A NEW ATTITUDE COUPLED WITH THE BASIC FUZZY THINKING TO DISTANCE BETWEEN TWO FUZZY NUMBERS

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Abstract. Fuzzy measures are suitable in analyzing human subjective evaluation processes. Several different strategies have been proposed for distance of fuzzy numbers. The distances introduced for fuzzy numbers can be categorized in two groups:
1. The crisp distances which explain crisp values for the distance between two fuzzy numbers.
2. The fuzzy distance which introduce a fuzzy distance for normal fuzzy numbers. It was introduced by Voxman [29] for the first time through using α-cut. However, both mentioned concepts can lead to unsatisfactory results from the applications point of view, but there is no method, which gives a satisfactory result to all situations. In this paper, a new attitude coupled with fuzzy thinking to the fuzzy distance function on the set of fuzzy numbers is proposed. In this new fuzzy distance, we considered both mentioned attitudes, then we introduced new fuzzy distance based on a combination (hybrid) of those two. Some properties of the proposed fuzzy distance have been discussed. Finally, several examples have been provided to explain the application of the proposed method and compare this methods with others.

1. Introduction

The issue of fuzzy distances has been explored in the past. Altman [5], for example, presented a fuzzy distance function, but his function yielded a non-fuzzy number, with which it is hard to perform subsequent mathematical operations. In the fuzzy set literature, however, Rosenfeld [22], Voxman [29], Bloch [7] and Guha and Chakrborty [18], among others have examined the issue. Most of these extend the concept of distance to subsets of a metric space, and they argue for the representation of fuzzy distance as fuzzy numbers. These studies tend to suggest many potential applications for different areas, including pattern recognition, image processing, robotics, computer graphics and engineering.
First motion of distance between two fuzzy sets was proposed by Kacprzyk [19]. Recently the researchers pay more attention to the distance between two fuzzy numbers [10] and there are many articles which have been used the fuzzy Distance [7, 8, 25, 26].
The distances introduced for fuzzy numbers can be categorized in two groups:
1. The crisp distances which explain crisp values for the distance between two fuzzy

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numbers. They were introduced earlier and have been used in clustering (Dursa and Giordani [15]); as well as Cheng [10], and Tran and Duckstein [28] who used it for ranking fuzzy numbers.
2. The fuzzy distance which introduce a fuzzy distance for normal fuzzy numbers. It was introduced by Voxman [29] for the first time through using α-cut."if we are not certain about the numbers themselves how can we are certain about the distances among them", hence it is not reasonable to define crisp distance between fuzzy objects. With this point of view Voxman first introduced the concept of fuzzy distance measure between two normal fuzzy numbers using the concept of α-cut and in 2006, Chakraborty et al. [18] were proposed another fuzzy distance in which the general fuzzy number was calculated by LR- Type fuzzy. This method uses α-cut and its main logical disadvantage is that this distance will be negative in some cases when calculated from the left point (Rouhparvar et al. [23]). To solve this problem Guha and Chakraborty [18] introduced a new distance for general fuzzy numbers through using the α cut concept if the distance is negative from the left point, the calculated distance will be estimated and changes to the difference between the centers of those fuzzy numbers. All the above mentioned distances have used the α-cut concept to calculate the fuzzy distance.
There are some other distances which have used other fuzzy concept. Chen and Wang [9] discussed the fuzzy distance for Trapezoidal fuzzy numbers. They used the graded mean integration which was discussed by Chen and Hsieh [11]. Shan-Huo and Chien-Chung [24] proposed a fuzzy distance based on fuzzy absolute value for triangular fuzzy number.
Therefore, we believe that a fuzzy thinking should be used to introduce the distance between fuzzy numbers in such a way that it is associated with their uncertainties. Thus, at first with a new attitude to the fuzzy distance function, we offer a set of fuzzy numbers as the fuzzy metric space. Then, we will present a new fuzzy distance as a combination (hydride) of two previous attitudes to the fuzzy distance.
Finally, in order to explain the new attitude, several basic properties and examples have been provided. In future literature, with this new fuzzy distance, a new attitude to fuzzy derivatives will be presented and we will have a new attitude to the fuzzy differential equations.
The paper is organized as follows. In section 2, we present the basic definitions and concepts related to the subject. In section 3, a new definition of fuzzy distance function and fuzzy metric space are provided and its basic properties are investigated in section 4. In section 5, the proposed method has been explained with examples and shows the results of comparing our method with others. Finally, conclusions and future research are drawn in section 6.

2. **Preliminaries**

In this section, some notations and background about the concept are brought.

Definition 2.1. [17] Let A be a fuzzy set in $\mathbb{R}$ (A=$\{(x, \mu_A(x)) | x \in \mathbb{R}\}$). Then,
i) A is called normal if there exists an $x \in \mathbb{R}$ such that $\mu_A(x) = 1$. Otherwise, A is subnormal,

ii) The support of A, denoted supp(A), is the subset of $\mathbb{R}$ whose elements all have nonzero membership grades in A. In other words,, $supp(A) = \{x \in \mathbb{R} | \mu_A(x) > 0\}$,
iii) An α-level set (or α-cut) of a fuzzy set A in $\mathbb{R}$ is a non-fuzzy set denoted by A_α and defined by

$$A_\alpha = \begin{cases} \{x \in \mathbb{R} | \mu_A(x) > \alpha\}, & \alpha > 0, \\ cl(suppA), & \alpha = 0, \end{cases} \tag{1}$$

where cl(suppA) denotes the closure of the support of A.

Definition 2.2. [17] A fuzzy set A in $\mathbb{R}$ is called a fuzzy number if it satisfies the following conditions
i) A is normal,
ii) $A_\alpha$is a closed interval for every $\alpha \in (0, 1]$,
iii) the support of A is bounded.

According to the above definition of fuzzy number and our emphasis on non-triangular and trapezoidal fuzzy numbers, we use definition of pseudo-geometric fuzzy numbers as follows:

Definition 2.3. [20] A fuzzy number $\widetilde{A}$ is called a pseudo-trapezoidal fuzzy number if its membership function $\mu_{\widetilde{A}}(x)$ is given by

$$\mu_{\widetilde{A}}(x) = \begin{cases} l_{\widetilde{A}}(x), & \underline{a} \leq x \leq a_1, \\ 1, & a_1 \leq x \leq a_2, \\ r_{\widetilde{A}}(x), & a_2 \leq x \leq \overline{a}, \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

Where $l_{\widetilde{A}}(x)$ and $r_{\widetilde{A}}(x)$ are nondecreasing and non increasing functions, respectively. The pseudo-trapezoidal fuzzy number $\widetilde{A}$ is denoted by

$$\widetilde{A} = (\underline{a}, a_1, a_2, \overline{a}, l_{\widetilde{A}}(x), r_{\widetilde{A}}(x)),$$

and the trapezoidal fuzzy number by

$$(\underline{a}, a_1, a_2, \overline{a}, -, -),$$

that, $-, -$ means $l_{\widetilde{A}}(x)$ and $r_{\widetilde{A}}(x)$ are linear.
A pseudo-triangular fuzzy number is a particular pseudo-trapezoidal fuzzy number, when the $a_1 = a_2$. The pseudo-triangular fuzzy number $\widetilde{A}$ is denoted by

$$\widetilde{A} = (\underline{a}, a, \overline{a}, l_{\widetilde{A}}(x), r_{\widetilde{A}}(x)),$$

and the triangular fuzzy number by

$$(\underline{a}, a, \overline{a}, -, -).$$

Definition 2.4. [16] Two fuzzy numbers $\widetilde{A}$ and $\widetilde{B}$ are said to be equal (denoted $\widetilde{A} = \widetilde{B}$) if and only if

$$\forall x \in \mathbb{R}, \qquad \mu_{\widetilde{A}}(x) = \mu_{\widetilde{B}}(x). \tag{3}$$

Definition 2.5. [14] Let $X = (x_1, x_2, x_3, ..., x_n)$ and $Y = (y_1, y_2, y_3, ..., y_n)$ be two points in Euclidean n-Space, then the distance from X to Y, or from Y to X is given by:

$$d(X, Y) = (\Sigma_1^n (x_i - y_i)^2)^{\frac{1}{2}}.$$

3. A New Attitude to Distance Between Fuzzy Numbers

In the first step, we will point to a fuzzy numbers set of reference [4] as a fuzzy linear (total) ordered set (quasi-linear set). In other words, we suggest the fuzzy numbers set as a comparable set. Then, we propose a new fuzzy distance using the new attitude combined with fuzzy thinking. Finally, the new attitude based on a combination (hybrid) of two general attitude is to introduce fuzzy distance.

3.1. Description of the New Attitude to Ranking Fuzzy Numbers.

Ranking of fuzzy numbers is an important issue in the study of fuzzy set theory. In order to rank fuzzy numbers, one fuzzy number needs to be compared with the others but it is difficult to determine clearly which of them is larger or smaller. Numerous methods have been proposed in previous studies to rank fuzzy numbers. There is not a unique method for comparing fuzzy numbers. As a result, it is reasonable to expect that different ranking methods can produce different ranking order for the same sample of fuzzy numbers and some of them seem to be good in a particular context but not in general. Intricacies like these make ranking fuzzy numbers rather difficult.
It should be noted that, all content provided in this subsection has been given from reference [4] and the proof of all theorems and lemmas has also been mentioned in that.

Definition 3.1. Let $\widetilde{A}$ be a fuzzy number. Then,
i) $a.c(\widetilde{A}) = \frac{1}{2}\{min(core(\tilde{A})) + max(core(\tilde{A}))\}$,
ii) $\underline{support}(\widetilde{A}) = \{x \in \mathbb{R} \mid x \leq a.c(\widetilde{A})\}$,

iii) $\overline{support}(\widetilde{A}) = \{x \in \mathbb{R} \mid x \geq a.c(\widetilde{A})\}$,
iv) $\underline{p}_{\widetilde{A}} = \{x \in support(\widetilde{A}) \mid inf(support(\widetilde{A})) \leq x \leq min(core(\widetilde{A}))\}$,
v) $\overline{p}_{\widetilde{A}} = \{x \in support(\widetilde{A}) \mid max(core(\widetilde{A})) \leq x \leq sup(support(\widetilde{A}))\}$,
vi) $\mu_{\widetilde{A}}(x) = \begin{cases} \underline{\mu}_{\widetilde{A}}(x), & x \in \underline{p}_{\widetilde{A}}, \\ 1, & x \in core(\widetilde{A}), \\ \overline{\mu}_{\widetilde{A}}(x), & x \in \overline{p}_{\widetilde{A}}, \\ 0, & \text{otherwise.} \end{cases}$

Definition 3.2. Let $\widetilde{A}, \widetilde{B}$ be two pseudo-geometric fuzzy numbers. Then,
1) $\widetilde{A} \prec_{a.s} \widetilde{B}$ iff $a.c(\widetilde{A}) < a.c(\widetilde{B})$ and $inf(supp(\widetilde{B})) \leq inf(supp(\widetilde{A}))$,
2) $\widetilde{A} \prec_{b.s} \widetilde{B}$ iff $a.c(\widetilde{A}) < a.c(\widetilde{B})$ and $inf(supp(\widetilde{A})) \leq inf(supp(\widetilde{B})) \leq sup(supp(\widetilde{A}))$,
3) $\widetilde{A} \prec_{n.s} \widetilde{B}$ iff $a.c(\widetilde{A}) < a.c(\widetilde{B})$ and $inf(supp(\widetilde{B})) \geq sup(supp(\widetilde{A}))$,
4) $\widetilde{A} \prec_{a.b.s} \widetilde{B}$ iff $\widetilde{A} \prec_{a.s} \widetilde{B}$ and $\widetilde{A} \prec_{b.s} \widetilde{B}$,
5) $\widetilde{A} \prec_{b.n.s} \widetilde{B}$ iff $\widetilde{A} \prec_{b.s} \widetilde{B}$ and $\widetilde{A} \prec_{n.s} \widetilde{B}$,
6) $\widetilde{\mathbf{A}} \prec \widetilde{\mathbf{B}}$ **iff** $\widetilde{\mathbf{A}} \prec_{a.s} \widetilde{\mathbf{B}}$ **or** $\widetilde{\mathbf{A}} \prec_{b.s} \widetilde{\mathbf{B}}$ **or** $\widetilde{\mathbf{A}} \prec_{n.s} \widetilde{\mathbf{B}}$.

Remark 3.3. The $\widetilde{\mathbf{A}} \prec_{a.s} \widetilde{\mathbf{B}}$ and $\widetilde{\mathbf{A}} \prec_{n.s} \widetilde{\mathbf{B}}$ can not occur together.

Definition 3.4. Let $\widetilde{A}, \widetilde{B}$ be two pseudo-geometric fuzzy numbers. Then,
1) $\widetilde{A} \cong \widetilde{B}$ iff $a.c(\widetilde{A}) = a.c(\widetilde{B})$ and $inf(supp(\widetilde{B})) = inf(supp(\widetilde{A}))$,

2) $\widetilde{A} \precsim_{a.s} \widetilde{B}$ iff $a.c(\widetilde{A}) = a.c(\widetilde{B})$ and $inf(supp(\widetilde{B})) < inf(supp(\widetilde{A}))$,
3) $\widetilde{A} \precsim_{b.s} \widetilde{B}$ iff $a.c(\widetilde{A}) = a.c(\widetilde{B})$ and $inf(supp(\widetilde{B})) > inf(supp(\widetilde{A}))$,
4) $\widetilde{\mathbf{A}} \simeq \widetilde{\mathbf{B}}$ **iff** $\widetilde{\mathbf{A}} \cong \widetilde{\mathbf{B}}$ **or** $\widetilde{\mathbf{A}} \precsim_{a.s} \widetilde{\mathbf{B}}$ **or** $\widetilde{\mathbf{A}} \precsim_{b.s} \widetilde{\mathbf{B}}$.

Definition 3.5. (Fuzzy less than or approximation) Let $\widetilde{A}, \widetilde{B}$ be two pseudo-geometric fuzzy numbers. Then,

$$\widetilde{\mathbf{A}} \precsim \widetilde{\mathbf{B}} \text{ iff } \widetilde{\mathbf{A}} \prec \widetilde{\mathbf{B}} \text{ or } \widetilde{\mathbf{A}} \simeq \widetilde{\mathbf{B}}.$$

Example 3.6. Consider the four fuzzy numbers $A = (-4, 1, 2, -, -)$, $B = (\frac{-7}{4}, \frac{1}{4}, \frac{5}{4}, -, -)$, $C = (-7, 2, 3, -, -)$ and $D = (-2, 0, 1, 1, -, -)$, taken from paper[1], with the following Figure 1.

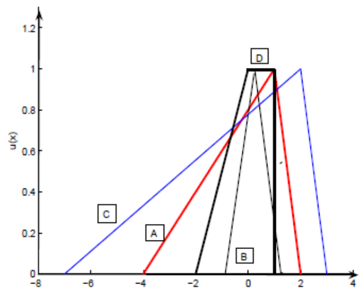

FIGURE 1. The Fuzzy Numbers of Example 3.6

Intuitively, the ranking order is $B \prec D \prec A \prec C$. Using our new approach, $B \prec_{a.s} D \prec_{a.s} A \prec_{a.s} C$. Thus the ranking order is $B \prec D \prec A \prec C$.

Although the proposed attitude for ranking fuzzy numbers is different from the other methods, however we use a comparative example to illustrate the advantage of the proposed method.

Example 3.7. Consider three triangular fuzzy numbers, $A = (5, 6, 7, -, -)$, $B = (5.9, 6, 7, -, -)$ and $C = (6, 6, 7, -, -)$, taken from paper[1], with the following Figure 2.

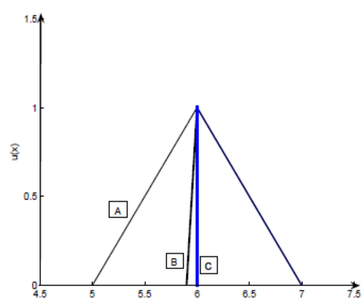

FIGURE 2. The Fuzzy Numbers of Example 3.7

Using our method, $A \precsim_{a.s} B \precsim_{b.s} C$.

Thus the ranking order is

$$A \simeq B \simeq C.$$

To compare with some other methods, the reader can refer to the following Table 1.

Fuzzy number	proposed method	Abbasbandy and Hajjari	Chu and Tsao	Cheng Distance	CV index
A	-	6.0000	3	6.021	0.028
B	-	6.0750	3.126	6.349	0.0098
C	-	6.0834	3.085	6.3519	0.0089
Results	$A \simeq B \simeq C$	$A \prec B \prec C$	$A \prec C \prec B$	$A \prec B \prec C$	$C \prec B \prec A$

TABLE 1. Comparative results of example 3.7

It should be noted that, in the above example, if the ambiguities of the fuzzy numbers be removed, then we do not have any equality between the numbers except in the proposed method.

Definition 3.8. A binary relation (here denoted by infix $\precsim$) on a set $\widetilde{X}$ of fuzzy sets is a fuzzy partial order if and only if it is
1) reflexive,
2) fuzzy anti-symmetric,
3) transitive.
The pair $(\widetilde{X}, \precsim)$ is called a fuzzy partially ordered set (fuzzy poset). Incidentally, for all $\widetilde{A}, \widetilde{B}, \widetilde{C} \in \widetilde{X}$:
a. $\widetilde{A} \precsim \widetilde{A}$, the reflexive property,
b. If $\widetilde{A} \precsim \widetilde{B}$ and $\widetilde{B} \precsim \widetilde{A}$ then $\widetilde{A} \simeq \widetilde{B}$, the fuzzy anti-symmetric property,
c. If $\widetilde{A} \precsim \widetilde{B}$ and $\widetilde{B} \precsim \widetilde{C}$ then $\widetilde{A} \precsim \widetilde{C}$, the transitive property.

(Note: A binary relation on a set $\widetilde{X}$ of fuzzy sets is a collection of ordered pairs of elements $\widetilde{X}$)

Definition 3.9. Any two elements $\widetilde{A}$ and $\widetilde{B}$ of a set $\widetilde{X}$ of fuzzy sets that is fuzzy partially ordered by a binary relation $\precsim$, are fuzzy comparable if either $\widetilde{A} \precsim \widetilde{B}$ or $\widetilde{B} \precsim \widetilde{A}$. otherwise they are called fuzzy incomparable.

Definition 3.10. A binary relation $\precsim$ on a set $\widetilde{X}$ of fuzzy sets is a fuzzy total order if and only if it is
1) a fuzzy partial order,
2) for any pair of elements $\widetilde{A}$ and $\widetilde{B}$ of $\widetilde{X}$, $\widetilde{A} \precsim \widetilde{B}$ or $\widetilde{B} \precsim \widetilde{A}$ (the totality property).
A fuzzy total order is also called a fuzzy linear order. Thus, a fuzzy totally ordered set is exactly a fuzzy poset in which every pair of elements is fuzzy comparable.

Currently, we offer a set of fuzzy numbers as a fuzzy linear ordered set and then we will explain its several properties.

Lemma 3.11. *Let $F_c(\mathbb{R})$ be a set of the pseudo-geometric fuzzy numbers, then* $\forall \widetilde{A}, \widetilde{B} \in F_c(\mathbb{R})$;
i) $\widetilde{A} \precsim \widetilde{B}$ *or* $\widetilde{B} \precsim \widetilde{A}$,
ii) $\widetilde{A} \precsim \widetilde{A}$,
iii) If $\widetilde{A} \precsim \widetilde{B}$ *and* $\widetilde{B} \precsim \widetilde{A}$ *then* $\widetilde{A} \simeq \widetilde{B}$,
iv) If $\widetilde{A} \precsim \widetilde{B}$ *and* $\widetilde{B} \precsim \widetilde{C}$ *then* $\widetilde{A} \precsim \widetilde{C}$.

Theorem 3.12. *Let $F_c(\mathbb{R})$ be a set of the pseudo-geometric fuzzy numbers, then the pair $(F_c(\mathbb{R}), \precsim)$ is a fuzzy totally ordered set.*

By $(F_c(\mathbb{R}), \precsim)$ we mean a fuzzy linear ordered set that is comparable in both components with the relation $\precsim$. In other words, $F_c(\mathbb{R})$ can be considered as a quasi-linear of its members.
In addition, we can suggest a parameter as an ambiguity rank, which can be as a comparison parameter with a crisp space. It is clear that each problem involved in at least a fuzzy number is fuzzy problem and, if all the fuzzy numbers involved is crisp, it is crisp problem. In this regard, the parameter as ambiguity rank for both arbitrary fuzzy numbers will be assigned. Then, using them, we define an ambiguity rank for the fuzzy problems, in order to compare with the crisp problems.

Definition 3.13. Let $F_c(\mathbb{R})$ be a set of pseudo-geometric fuzzy numbers. Then, ambiguity rank of $\widetilde{A} \precsim \widetilde{B}$ is defined by $ar = (l_{1\widetilde{A},\widetilde{B}}, r_{1\widetilde{A},\widetilde{B}}, l_{2\widetilde{A},\widetilde{B}}, r_{2\widetilde{A},\widetilde{B}})$ as follows:
$l_{1\widetilde{A},\widetilde{B}} = d(sup\{d(\underline{\mu}^{-1}_{\widetilde{A}}(\alpha), a.c(\widetilde{A})) \mid \alpha \in (0,1]\}, sup\{d(\underline{\mu}^{-1}_{\widetilde{B}}(\alpha), a.c(\widetilde{B})) \mid \alpha \in (0,1]\})$,
$r_{1\widetilde{A},\widetilde{B}} = d(sup\{d(\overline{\mu}^{-1}_{\widetilde{A}}(\alpha), a.c(\widetilde{A})) \mid \alpha \in (0,1]\}, sup\{d(\overline{\mu}^{-1}_{\widetilde{B}}(\alpha), a.c(\widetilde{B})) \mid \alpha \in (0,1]\})$,
$l_{2\widetilde{A},\widetilde{B}} = sup\{d(\underline{\mu}_{\widetilde{A}}(x), \underline{\mu}_{\widetilde{B}}(y)) \mid \forall x \in \underline{support}(\widetilde{A}), \forall y \in \underline{support}(\widetilde{B}); d(x, a.c(\widetilde{A})) = d(y, a.c(\widetilde{B}))\}$,
$r_{2\widetilde{A},\widetilde{B}} = sup\{d(\overline{\mu}_{\widetilde{A}}(x), \overline{\mu}_{\widetilde{B}}(y)) \mid \forall x \in \overline{support}(\widetilde{A}), \forall y \in \overline{support}(\widetilde{B}); d(x, a.c(\widetilde{A})) = d(y, a.c(\widetilde{B}))\}$.
We denote $\precsim$ with ambiguity rank by:

$$\begin{gathered} \widetilde{A} \precsim_{ar} \widetilde{B}, \\ s.t; \\ ar = (.,.,.,.). \end{gathered} \tag{4}$$

Definition 3.14. Let $F_c(\mathbb{R})$ be a set of pseudo-geometric fuzzy numbers. Then, ambiguity rank of the pair $(F_c(\mathbb{R}), \precsim)$ is defined by
$F_c ar = (l_{1F_c(\mathbb{R})}, r_{1F_c(\mathbb{R})}, l_{2F_c(\mathbb{R})}, r_{2F_c(\mathbb{R})})$ as follows:

$$\begin{aligned} l_{1F_c(\mathbb{R})} &= sup\{l_{1\widetilde{A},\widetilde{B}} \mid \widetilde{A}, \widetilde{B} \in F_c(\mathbb{R})\}, \\ r_{1F_c(\mathbb{R})} &= sup\{r_{1\widetilde{A},\widetilde{B}} \mid \widetilde{A}, \widetilde{B} \in F_c(\mathbb{R})\}, \\ l_{2F_c(\mathbb{R})} &= sup\{l_{2\widetilde{A},\widetilde{B}} \mid \widetilde{A}, \widetilde{B} \in F_c(\mathbb{R})\}, \\ r_{2F_c(\mathbb{R})} &= sup\{r_{2\widetilde{A},\widetilde{B}} \mid \widetilde{A}, \widetilde{B} \in F_c(\mathbb{R})\}. \end{aligned}$$

Remark 3.15. Another natural characteristic of a fuzzy totally ordered set is the number of elements it contains. We call this the order of fuzzy totally ordered set. This number is, of course, most interesting when it is finite. In that case we say that fuzzy totally ordered set is finite. Then, the ambiguity rank of a fuzzy totally ordered set will be defined as the change sup to max in definition (3.14).

Definition 3.16. Let $\widetilde{A}, \widetilde{B}$ be two fuzzy numbers. Then,
$\widetilde{A} \sim_1 \widetilde{B}$ if and only if
1. $\forall x \in \underline{support}(\widetilde{A}), \forall y \in \underline{support}(\widetilde{B})$;

$$if\ d(x, a.c(\widetilde{A})) = d(y, a.c(\widetilde{B}))\ then\ \underline{\mu}_{\widetilde{A}}(x) = \underline{\mu}_{\widetilde{B}}(x),$$

2. $\forall x \in \overline{support}(\widetilde{A}), \forall y \in \overline{support}(\widetilde{B})$;

$$if\ d(x, a.c(\widetilde{A})) = d(y, a.c(\widetilde{B}))\ then\ \overline{\mu}_{\widetilde{A}}(x) = \overline{\mu}_{\widetilde{B}}(x).$$

Lemma 3.17. *Let $\widetilde{A}, \widetilde{B}$ be two fuzzy numbers. Then,*

$$\widetilde{A} \simeq \widetilde{B}\ and\ \widetilde{A} \sim_1 \widetilde{B}\ if\ and\ only\ if\ \widetilde{A} = \widetilde{B}.$$

Remark 3.18. A type-1 fuzzy pair is a necessary condition for equality, but it is not sufficient condition. As an example,
Consider the fuzzy numbers
$A_1 = (1, 2, 4, 1 - (x-2)^2)^{\frac{1}{2}}, (1 - \frac{1}{4}(x-2)^2)^{\frac{1}{2}}),$
$A_2 = (1, 2, 3, -, -),$
$A_3 = (6, 7, 9, 1 - (x-7)^2)^{\frac{1}{2}}, (1 - \frac{1}{4}(x-7)^2)^{\frac{1}{2}}),$
$A_4 = (3, 4, 5, -, -),$
with the following Figure 3. We have,

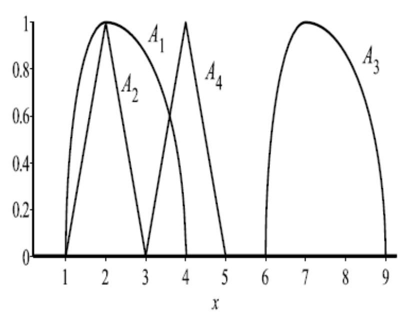

FIGURE 3. The Fuzzy Numbers of Remark 3.18

$$A_2 \sim_1 A_4 \qquad and \qquad A_1 \sim_1 A_3,$$

but,

$$A_2 \neq A_4 \qquad and \qquad A_1 \neq A_3.$$

Lemma 3.19. *Let $(F_c(\mathbb{R}), \precsim)$ be a fuzzy totally ordered set of the pseudo-geometric fuzzy numbers, with the ambiguity rank $F_c ar = (0,0,0,0)$. Then,*

$$\forall \widetilde{A}, \widetilde{B} \in F_c(\mathbb{R}); \qquad \widetilde{A} \sim_1 \widetilde{B}.$$

Now, with regard to the items mentioned above, we provide relationship between fuzzy linear ordered set and crisp linear ordered set, see the following theorem.

Theorem 3.20. *Let $(F_c(\mathbb{R}), \precsim)$ be a fuzzy totally ordered set of pseudo-geometric fuzzy numbers, with the ambiguity rank $F_c ar = (0,0,0,0)$. Then, $(F_c(\mathbb{R}), \precsim)$ is a totally ordered set.*

3.2. **Description of the New Attitude to Fuzzy Distance.**
By the proposed fuzzy ranking method, we will have a new attitude to the fuzzy distance. But before that it is necessary to point to applying a fuzzy arithmetic based on TA for pseudo-geometric fuzzy numbers, along with several related theorems and lemmas in [2, 3].
As regards to fuzzy arithmetic operations using of the extension principle (in the domain of membership function) or the interval arithmetics (in the domain of α-cuts), we have some problem in subtraction operator, division operator and obtaining the membership functions of operators. Although with the revised definitions on subtraction and division in [27] , usage of an interval arithmetic for fuzzy operators have been permitted, because it always exists, but its not efficient, it means that result's support is major agent (dependence effect) and also complex calculations of interval arithmetic in determining the membership function of operators based on the extension principle , are not yet resolved.
Therefore, we eliminated such deficiency with the fuzzy arithmetic operations based on TA for addition, subtraction, multiplication and division as follows:

Definition 3.21. [2] Consider two pseudo-trapezoidal fuzzy number

$$\widetilde{A} = (\underline{a}, a_1, a_2, \overline{a}, l_{\widetilde{A}}(x), r_{\widetilde{A}}(x)),$$
$$\widetilde{B} = (\underline{b}, b_1, b_2, \overline{b}, l_{\widetilde{B}}(x), r_{\widetilde{B}}(x)).$$

with the following α-cut forms:

$$\begin{array}{llll} \tilde{A} = \bigcup_\alpha A_\alpha, & A_\alpha = [\underline{A}_\alpha, \overline{A}_\alpha], & 0 < \alpha \leq 1, & A_1 = [a_1,\ a_2], \\ \tilde{B} = \bigcup_\alpha B_\alpha, & B_\alpha = [\underline{B}_\alpha, \overline{B}_\alpha], & 0 < \alpha \leq 1, & B_1 = [b_1,\ b_2]. \end{array}$$

Let

$$\phi = \frac{a_1 + a_2}{2}, \qquad \psi = \frac{b_1 + b_2}{2}.$$

In the following, we define fuzzy arithmetic operations based on TA for addition, subtraction, and multiplication

$$\widetilde{A+B} = \bigcup_\alpha (\widetilde{A+B})_\alpha,$$

$$(\widetilde{A+B})_\alpha = [\frac{\phi+\psi}{2} + (\frac{\underline{A}_\alpha + \underline{B}_\alpha}{2}),\ \frac{\phi+\psi}{2} + (\frac{\overline{A}_\alpha + \overline{B}_\alpha}{2})]. \tag{5}$$

$$\widetilde{-A}=\bigcup_{\alpha}(\widetilde{-A})_{\alpha},\qquad (\widetilde{-A})_{\alpha}=[-2\phi+\underline{A}_{\alpha},\ -2\phi+\overline{A}_{\alpha}]. \tag{6}$$

$$\widetilde{A-B}=\tilde{A}+\widetilde{-B},$$
$$\widetilde{A-B}=\bigcup_{\alpha}(\widetilde{A-B})_{\alpha},$$
$$(\widetilde{A-B})_{\alpha}=[\frac{\phi-3\psi}{2}+(\frac{\underline{A}_{\alpha}+\underline{B}_{\alpha}}{2}),\ \frac{\phi-3\psi}{2}+(\frac{\overline{A}_{\alpha}+\overline{B}_{\alpha}}{2})]. \tag{7}$$

$$\widetilde{A.B}=\bigcup_{\alpha}(\widetilde{A.B})_{\alpha},$$
$$(\widetilde{A.B})_{\alpha}=\begin{cases}[(\frac{\psi}{2})\underline{A}_{\alpha}+(\frac{\phi}{2})\underline{B}_{\alpha},\ (\frac{\psi}{2})\overline{A}_{\alpha}+(\frac{\phi}{2})\overline{B}_{\alpha}], & \phi\geq 0,\psi\geq 0,\\ [(\frac{\psi}{2})\overline{A}_{\alpha}+(\frac{\phi}{2})\underline{B}_{\alpha},\ (\frac{\psi}{2})\underline{A}_{\alpha}+(\frac{\phi}{2})\overline{B}_{\alpha}], & \phi\geq 0,\ \psi\leq 0,\\ [(\frac{\psi}{2})\overline{A}_{\alpha}+(\frac{\phi}{2})\overline{B}_{\alpha},\ (\frac{\psi}{2})\underline{A}_{\alpha}+(\frac{\phi}{2})\underline{B}_{\alpha}], & \phi\leq 0,\ \psi\leq 0,\\ [(\frac{\psi}{2})\underline{A}_{\alpha}+(\frac{\phi}{2})\overline{B}_{\alpha},\ (\frac{\psi}{2})\overline{A}_{\alpha}+(\frac{\phi}{2})\underline{B}_{\alpha}], & \phi\leq 0,\ \psi\geq 0.\end{cases} \tag{8}$$

$$\widetilde{A^{-1}}=\bigcup_{\alpha}(\widetilde{A^{-1}})_{\alpha},\qquad (\widetilde{A^{-1}})_{\alpha}=[(\frac{1}{\phi^2})\underline{A}_{\alpha},\ (\frac{1}{\phi^2})\overline{A}_{\alpha}]. \tag{9}$$

$$\widetilde{A.B^{-1}}=\bigcup_{\alpha}(\widetilde{A.B^{-1}})_{\alpha},$$
$$(\widetilde{A.B^{-1}})_{\alpha}=\begin{cases}[(\frac{1}{2\psi})\underline{A}_{\alpha}+(\frac{\phi}{2\psi^2})\underline{B}_{\alpha},\ (\frac{1}{2\psi})\overline{A}_{\alpha}+(\frac{\phi}{2\psi^2})\overline{B}_{\alpha}], & \phi\geq 0,\ \psi>0,\\ [(\frac{1}{2\psi})\overline{A}_{\alpha}+(\frac{\phi}{2\psi^2})\underline{B}_{\alpha},\ (\frac{1}{2\psi})\underline{A}_{\alpha}+(\frac{\phi}{2\psi^2})\overline{B}_{\alpha}], & \phi\geq 0,\ \psi<0,\\ [(\frac{1}{2\psi})\overline{A}_{\alpha}+(\frac{\phi}{2\psi^2})\overline{B}_{\alpha},\ (\frac{1}{2\psi})\underline{A}_{\alpha}+(\frac{\phi}{2\psi^2})\underline{B}_{\alpha}], & \phi\leq 0,\ \psi<0,\\ [(\frac{1}{2\psi})\underline{A}_{\alpha}+(\frac{\phi}{2\psi^2})\overline{B}_{\alpha},\ (\frac{1}{2\psi})\overline{A}_{\alpha}+(\frac{\phi}{2\psi^2})\underline{B}_{\alpha}], & \phi\leq 0,\ \psi>0.\end{cases} \tag{10}$$

Applying a fuzzy arithmetic based on TA for pseudo-triangular fuzzy numbers is a special case of the above when $a_1=a_2=a$ and $b_1=b_2=b$ (i.e. $\phi=a,\psi=b$).

Remark 3.22. Division on fuzzy numbers that, central core is zero ($a.c(\ .\ )=0$), is not definable.

Example 3.23. *Let*
$\tilde{A}=\bigcup_{\alpha}[\underline{A}_{\alpha},\ \overline{A}_{\alpha}],\qquad \underline{A}_{\alpha}=3-\sqrt{4-4\alpha},\qquad \overline{A}_{\alpha}=6-\alpha,$
$\tilde{B}=\bigcup_{\alpha}[\underline{B}_{\alpha},\ \overline{B}_{\alpha}],\qquad \underline{B}_{\alpha}=2+ln\alpha,\qquad \overline{B}_{\alpha}=4-ln\alpha.$

Then using the elementary fuzzy arithmetic operations based on the TA, we get:

$\widetilde{A+B}=\bigcup_{\alpha}(\widetilde{A+B})_{\alpha},\qquad (\widetilde{A+B})_{\alpha}=[\frac{7}{2}+\frac{5-\sqrt{4-4\alpha}+ln\alpha}{2},\frac{7}{2}+\frac{10-\alpha-ln\alpha}{2}],$
$\widetilde{-B}=\bigcup_{\alpha}(\widetilde{-B})_{\alpha},\qquad (\widetilde{-B})_{\alpha}=[-4+ln\alpha,-2-ln\alpha],$
$\widetilde{A-B}=\bigcup_{\alpha}(\widetilde{A-B})_{\alpha},\qquad (\widetilde{A-B})_{\alpha}=[\frac{-5}{2}+\frac{5-\sqrt{4-4\alpha}+ln\alpha}{2},\frac{-5}{2}+\frac{10-\alpha-ln\alpha}{2},$
$\widetilde{A.B}=\bigcup_{\alpha}(\widetilde{A.B})_{\alpha},\qquad (\widetilde{A.B})_{\alpha}=[\frac{3}{2}(3-\sqrt{4-4\alpha})+2(2+ln\alpha),\frac{3}{2}(6-\alpha)+2(4-ln\alpha)],$
$\widetilde{B^{-1}}=\bigcup_{\alpha}(\widetilde{B^{-1}})_{\alpha},\qquad (\widetilde{B^{-1}})_{\alpha}=[\frac{2+ln\alpha}{9},\frac{4-ln\alpha}{9}],$
$\widetilde{A.B^{-1}}=\bigcup_{\alpha}(\widetilde{A.B^{-1}})_{\alpha},\qquad (\widetilde{A.B^{-1}})_{\alpha}=[\frac{1}{6}(3-\sqrt{4-4\alpha})+\frac{2}{9}(2+ln\alpha),\frac{1}{6}(6-\alpha)+\frac{2}{9}(4-ln\alpha)].$

Example 3.24. *Let*
$\tilde{A} = \bigcup_\alpha [\underline{A}_\alpha, \overline{A}_\alpha]$, $\quad \underline{A}_\alpha = 2\alpha + 1$, $\quad \overline{A}_\alpha = 3 + \sqrt{1-\alpha}$,
$\tilde{B} = \bigcup_\alpha [\underline{B}_\alpha, \overline{B}_\alpha]$, $\quad \underline{B}_\alpha = (2\alpha + 2)^2$, $\quad \overline{B}_\alpha = (5-\alpha)^2$.

Then using the elementary fuzzy arithmetic operations based on the TA, we get:

$\widetilde{A+B} = \bigcup_\alpha (\widetilde{A+B})_\alpha$, $\quad (\widetilde{A+B})_\alpha = [\frac{19}{2} + \frac{2\alpha+1+(2\alpha+2)^2}{2}, \frac{19}{2} + \frac{3+\sqrt{1-\alpha}+(5-\alpha)^2}{2}]$,
$\widetilde{-B} = \bigcup_\alpha (\widetilde{-B})_\alpha$, $\quad (\widetilde{-B})_\alpha = [-32 + (2\alpha+2)^2,\ -32 + (5-\alpha)^2]$,
$\widetilde{A-B} = \bigcup_\alpha (\widetilde{A-B})_\alpha$, $\quad (\widetilde{A-B})_\alpha = [\frac{-45}{2} + \frac{2\alpha+1+(2\alpha+2)^2}{2}, \frac{-45}{2} + \frac{3+\sqrt{1-\alpha}+(5-\alpha)^2}{2}]$,
$\widetilde{A.B} = \bigcup_\alpha (\widetilde{A.B})_\alpha$, $\quad (\widetilde{A.B})_\alpha = [8(2\alpha+1) + \frac{3}{2}(2\alpha+2)^2,\ 8(3+\sqrt{1-\alpha}) + \frac{3}{2}(5-\alpha)^2]$,
$\widetilde{B^{-1}} = \bigcup_\alpha (\widetilde{B^{-1}})_\alpha$, $\quad (\widetilde{B^{-1}})_\alpha = [\frac{1}{256}(2\alpha+2)^2,\ \frac{1}{256}(5-\alpha)^2]$,

$\widetilde{A.B^{-1}} = \bigcup_\alpha (\widetilde{A.B^{-1}})_\alpha$, $\quad (\widetilde{A.B^{-1}})_\alpha = [\frac{1}{32}(2\alpha+1) + \frac{3}{512}(2\alpha+2)^2,\ \frac{1}{32}(3 + \sqrt{1-\alpha}) + \frac{3}{512}(5-\alpha)^2]$.

Theorem 3.25. [2] *Let $F_C(\mathbb{R})$ be a set of pseudo-geometric fuzzy numbers defined on the set of real numbers, then*
$\forall \tilde{A}\ \exists!\ \tilde{0}_{\tilde{A}}$ *such that* $\widetilde{A+0}_{\tilde{A}} = \widetilde{0_{\tilde{A}}+A} = \tilde{A}$ *and* $\widetilde{A-A} = \tilde{0}_{\tilde{A}}$,
$\forall \tilde{A}\ \exists!\ \tilde{1}_{\tilde{A}}$ *such that* $\widetilde{A.1}_{\tilde{A}} = \widetilde{1_{\tilde{A}}.A} = \tilde{A}$ *and* $\widetilde{A.A^{-1}} = \tilde{1}_{\tilde{A}}$, $\quad a.c(\tilde{A}) \neq 0$.

Proof. See in the reference mentioned. □

Lemma 3.26. *Let $F_c(\mathbb{R})$ be a set of the pseudo-geometric fuzzy numbers, then*
$\forall \widetilde{A}, \widetilde{B}, \widetilde{C} \in F_c(\mathbb{R})$;
1) $\widetilde{A} \sim_1 \widetilde{B} \Rightarrow \widetilde{A+B} \sim_1 \widetilde{A}$ *or* $\widetilde{A+B} \sim_1 \widetilde{B}$,
2) $\widetilde{A+B} = \widetilde{B+A}$,
3) $\widetilde{A.B} = \widetilde{B.A}$,
4) $A + \widetilde{(B+C)} \simeq \widetilde{(A+B)} + C$,
5) $A.\widetilde{(B.C)} \simeq \widetilde{(A.B)}.C$,
6) $A.\widetilde{(B+C)} \simeq \widetilde{(A.B) + (A.C)}$,
7) $\widetilde{(B+C)}.A \simeq \widetilde{(B.A) + (C.A)}$,
8) $\widetilde{-(-A)} = \widetilde{A}$,
9) $\widetilde{-1.A} \simeq \widetilde{-A}$, $-1.\widetilde{(-A)} \simeq \widetilde{A}$, $-1.\widetilde{(-1.A)} \simeq \widetilde{A}$.

Proof. See in the [2]. □

Lemma 3.27. *Let $F_c(\mathbb{R})$ be a set of pseudo-geometric fuzzy numbers, and* $F_0(\mathbb{R}) = \{\widetilde{A} \in F_c(\mathbb{R}) \mid a.c(\widetilde{A}) = 0\}$. *Then,*
$\forall \widetilde{A}, \widetilde{B}, \widetilde{C} \in F_c(\mathbb{R})$, $\quad \tilde{0} \in F_0(\mathbb{R})$;
1) if $\quad \widetilde{A} \precsim \widetilde{B} \quad$ *then* $\quad \widetilde{A \pm C} \precsim \widetilde{B \pm C}$,

2) if $\widetilde{A} \precsim \widetilde{B}$ *then*

$$\begin{cases} \widetilde{A.C} \precsim \widetilde{B.C}, & \widetilde{0} \prec \widetilde{C}, \\ \widetilde{B.C} \precsim \widetilde{A.C}, & \widetilde{C} \prec \widetilde{0}, \end{cases}$$

3) if $\widetilde{A} \precsim \widetilde{B}$ *and* $\widetilde{C} \precsim \widetilde{D}$ *then*

$$\widetilde{A+C} \precsim \widetilde{B+D},$$

4) if $\widetilde{0} \prec \widetilde{A} \precsim \widetilde{B}$ *and* $\widetilde{0} \prec \widetilde{C} \precsim \widetilde{D}$ *then*

$$\widetilde{A.C} \precsim \widetilde{B.D}.$$

Proof. See [4]. □

Example 3.28. Consider the fuzzy numbers
$A_1 = (1, 2, 4, 1-(x-2)^2)^{\frac{1}{2}}, (1-\frac{1}{4}(x-2)^2)^{\frac{1}{2}}),$
$A_2 = (1, 2, 4, -, -),$
$A_3 = (6, 7, 9, 1-(x-7)^2)^{\frac{1}{2}}, (1-\frac{1}{4}(x-7)^2)^{\frac{1}{2}}),$
$A_4 = (3, 4, 5, -, -),$
$A_5 = (1, 4, 6, \frac{x-1}{3}, (1-\frac{1}{4}(x-4)^2)^{\frac{1}{2}}).$
The fuzzy numbers are visualized in the following:
We have

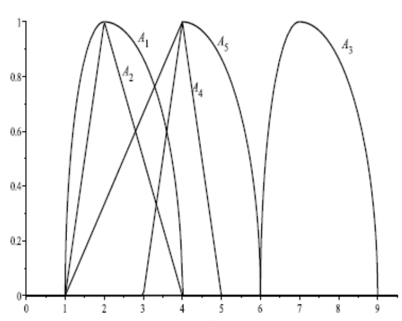

FIGURE 4. The Fuzzy Numbers of Example 3.28

$A_1 \sim_1 A_3$ and $A_1 \neq A_3$,
$A_1 + A_3 = (8, 9, 11, 1-(x-9)^2)^{\frac{1}{2}}, (1-\frac{1}{4}(x-9)^2)^{\frac{1}{2}})$ and $A_1 + A_3 \sim_1 A_1 (or A_3)$,
$-A_2 = (-3, -2, 0, -, -)$, $-1(-A_2) = (1, 2, \frac{5}{2}, -, -)$ and $-(-A_2) = (1, 2, 4, -, -)$,
namely, $-1(-A_2) \simeq A_2$ and $-(-A_2) = A_2$,
$A_1 + A_1 = (3, 4, 6, 1-(x-4)^2)^{\frac{1}{2}}, (1-\frac{1}{4}(x-4)^2)^{\frac{1}{2}}),$
$A_1 + A_3 = (8, 9, 11, 1-(x-9)^2)^{\frac{1}{2}}, (1-\frac{1}{4}(x-9)^2)^{\frac{1}{2}})$, then,
$A_1 + A_1 \precsim_{ar} A_1 + A_3, \qquad ar_{1A_1A_3} = (0, 0, 0, 0),$
$A_1 \simeq A_2, \qquad ar_{1A_1A_2} = (0, 0, 0.4142135622, 0.4142135620),$
$A_1 \precsim_{ar} A_3, \qquad ar_{1A_1A_3} = (0, 0, 0, 0),$
$A_4 \simeq A_5, \qquad ar_{1A_4A_5} = (2, 1, 2, 1.236067975)$, etc.

Theorem 3.29. *Let $F_c(\mathbb{R})$ be a set of the pseudo-geometric fuzzy numbers, then* $\forall \widetilde{A}, \widetilde{B} \in F_c(\mathbb{R})$;
1) If $\widetilde{A} \precsim \widetilde{B}$ *then* $\exists \widetilde{X}, \widetilde{Y} \in F_c(\mathbb{R})$; $\widetilde{A+X} \simeq \widetilde{B}$ *and* $\widetilde{B+Y} \simeq \widetilde{A}$,
2) If $\widetilde{A} \precsim \widetilde{B}$ *and* $\widetilde{A} \sim_1 \widetilde{B}$ *then* $\exists \widetilde{X}, \widetilde{Y} \in F_c(\mathbb{R})$; $\widetilde{A+X} = \widetilde{B}$ *and* $\widetilde{B+Y} = \widetilde{A}$.

Proof. See [4]. □

Finally, by fuzzy rankings and applying fuzzy arithmetic proposed in [4, 2], we have a new attitude to a fuzzy metric space as follows:

Definition 3.30. Let $\widetilde{X}$ be a set of fuzzy numbers on the universal set $\mathbb{R}$, $F_c(\mathbb{R})$ be a set of the pseudo-geometric fuzzy numbers, and $F_0(\mathbb{R}) = \{\widetilde{A} \in F_c(\mathbb{R}) \mid a.c(\widetilde{A}) = 0\}$. Then, a function $\tilde{D} : \widetilde{X} \times \widetilde{X} \to F_c(\mathbb{R})$ is called fuzzy metric on $\widetilde{X}$ if, for all $\tilde{x}, \tilde{y}, \tilde{z} \in \tilde{X}$ and $\tilde{0} \in F_0(\mathbb{R})$, it holds:
1. $\tilde{D}(\tilde{x}, \tilde{y}) \succ \tilde{0}$ (fuzzy non-negativity),
2. $\tilde{D}(\tilde{x}, \tilde{y}) \simeq \tilde{0}$ if and only if $\tilde{x} \simeq \tilde{y}$ (fuzzy separation or fuzzy self-identity axiom),
3. $\tilde{D}(\tilde{x}, \tilde{y}) = \tilde{D}(\tilde{y}, \tilde{x})$ (symmetry),
4. $\tilde{D}(\tilde{x}, \tilde{y}) \precsim \widetilde{D(\tilde{x}, \tilde{z}) + D(\tilde{z}, \tilde{y})}$ (fuzzy triangle inequality).

$(\widetilde{X}, \tilde{D})$ is called fuzzy metric space.

To introduce a pseudo-geometric fuzzy number as a fuzzy metric space, we offer the following Lemma and definition.

Definition 3.31. Let $\widetilde{A}$ be a pseudo-geometric fuzzy number. The absolute of $\widetilde{A}$ (here denoted by infix $\mid \widetilde{A} \mid$) is defined as follows:

$$\mid \widetilde{A} \mid = \mid a.c(\widetilde{A}) \mid \tag{11}$$

Definition 3.32. Let $\widetilde{A}$ be a pseudo-geometric fuzzy number. The fuzzy absolute of $\widetilde{A}$ (here denoted by infix $\int \widetilde{A} \int$) is defined as follows:

$$\int \widetilde{A} \int = \begin{cases} \widetilde{A}, & \widetilde{A} \succsim \widetilde{0}, \\ \widetilde{-A}, & \widetilde{A} \prec \widetilde{0}. \end{cases} \tag{12}$$

Lemma 3.33. *Let $F_c(\mathbb{R})$ be a set of the pseudo-geometric fuzzy numbers, then* $\forall \widetilde{A}, \widetilde{B}, \widetilde{C} \in F_c(\mathbb{R})$;
i) $\mid \widetilde{A} \mid = \mid \widetilde{-A} \mid$, $\int \widetilde{A} \int = \int \widetilde{-A} \int$,
ii) $\mid \widetilde{A} \mid \geq 0$, $\int \widetilde{A} \int \succsim \widetilde{0}$,
iii) If $\widetilde{A} \simeq \widetilde{B}$ *then* $\mid \widetilde{A} \mid = \mid \widetilde{B} \mid$ *and* $\int \widetilde{A} \int \simeq \int \widetilde{B} \int$

Proof. It is obvious. □

We now show that the set of the pseudo-geometric fuzzy numbers is a fuzzy metric space with a new attitude to the proposed fuzzy metric space. For this purpose, we purposed the following fuzzy distance function on the set of the pseudo-geometric fuzzy numbers set.

Lemma 3.34. *Let $\tilde{D_\star}(.,.)$ be a fuzzy function as follows:*

$$\tilde{D_\star} : F_c(\mathbb{R}) \times F_c(\mathbb{R}) \to F_c(\mathbb{R})$$

$$\tilde{D_\star}(\tilde{A},\tilde{B}) = \begin{cases} \mid \widetilde{A-B} \mid, & \widetilde{A} \sim_1 \widetilde{B}, \\ \int \widetilde{A-B} \int, & \widetilde{A} \nsim_1 \widetilde{B}. \end{cases} \tag{13}$$

Then, $(F_c(\mathbb{R}), \tilde{D_\star})$ is a fuzzy metric space.

Proof. We have the above cases, according to the related definitions. □

4. **Theorems and Properties**

In what follows present some properties of proposed distance that are useful in fuzzy mathematics problems. We have a new attitude to fuzzy neighbors and using this to the fuzzy distance, that will have a major application in future discussions (especially fuzzy derivatives). In addition, we introduce fuzzy distance ambiguity rank as a comparison parameter with corresponding crisp distance.

Lemma 4.1. *Let $F_c(\mathbb{R})$ be a set of the pseudo-geometric fuzzy numbers, and $\tilde{D_\star}(.,.)$ be a fuzzy metric from the lemma (3.34). Then*
$\forall \widetilde{A}, \widetilde{B}, \widetilde{C} \in F_c(\mathbb{R})$;
1) $\tilde{D_\star}(0,\tilde{A}) \simeq \int \tilde{A} \int$,
2) $\tilde{D_\star}(\tilde{0}_A,\tilde{A}) = \mid \tilde{A} \mid$,
3) $\tilde{D_\star}(\tilde{A},\tilde{A}) = 0$,
4) $\tilde{D_\star}(\widetilde{A+C},\widetilde{B+C}) \simeq \tilde{D_\star}(\tilde{A},\tilde{B})$,
5) If $\widetilde{A} \sim_1 \widetilde{B}$ *then* $\tilde{D_\star}(\widetilde{A+C},\widetilde{B+C}) = \tilde{D_\star}(\tilde{A},\tilde{B})$,
6) If $\widetilde{A} \sim_1 \widetilde{B}$ *then* $\tilde{D_\star}(\tilde{A},\tilde{B}) \sim_1 \widetilde{A}$ *or* $\tilde{D_\star}(\tilde{A},\tilde{B}) \sim_1 \widetilde{B}$.

Proof. We have the above cases, according to the related definitions. □

Definition 4.2. $\forall \widetilde{A}, \widetilde{\epsilon} \in F_c(\mathbb{R})$;
i) $\widetilde{N}_{\widetilde{\epsilon}}(\widetilde{A}) = \{\widetilde{B} \in F_c(\mathbb{R}) \mid \tilde{D_\star}(\tilde{A},\tilde{B}) \prec \widetilde{\epsilon}\}$, (Fuzzy neighborhood)
ii) $\widetilde{N1}_{\widetilde{\epsilon}}(\widetilde{A}) = \{\widetilde{B} \in F_c(\mathbb{R}) \mid \widetilde{B} \sim_1 \widetilde{A} \; and \; \tilde{D_\star}(\tilde{A},\tilde{B}) \prec \widetilde{\epsilon}\}$. (Type-1 fuzzy par neighborhood)

Definition 4.3. Let $F_c(\mathbb{R})$ be a set of pseudo-geometric fuzzy numbers. Then, ambiguity rank of $\tilde{D_\star}(\tilde{A},\tilde{B})$ is defined with $ar = (l_{1\widetilde{A},\widetilde{B}}, r_{1\widetilde{A},\widetilde{B}}, l_{2\widetilde{A},\widetilde{B}}, r_{2\widetilde{A},\widetilde{B}})$ as follows:
$l_{1\widetilde{A},\widetilde{B}} = d(sup\{d(\underline{\mu}^{-1}_{\widetilde{A}}(\alpha), a.c(\widetilde{A})) \mid \alpha \in (0,1]\}, sup\{d(\underline{\mu}^{-1}_{\widetilde{B}}(\alpha), a.c(\widetilde{B})) \mid \alpha \in (0,1]\})$,
$r_{1\widetilde{A},\widetilde{B}} = d(sup\{d(\overline{\mu}^{-1}_{\widetilde{A}}(\alpha), a.c(\widetilde{A})) \mid \alpha \in (0,1]\}, sup\{d(\overline{\mu}^{-1}_{\widetilde{B}}(\alpha), a.c(\widetilde{B})) \mid \alpha \in (0,1]\})$,
$l_{2\widetilde{A},\widetilde{B}} = sup\{d(\underline{\mu}_{\widetilde{A}}(x), \underline{\mu}_{\widetilde{B}}(y)) \mid \forall x \in \underline{support}(\widetilde{A}), \forall y \in \underline{support}(\widetilde{B}); d(x, a.c(\widetilde{A})) = d(y, a.c(\widetilde{B}))\}$,
$r_{2\widetilde{A},\widetilde{B}} = sup\{d(\overline{\mu}_{\widetilde{A}}(x), \overline{\mu}_{\widetilde{B}}(y)) \mid \forall x \in \overline{support}(\widetilde{A}), \forall y \in \overline{support}(\widetilde{B}); d(x, a.c(\widetilde{A})) = d(y, a.c(\widetilde{B}))\}$.

We denote the $\tilde{D}_{\star ar}(\ .\ ,\ .\ )$ with ambiguity rank by:

$$\begin{aligned} &\tilde{D}_{\star ra}(\ .\ ,\ .\ ),\\ &s.t;\\ &ar = (.,.,.,.). \end{aligned} \tag{14}$$

Definition 4.4. Let $F_c(\mathbb{R})$ be a set of pseudo-geometric fuzzy numbers. Then, ambiguity rank of the pair $(F_c(\mathbb{R}), \tilde{D}_\star)$ is defined with
$F_c ar = (l_{1F_c(\mathbb{R})}, r_{1F_c(\mathbb{R})}, l_{2F_c(\mathbb{R})}, r_{2F_c(\mathbb{R})})$ as follows:

$$\begin{aligned} l_{1F_c(\mathbb{R})} &= sup\{l_{1\widetilde{A},\widetilde{B}} \mid\ \widetilde{A},\widetilde{B}\ \in\ F_c(\mathbb{R})\},\\ r_{1F_c(\mathbb{R})} &= sup\{r_{1\widetilde{A},\widetilde{B}} \mid\ \widetilde{A},\widetilde{B}\ \in\ F_c(\mathbb{R})\},\\ l_{2F_c(\mathbb{R})} &= sup\{l_{2\widetilde{A},\widetilde{B}} \mid\ \widetilde{A},\widetilde{B}\ \in\ F_c(\mathbb{R})\},\\ r_{2F_c(\mathbb{R})} &= sup\{r_{2\widetilde{A},\widetilde{B}} \mid\ \widetilde{A},\widetilde{B}\ \in\ F_c(\mathbb{R})\}. \end{aligned}$$

Remark 4.5. Another natural characteristic of a fuzzy metric space is the number of elements it contains. We call this the order of fuzzy metric space. This number is, of course, most interesting when it is finite. In that case we say that fuzzy metric space is finite. Then, ambiguity rank of fuzzy metric space will be defined with the change sup to max in the definition (4.4).

Lemma 4.6. *Let $(F_c(\mathbb{R}), \tilde{D}_\star)$ be a fuzzy metric space of the pseudo-geometric fuzzy numbers, with the ambiguity rank $F_c ar = (0,0,0,0)$. Then,*

$$\forall \widetilde{A},\widetilde{B} \in F_c(\mathbb{R}); \qquad \widetilde{A} \sim_1 \widetilde{B}.$$

Proof. Suppose $\widetilde{A},\widetilde{B} \in F_c(\mathbb{R})$.
Then, we have from $(l_{1F_c(\mathbb{R})}, r_{1F_c(\mathbb{R})}) = (0,0) that:$
$sup\{d(\underline{\mu}_{\widetilde{A}}^{-1}(\alpha), a.c(\widetilde{A})) \mid \alpha \in (0,1]\} = sup\{d(\underline{\mu}_{\widetilde{B}}^{-1}(\alpha), a.c(\widetilde{B})) \mid \alpha \in (0,1]\}$,
$sup\{d(\overline{\mu}_{\widetilde{A}}^{-1}(\alpha), a.c(\widetilde{A})) \mid \alpha \in (0,1]\} = sup\{d(\overline{\mu}_{\widetilde{B}}^{-1}(\alpha), a.c(\widetilde{B})) \mid \alpha \in (0,1]\}$.
So,
$\mid\ \{x \in \underline{support}(\widetilde{A}) \mid \underline{\mu}_{\widetilde{A}}(x) > 0\}\ \mid=\mid\ \{x \in \underline{support}(\widetilde{B}) \mid \underline{\mu}_{\widetilde{B}}(x) > 0\}\ \mid$,
$\mid\ \{x \in \overline{support}(\widetilde{A}) \mid \overline{\mu}_{\widetilde{A}}(x) > 0\}\ \mid=\mid\ \{x \in \overline{support}(\widetilde{B}) \mid \overline{\mu}_{\widetilde{B}}(x) > 0\}\ \mid$.
(Note: $\mid \cdot \mid = sup(\cdot) - inf(\cdot)$)
Therefore, have Prerequisite for $\widetilde{A} \sim_1 \widetilde{B}$.
Also, we have from $(l_{2F_c(\mathbb{R})}, r_{2F_c(\mathbb{R})}) = (0,0) that:$
$\forall x \in \underline{support}(\widetilde{A}), \forall y \in \underline{support}(\widetilde{B})$;

$$d(x, a.c(\widetilde{A})) = d(y, a.c(\widetilde{B})) \Rightarrow d(\underline{\mu}_{\widetilde{A}}(x), \underline{\mu}_{\widetilde{B}}(y)) = 0.$$

Namely,

$$\underline{\mu}_{\widetilde{A}}(x) = \underline{\mu}_{\widetilde{B}}(y). \tag{15}$$

And, $\forall x \in \overline{support}(\widetilde{A}), \forall y \in \overline{support}(\widetilde{B});$

$$d(x, a.c(\widetilde{A})) = d(y, a.c(\widetilde{B})) \Rightarrow d(\overline{\mu}_{\widetilde{A}}(x), \overline{\mu}_{\widetilde{B}}(y)) = 0.$$

Namely,

$$\overline{\mu}_{\widetilde{A}}(x) = \overline{\mu}_{\widetilde{B}}(y). \tag{16}$$

Finally, from (15), (16) and the definition (3.16) we have $\widetilde{A} \sim_1 \widetilde{B}$. □

Now, according to the cases mentioned above, we provide the relation between fuzzy metric space and crisp metric space.

Theorem 4.7. *Let $(F_c(\mathbb{R}), \tilde{D_\star})$ be a fuzzy metric space of the pseudo-geometric fuzzy numbers, with the ambiguity rank $F_c ar = (0, 0, 0, 0)$. Then, the pair $(F_c(\mathbb{R}), \tilde{D_\star})$ is metric space.*

Proof. From the lemma (4.6), have $\widetilde{A} \sim_1 \widetilde{B}$. As a result, $\forall \widetilde{A}, \widetilde{B} \in F_c(\mathbb{R})$

$$\tilde{D_\star}(\tilde{A}, \tilde{B}) = \mid \widetilde{A - B} \mid .$$

Namely, $\tilde{D_\star}$ is a metric, because fuzzy absolute is defined by the crisp absolute. □

5. Illustrative Example and Comparison Study

It is reasonable to expect that different methods for distance can produce different results for the same sample of fuzzy numbers and some of them seem to be good in a particular context but not in general. The proposed distance has been compared with other methods in this section.

Example 5.1. Consider the two fuzzy numbers A=(0,0,0, -, -), B=(0,0,0.33,-,-), taken from paper[6], with the following Figure 5.

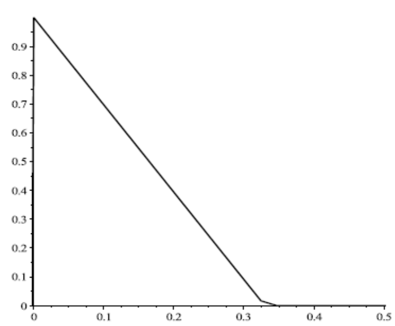

FIGURE 5. The Fuzzy Numbers of Example 5.1

Example 5.2. Consider the four fuzzy numbers $A_1 = (0, 2, 3, -, -)$, $A_2 = (-1, 0, 1, -, -), A_3 = (-3, -2, -1, -, -)$ and $A_4 = (-5, -4, 0, -, -)$, taken from paper[21], with the following Figure 6.

proposed method	Chakraborty and Chakraborty[12]	Guha and Chakraborty[18]	Ali Beigi and Hajjari
$(0, 0, \frac{0.33}{2}, -, -)$	$(0, 0, 0, -, -)$	$(0, 0, \frac{0.33}{2}, -, -)$	$(0, 0, 0.33, -, -)$

TABLE 2. Comparative Results of Example 5.1

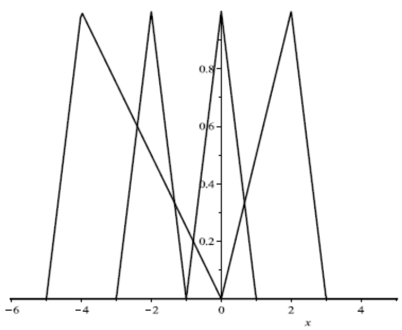

FIGURE 6. The Fuzzy Numbers of Example 5.2

Distance fuzzy numbers	Proposed method	Voxman[29]	Guha and Cha.[18]	Ali Beigi and Hajjari
$D(A1, A_1)$	$(0, 0, 0, -, -)$	$(0, 0, 3, -, -)$	$(0, 0, 1.5, -, -)$	$(0, 0, 0, -, -)$
$D(A1, A_2)$	$(0.5, 2, 3, -, -)$	$(2, 2, 4, -, -)$	$(0.5, 2, 3, -, -)$	$(1, 2, 4, -, -)$
$D(A1, A_3)$	$(2.5, 4, 5, -, -)$	$(1, 4, 6, -, -)$	$(2.5, 4, 5, -, -)$	$(1, 4, 6, -, -)$
$D(A1, A_4)$	$(4.5, 6, 8.5, -, -)$	$(0, 6, 8, -, -)$	$(3, 6, 7, -, -)$	$(0, 6, 8, -, -)$
$D(A2, A_2)$	$(0, 0, 0, -, -)$	$(0, 0, 2, -, -)$	$(0, 0, 1.5, -, -)$	$(0, 0, 0, -, -)$
$D(A2, A_3)$	$(2, 2, 2, -, -)$	$(0, 2, 4, -, -)$	$(1, 2, 3, -, -)$	$(0, 2, 4, -, -)$
$D(A2, A_4)$	$(3, 4, 6.5, -, -)$	$(4, 4, 6, -, -)$	$(2.5, 4, 5, -, -)$	$(1, 4, 6, -, -)$
$D(A3, A_3)$	$(0, 0, 0, -, -)$	$(0, 0, 2, -, -)$	$(0, 0, 1, -, -)$	$(0, 0, 0, -, -)$
$D(A3, A_4)$	$(1, 2, 4.5, -, -)$	$(2, 2, 4, -, -)$	$(0, 2, 3, -, -)$	$(3, 3, 4, -, -)$
$D(A4, A_4)$	$(0, 0, 0, -, -)$	$(0, 0, 5, -, -)$	$(0, 0, 2.5, -, -)$	$(0, 0, 0, -, -)$

TABLE 3. Comparative Results of Example 5.2

Example 5.3. Consider two fuzzy numbers,
$A = (1, 2, 5, -, -)$ and $B = (1, 2, 4, l_B(x), r_B(x))$, as
$l_B(x) = (1 - (x-2)^2)^{\frac{1}{2}}, r_B(x) = (1 - \frac{1}{4}(x-2)^2)^{\frac{1}{2}}$, taken from paper [13], with the following Figure 7.

Using our method, we have $D_\star(A, B)$ as a pseudo-geometric fuzzy number as follows,

$$(-1, 0, 2.5, l_{D_\star}(x), r_{D_\star}(x)),$$
$$l_{D_\star}^{-1}(\alpha) = \frac{(1-\alpha) - (1-\alpha^2)^{\frac{1}{2}}}{2},$$
$$r_{D_\star}^{-1}(\alpha) = \frac{(3-3\alpha) + (4-4\alpha^2)^{\frac{1}{2}}}{2},$$

$\alpha \in (0, 1]$.

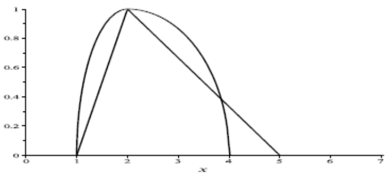

Figure 7. The Fuzzy Numbers of Example 5.3

6. **Conclusion and Future Research**

Distance measures for fuzzy sets are an important tool and have been applied to many fields. Several different strategies have been proposed for distance of fuzzy numbers. The distances introduced for fuzzy numbers can be categorized in two groups: "crisp distance and fuzzy distance". However, both mentioned concepts can lead to unsatisfactory results from the applications point of view, but there is no method, which gives a satisfactory result to all situations. Therefore, we believe that a fuzzy thinking should be used to introduce the distance between fuzzy numbers in such a way that it is associated with their uncertainties.
Therefore, using a new attitude to the fuzzy distance function, we introduce a new look at a pseudo-geometric fuzzy numbers set as a fuzzy metrics pace. Then, we defined new fuzzy distance as combination of two proposed attitudes for fuzzy distances. Finally, in order to description new proposed attitude, several basic properties and illustrative examples presented for distance between the fuzzy numbers. In future literature, with this new fuzzy distance, a new attitude to fuzzy derivatives will be presented and we will have a new attitude to the fuzzy differential equations.

References

[1] S. Abbasbandy and T. Hajjari, *A new approach for ranking of trapezoidal fuzzy numbers*, Computers and Mathematics with Applications, **57** (2009), 413-419.

[2] F. Abbasi, T. Allahviranloo and S. Abbasbandy,*A new attitude coupled with fuzzy thinking to fuzzy rings and fields*, Journal of Intelligent and Fuzzy Systems, **29** (2015), 851-861.

[3] F. Abbasi, T. Allahviranloo and S. Abbasbandy,*A new attitude coupled with fuzzy thinking to fuzzy group and subgroup*, Journal of Fuzzy Set Valued Analysis, **4** (2015), 1-18.

[4] F. Abbasi, T. Allahviranloo and S. Abbasbandy, *A new attitude coupled with the basic thinking to ordering for ranking fuzzy numbers*, International Journal of Industrial Mathematics, **8(4)** (2016), 365-375.

[5] D. Altman, *Fuzzy set theoretic approaches for handling imprecision in spatial analysis*, International Journal of Geographical Information Systems, **8** (1994), 271-289.

[6] M. Ali Beigi, T. Hajjari and E. Ghasem Khani,*An Algorithm to Determine Fuzzy Distance Measure*, 13th Iranian Conference on Fuzzy Systems (IFSC), 2013.

[7] I. Bloch,*On fuzzy distances and their use in image processing under imprecisionm*, Pattern Recognition, **32(11)** (1999), 1873-1895.

[8] C. Chakraborty and D. Chakraborty, *Atheoretical development on a fuzzy distance measure for fuzzy numbers*, Mathematical and Computer Modelling, **43(3-4)** (2006), 254-261.

[9] S. H. Chen and C. C. Wang,*Fuzzy distance of trapezoidal fuzzy numbers*, In: Proceedings of the 9th Joint Conference on Information Sciences, JCIS 2006.

[10] C.H. Cheng,*A new approach for ranking fuzzy numbers by distance method*, Fuzzy Sets and Systems, **95(3)** (1998), 307-317.
[11] S. H. Chen and C. H. Hsieh,*Graded mean integration representation of generalized fuzzy number*, Proceeding of TFSA, 1998.
[12] C. Chakraborty and D. Chakraborty, *A theoretical development on a fuzzy distance measure for fuzzy numbers*, Mathematical and Computer Modeling, **43**(2006), 254-261, .
[13] T. C. Chu and C. T. Tsao,*Ranking fuzzy numbers with an area between the centroid point and original point*, Computers Math. Applications, **43** (2002), 111-117.
[14] M. M. Deza and E. Deza, *Encyclopedia of Distances*, 2009.
[15] P. D'Urso and P. Giordani, *A weighted fuzzy c-means clustering model for fuzzy data*, Computational Statistics and Data Analysis, **50(6)** (2006), 1496-1523.
[16] D. Dubois and H. Prade, *Fuzzy Sets and Systems: Theory and Applications*, 1980.
[17] R. Fullór,*Fuzzy reasoning and fuzzy optimization*, On leave from Department of Operations Research, Eötvös Lorand University, Budapest, 1998.
[18] D. Guha and D. Chakraborty,*A new approach to fuzzy distance measure and similarity measure between two generalized fuzzy numbers*, Applied Soft Computing, **10(1)** (2010), 90-99.
[19] J. Kacprzyk,*Multistage fuzzy control: a prescriptive approach*, John Wiley and Sons, Inc, 1997.
[20] G. J. Klir and B. Yuan,*Fuzzy sets and fuzzy logic: theory and applications*, Prentice-Hall PTR, Upper Saddlie River, 1995.
[21] S. Nezhad, A. Noroozi and A. Makui,*Fuzzy distance of triangular fuzzy numbers*, Journal of Intelligent and Fuzzy Systems, 2012.
[22] A. Rosenfeld,*Distance between Fuzzy Sets. Pattern Recognition Letters*, **3** (1985), 229-233.
[23] H. Rouhparvar, A. Panahi and A. Noorafkan Zanjani,*Fuzzy distance measure for fuzzy numbers*, Australian Journal of Basic and Applied Sciences, **5(6)** (2011), 258-265.
[24] C. Shan-Huo and W. Chien-Chung, *Fuzzy distance using fuzzy absolute value*, In: Machine Learning and Cybernetics, International Conference, 2009.
[25] K. Sridharan and H. E. Stephanou,*Fuzzy distance functions for motion planning*, In: Tools with Artificial Intelligence. TAI '92, Proceedings., Fourth International Conference, 1992.
[26] S. R. Sudharsanan, *Fuzzy distance approach to routing algorithms for optimal web path estimation*, In: Fuzzy Systems, The 10th IEEE International Conference, 2001.
[27] L. Stefanini, *A generalization of hukuhara difference and division for interval and fuzzy arithmetic*, Fuzzy Sets and Systems, **161** (2010), 1564-1584.
[28] L. Tran and L. Duckstein, *Comparison of fuzzy numbers using a fuzzy distance measure*, Fuzzy Sets and Systems, **130(3)** (2002), 331-341.
[29] W. Voxman, *Some remarks on distances between fuzzy numbers, Fuzzy Sets and Systems*, **100(1-3)** (1998), 353365.

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