcup_{i\in I} f_i[X_i](f_i[X_i] = \{f_i(x_i) \mid x_i \in X_i\})$.

Proof. Sufficiency is easily checked.

Necessity. The cardinal number of X is denoted by $|X|$.

If $|X| < 2$, then the assertion is trivial.

If $|X| \geq 2$ and $X \neq \bigcup_{i\in I} f_i[X_i]$, then there exists $z_1 \in X \setminus \bigcup_{i\in I} f_i[X_i]$. Take $z_2 \in \bigcup_{i\in I} f_i[X_i]$, and let $Z = \{z_1, z_2\}$. Then endow Z with the indiscrete L-fuzzy Q-convergence structure c_Z. Define $\alpha : (X, c_X) \to (Z, c_Z)$ by $\alpha(x) = z_2$ for all $x \in X$ and $\beta : (X, c_X) \to (Z, c_Z)$ by $\beta(x) = z_2$, for $x \in \bigcup_{i\in I} f_i[X_i]$, $\beta(x) = z_1$, otherwise. Obviously, α, β are continuous. It is easy to check that $\alpha \circ f_i = \beta \circ f_i$ for all $i \in I$. However, $\alpha \neq \beta$, which contradicts the fact that $\{f_i : (X_i, c_i) \to (X, c_X)\}_{i\in I}$ is an epi-sink. □

Theorem 3.4. *Let* $\{(X_i, c_i)\}$ *be a family of* L*-fuzzy Q-convergence spaces and* X *be a nonempty set. If* $\{f_i : (X_i, c_i) \to X\}_{i\in I}$ *is a sink, then* $c_X : \mathcal{F}_L(X) \to L^X$ *defined by*

$$c_X(\mathcal{F})(x) = \begin{cases} \bigvee\limits_{f_i(x_i)=x} \bigvee\limits_{f_i^{\Rightarrow}(\mathcal{F}_i)\leq\mathcal{F}} c_i(\mathcal{F}_i)(x_i), & \forall\lambda \in J(L), \mathcal{F} \not\geq \hat{q}(x_\lambda); \\ \bigvee\limits_{\mathcal{F}\geq\hat{q}(x_\lambda)} \lambda, & \exists\lambda \in J(L), \mathcal{F} \geq \hat{q}(x_\lambda). \end{cases}$$

is the unique final structure with respect to the sink $\{f_i : (X_i, c_i) \to X\}_{i\in I}$.

Further, if $\{f_i : (X_i, c_i) \to (X, c_X)\}_{i\in I}$ *is a final epi-sink, then it holds that for each* $\mathcal{F} \in \mathcal{F}_L(X)$ *and* $x \in X$,

$$c_X(\mathcal{F})(x) \leq \bigvee_{f_i(x_i)=x} \bigvee_{f_i^{\Rightarrow}(\mathcal{F}_i)\leq\mathcal{F}} c_i(\mathcal{F}_i)(x_i),$$

where $x_i \in X_i$.

Proof. First, it is easy to check that c_X is an L-fuzzy Q-convergence structure on X.

Second, let (Y, c_Y) be an L-fuzzy Q-convergence space and $f : X \to Y$ be a mapping satisfying that $f \circ f_i : (X_i, c_i) \to (Y, c_Y)$ is continuous for all $i \in I$. We need only to show the continuity of f.

Take $\mathcal{F} \in \mathcal{F}_L(X)$ and $x \in X$. If $\mathcal{F} \geq \hat{q}(x_\lambda)$, then $f^{\Rightarrow}(\mathcal{F}) \geq f^{\Rightarrow}(\hat{q}(x_\lambda)) = \hat{q}(f(x)_\lambda)$. It follows that $c_Y(f^{\Rightarrow}(\mathcal{F}))(f(x)) \geq \bigvee\limits_{\mathcal{F}\geq\hat{q}(x_\lambda)} c_Y(\hat{q}(f(x)_\lambda))(f(x)) \geq \bigvee\limits_{\mathcal{F}\geq\hat{q}(x_\lambda)} \lambda = c_X(\mathcal{F})(x)$. If $\forall\lambda \in J(L)$, $\mathcal{F} \not\geq \hat{q}(x_\lambda)$, then

$$c_X(\mathcal{F})(x) = \bigvee_{f_i(x_i)=x} \bigvee_{f_i^{\Rightarrow}(\mathcal{F}_i)\leq\mathcal{F}} c_i(\mathcal{F}_i)(x_i)$$

$$\leq \bigvee_{f_i(x_i)=x} \bigvee_{f_i^{\Rightarrow}(\mathcal{F}_i)\leq\mathcal{F}} c_Y(f\circ f_i)^{\Rightarrow}(\mathcal{F}_i)(f\circ f_i(x_i))$$
$$\leq c_Y(f^{\Rightarrow}(\mathcal{F}))(f(x)).$$

As a consequence, f is continuous if and only if $f\circ f_i$ is continuous for all $i\in I$. Hence, we obtain that c_X is the final structure on X with respect to the sink $\{f_i:(X_i,c_i)\to X\}_{i\in I}$.

Finally, if $\{f_i:(X_i,c_i)\to(X,c_X)\}_{i\in I}$ is a final epi-sink, then by Lemma 3.3, $X=\bigcup_{i\in I} f_i[X_i]$. Thus for each $x\in X$ with $\mathcal{F}\geq\hat{q}(x_\lambda)$, there exist $i_0\in I$ and $x_{i_0}\in X_{i_0}$ such that $x=f_{i_0}(x_{i_0})$ and $f_{i_0}^{\Rightarrow}(\hat{q}(x_{i_0})_\lambda)=\hat{q}(f_{i_0}(x_{i_0})_\lambda)=\hat{q}(x_\lambda)\leq\mathcal{F}$. Thus

$$\bigvee_{f_i(x_i)=x} \bigvee_{f_i^{\Rightarrow}(\mathcal{F}_i)\leq\mathcal{F}} c_i(\mathcal{F}_i)(x_i)$$
$$\geq \bigvee_{f_{i_0}^{\Rightarrow}(\mathcal{F}_{i_0})\leq\mathcal{F}} c_{i_0}(\mathcal{F}_{i_0})(x_{i_0})$$
$$\geq \bigvee_{\mathcal{F}\geq\hat{q}(x_\lambda)} c_{i_0}(\hat{q}(x_{i_0})_\lambda)(x_{i_0})$$
$$\geq \bigvee_{\mathcal{F}\geq\hat{q}(x_\lambda)} \lambda = c_X(\mathcal{F})(x).$$

So we obtain that

$$c_X(\mathcal{F})(x)\leq \bigvee_{f_i(x_i)=x} \bigvee_{f_i^{\Rightarrow}(\mathcal{F}_i)\leq\mathcal{F}} c_i(\mathcal{F}_i)(x_i)$$

for all $\mathcal{F}\in\mathcal{F}_L(X)$ and $x\in X$ whenever $\{f_i:(X_i,c_i)\to(X,c_X)\}_{i\in I}$ is a final epi-sink. □

Theorem 3.5. *The category of L-fuzzy Q-convergence spaces is Cartesian closed.*

Proof. Suppose that $\{f_i:(X_i,c_i)\to(X,c_X)\}_{i\in I}$ is a final epi-sink in L-**QFCS**. For each L-fuzzy Q-convergence space (Y,c_Y), since $\{f_i:(X_i,c_i)\to(X,c_X)\}_{i\in I}$ is a final epi-sink, by Lemma 3.3, we know $X=\bigcup\limits_{i\in I} f_i[X_i]$. Thus it is easy to check that $Y\times X=\bigcup\limits_{i\in I} id_Y\times f_i[Y\times X_i]$. By Lemma 3.3 again, we know the family $\{id_Y\times f_i:(Y\times X_i,c_Y\times c_i)\to(Y\times X,c_Y\times c_X)\}_{i\in I}$ is a final epi-sink in L-**QFCS**.

Next, we show $c_Y\times c_X$ is the final structure with respect to the sink $\{id_Y\times f_i:(Y\times X_i,c_Y\times c_i)\to(Y\times X,c_Y\times c_X)\}_{i\in I}$.

For each L-fuzzy Q-convergence space (Z,c_Z) and each mapping $f:Y\times X\to Z$. We need to show that the continuity of $f\circ(id_Y\times f_i)$ for all $i\in I$ implies the continuity of f. Take any $\mathcal{H}\in\mathcal{F}_L(Y\times X)$ and $(y,x)\in Y\times X$, then

$$(c_Y\times c_X)(\mathcal{H})(y,x)$$
$$=c_Y(P_Y^{\Rightarrow}(\mathcal{H}))(y)\wedge c_X(P_X^{\Rightarrow}(\mathcal{H}))(x)$$
$$\leq \bigvee_{f_i(x_i)=x} \bigvee_{f_i^{\Rightarrow}(\mathcal{F}_i)\leq P_X^{\Rightarrow}(\mathcal{H})} c_Y(P_Y^{\Rightarrow}(\mathcal{H}))(y)\wedge c_i(\mathcal{F}_i)(x_i)$$

$$\leq \bigvee_{f_i(x_i)=x} \bigvee_{f_i^{\Rightarrow}(\mathcal{F}_i)\leq P_X^{\Rightarrow}(\mathcal{H})} (c_Y \times c_i)(P_Y^{\Rightarrow}(\mathcal{H}) \times \mathcal{F}_i)(y, x_i)$$

$$\leq \bigvee_{f_i(x_i)=x} \bigvee_{f_i^{\Rightarrow}(\mathcal{F}_i)\leq P_X^{\Rightarrow}(\mathcal{H})} c_Z(f \circ (id_Y \times f_i))^{\Rightarrow}(P_Y^{\Rightarrow}(\mathcal{H}) \times \mathcal{F}_i)(f(y, f_i(x_i)))$$

$$= \bigvee_{f_i^{\Rightarrow}(\mathcal{F}_i)\leq P_X^{\Rightarrow}(\mathcal{H})} c_Z(f^{\Rightarrow}(P_Y^{\Rightarrow}(\mathcal{H}) \times f_i^{\Rightarrow}(\mathcal{F}_i))(f(y, x))$$

$$\leq c_Z(f^{\Rightarrow}(P_Y^{\Rightarrow}(\mathcal{H}) \times P_X^{\Rightarrow}(\mathcal{H}))(f(y, x))$$

$$\leq c_Z(f^{\Rightarrow}(\mathcal{H}))(f(y, x)).$$

This shows the continuity of f. Then it follows that the family $\{id_Y \times f_i : (Y \times X_i, c_Y \times c_i) \to (Y \times X, c_Y \times c_X)\}_{i\in I}$ is a final epi-sink in L-**QFCS**. By Lemma 3.2 and Theorem 2.11, we know that the category L-**QFCS** is Cartesian closed. □

Remark 3.6. Cartesian-closedness of L-**FQCS** implies the existence of the structures of function spaces. We wish to find their concrete forms in our future work.

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