A NEW APPROACH FOR SOLVING FUZZY LINEAR VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

M. GHANBARI

Abstract. In this paper, a fuzzy numerical procedure for solving fuzzy linear Volterra integro-differential equations of the second kind under strong generalized differentiability is designed. Unlike the existing numerical methods, we do not replace the original fuzzy equation by a 2×2 system of crisp equations, that is the main difference between our method and other numerical methods. Error analysis and numerical examples are given to show the convergency and efficiency of the proposed method, respectively.

1. Introduction

The fuzzy differential and integral equations are important part of the fuzzy analysis theory and also they have many applications in control theory.

The concept of integration of fuzzy functions was first introduced by Dubois and Prade [9, 10]. In 1983, Puri and Ralescu [19] introduced the notion of H-differentiability to extend the differential of set-valued functions to that of fuzzy functions. Afterwards, Seikkala [23] introduced the notion of fuzzy derivative, which is a generalization of the Hukuhara derivative, and the fuzzy integral, which is the same as that proposed by Dubois and Prade [9, 10]. Also, Bede et al. [5, 6] have introduced a more general definition of the derivative for fuzzy mappings, namely " strong generalized differentiability", to solve shortcomings of H-differentiability or its generalization.

Also, it is well-known that an important class of fuzzy integral equations is fuzzy Volterra integral equations. Existence and uniqueness of solutions of such equations has been studied by several authors [4, 17, 18, 24]. In 2010, Hajighasemi et al. [15] studied the existence and uniqueness of the solution of fuzzy Volterra integro-differential equations of the second kind under strong generalized differentiability. In 2012, Alikhani et al. [1] introduced new solutions for fuzzy differential equations as mixed solutions, and proved the existence and uniqueness of global solutions for fuzzy initial value problems involving integro-differential operators of Volterra type. In 2013, Salahshour and Allahviranloo [22] solved the fuzzy Volterra integral equations with separable kernel by using fuzzy differential transform method. In 2015, Alikhani and Bahrami [2] by introducing a new concept of upper and lower solutions, presented the existence and uniqueness of global solutions of an initial

Received: March 2015; Revised: January 2016; Accepted: June 2016

Key words and phrases: Fuzzy number, Fuzzy linear Volterra integro-differential equation, Generalized differentiability, Fuzzy trapezoidal rule.

value problem for first-order nonlinear fuzzy integro-differential equations under generalized differentiability.

In this paper, we present a fuzzy numerical method for solving fuzzy linear Volterra integro-differential equations of the second kind under strong generalized differentiability. Since there are many real-world problems in engineering and mechanics that can be put in the form of these equations, it is important to develop numerical methods for solving such integral equations. The proposed method can be used directly for a fuzzy equation. This means that we do not need to replace the original fuzzy equation by a 2×2 system of crisp equations, that is why we call it a "*fuzzy numerical method*". This is a fundamental difference between our numerical method and other numerical methods.

In this paper, we consider the fuzzy linear Volterra integro-differential equation

$$f'(t) = y(t) + \lambda \int_a^t k(t,s) \cdot f(s)\, ds, \qquad \lambda > 0, \qquad a \leqslant t \leqslant b, \tag{1}$$

with the fuzzy initial condition $f(a) = f_0$, where $y : [a,b] \longrightarrow E$ is a fuzzy function (E is the set of all fuzzy numbers), $k(t,s)$ is an arbitrary real-valued kernel function over $\Delta = \{(t,s) \mid a \leqslant t \leqslant b,\ a \leqslant s \leqslant t\}$ and f_0 is a fuzzy number. By integrating of both sides of the equation (1) and then by approximating the appeared definite integrals, by means of the fuzzy trapezoidal rule [7], we present a numerical method for solving equation (1).

This paper is organized as following: In section 2, the basic concepts are given which are used throughout the paper. In section 3, the proposed numerical method is described. In Section 4, we present an error analysis of the proposed method. In Section 5 two numerical examples are given to show the efficiency of the method. Finally, conclusion and future research is drawn in section 6.

2. Preliminaries

There are various definitions for the concept of fuzzy numbers [11, 12, 16]. In this paper, we define a fuzzy number as follows:

Definition 2.1. [16] A fuzzy number u in parametric form is represented by an ordered pair of functions $(\underline{u}(r),\ \overline{u}(r))$, $0 \leqslant r \leqslant 1$, satisfying the following requirements:

(1) $\underline{u}(r)$ is a bounded left-continuous non-decreasing function over $[0,1]$.
(2) $\overline{u}(r)$ is a bounded left-continuous non-increasing function over $[0,1]$.
(3) $\underline{u}(r) \leqslant \overline{u}(r),\ 0 \leqslant r \leqslant 1$.

In this paper, we denote the set of all fuzzy numbers by E. A crisp number α is simply represented by $\underline{u}(r) = \overline{u}(r) = \alpha$, $0 \leqslant r \leqslant 1$. We recall that for $a \leqslant b \leqslant c$, $a, b, c \in \mathbb{R}$, the triangular fuzzy number $u = (a, b, c)$ determined by a, b, c is denoted as $u = (\underline{u}(r), \overline{u}(r))$ such that $\underline{u}(r) = a + (b-a)r$ and $\overline{u}(r) = c - (c-b)r$. For arbitrary fuzzy numbers $u = (\underline{u}(r), \overline{u}(r))$, $v = (\underline{v}(r), \overline{v}(r))$ and crisp number k we define fuzzy addition and scalar multiplication as

(1) $(\underline{u+v})(r) = (\underline{u}(r) + \underline{v}(r))$,
(2) $(\overline{u+v})(r) = (\overline{u}(r) + \overline{v}(r))$,

(3) $(\underline{k \cdot u})(r) = k \cdot \underline{u}(r), \quad (\overline{k \cdot u})(r) = k \cdot \overline{u}(r), \quad k \geqslant 0,$
(4) $(\underline{k \cdot u})(r) = k \cdot \overline{u}(r), \quad (\overline{k \cdot u})(r) = k \cdot \underline{u}(r), \quad k < 0.$

Theorem 2.2. [3]

a: *If we define* $0 = (\underline{0}(r), \overline{0}(r)) = (0, 0)$, *then* $0 \in E$ *is a neutral element with respect to addition, i.e.* $u + 0 = 0 + u = u$, *for all* $u \in E$.
b: *With respect to* 0, *none of* $u \in E \setminus \mathbb{R}$, *has inverse in* E *(with respect to* $+$*).*
c: *For any* $a, b \in \mathbb{R}$ *with* $a, b \geqslant 0$ *or* $a, b \leqslant 0$ *and any* $u \in E$, *we have*

$$(a + b) \cdot u = (a \cdot u) + (b \cdot u).$$

For general $a, b \in \mathbb{R}$, *the above property does not hold.*
d: *For any* $\lambda \in \mathbb{R}$ *and any* $u, v \in E$, *we have* $\lambda \cdot (u + v) = (\lambda \cdot u) + (\lambda \cdot v)$.
e: *For any* λ, $\mu \in \mathbb{R}$ *and any* $u \in E$, *we have* $\lambda \cdot (\mu \cdot u) = (\lambda \cdot \mu) \cdot u$.

Definition 2.3. [14] For arbitrary fuzzy numbers $u = (\underline{u}, \overline{u})$ and $v = (\underline{v}, \overline{v})$ the quantity

$$D(u, v) = \sup_{0 \leqslant r \leqslant 1} \max\{|\underline{u}(r) - \underline{v}(r)|, |\overline{u}(r) - \overline{v}(r)|\}, \tag{2}$$

is the distance between u and v.

It is shown [12, 20, 25] that (E, D) is a complete metric space. Also, the following properties are well-known (see e.g. [12, 25])

(1) $D(u + w, v + w) = D(u, v), \quad \forall u, v, w \in E,$
(2) $D(\lambda \cdot u, \lambda \cdot v) = |\lambda| \cdot D(u, v), \quad \forall \lambda \in \mathbb{R},\ u, v \in E,$
(3) $D(u + v, w + e) \leqslant D(u, w) + D(v, e), \quad \forall u, v, w, e \in E.$

Definition 2.4. A function $f : \mathbb{R} \to E$ is called a fuzzy function. Also, if for arbitrary fixed $x_0 \in \mathbb{R}$ and $\varepsilon > 0$, there exists $\xi > 0$ such that

$$| x - x_0 | < \xi \Longrightarrow D\left(f(x), f(x_0)\right) < \varepsilon, \tag{3}$$

then f is said to be continuous.

Definition 2.5. [14] Let the fuzzy function $f(x)$ is continuous on $[a, b]$ with the parametric form $(\underline{f}(x; r), \overline{f}(x; r))$. Then, we define the definite integral of $f(x)$ over $[a, b]$ as

$$\int_a^b f(x)\,dx = \left(\int_a^b \underline{f}(x; r)\,dx, \int_a^b \overline{f}(x; r)\,dx \right). \tag{4}$$

In the following remark, we present a numerical approximation for a fuzzy definite integral.

Remark 2.6. We know that in the classic Newton-Cotes formulas, only for $n \leqslant 7$ and $n = 9$, all of weight functions are positive [21]. In our opinion, we can apply the classic Newton-Cotes formulas with the positive weights for approximate computing the fuzzy definite integrals as follows. Let S_n be a simple (no composite) quadrature rule such that all of weight functions are positive, i.e., $n \leqslant 7$ or $n = 9$. We have

$$\int_a^b f(x)\,dx \approx S_n \left(\int_a^b f(x)\,dx \right) = \sum_{i=0}^{n} w_i \cdot f(x_i), \tag{5}$$

where $\{a = x_0, x_1, \ldots, x_n = b\}$ is an uniform partition on the closed interval $[a, b]$ given by

$$x_i = a + ih, \qquad\qquad i = 0, 1, \ldots, n,$$

with the step length $h = (b-a)/n$ and w_i, $i = 0, 1, \ldots, n$ are the weight functions. Since all of weights are positive, then we can write

$$\sum_{i=0}^{n} w_i \cdot f(x_i) = \left(\sum_{i=0}^{n} w_i \cdot \underline{f}(x_i; r), \sum_{i=0}^{n} w_i \cdot \overline{f}(x_i; r) \right). \tag{6}$$

Hence

$$\int_a^b f(x)\,dx \approx \left(\sum_{i=0}^{n} w_i \cdot \underline{f}(x_i; r), \sum_{i=0}^{n} w_i \cdot \overline{f}(x_i; r) \right). \tag{7}$$

Definition 2.7. Consider $u, v \in E$. If there exists $w \in E$ such that $u = v + w$, then w is called the H-difference of u and v and it is denoted by $u \ominus v = w$. Also, it should be noted that $u \ominus v \neq u + (-1) \cdot v$.

Remark 2.8. [13] Let u, v and w be in E. By H-difference the following statements are true:

(1) If $u + v = w$, then $w \ominus u = v$ and $w \ominus v = u$,
(2) $u \ominus (v \ominus w) = (u \ominus v) + w$,
(3) $\ominus(\ominus u) = u$,
(4) $u \ominus v = u + (\ominus v)$,
(5) $u \ominus u = 0$, where $0 = \big(\underline{0}(r), \overline{0}(r)\big) = (0, 0)$,
(6) $\ominus(u + v) = (\ominus u) + (\ominus v)$.

Now, let us recall the definition of strong generalized differentiability introduced in [5, 6].

Definition 2.9. Let $f : (a, b) \longrightarrow E$ and $x_0 \in (a, b)$. We say that f is strongly generalized differentiable at x_0 (Bede-Gal differentiability), if there exists an element $f'(x_0) \in E$, such that

(i): for all $h > 0$ sufficiently small, $\exists f(x_0 + h) \ominus f(x_0)$, $\exists f(x_0) \ominus f(x_0 - h)$ and the following limits hold (in the metric D):

$$\lim_{h \searrow 0} \frac{f(x_0 + h) \ominus f(x_0)}{h} = \lim_{h \searrow 0} \frac{f(x_0) \ominus f(x_0 - h)}{h} = f'(x_0),$$

or

(ii): for all $h > 0$ sufficiently small, $\exists f(x_0) \ominus f(x_0 + h)$, $\exists f(x_0 - h) \ominus f(x_0)$ and the following limits hold (in the metric D):

$$\lim_{h \searrow 0} \frac{f(x_0) \ominus f(x_0 + h)}{(-h)} = \lim_{h \searrow 0} \frac{f(x_0 - h) \ominus f(x_0)}{(-h)} = f'(x_0),$$

or

(iii): for all $h > 0$ sufficiently small, $\exists f(x_0 + h) \ominus f(x_0)$, $\exists f(x_0 - h) \ominus f(x_0)$ and the following limits hold (in the metric D):

$$\lim_{h \searrow 0} \frac{f(x_0 + h) \ominus f(x_0)}{h} = \lim_{h \searrow 0} \frac{f(x_0 - h) \ominus f(x_0)}{(-h)} = f'(x_0),$$

or

(iv): for all $h > 0$ sufficiently small, $\exists f(x_0) \ominus f(x_0+h)$, $\exists f(x_0) \ominus f(x_0-h)$ and the following limits hold (in the metric D):

$$\lim_{h \searrow 0} \frac{f(x_0) \ominus f(x_0+h)}{(-h)} = \lim_{h \searrow 0} \frac{f(x_0) \ominus f(x_0-h)}{h} = f'(x_0),$$

(the denominators of h and $-h$ denote multiplication by $\frac{1}{h}$ and $\frac{-1}{h}$, respectively).

Theorem 2.10. [6] *Let $f : (a,b) \longrightarrow E$ be strongly differentiable on each point $x \in (a,b)$ in the sense of Definition 2.9**(iii)** or 2.9**(iv)**. Then $f'(x) \in \mathbb{R}$ for all $x \in (a,b)$.*

According to the Theorem 2.10, we restrict our attention to functions that are **(i)**- or **(ii)**-differentiable on their domain except for a finite number of points. Also, we have:

Theorem 2.11. [8] *Let $f : \mathbb{R} \longrightarrow E$ be a fuzzy function and denoted as $f(x) = (\underline{f}(x;r), \overline{f}(x;r))$, for each $r \in [0,1]$. Then*

(1) *If f is differentiable in the first form* **(i)**, *then $\underline{f}(x;r)$ and $\overline{f}(x;r)$ are differentiable functions and $f'(x) = (\underline{f}'(x;r), \overline{f}'(x;r))$.*

(2) *If f is differentiable in the second form* **(ii)**, *then $\underline{f}(x;r)$ and $\overline{f}(x;r)$ are differentiable functions and $f'(x) = (\overline{f}'(x;r), \underline{f}'(x;r))$.*

Two following theorems are outcomes from Theorem 18 of [6]:

Theorem 2.12. *Let $f : [a,b] \longrightarrow E$ be differentiable in the first form* **(i)** *on $[a,b]$ and its derivative $f'(x)$ be integrable over $[a,b]$. Then*

$$\int_a^t f'(x)\, dx = f(t) \ominus f(a), \qquad \forall t \in [a,b]. \tag{8}$$

Theorem 2.13. *Let $f : [a,b] \longrightarrow E$ be differentiable in the second form* **(ii)** *on $[a,b]$ and its derivative $f'(x)$ is integrable over $[a,b]$. Then*

$$\int_a^t f'(x)\, dx = (-1) \cdot (f(a) \ominus f(t)), \qquad \forall t \in [a,b]. \tag{9}$$

In this paper, using the above notations and definitions, we present a fuzzy numerical method to solve the fuzzy linear Volterra integro-differential equations. The main advantage of our method is that it can be applied directly for a fuzzy equation and we do not need to replace the original fuzzy equation by a 2×2 system of crisp equations.

3. The Proposed Method

Consider the fuzzy linear Volterra integro-differential equation (1). Let $a = t_0 < t_1 < \cdots < t_n = b$, be a uniform partition on $[a,b]$ with the step size $h = (b-a)/n$, such that

$$|k(t_i, t_i)| \neq \frac{4}{\lambda h^2}, \qquad i = 0, 1, \ldots, n. \tag{10}$$

Also, we suppose that the solution of equation (1) is **(i)**- or **(ii)**-differentiable in the sense of definition 2.9 on each of the subintervals $[t_{i-1}, t_i]$, $i = 1, 2, \ldots, n$ with the same kind of differentiability on each subinterval. Supposing that the equation (1) has a fuzzy solution, then we have two cases as follow

3.1. **Case 1:**

Firstly, we suppose that f is **(i)**-differentiable on $[t_{i-1}, t_i]$, $1 \leqslant i \leqslant n$. By integrating both sides of equation (1) from t_{i-1} to t_i, we have

$$\int_{t_{i-1}}^{t_i} f'(t)\,dt = \int_{t_{i-1}}^{t_i} y(t)\,dt + \lambda \int_{t_{i-1}}^{t_i} \int_a^t k(t,s) \cdot f(s)\,ds\,dt. \tag{11}$$

By Theorem 2.12, we can write

$$f(t_i) \ominus f(t_{i-1}) = \int_{t_{i-1}}^{t_i} y(t)\,dt + \lambda \int_{t_{i-1}}^{t_i} \int_a^t k(t,s) \cdot f(s)\,ds\,dt, \tag{12}$$

or

$$f(t_i) = f(t_{i-1}) + \int_{t_{i-1}}^{t_i} y(t)\,dt + \lambda \int_{t_{i-1}}^{t_i} \int_{t_0}^t k(t,s) \cdot f(s)\,ds\,dt. \tag{13}$$

Approximating the definite integrals in equation (3) by fuzzy trapezoidal rule [7] and replacing the approximation sign ($\approx$) by the equality sign ($=$), we obtain

$$\begin{aligned} f(t_i) &= f(t_{i-1}) + \frac{h}{2}\left[y(t_{i-1}) + y(t_i)\right] \\ &+ \frac{\lambda h}{2}\left[\int_{t_0}^{t_{i-1}} k(t_{i-1},s) \cdot f(s)\,ds + \int_{t_0}^{t_i} k(t_i,s) \cdot f(s)\,ds\right]. \end{aligned} \tag{14}$$

Again, we apply the fuzzy trapezoidal rule for the definite integrals in equation (14). By abbreviations $f_i := f(t_i)$, $y_i := y(t_i)$ and $k_{ij} := k(t_i, t_j)$, we have

$$\begin{aligned} f_i &= f_{i-1} + \frac{h}{2}\left[y_{i-1} + y_i\right] + \frac{\lambda h}{2}\left[\frac{h}{2} k_{i-1,0} f_0 + h \sum_{j=1}^{i-2} k_{i-1,j} f_j + \frac{h}{2} k_{i-1,i-1} f_{i-1}\right] \\ &+ \frac{\lambda h}{2}\left[\frac{h}{2} k_{i0} f_0 + h \sum_{j=1}^{i-1} k_{ij} f_j + \frac{h}{2} k_{ii} f_i\right]. \end{aligned} \tag{15}$$

Setting

$$\begin{aligned} a_{i-1} &:= f_{i-1} + \frac{h}{2}\left[y_{i-1} + y_i\right] + \frac{\lambda h}{2}\left[\frac{h}{2} k_{i-1,0} f_0 + h \sum_{j=1}^{i-2} k_{i-1,j} f_j + \frac{h}{2} k_{i-1,i-1} f_{i-1}\right] \\ &+ \frac{\lambda h}{2}\left[\frac{h}{2} k_{i0} f_0 + h \sum_{j=1}^{i-1} k_{ij} f_j\right], \end{aligned} \tag{16}$$

we have

$$f_i = a_{i-1} + \frac{\lambda h^2}{4} k_{ii} f_i. \tag{17}$$

Obviously, since a_{i-1} is a linear combination of fuzzy numbers, therefore it is a fuzzy number. By equation (17) and Definition 2.7, we conclude

$$f_i \ominus \frac{\lambda h^2}{4} k_{ii} f_i = a_{i-1}. \tag{18}$$

It should be noted that the above H-difference is always available. Because we assumed that the equation (1) has a fuzzy solution.

Theorem 3.1. *In the equation (18) we have* $|k_{ii}| < \frac{4}{\lambda h^2}$.

Proof. Consider two cases: $k_{ii} > 0$ and $k_{ii} < 0$. For $k_{ii} = 0$, the proof is obvious.
1) Suppose that $k_{ii} > 0$. From equation (18) and by the abbreviations $\underline{f_i}(r) := \underline{f}(t_i; r)$ and $\overline{f_i}(r) := \overline{f}(t_i; r)$ we conclude

$$\begin{cases} \underline{f_i}(r) - \frac{\lambda h^2}{4} k_{ii} \underline{f_i}(r) = \underline{a_{i-1}}(r), \\ \\ \overline{f_i}(r) - \frac{\lambda h^2}{4} k_{ii} \overline{f_i}(r) = \overline{a_{i-1}}(r). \end{cases}$$

On the other hand, since a_{i-1} is a fuzzy number, we have $\underline{a_{i-1}}(r) \leqslant \overline{a_{i-1}}(r)$ which implies that $\left(1 - \frac{\lambda h^2}{4} k_{ii}\right) \geqslant 0$ or $k_{ii} \leqslant \frac{4}{\lambda h^2}$. Therefore, in this case, regarding to the condition (10) we conclude that $0 < k_{ii} < \frac{4}{\lambda h^2}$.

2) Suppose that $k_{ii} < 0$. From equation (18) we conclude

$$\begin{cases} \underline{f_i}(r) - \frac{\lambda h^2}{4} k_{ii} \overline{f_i}(r) = \underline{a_{i-1}}(r), \\ \\ \overline{f_i}(r) - \frac{\lambda h^2}{4} k_{ii} \underline{f_i}(r) = \overline{a_{i-1}}(r). \end{cases} \tag{19}$$

Similarly, since a_{i-1} is a fuzzy number, then $\underline{a_{i-1}}(r) \leqslant \overline{a_{i-1}}(r)$ that implies $\left(1 + \frac{\lambda h^2}{4} k_{ii}\right) \geqslant 0$ or $-\frac{4}{\lambda h^2} \leqslant k_{ii}$. Therefore, in this case, regarding to the condition (10) we conclude that must $-\frac{4}{\lambda h^2} < k_{ii} < 0$.
Therefore, the required result is obtained. □

Theorem 3.2. *If* $|k_{ii}| > \frac{4}{\lambda h^2}$*, then the H-difference of equation (3.1) does not exist (i.e., it is not a fuzzy number). Unless,* f_i *is a crisp number.*

Proof. There are two cases: $k_{ii} > \frac{4}{\lambda h^2}$ and $k_{ii} < -\frac{4}{\lambda h^2}$.
1) If $k_{ii} > \frac{4}{\lambda h^2}$ we have

$$f_i \ominus \frac{\lambda h^2}{4} k_{ii} f_i = \left((1 - \frac{\lambda h^2}{4} k_{ii}) \underline{f_i}(r), (1 - \frac{\lambda h^2}{4} k_{ii}) \overline{f_i}(r) \right). \tag{20}$$

On the other hand, the inequality $k_{ii} > \frac{4}{\lambda h^2}$ yields $(1 - \frac{\lambda h^2}{4} k_{ii}) < 0$ and since f_i is a fuzzy number then $\underline{f_i}(r) \leqslant \overline{f_i}(r)$ and consequently

$$(1 - \frac{\lambda h^2}{4} k_{ii}) \underline{f_i}(r) \geqslant (1 - \frac{\lambda h^2}{4} k_{ii}) \overline{f_i}(r). \tag{21}$$

The equations (20) and (21) imply that $\left(f_i \ominus \frac{\lambda h^2}{4} k_{ii} f_i\right)$ is not a fuzzy number, unless f_i be a crisp number.

2) If $k_{ii} < -\frac{4}{\lambda h^2}$ we have

$$f_i \ominus \frac{\lambda h^2}{4} k_{ii} f_i = \left(\underline{f_i}(r) - \frac{\lambda h^2}{4} k_{ii} \overline{f_i}(r), \overline{f_i}(r) - \frac{\lambda h^2}{4} k_{ii} \underline{f_i}(r)\right). \tag{22}$$

On the other hand, the inequality $k_{ii} > \frac{4}{\lambda h^2}$ yields $(1 + \frac{\lambda h^2}{4} k_{ii}) < 0$ and noting that f_i is a fuzzy number we have

$$\begin{aligned} \underline{f_i}(r) \leqslant \overline{f_i}(r) \quad &\Longrightarrow \quad (1 + \frac{\lambda h^2}{4} k_{ii}) \underline{f_i}(r) \geqslant (1 + \frac{\lambda h^2}{4} k_{ii}) \overline{f_i}(r) \\ &\Longrightarrow \quad \underline{f_i}(r) - \frac{\lambda h^2}{4} k_{ii} \overline{f_i}(r) \geqslant \overline{f_i}(r) - \frac{\lambda h^2}{4} k_{ii} \underline{f_i}(r). \end{aligned}$$

By the last inequality and equation (22), it is clear that $\left(f_i \ominus \frac{\lambda h^2}{4} k_{ii} f_i\right)$ is not a fuzzy number, unless f_i is a crisp number.
This completes the proof of Theorem 3.2. □

To present the fuzzy solution $f(t_i)$ via the equation (18), we consider three cases: $k_{ii} = 0$, $k_{ii} > 0$, $k_{ii} < 0$.

Case 1: If $k_{ii} = 0$, then from equation (18) we conclude

$$f_i = a_{i-1}. \tag{23}$$

Therefore, we obtain the fuzzy number $f(t_i)$ by using information on all previous points, i.e., $f(t_0)$, $f(t_1)$, ..., $f(t_{i-1})$ via equation (16).

Case 2: If $k_{ii} > 0$, then by Theorem 3.1 we have

$$0 < k_{ii} < \frac{4}{\lambda h^2}. \tag{24}$$

It should be noted that if $k_{ii} > \frac{4}{\lambda h^2}$, then $\left(f_i \ominus \frac{\lambda h^2}{4} k_{ii} f_i\right)$ is not a fuzzy number and consequently (18) implies that a_{i-1} is not a fuzzy number, that is a contradiction. Therefore, in this case, from equations(24) and (18) we have

$$f_i = \left(\frac{4}{4 - \lambda h^2 k_{ii}}\right) \cdot a_{i-1}. \tag{25}$$

Obviously, since a_{i-1} is a fuzzy number, f_i is a fuzzy number, too.

Case 3: If $k_{ii} < 0$, then by Theorem 3.1 we have

$$-\frac{4}{\lambda h^2} < k_{ii} < 0. \tag{26}$$

Especially, since $k_{ii} < 0$ from equation (18), we obtain the 2×2 system (19). On the other hand, the condition (10) implies that the system (19) has unique solution

$$\begin{cases} \underline{f_i}(r) = \frac{16}{16-\lambda^2 h^4 k_{ii}^2} \left[\underline{a_{i-1}}(r) + \frac{\lambda h^2}{4} k_{ii} \overline{a_{i-1}}(r)\right], \\ \overline{f_i}(r) = \frac{16}{16-\lambda^2 h^4 k_{ii}^2} \left[\overline{a_{i-1}}(r) + \frac{\lambda h^2}{4} k_{ii} \underline{a_{i-1}}(r)\right]. \end{cases} \tag{27}$$

In the following theorem, we show that the functions in equation (27) present always a fuzzy number.

Theorem 3.3. *The functions in equation (27) are always the parametric form of a fuzzy number.*

Proof. It is sufficient to show that the functions in equation (27) satisfy the Definition 2.1. Since a_{i-1} is a fuzzy number, $\underline{a_{i-1}}(r)$ and $\overline{a_{i-1}}(r)$ satisfying all conditions of Definition 2.1. Thus, from equation (27), it is clear that $\underline{f_i}(r)$ and $\overline{f_i}(r)$ are bounded left-continuous. Also, from equation (26) we conclude that

$$16 - \lambda^2 h^4 k_{ii}^2 = (4 - \lambda h^2 k_{ii})(4 + \lambda h^2 k_{ii}) > 0, \tag{28}$$

and consequently $\underline{f_i}(r)$ and $\overline{f_i}(r)$ are non-decreasing and non-increasing functions, respectively. Now, we show that $\underline{f_i}(r) \leqslant \overline{f_i}(r)$. Suppose that there exists $r \in [0,1]$ such that $\underline{f_i}(r) > \overline{f_i}(r)$, then using equations (27) and (28) we have

$$\left(\frac{4 - \lambda h^2 k_{ii}}{4}\right) \underline{a_{i-1}}(r) > \left(\frac{4 - \lambda h^2 k_{ii}}{4}\right) \overline{a_{i-1}}(r).$$

and since $k_{ii} < 0$, we have

$$\underline{a_{i-1}}(r) > \overline{a_{i-1}}(r),$$

which is a contradiction. Therefore $\underline{f_i}(r) \leqslant \overline{f_i}(r)$, which concludes the proof. □

Now, we consider the case that f is **(ii)**-differentiable on $[t_{i-1}, t_i]$.

3.2. **Case 2:**

Suppose that f is **(ii)**-differentiable on the subinterval $[t_{i-1}, t_i]$, $1 \leqslant i \leqslant n$. As before, by integrating both sides of equation (1) from t_{i-1} to t_i, we have

$$\int_{t_{i-1}}^{t_i} f'(t)\, dt = \int_{t_{i-1}}^{t_i} y(t)\, dt + \lambda \int_{t_{i-1}}^{t_i} \int_{t_0}^{t} k(t,s) \cdot f(s)\, ds\, dt. \tag{29}$$

By Theorem 2.13, we can write

$$(-1) \cdot (f(t_{i-1}) \ominus f(t_i)) = \int_{t_{i-1}}^{t_i} y(t)\, dt + \lambda \int_{t_{i-1}}^{t_i} \int_{t_0}^{t} k(t,s) \cdot f(s)\, ds\, dt, \tag{30}$$

or

$$f(t_{i-1}) = f(t_i) + (-1) \cdot \int_{t_{i-1}}^{t_i} y(t)\, dt + (-\lambda) \cdot \int_{t_{i-1}}^{t_i} \int_{t_0}^{t} k(t,s) \cdot f(s)\, ds\, dt. \tag{31}$$

Similarly, by approximating the definite integrals in equation (31) via fuzzy trapezoidal rule [7] and replacing the approximation sign ($\approx$) by the equality sign ($=$) we obtain

$$\begin{aligned} f_{i-1} &= f_i + \left(\frac{-h}{2}\right)[y_{i-1}+y_i] \\ &+ \left(\frac{-\lambda h}{2}\right)\left[\frac{h}{2}k_{i-1,0}f_0 + h\sum_{j=1}^{i-2}k_{i-1,j}f_j + \frac{h}{2}k_{i-1,i-1}f_{i-1}\right] \\ &+ \left(\frac{-\lambda h}{2}\right)\left[\frac{h}{2}k_{i0}f_0 + h\sum_{j=1}^{i-1}k_{ij}f_j + \frac{h}{2}k_{ii}f_i\right]. \end{aligned} \tag{32}$$

Then, we can write

$$\begin{aligned} f_i + \left(\frac{-\lambda h^2}{4}k_{ii}\right)f_i &= f_{i-1} \ominus \left(\frac{-h}{2}\right)[y_{i-1}+y_i] \\ &\ominus \left(\frac{-\lambda h}{2}\right)\left[\frac{h}{2}k_{i-1,0}f_0 + h\sum_{j=1}^{i-2}k_{i-1,j}f_j + \frac{h}{2}k_{i-1,i-1}f_{i-1}\right] \\ &\ominus \left(\frac{-\lambda h}{2}\right)\left[\frac{h}{2}k_{i0}f_0 + h\sum_{j=1}^{i-1}k_{ij}f_j\right]. \end{aligned}$$

Therefore, by setting

$$\begin{aligned} b_{i-1} &:= f_{i-1} \ominus \left(\frac{-h}{2}\right)[y_{i-1}+y_i] \\ &\ominus \left(\frac{-\lambda h}{2}\right)\left[\frac{h}{2}k_{i-1,0}f_0 + h\sum_{j=1}^{i-2}k_{i-1,j}f_j + \frac{h}{2}k_{i-1,i-1}f_{i-1}\right] \\ &\ominus \left(\frac{-\lambda h}{2}\right)\left[\frac{h}{2}k_{i0}f_0 + h\sum_{j=1}^{i-1}k_{ij}f_j\right], \end{aligned}$$

we conclude

$$f_i + \left(\frac{-\lambda h^2}{4}k_{ii}\right)f_i = b_{i-1}. \tag{33}$$

It should be noted that b_{i-1} is a fuzzy number. Because by our assumption the equation (1) has a fuzzy solution and consequently all the above H-differences exist (i.e., are fuzzy numbers). To present the fuzzy solution $f(t_i)$ via the equation (33), we consider three cases: $k_{ii} = 0$, $k_{ii} > 0$, $k_{ii} < 0$.

Case 1: If $k_{ii} = 0$, then

$$f_i = b_{i-1}, \tag{34}$$

which is obviously a fuzzy number.

Case 2: If $k_{ii} < 0$, then we can rewrite the equation (33) as

$$\left(1 - \frac{\lambda h^2}{4} k_{ii}\right) \cdot f_i = b_{i-1}, \tag{35}$$

and consequently

$$f_i = \left(\frac{4}{4 - \lambda h^2 k_{ii}}\right) \cdot b_{i-1}. \tag{36}$$

Note that, since $k_{ii} < 0$ we have $(4 - \lambda h^2 k_{ii}) > 0$. Also clearly, f_i is a fuzzy number.

Case 3: If $k_{ii} > 0$, then from equation (33), we obtain the 2×2 system

$$\begin{cases} \underline{f_i}(r) - \frac{\lambda h^2}{4} k_{ii} \overline{f_i}(r) = \underline{b_{i-1}}(r), \\ \overline{f_i}(r) - \frac{\lambda h^2}{4} k_{ii} \underline{f_i}(r) = \overline{b_{i-1}}(r). \end{cases} \tag{37}$$

Condition (10) implies that the above system (37) has the unique solution

$$\begin{cases} \underline{f_i}(r) = \frac{16}{16 - \lambda^2 h^4 k_{ii}^2} \left[\underline{b_{i-1}}(r) + \frac{\lambda h^2}{4} k_{ii} \overline{b_{i-1}}(r)\right], \\ \overline{f_i}(r) = \frac{16}{16 - \lambda^2 h^4 k_{ii}^2} \left[\overline{b_{i-1}}(r) + \frac{\lambda h^2}{4} k_{ii} \underline{b_{i-1}}(r)\right]. \end{cases} \tag{38}$$

Since $k_{ii} > 0$, we can not say that $\underline{f_i}(r)$ and $\overline{f_i}(r)$ in equation (38), are always parametric form of a fuzzy number. Indeed, $\underline{f_i}(r)$ and $\overline{f_i}(r)$ may not be non-decreasing and non-increasing functions, respectively.

Theorem 3.4. *In the equation (38), $\underline{f_i}(r)$ and $\overline{f_i}(r)$ have all properties of parametric form of a fuzzy number, expect the conditions non-decreasing and non-increasing.*

Proof. Since b_{i-1} is a fuzzy number, $\underline{b_{i-1}}(r)$ and $\overline{b_{i-1}}(r)$ are bounded left-continuous functions and $\underline{b_{i-1}}(r) \leqslant \overline{b_{i-1}}(r)$, $0 \leqslant r \leqslant 1$. Therefore, clearly $\underline{f_i}(r)$ and $\overline{f_i}(r)$ are bounded left-continuous functions, too. Thus, it is sufficient to that show that

$$\underline{f_i}(r) \leqslant \overline{f_i}(r), \qquad 0 \leqslant r \leqslant 1.$$

Suppose that there exists $r \in [0,1]$ such that $\underline{f_i}(r) > \overline{f_i}(r)$. Hence, from equation (38) we have

$$\frac{16}{16 - \lambda^2 h^4 k_{ii}^2} \left[\underline{b_{i-1}}(r) + \frac{\lambda h^2}{4} k_{ii} \overline{b_{i-1}}(r)\right] > \frac{16}{16 - \lambda^2 h^4 k_{ii}^2} \left[\overline{b_{i-1}}(r) + \frac{\lambda h^2}{4} k_{ii} \underline{b_{i-1}}(r)\right]. \tag{39}$$

Here, we have two cases: $k_{ii} > \frac{4}{\lambda h^2}$ and $k_{ii} < \frac{4}{\lambda h^2}$. Note that condition (10) guarantee $k_{ii} \neq \frac{4}{\lambda h^2}$.
1) If $k_{ii} > \frac{4}{\lambda h^2}$, then

$$16 - \lambda^2 h^4 k_{ii}^2 = (4 - \lambda h^2 k_{ii})(4 + \lambda h^2 k_{ii}) < 0,$$

and consequently from equation (39) we have

$$\left(\frac{4 - \lambda h^2 k_{ii}}{4}\right) \underline{b_{i-1}}(r) < \left(\frac{4 - \lambda h^2 k_{ii}}{4}\right) \overline{b_{i-1}}(r),$$

then

$$\underline{b_{i-1}}(r) > \overline{b_{i-1}}(r),$$

which is a contradiction.
2) Suppose that $k_{ii} < \frac{4}{\lambda h^2}$. Since in this case $k_{ii} > 0$, thus

$$16 - \lambda^2 h^4 k_{ii}^2 = (4 - \lambda h^2 k_{ii})(4 + \lambda h^2 k_{ii}) > 0.$$

and so, from equation (39) we have

$$\left(\frac{4 - \lambda h^2 k_{ii}}{4}\right) \underline{b_{i-1}}(r) > \left(\frac{4 - \lambda h^2 k_{ii}}{4}\right) \overline{b_{i-1}}(r),$$

then

$$\underline{b_{i-1}}(r) > \overline{b_{i-1}}(r),$$

which is a contradiction. Therefore, always $\underline{f_i}(r) \leqslant \overline{f_i}(r),\ 0 \leqslant r \leqslant 1$. Note that since $k_{ii} > 0$ then $\underline{f_i}(r)$ and $\overline{f_i}(r)$ may not be non-decreasing or non-increasing functions. In fact, since the sum of a non-decreasing function with a non-increasing function may be neither non-decreasing nor non-increasing, then we can not say that $\underline{f_i}(r)$ and $\overline{f_i}(r)$ are non-decreasing or non-increasing functions.
This complete the proof of Theorem 3.4. □

Remark 3.5. If the equation (38) be parametric form of a fuzzy number, then we have a "*strong fuzzy solution*". Otherwise, if the equation (38) is not parametric form of a fuzzy number, based on Theorem 3.4, we define

$$\underline{f_i}^*(r) = \min\{\underline{f_i}(r), \underline{f_i}(1)\}, \tag{40}$$

$$\overline{f_i}^*(r) = \max\{\overline{f_i}(r), \overline{f_i}(1)\}, \tag{41}$$

as a "*week fuzzy solution*".

In following, we present a new definition that will be used.

Definition 3.6. Let $g(x, y)$ be a continuous real function on

$$\triangle = \{(x, y)|a \leqslant x \leqslant b, \quad a \leqslant y \leqslant x\},$$

and let $P = \{a = t_0, t_1, \ldots, t_n = b\}$ be a uniform partition with the step size $h = \frac{b-a}{n}$ on the interval $[a, b]$. We say that the partition P is a "*g-partition*" if

$$|g(x, x)| < \frac{4}{h^2}, \quad \forall x \in [a, b].$$

Example 3.7. Consider the fuction $g(x, y) = x - y - 100$ and

$$\triangle = \{(x, y)|0 \leqslant x \leqslant 1, \quad 0 \leqslant y \leqslant x\}.$$

Then, any arbitraray uniform partition with the step size $h < 0.2$ is a g-partition.

Theorem 3.8. *Let $P = \{a = t_0, t_1, \dots, t_n = b\}$ be a uniform partition with the step size $h = \frac{b-a}{n}$ on $[a, b]$. If in equation (1), f is $(\mathbf{i})$-differentiable on $[a, b]$ and P is a λk-partition, then the proposed method gives always strong fuzzy solution for the problem (1).*

Proof. Since the partition P is λk-partition then

$$|k(x, x)| < \frac{4}{\lambda h^2}, \qquad \forall x \in [a, b],$$

and regarding to section 3.1, the proof is straightforward. □

4. **Error Analysis**

In this section, we present a general error analysis of our proposed method. For this end, it is obvious that according to trapezoidal rule we have

$$\int_{t_{i-1}}^{t_i} f(t)dt = \frac{h}{2}(f_{i-1} + f_i) - \frac{h^3}{12}f''(\xi), \qquad \xi \in [t_{i-1}, t_i]. \tag{42}$$

In other words, the error term in interval $[t_{i-1}, t_i]$ is

$$E(i) = \int_{t_{i-1}}^{t_i} f(t)dt - \frac{h}{2}(f_{i-1} + f_i) = -\frac{h^3}{12}f''(\xi), \qquad \xi \in [t_{i-1}, t_i]. \tag{43}$$

Now, we can present the following theorem.

Theorem 4.1. *For the fuzzy linear Volterra integro-differential equation (1), let $f_i = f(t_i)$ be the value of exact solution and $f_i^* = f^*(t_i)$ be the approximate solution obtained via our method, $i = 1, 2, \dots, n$. Also, suppose that $E(i) = f_i - f_i^*$ is be the error term in interval $[t_{i-1}, t_i]$. Then, we have*

$$|E(i)| \leq \frac{h^3}{12}(M_1 + \lambda M_2 + \frac{b-a}{2}\lambda M_3 + \frac{b-a}{2}\lambda M_4), \qquad i = 1, 2, \dots, n, \tag{44}$$

where

$$M_1 = \max_{t \in [a,b]} |y''(t)|, \qquad M_2 = \max_{t \in [a,b]} |\psi''(t)|, \tag{45}$$

$$M_3 = \max_{s \in [a,b]} |R''_{i-1}(s)|, \qquad M_4 = \max_{s \in [a,b]} |R''_i(s)|, \tag{46}$$

and

$$\psi(t) = \int_a^t k(t, s)f(s)ds, \quad R''_{i-1}(s) = k(t_{i-1}, s)f(s), \quad R''_i(s) = k(t_i, s)f(s). \tag{47}$$

Proof. We prove theorem when f is (i)-differentiable on $[t_i, t_{i-1}]$ (Case 1). The proof when f is (ii)-differentiable on $[t_i, t_{i-1}]$ is similar.

For this end, in equations (13) and (14), based on equations (42) and (45)-(47) we have

$$\int_{t_{i-1}}^{t_i} y(t)dt = \frac{h}{2}(y_{i-1} + y_i) - \frac{h^3}{12}y''(\xi_1), \qquad \xi_1 \in [t_{i-1}, t_i], \tag{48}$$

$$\int_{t_{i-1}}^{t_i} \int_{a=t_0}^{t} k(t,s)f(s)dsdt = \frac{h}{2}\left[\int_{t_0}^{t_{i-1}} k(t_{i-1}, s)f(s)ds + \int_{t_0}^{t_i} k(t_i, s)f(s)ds\right]$$
$$-\frac{h^3}{12}\psi''(\xi_2), \qquad \xi_2 \in [t_{i-1}, ti], \tag{49}$$

$$\int_{t_0}^{t_{i-1}} k(t_{i-1}, s)\, f(s)ds = \frac{h}{2}k_{i-1,0}\, f_0 + h\sum_{j=1}^{i-2} k_{i-1,j}\, f_j + \frac{h}{2}k_{i-1,i-1}\, f_{i-1}$$
$$-\frac{(i-1)h^3}{12}R''_{i-1}(\xi_3), \qquad \xi_3 \in [t_0, t_{i-1}], \tag{50}$$

$$\int_{t_0}^{t_i} k(t_i, s)\, f(s)ds = \frac{h}{2}k_{i,0}\, f_0 + h\sum_{j=1}^{i-1} k_{i,j}\, f_j + \frac{h}{2}k_{i,i}\, f_i$$
$$-\frac{(i)h^3}{12}R''_i(\xi_4), \qquad \xi_4 \in [t_0, t_i], \tag{51}$$

According to proposed method and equations (48)-(51), the error term in interval $[t_i, t_{i-1}]$ is obtained as

$$E(i) = -\frac{h^3}{12}y''(\xi_1) - \frac{\lambda h^3}{12}\psi''(\xi_2) - \frac{\lambda h}{2}\left(\frac{(i-1)h^3}{12}R''_{i-1}(\xi_3) + \frac{(i)h^3}{12}R''_i(\xi_4)\right), \tag{52}$$

or, in other words

$$E(i) = -\frac{h^3}{12}\left(y''(\xi_1) + \lambda\psi''(\xi_2)\right) - \frac{\lambda h^4}{24}\left((i-1)R''_{i-1}(\xi_3) + iR''_i(\xi_4)\right). \tag{53}$$

Obviously

$$|E(i)| \leq \frac{h^3}{12}\left(|y''(\xi_1)| + \lambda|\psi''(\xi_2)|\right) + \frac{\lambda h^4}{24}\left((i-1)|R''_{i-1}(\xi_3)| + i|R''_i(\xi_4)|\right), \tag{54}$$

and consequently

$$|E(i)| \leq \frac{h^3}{12}\left(M_1 + \lambda M_2\right) + \frac{(b-a)\lambda h^3}{24}\left(M_3 + M_4\right). \tag{55}$$

This completes the proof of Theorem. □

Remark 4.2. In Theroem 4.1, if $M = M_1 + M_2 + M_3 + M_4$, then we will obtain

$$|E(i)| \leq \frac{Mh^3}{12}\left[\lambda(b-a+1)+1\right]. \tag{56}$$

Obviously if $h \longrightarrow 0$, then $|E(i)| \longrightarrow 0$ for $i = 1, 2, \ldots, n$.

5. **Numerical Results**

Now we apply the proposed method for two examples and compare approximate solutions with exact solutions.

Example 5.1. Consider the fuzzy linear Volterra integro-differential equation

$$f'(t) = C + \int_0^t f(s)\,ds, \qquad 0 \leqslant t \leqslant \frac{\pi}{4},$$

with the initial condition $f(0) = (r, 2-r)$, where C is a fuzzy number as $C = (r, 2-r)$.

Case 1: Suppose that the exact solution $f(t)$ be **(i)**-differentiable on the interval $[0, \frac{\pi}{4}]$, the exact solution of the problem is

$$f(t) = \left(\underline{f}(t;r), \overline{f}(t;r)\right) = \left(r \cdot \exp(t),\, (2-r)\cdot \exp(t)\right). \tag{57}$$

It can be easily verified that equation (57) is a fuzzy function on $[0, \frac{\pi}{4}]$ and is **(i)**-differentiable on the interval $[0, \frac{\pi}{4}]$. We present the parametric forms of the approximate solutions obtained by our method with $n = 8$ in Table 1. Comparison of the exact and approximate solutions shows that the obtained results are very closed to the exact solutions.

t_i	Approximate solutions	Exact solutions
$\pi/32$	$(1.1032\,r,\, 2.2065 - 1.1032\,r)$	$(1.1032\,r,\, 2.2063 - 1.1032\,r)$
$2\pi/32$	$(1.2171\,r,\, 2.4343 - 1.2171\,r)$	$(1.2170\,r,\, 2.4339 - 1.2170\,r)$
$3\pi/32$	$(1.3428\,r,\, 2.6856 - 1.3428\,r)$	$(1.3425\,r,\, 2.6850 - 1.3425\,r)$
$4\pi/32$	$(1.4814\,r,\, 2.9629 - 1.4814\,r)$	$(1.4810\,r,\, 2.9619 - 1.4810\,r)$
$5\pi/32$	$(1.6344\,r,\, 3.2688 - 1.6344\,r)$	$(1.6337\,r,\, 3.2675 - 1.6337\,r)$
$6\pi/32$	$(1.8031\,r,\, 3.6063 - 1.8031\,r)$	$(1.8023\,r,\, 3.6045 - 1.8023\,r)$
$7\pi/32$	$(1.9893\,r,\, 3.9786 - 1.9893\,r)$	$(1.9882\,r,\, 3.9764 - 1.9882\,r)$
$\pi/4$	$(2.1947\,r,\, 4.3893 - 2.1947\,r)$	$(2.1933\,r,\, 4.3866 - 2.1933\,r)$

TABLE 1. The Parametric Forms of Approximate Solutions Obtained by Our Method with $n = 8$, for the Case 1 of the Example 5.1

Case 2: Suppose that the exact solution $f(t)$ is **(ii)**-differentiable on the interval $[0, \frac{\pi}{4}]$, the exact solution of the problem is

$$f(t) = \left(\underline{f}(t;r), \overline{f}(t;r)\right),$$

where

$$\underline{f}(t;r) = \exp(t) + (1-r)\sin t + (r-1)\cos t,$$
$$\overline{f}(t;r) = \exp(t) + (r-1)\sin t + (1-r)\cos t.$$

It can be shown that the above equations construct a fuzzy function on $[0, \frac{\pi}{4}]$ which is **(ii)**-differentiable on the interval $[0, \frac{\pi}{4}]$. We present the parametric forms of the approximate solutions obtained by our method with $n = 8$ in Table 2. Comparison of the exact and approximate solutions shows that the obtained results are very closed to the exact solutions.

t_i	Approximate solutions	Exact solutions
$\pi/32$	$(0.897\,r + 0.206,\ 2.001 - 0.897\,r)$	$(0.897\,r + 0.206,\ 2.000 - 0.897\,r)$
$2\pi/32$	$(0.786\,r + 0.431,\ 2.003 - 0.786\,r)$	$(0.786\,r + 0.431,\ 2.003 - 0.786\,r)$
$3\pi/32$	$(0.667\,r + 0.676,\ 2.010 - 0.667\,r)$	$(0.667\,r + 0.675,\ 2.009 - 0.667\,r)$
$4\pi/32$	$(0.542\,r + 0.940,\ 2.023 - 0.542\,r)$	$(0.541\,r + 0.940,\ 2.022 - 0.541\,r)$
$5\pi/32$	$(0.411\,r + 1.223,\ 2.045 - 0.411\,r)$	$(0.411\,r + 0.223,\ 2.044 - 0.411\,r)$
$6\pi/32$	$(0.277\,r + 1.527,\ 2.080 - 0.277\,r)$	$(0.276\,r + 0.526,\ 2.078 - 0.276\,r)$
$7\pi/32$	$(0.139\,r + 1.850,\ 2.129 - 0.139\,r)$	$(0.139\,r + 0.850,\ 2.127 - 0.139\,r)$
$\pi/4$	$(0.001\,r + 2.194,\ 2.196 - 0.001\,r)$	$(0.000\,r + 2.193,\ 2.193 - 0.000\,r)$

TABLE 2. The Parametric Forms of Approximate Solutions Obtained by Our Method with $n = 8$, for the Case 2 of the Example 5.1

Example 5.2. Consider the fuzzy linear Volterra integro-differential equation

$$f'(t) = y(t) + \int_0^t (-f(s))\, ds, \qquad 0 \leqslant t \leqslant \frac{\pi}{4},$$

with the initial condition $f(0) = (3r - 2, 2 - r)$, where y is a fuzzy function on the interval $[0, \frac{\pi}{4}]$ as

$$y(t) = \big(\underline{y}(t;r),\ \overline{y}(t;r)\big) = (2(r-2)\sin t,\ 2(2-3r)\sin t)\,.$$

As before, we consider two cases:

Case 1: Suppose that the exact solution $f(t)$ is **(i)**-differentiable on the interval $[0, \frac{\pi}{4}]$, the exact solution of the problem is

$$f(t) = \big(\underline{f}(t;r), \overline{f}(t;r)\big)\,,$$

where

$$\underline{f}(t;r) = -rt\,\sin t + (2-r)\cos t + 2(r-1)(\exp(t) + \exp(-t)),$$
$$\overline{f}(t;r) = -rt\,\sin t + (3r-2)\cos t + 2(1-r)(\exp(t) + \exp(-t)).$$

It can be verified that the above equations construct a fuzzy function which is **(i)**-differentiable on $[0, \frac{\pi}{4}]$. We present the parametric forms of the approximate solutions obtained by our method with $n = 7$ in Table 3. Comparison of the exact and approximate solutions shows that the obtained results are very closed to the exact solutions.

Case 2: Suppose that the exact solution $f(t)$ is **(ii)**-differentiable on the interval $[0, \frac{\pi}{4}]$, the exact solution of the problem is

$$f(t) = \big(\underline{f}(t;r), \overline{f}(t;r)\big)\,,$$

where

$$\underline{f}(t;r) = (3r-2)(\cos t - t\sin t),$$
$$\overline{f}(t;r) = (2-r)(\cos t - t\sin t).$$

t_i	Approximate solutions	Exact solutions
$\pi/28$	$(3.020\,r - 2.038,\, 2.038 - 1.0566\,r)$	$(3.019\,r - 2.038,\, 2.038 - 1.057\,r)$
$2\pi/28$	$(3.077\,r - 2.152,\, 2.152 - 1.226\,r)$	$(3.076\,r - 2.151,\, 2.151 - 1.226\,r)$
$3\pi/28$	$(3.175\,r - 2.342,\, 2.342 - 1.508\,r)$	$(3.174\,r - 2.341,\, 2.340 - 1.508\,r)$
$4\pi/28$	$(3.316\,r - 2.609,\, 2.609 - 1.902\,r)$	$(3.314\,r - 2.608,\, 2.608 - 1.902\,r)$
$5\pi/28$	$(3.504\,r - 2.954,\, 2.954 - 2.405\,r)$	$(3.501\,r - 2.953,\, 2.953 - 2.404\,r)$
$6\pi/28$	$(3.743\,r - 3.380,\, 3.380 - 3.016\,r)$	$(3.740\,r - 3.378,\, 3.378 - 3.015\,r)$
$\pi/4$	$(4.041\,r - 3.887,\, 3.887 - 3.733\,r)$	$(4.036\,r - 3.884,\, 3.884 - 3.733\,r)$

Table 3. The Parametric Forms of Approximate Solutions Obtained by Our Method with $n = 7$, for the Case 1 of the Example 5.2

It can be shown that the above defined $f(t)$ is a fuzzy function and **(ii)**-differentiable on $[0, \frac{\pi}{4}]$. We present the parametric forms of the approximate solutions obtained by our method with $n = 7$ in Table 4. Comparison of the exact and approximate solutions shows that the obtained results are useful.

t_i	Approximate solutions	Exact solutions
$\pi/28$	$(2.944\,r - 1.962,\, 1.962 - 0.981\,r)$	$(2.943\,r - 1.962,\, 1.962 - 0.981\,r)$
$2\pi/28$	$(2.776\,r - 1.850,\, 1.850 - 0.925\,r)$	$(2.775\,r - 1.850,\, 1.850 - 0.925\,r)$
$3\pi/28$	$(2.500\,r - 1.666,\, 1.666 - 0.833\,r)$	$(2.498\,r - 1.665,\, 1.665 - 0.833\,r)$
$4\pi/28$	$(2.121\,r - 1.414,\, 1.414 - 0.707\,r)$	$(2.119\,r - 1.413,\, 1.413 - 0.706\,r)$
$5\pi/28$	$(1.649\,r - 1.099,\, 1.099 - 0.550\,r)$	$(1.645\,r - 1.097,\, 1.097 - 0.548\,r)$
$6\pi/28$	$(1.092\,r - 0.728,\, 0.728 - 0.364\,r)$	$(1.086\,r - 0.724,\, 0.724 - 0.362\,r)$
$\pi/4$	$(0.462\,r - 0.308,\, 0.308 - 0.154\,r)$	$(0.455\,r - 0.304,\, 0.304 - 0.152\,r)$

Table 4. The Parametric Forms of Approximate Solutions Obtained by Our Method with $n = 7$, for the Case 2 of the Example 5.2

6. **Conclusion**

In this paper, we introduced a fuzzy numerical method for solving fuzzy linear Volterra integro-differential equations of the second kind under strong generalized differentiability. In the proposed method, we do not need to replace the original fuzzy problem by a 2×2 system of crisp problems, that is main advantage of the method. Also, we presented an error analysis of the proposed method that prove the convergency of the method. The numerical results showed the efficiency of our method. It should be noted that we can apply each of Newton-Cotes formulas with the positive weights to present a similar numerical method, which can be verified in the future researches.

Acknowledgements. This research is supported by Islamic Azad University of Aliabad Katoul Branch.

References

[1] R. Alikhani and F. Bahrami, *Global solutions of fuzzy integro-differential equations under generalized differentiability by the method of upper and lower solutions*, Information Sciences, **295** (2015), 600–608.

[2] R. Alikhani, F. Bahrami and A. Jabbari, *Existence of global solutions to nonlinear fuzzy Volterra integro-differential equations*, Nonlinear Analysis: Theory, Methods & Applications, **75(4)** (2012), 1810–1821.

[3] G. A. Anastassiou and S. G. Gal, *On a fuzzy trigonometric approximation theorem of Weierstrass-type*, J. Fuzzy Math., **9(3)** (2001), 701–708.

[4] K. Balachandran and P. Prakash, *On fuzzy volterra integral equations with deviting arguments*, Journal of Applied Mathematics and Stochastic Analysis, **2** (2004), 169–176.

[5] B. Bede and S. G. Gal, *Almost Periodic fuzzy-number valued functions*, Fuzzy Sets and Systems, **147** (2004), 385–403.

[6] B. Bede and S. G. Gal, *Generalizations of differentiablity of fuzzy number valued function with application to fuzzy differential equations*, Fuzzy Sets and Systems, **151** (2005), 581–599.

[7] B. Bede and S. G. Gal, *Quadrature rules for integrals of fuzzy-number-valued functions*, Fuzzy Sets and Systems, **145** (2004), 359–380.

[8] Y. Chalco-Cano and H. Roman-Flores, *On new solutions of fuzzy differential equations*, Chaos, Solitons and Fractals, **38** (2008), 112–119.

[9] D. Dubois and H. Prade, *Towards fuzzy differential calculus. I. Integration of fuzzy mappings*, Fuzzy Sets and Systems, **8(1)** (1982), 1–17.

[10] D. Dubois and H. Prade, *Towards fuzzy differential calculus. II. Integration on fuzzy intervals*, Fuzzy Sets and Systems, **8(2)** (1982), 105–116.

[11] D. Dubois and H. Prade, *Towards fuzzy differential calculus: III, differentiation*, Fuzzy Sets and Systems, **8** (1982), 225–233.

[12] S. G. Gal, *Approximation theory in fuzzy setting, in: G.A. Anastassiou (Ed.), Handbook of Analytic-Computational Methods in Applied Mathematics*, Chapman Hall, CRC Press, (2000), 617–666.

[13] M. Ghatee and S. M. Hashemi, *Ranking function-based solutions of fully fuzzified minimal coast flow problem*, Information Sciences, **177** (2007), 4271–4294.

[14] R. Goetschel and W. Voxman, *Elementary fuzzy calculus*, Fuzzy sets and Systems, **18** (1986), 31–43.

[15] S. Hajighasemi, T. Allahviranloo, M. Khezerloo, M. Khorasany and S. Salahshour, *Existence and uniqueness of solutions of fuzzy volterra integro-differential equations*, IPMU, Part II, CCIS 81, (2010), 491–500.

[16] O. Kaleva, *Fuzzy differential equations*, Fuzzy Sets and Systems, **24** (1987), 301–317.

[17] J. Y. Park and H. K. Han, *Existence and uniqueness theorem for a solution of fuzzy Volterra integral equations*, Fuzzy Sets and Systems, **105** (1999), 481–488.

[18] J. Y. Park and J. U. Jeong, *A note on fuzzy integral equations*, Fuzzy Sets and Systems, **108** (1999), 193–200.

[19] M. L. Puri and D. Ralescu, *Differentials of fuzzy functions*, J. Math. Anal. Appl., **91** (1983), 552–558.

[20] M. L. Puri and D. Ralescu, *Fuzzy random variables*, J. Math. Anal. Appl., **114** (1986), 409–422.

[21] A. Ralston and P. Rabinowitz, *First course in numerical analysis*, McGraw-Hill, 1978.

[22] S. Salahshour and T. Allahviranloo, *Application of fuzzy differential transform method for solving fuzzy Volterra integral equations*, Applied Mathematical Modelling, **37(3)** (2013), 1016–1027.

[23] S. Seikkala, *On the fuzzy initial value problem*, Fuzzy Sets and Systems, **24(3)** (1987), 319–330.

[24] S. J. Song, Q.Y. Liu and Q. C Xu, *Existence and comparison theorems to Volterra fuzzy integral equation in* (E^n, D), Fuzzy Sets and Systems, **104** (1999), 315–321.

[25] C. Wu and Z. Gong, *On Henstock integral of fuzzy-number-valued functions I*, Fuzzy Sets and Systems, **120** (2001), 523–532.

Mojtaba Ghanbari, Department of Mathematics, Aliabad Katoul Branch, Islamic Azad University, Aliabad Katoul, Iran
E-mail address: Mojtaba.Ghanbari@gmail.com