UNIVERSAL APPROXIMATION OF INTERVAL-VALUED FUZZY SYSTEMS BASED ON INTERVAL-VALUED IMPLICATIONS

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Abstract. It is firstly proved that the multi-input-single-output (MISO) fuzzy systems based on interval-valued R- and S-implications can approximate any continuous function defined on a compact set to arbitrary accuracy. A formula to compute the lower upper bounds on the number of interval-valued fuzzy sets needed to achieve a pre-specified approximation accuracy for an arbitrary multivariate continuous function is then presented. In addition, a method to design the interval-valued fuzzy systems based on R- and S-implications in order to approximate a given continuous function with a required approximation accuracy is represented. Finally, two numerical examples are provided to illustrate the proposed procedure.

1. Introduction

In order to strengthen the capability of modeling and manipulating inexact information in a logical manner, type-2 fuzzy sets were introduced by Zadeh [40,41]. Since type-2 fuzzy set owns more parameters than traditional fuzzy set, type-2 fuzzy systems (which are described with type-2 membership functions) can provide us with more design degrees of freedom, and then they have relatively higher robustness than traditional fuzzy controllers. Therefore, type-2 fuzzy systems may outperform traditional fuzzy controllers in uncertain environments [35]. Since interval type-2 fuzzy logic systems were first designed by Liang and Mendel [23], interval-valued fuzzy set theory has been successfully employed in different control applications in recent years [14-20,24,36,42,43]. It is worth to mention that Castillo and Melin provided a good over view on methods in design of interval type-2 fuzzy controllers and interval type-2 fuzzy logic applications in intelligent control [1]. Dereli et al. represented a concise over view of different industrial applications of type-2 fuzzy sets and systems [4]. In most of these applications, the main goal is to design a type-2 fuzzy system to approximate a desired control or decision (often experts).

Let $\mathbf{R}$ represent the set of all real numbers. As being special type-2 fuzzy system, the interval-valued MISO fuzzy system can be regarded as a mapping input space $U(\subseteq \mathbf{R}^m)$ to output space $V(\subseteq \mathbf{R})$ as follows:

$$y = f(\mathbf{x}) = D(I(F(\mathbf{x}))), \tag{1}$$

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where fuzzifier F is a mapping from a real-valued point $\mathbf{x} \in U(\subseteq \mathbf{R}^m)$ to an interval-valued fuzzy set $\widetilde{A}'$, I is fuzzy inference engine which maps an interval-valued fuzzy set $\widetilde{A}'$ in U into another interval-valued fuzzy set $\widetilde{B}'$ in $V(\subseteq \mathbf{R})$, and defuzzifier D is a mapping which maps interval-valued fuzzy set $\widetilde{B}'$ in V to crisp point $y \in V$. At present, Mamdani type and Takagi-Sugeno type of interval-valued fuzzy systems were investigated By Ying [38,39]. He chosen interval-valued min or product t-norm as implication operators in interval-valued Mamdani and Takagi-Sugeno fuzzy systems. However, there is deficiency to interval-valued knowledge representation and reasoning using min or product t-norm as implication operators [32].

In addition, to make these interval-valued fuzzy systems more practical, we need to tackle this drawback by omitting type reduction process or approximating the output of the type-reduction process. To do so, many approaches have been proposed [3,8,13,18,25]. However, to the best of our knowledge, there have been few methods possessing accurate and rapid responses in practical applications.

In order to guarantee that the interval-valued fuzzy systems would be used to solve any control and modeling problems, it is very necessary to investigate their ability to uniformly approximate any continuous function. Considering a system of many-valued logic capturing the tautologies of interval-valued residuated lattices is regarded as an expansion of classical logic, it has been studied as multi-valued logic systems [9]. Moreover, it has a strict logic foundation to approximately reasoning using interval-valued R- and S-implications as implication operators [10]. Therefore, we will extend the results in [21,22] and consider the approximation properties of MISO fuzzy systems based on interval-valued R- and S-implications in this paper. Having this in mind, the structure of this paper is as following. In Section 2, we give some definitions of basic notions and notations. In Section 3, the interpretation of fuzzy rules and inference methods for interval-valued fuzzy systems based on R- and S-implications are studied. Section 4 shows that the interval-valued fuzzy systems based on R- and S-implications are universal approximators. Section 5 presents sufficient condition for the interval-valued fuzzy systems based on R- and S-implications as universal approximators. In Section 6, two examples are provided to demonstrate how to design such an interval-valued fuzzy system in order to approximate a given continuous function with a required approximation accuracy.

2. **Preliminaries**

Let $L^I = \{[x_1, x_2] | x_1 \leq x_2,\ x_1, x_2 \in [0,1]\}$. In the sequel, we will denote $x = [x_1, x_2]$ and the first and second projection mapping pr_1 and pr_2 on L^I are defined as $pr_1 x = x_1$ and $pr_2 x = x_2$ for any $x \in L^I$. An ordering on L^I as $x \leq_{L^I} y$ if $x_1 \leq y_1$ and $x_2 \leq y_2$ is called component-wise order or Kulisch-Miranker order. It is easy to verify that the ordering just defined is a partially ordering on L^I. We define the set $D = \{[x,x] | x \in [0,1]\}$ for further usage. The largest and the smallest elements of L^I are denoted by $1_{L^I} = [1,1]$ and $0_{L^I} = [0,0]$, respectively. Notice that for any non-empty $A \subseteq L^I$ it holds that $\sup A = [\sup\{x_1 | x \in A\}, \sup\{x_2 | x \in A\}]$ and $\inf A = [\inf\{x_1 | x \in A\}, \inf\{x_2 | x \in A\}]$ [5]. Therefore, it can be verified that

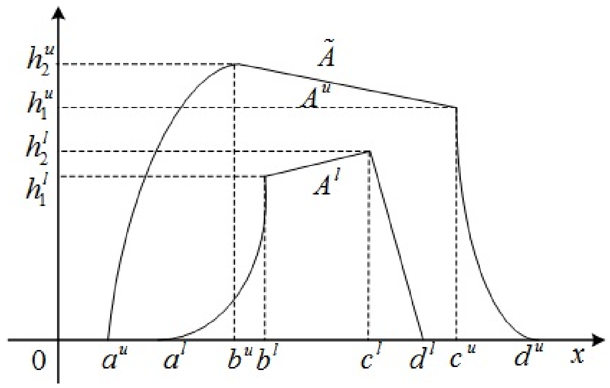

FIGURE 1. Lower Membership Function A^l and Upper Membership Function A^u of $\widetilde{A}$

the algebraic structure $(L^I, \vee, \wedge, 0_{L^I}, 1_{L^I})$ is a complete, bounded and distributive lattice.

Definition 2.1. [33] An interval-valued fuzzy set $\widetilde{A}$ is characterized by a membership function $\widetilde{A}(x) : X \to L^I$. The set of all interval-valued fuzzy sets on X is denoted by $IVFS(X)$.

Definition 2.2. [34] For an $\widetilde{A} \in IVFS(X)$, let $\widetilde{A}(x) = [A^l(x), A^u(x)]$ with $0 \leq A^l(x) \leq A^u(x) \leq 1$. The two ordinary fuzzy sets $A^l : X \to [0,1]$ and $A^u : X \to [0,1]$ are known as "the lower fuzzy set about $\widetilde{A}$" and "the upper fuzzy set about$\widetilde{A}$, respectively.

It is not difficult to find that an interval-valued pseudo-trapezoidal fuzzy set $\widetilde{A}$ on $\mathbf{R}$ can be represented by $\widetilde{A} = [A^l, A^u] = [(a^l, b^l, c^l, d^l; h_1^l, h_2^l), (a^u, b^u, c^u, d^u; h_1^u, h_2^u)]$ as shown in Figure 1, where $A^l(x)$ and $A^u(x)$ have the following forms:

$$A^l(x) = \begin{cases} L_{A^l}(x), & x \in [a^l, b^l) \\ \frac{h_2^l - h_1^l}{c^l - b^l}, & x \in [b^l, c^l] \\ R_{A^l}(x), & x \in (c^l, d^l] \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad A^u(x) = \begin{cases} L_{A^u}(x), & x \in [a^u, b^u) \\ \frac{h_2^u - h_u^l}{c^u - b^u}, & x \in [b^u, c^u] \\ R_{A^u}(x), & x \in (c^u, d^u] \\ 0, & \text{otherwise} \end{cases}$$

where $L_{A^l}(x)$ and $L_{A^u}(x)$ are nondecreasing upper semicontinuous functions. $R_{A^l}(x)$ and $R_{A^u}(x)$ are nonincreasing upper semicontinuous functions. In particular, if $b^l = c^l$ and $b^u = c^u$, then $\widetilde{A}$ is called interval-valued pseudo-triangle-shaped fuzzy set.

Definition 2.3. [11, 12] An interval-valued fuzzy set $\widetilde{A}$ is normal if $\sup\{\widetilde{A}(x) | x \in X\} = 1_{L^I}$.

Since an interval-valued pseudo-trapezoidal fuzzy set $\widetilde{A}$ can be represented by $\widetilde{A} = (A^l, A^u)$, $\widetilde{A}$ is normal if there exists at least an $x \in X$ such that $A^l(x) =$

$A^u(x) = 1$. This implies the fact that an interval-valued pseudo-trapezoidal fuzzy set $\widetilde{A}$ is normal if and only if $h_1^l = h_2^l = h_1^u = h_2^u = 1$.

Definition 2.4. For an $\widetilde{A} \in IVFS(X)$, the Support and Kernel of $\widetilde{A}$, respectively denoted as Supp$\widetilde{A}$ and Ker$\widetilde{A}$, are defined as:

$$\text{Supp}\widetilde{A} = \{x \in X | \widetilde{A}(x) > 0_L^I\}, \qquad \text{Ker}\widetilde{A} = \{x \in X | \widetilde{A}(x) = 1_{L^I}\}.$$

As an extension of fuzzy partition in [26], we can define an interval-valued fuzzy partition on X as follows.

Definition 2.5. Let $\{\widetilde{A}_k\}_{k=1}^n$ be a collection of interval-valued fuzzy sets on X. We say $\{\widetilde{A}_k\}_{k=1}^n$ forms a complete interval-valued fuzzy partition on X if $X \subseteq \bigcup\limits_{k=1}^{n} \text{Supp}\widetilde{A}_k$.

Definition 2.6. An interval-valued fuzzy partition $\{\widetilde{A}_k\}_{k=1}^n$ is said to be consistent if whenever for some k and some $x \in X$, $\widetilde{A}_k(x) = 1_{L^I}$ then $\widetilde{A}_j(x) = 0_{L^I}$ for $j \neq k$. Further, it is called a Ruspini Partition if $\sum\limits_{k}^{n}(A_{2k}^u(x) + A_{2k+1}^l(x)) = 1$, $\sum\limits_{k}^{n}(A_{2k}^l(x) + A_{2k+1}^u(x)) = 1$ for any $x \in X$.

Definition 2.7. [5] A function $\mathcal{N} : L^I \to L^I$ is called an interval-valued fuzzy negation if

$\mathcal{N}1$: $\mathcal{N}(0_{L^I}) = 1_{L^I}$, $\mathcal{N}(1_{L^I}) = 0_{L^I}$;

$\mathcal{N}2$: $\mathcal{N}(y) \leq_{L^I} \mathcal{N}(x)$ if $x \leq_{L^I} y$, $\forall\, x, y \in L^I$.

Furthermore, an interval-valued fuzzy negation $\mathcal{N}$ is strict if it satisfies the following properties:

$\mathcal{N}3$: $\mathcal{N}$ is continuous;

$\mathcal{N}4$: $\mathcal{N}(x) >_{L^I} \mathcal{N}(y)$ if $x <_{L^I} y$.

An interval-valued fuzzy negation is strong if it is involutive, i.e.,

$\mathcal{N}5$: $\mathcal{N}(\mathcal{N}(x)) = x, \forall\, x \in L^I$.

Notice that $(L^I, \vee, \wedge, \mathcal{N}, 0_{L^I}, 1_{L^I})$ is a bounded and distributive lattice and keeps De Morgan identities when $\mathcal{N}$ is involutive.

Definition 2.8. [5] An associative, symmetric and isotonic operation $\mathcal{T} : L^I \times L^I \to L^I$ is called a t-norm on L^I if it satisfies $\mathcal{T}(x, 1_{L^I}) = x$ for any $x \in L^I$.

Definition 2.9. [5] An associative, symmetric and isotonic operation $\mathcal{S} : L^I \times L^I \to L^I$ is called an s-norm on L^I if it satisfies $\mathcal{S}(x, 0_{L^I}) = x$ for any $x \in L^I$. Especially, we call $\mathcal{S}(x, y) = [S_1(x_1, x_2), S_2\ (y_1, y_2)]$ as s-representable s-norm associated with S_1 and S_2, where S_1 and S_2 are two s-norms on $[0, 1]$ and $S_1 \leq S_2$.

Definition 2.10. [2] An interval-valued fuzzy implication $\to$ is a mapping from $L^I \times L^I$ to L^I which is antitonic in its first and isotonic in its second component, and which satisfies $0_{L^I} \to 0_{L^I} = 1_{L^I}$ $0_{L^I} \to 1_{L^I} = 1_{L^I}$, $1_{L^I} \to 1_{L^I} = 1_{L^I}$, $1_{L^I} \to 0_{L^I} = 0_{L^I}$.

Moreover one demands any interval-valued fuzzy implication that must be an extension of the fuzzy implication, that is, If $[x, x] \to [y, y] = [a, b]$, then $a = b$.

Definition 2.11. [2] For every $x, y \in L^I$, an interval-valued R-implication is defined by $x \to_R y = \bigvee\{z \in L^I | \mathcal{T}(x, z) \leq_{L^I} y\}$, where $\mathcal{T}$ is a t-norm on L^I.

We say that an interval-valued t-norm satisfies the residuation principle if and only if $\mathcal{T}(x, y) \leq_{L^I} z \Leftrightarrow z \leq_{L^I} x \to_R y$ for any $x, y, z \in L^I$.

Lemma 2.12. [5, 6, 7] *Let $\mathcal{T}$ be an interval-valued t-norm. Then, $\mathcal{T}$ satisfies the residuation principle if and only if*

$$\sup_{z \in Z} \mathcal{T}(x, z) = \mathcal{T}(x, \sup_{z \in Z} z) \tag{2}$$

for any $x \in L^I$ and any nonempty subset Z of L^I.

Remark 2.13. In general from continuity it cannot be deduced that an interval-valued t-norms satisfies the residuation principle. For example, $\mathcal{T}(x, y) = [(x_1 + y_1 + x_2 + y_2 - x_2 y_2 - 2) \vee 0, (x_2 + y_2 - 1) \vee 0]$ is continuous. However, $\mathcal{T}$ does not satisfies the residuation principle [5].

Lemma 2.14. [6, 7] *Let $\mathcal{T}$ be an interval-valued t-norm such that $pr_2\mathcal{T}(x, [y_2, y_2]) = pr_2\mathcal{T}(x, [0,\ y_2])$ for $x \in D$, $y_2 \in [0, 1]$. Then, $\mathcal{T}$ satisfies the residuation principle if and only if there exists two left-continuous t-norm T_1 and T_2 on [0,1] and a real number $a \in [0, 1]$ such that, for any $x, y \in L^I$*

$$\mathcal{T}(x, y) = [T_1(x_1, y_1), \max(T_2(a, T_2(x_2, y_2)), T_2(x_1, y_2), T_2(y_1, x_2))] \tag{3}$$

and for all $x_1, y_1 \in [0, 1]$

$$\begin{cases} T_1(x_1, y_1) = T_2(x_1, y_1), & \text{if } T_2(x_1, y_1) > T_2(a, T_2(x_1, y_1)) \\ T_1(x_1, y_1) \leq T_2(x_1, y_1), & \text{otherwise} \end{cases}. \tag{4}$$

Lemma 2.15. *Let interval-valued t-norm $\mathcal{T}$ satisfy the residuation principle and $pr_2\mathcal{T}(x, [y_2,\ y_2]) = pr_2\mathcal{T}(x, [0, y_2])$ for $x \in D$, $y_2 \in [0, 1]$. Then interval-valued R-implication $\to_{R_\mathcal{T}}$ associated to $\mathcal{T}$ has the form*

$$x \to_{R_\mathcal{T}} y = [I_{R_{T_1}}(x_1, y_1) \wedge I_{R_{T_2}}(x_2, y_2), I_{R_{T_2}}(x_1, y_2) \wedge I_{R_{T_2}}(x_2, I_{R_{T_2}}(a, y_2))], \tag{5}$$

where $I_{R_{T_1}}$ and $I_{R_{T_2}}$ are the R-implication associated to the left-continuous t-norms T_1 and T_2, respectively.

Proof. For any $x, y, z \in L^I$, we have

$$\begin{aligned} \mathcal{T}(x, y) \leq_{L^I} z \Longleftrightarrow\ & T_1(x_1, y_1) \leq z_1 \text{ and } T_2(a, T_2(x_2, y_2)) \leq z_2 \\ & \text{and } T_2(x_1, y_2) \leq z_2 \text{ and } T_2(y_1, x_2) \leq z_2 \\ \Longleftrightarrow\ & y_1 \leq I_{R_{T_1}}(x_1, z_1) \text{ and } y_2 \leq I_{R_{T_2}}(x_2, I_{R_{T_2}}(a, z_2)) \\ \Longleftrightarrow\ & y_2 \leq I_{R_{T_2}}(x_1, z_2) \text{ and } y_1 \leq I_{R_{T_2}}(x_2, z_2), \end{aligned}$$

where $I_{R_{T_1}}$ and $I_{R_{T_2}}$ are the R-implication associated to the left-continuous t-norms T_1 and T_2, respectively.

Define the mapping $\to_{R_\mathcal{T}}: L^I \times L^I \to L^I$ as follows

$$x \to_{R_\mathcal{T}} y = [I_{R_{T_1}}(x_1, y_1) \wedge I_{R_{T_2}}(x_2, y_2), I_{R_{T_2}}(x_1, y_2) \wedge I_{R_{T_2}}(x_2, I_{R_{T_2}}(a, y_2))].$$

Then, it is clear to see that $\to_{R_\mathcal{T}}$ is the residuum of $\mathcal{T}$. □

As some special cases of Lemma 2.14, Equation (4) can be simplified as shown in the following corollaries.

Corollary 2.16. *Let $\mathcal{T}_L$ be interval-valued Łukasiewicz t-norm. The interval-valued R-implication $\to_{R_{\mathcal{T}_L}}$ associated to $\mathcal{T}$ is*

$$x \to_{R_{\mathcal{T}_L}} y = [(1 - x_1 + y_1) \wedge (1 - x_2 + y_2) \wedge 1, (1 - x_1 + y_2) \wedge 1]. \tag{6}$$

Definition 2.17. [33] An interval-valued S-implication is defined by $x \to_S y = \mathcal{S}(\mathcal{N}(x), y)$, where $\mathcal{S}$ is an s-norm and $\mathcal{N}$ an involutive negation on L^I.

Definition 2.18. [2] An interval-valued S-implication is defined by $x \to_S y = \mathcal{S}(\mathcal{N}(x), y)$, where $\mathcal{S}$ is an s-norm and $\mathcal{N}$ an involutive negation on L^I.

A fuzzy relation in $X_1 \times X_2 \times \cdots \times X_n$ is a fuzzy subset of the product space. Similarly, an interval-valued fuzzy relation in $X_1 \times X_2 \times \cdots \times X_n$ is an interval-valued fuzzy subset of the product space. We concentrate on binary interval-valued fuzzy relations.

Definition 2.19. [2, 11, 33] Let $\widetilde{R}$ and $\widetilde{S}$ be two interval-valued fuzzy relations on $X \times Y$ and $Y \times Z$. Then the sup-t composition of $\widetilde{R}$ and $\widetilde{S}$ is defined as follows:

$$\widetilde{R} \circ_\mathcal{T} \widetilde{S}(x, z) = \max_{y \in Y} \mathcal{T}(\widetilde{R}(x, y), \widetilde{S}(y, z)). \tag{7}$$

Remark 2.20. By equation (7), we can obtain the sup-t composition of an interval-valued fuzzy set $\widetilde{A}$ and relation $\widetilde{R}$ as $(\widetilde{A} \circ_\mathcal{T} \widetilde{R})(y) = \max_{x \in X} \mathcal{T}(\widetilde{A}(x), \widetilde{R}(x, y))$. This is a smooth extension of Zadeh's max-t composition rules (CRI).

3. Relational Interpretation of Fuzzy Rule

There is no doubt that fuzzy IF-THEN rules (short, fuzzy rules) are the central concept in fuzzy inference engine. The fuzzy rule base of MISO interval-valued fuzzy system consists of rules in the following forms [38,39]:

$$\mathrm{R}_j: \text{ IF } x_1 \text{ is } \widetilde{A}_j^1 \text{ AND } x_2 \text{ is } \widetilde{A}_j^2 \text{ AND} \cdots \text{ AND } x_m \text{ is } \widetilde{A}_j^m \text{ THEN } y \text{ is } \widetilde{B}_j, \tag{8}$$

where $x_i (i = 1, 2, \cdots, m)$ and y are variables and $\widetilde{A}_j^i (j = 1, 2, \cdots, n)$ and $\widetilde{B}_j$ are specific linguistic expressions expressing properties of values of x_i and y, respectively. The statement between "IF" and "THEN" is called the antecedent of R_j and the statement after "THEN" is called the consequent of R_j.

In this paper, the fuzzy rule is interpreted by a fuzzy relation on $U \times V$ derived from $\widetilde{A}_j^1 \times \widetilde{A}_j^2 \times \cdots \times \widetilde{A}_j^m$ and $\widetilde{B}_j$. We now construct an interval-valued fuzzy relation $\widetilde{R}$ from a given rule R. Considering the conjunctive normal form (CNF) interpretation is well-known in fuzzy logic [28-30], we will utilize CNF interpretation to construct the fuzzy relation $\widetilde{R}$ in this paper.

Let $\widetilde{A}_j = \widetilde{A}_j^1 \times \widetilde{A}_j^2 \times \cdots \times \widetilde{A}_j^m$ and $\mathbf{x} = (x_1, x_2, \cdots, x_m)$. Then each fuzzy rule R_i can be regarded as a fuzzy relation $\widetilde{R}_j$ with a membership function $\widetilde{R}_j(\mathbf{x}, y) = \widetilde{A}_j(\mathbf{x}) \to \widetilde{B}_j(y)$. Further, interval-valued t-norms are employed to evaluate the ANDs in the fuzzy rules. For interval-valued fuzzy systems and a given input $\widetilde{A}'$, the inference approaches are presented as follows according to the generalized modus ponens:

$$\widetilde{B}_1' = \bigvee_{j=1}^{m} \left(\widetilde{A}' \circ_{\mathcal{T}} (\widetilde{A}_j \to \widetilde{B}_j)\right) \tag{9}$$

$$\widetilde{B}_2' = \bigwedge_{j=1}^{m} \left(\widetilde{A}' \circ_{\mathcal{T}} (\widetilde{A}_j \to \widetilde{B}_j)\right) \tag{10}$$

$$\widetilde{B}_3' = \widetilde{A}' \circ_{\mathcal{T}} \bigvee_{j=1}^{m} \left(\widetilde{A}_j \to \widetilde{B}_j\right) \tag{11}$$

$$\widetilde{B}_4' = \widetilde{A}' \circ_{\mathcal{T}} \bigwedge_{j=1}^{m} \left(\widetilde{A}_j \to \widetilde{B}_j\right) \tag{12}$$

Remark 3.1. The second interpretation considers the rule (8) as a conditional clause of natural language and then the rule is interpreted by the theory of linguistic semantics [27-30]. And then there are many types of fuzzy inference mechanisms have been proposed in the literature [31]. We restrict this study only to fuzzy relation based inference mechanisms.

In order to ensure the rules cover all the possible situations that the interval-valued fuzzy system may face, we need attach some additional conditions to the interval-valued fuzzy rules.

Definition 3.2. A set of interval-valued fuzzy rules is complete if for any $\mathbf{x} \in U$, there exists at least one rule in the fuzzy rule base, say R_j, such that $\widetilde{A}_j^k(x_j) \neq 0_{L^I}$.

Definition 3.3. A set of interval-valued fuzzy rules is consistent if there are no rules with the same IF parts but different THEN parts.

Assumption: $\widetilde{A}_j^i$ and $\widetilde{B}_j$ are normal, continuous, complete and consistent interval-valued pseudo-trapezoid-shaped (the discussion for that of interval-valued pseudo-triangle-shaped fuzzy sets membership function is similar) which often form an interval-valued Ruspini partition in the fuzzy rule base as a form (7), and the fuzzy rules is complete, consistent.

In this paper, we always make this assumption on interval-valued fuzzy systems. Remember that the fuzzyifier maps a crisp point $\mathbf{x}_0 = (x_1^0, x_2^0, \cdots, x_m^0) \in U$ into an interval-valued fuzzy set in $IVFS(U)$. The interval-valued singleton fuzzifier is denoted as

$$\widetilde{A}'(\mathbf{x}) = \begin{cases} 1_{L^I}, & \text{if } \mathbf{x} = \mathbf{x}_0 \\ 0_{L^I}, & \text{otherwise} \end{cases}. \tag{13}$$

With the above input $\widetilde{A}'$, it is easy to see that the algorithms (9)-(12) reduce to

$$\widetilde{B}_1' = \widetilde{B}_3' = \bigvee_{j=1}^{m} \widetilde{A}_j \to \widetilde{B}_j \tag{14}$$

$$\widetilde{B}'_2 = \widetilde{B}'_4 = \bigwedge_{j=1}^{m} \widetilde{A}_j \to \widetilde{B}_j. \tag{15}$$

Lemma 3.4. *Assume that the number of interval-valued fuzzy rules in (8) are greater than two. If the operator $\to$ is chosen as an interval-valued t-norm in inference algorithms (9) and (11), then $\widetilde{B}' \equiv 0_{L^I}$.*

Proof. Since the antecedent interval-valued fuzzy sets of interval-valued fuzzy rules are complete and form an interval-valued Ruspini partition, there exists j such that $\widetilde{A}_j(\mathbf{x}_0) = 0_{L^I}$ for any input $\mathbf{x}_0 \in U$. Therefore, $\widetilde{B}'(y) = \bigwedge_{j=1}^{m} \mathcal{T}\left(\widetilde{A}_j(\mathbf{x}_0), \widetilde{B}_j(y)\right) = 0_{L^I}$. □

Remark 3.5. If the operator $\to$ is chosen as an interval-valued t-norm in inference algorithms (9)-(12), then the control is impossible for any processes. Thus, the inference methods (10) and (12) should be used.

Lemma 3.6. *Assume that the number of interval-valued fuzzy rules in (7) are greater than two. If the operator $\to$ is chosen as an interval-valued implication in inference algorithms (9) and (11), then $\widetilde{B}' \equiv 1_{L^I}$.*

Proof. Since an interval-valued implication is isotonic in its second component, for any $x \in L^I$ we have $0_{L^I} \to x \geq 0_{L^I} \to 0_{L^I} = 1_{L^I}$. Therefore, $0_{L^I} \to x = 1_{L^I}$ holds for any $x \in L^I$ if $\to$ is chosen as an interval-valued implication in inference algorithms (9) and (11). Thus, $\widetilde{B}' \equiv 1_{L^I}$. □

Remark 3.7. i. If the operator $\to$ is chosen as an interval-valued implication, then we should use inference methods (9) and (11) instead of (10) and (12).

ii. From the view of semantics of formal logic, fuzzy inference based on interval-valued implications admit the truth-values of the rules are non-increasing about the antecedents of the rules. In the actions of control, however, we always assume that the corresponding rules shall not be fired or have little influence on the control results if the truth-values of the antecedents of the rules are false or almost false. In this case, it is reasonable to use the inference form (9) or (11) instead of (10) or (12) in order to eliminate the effect of these rules.

Since $\widetilde{A}_j$ is complete and consistent, we assume that there only $\widetilde{A}_{j_1}(\mathbf{x}_0), \cdots, \widetilde{A}_{j_l}(\mathbf{x}_0)$ are not zero. Thus equation (15) can be rewritten as follows:

$$\widetilde{B}'(y) = \bigwedge_{i=1}^{l} \left(\widetilde{A}_{j_i}(\mathbf{x}_0) \to \widetilde{B}_{j_i}(y)\right). \tag{16}$$

Since $\widetilde{B}_j$ is consistent, for any $y \in V$, there are at most two elements, say $\widetilde{B}_j$ and $\widetilde{B}_{j+1}$ such that $\widetilde{B}_j(y) \neq 0_{L^I}$, $\widetilde{B}_{j+1}(y) \neq 0_{L^I}$. Therefore, Equation (16) can be expressed as follows:

$$\widetilde{B}'(y) = \bigwedge_{i\in I_1} \left(\widetilde{A}_{j_i}(\mathbf{x}_0) \to \widetilde{B}_j(y)\right) \wedge \bigwedge_{i\in I_2} \left(\widetilde{A}_{j_i}(\mathbf{x}_0) \to \widetilde{B}_{j+1}(y)\right) \wedge \bigwedge_{i\in I_3} \left(\widetilde{A}_{j_i}(\mathbf{x}_0) \to 0_{L^I}\right), \tag{17}$$

where I_1, I_2 and I_3 are finite subsets of $\{1, 2, \ldots, l\}$.

As what mentioned above, for any singleton input $\mathbf{x}_0$, it is reasonable to require that, if the rules R_{k_1}, $\cdots$, R_{k_l} are fired, then the membership functions $\widetilde{B}_{k_1}(y), \cdots, \widetilde{B}_{k_l}(y)$ of the consequent part of these rules have at most two adjacent different elements, say $\widetilde{B}_j$ and $\widetilde{B}_{j+1}$. In this case, we can simplify equation (17) as follows:

$$\widetilde{B}'(y) = \bigwedge_{i \in I_1} \left(\widetilde{A}_{j_i}(\mathbf{x}_0) \to \widetilde{B}_j(y)\right) \wedge \bigwedge_{i \in I_2} \left(\widetilde{A}_{j_i}(\mathbf{x}_0) \to \widetilde{B}_{j+1}(y)\right). \tag{18}$$

Since interval-valued implication is antitonic in its first variable and I_1, I_2 are finite subsets of $\{1, 2, \ldots, l\}$, there exist $j_1 \in I_1$ and $j_2 \in I_2$ such that $\bigwedge_{i \in I_1} \left(\widetilde{A}_{j_i}(\mathbf{x}_0) \to \widetilde{B}_j(y)\right) = \bigvee_{i \in I_1} \widetilde{A}_{j_i}(\mathbf{x}_0) \to \widetilde{B}_j(y) = \widetilde{A}_{j_1}(\mathbf{x}_0) \to \widetilde{B}_j(y)$ and $\bigwedge_{i \in I_2} \left(\widetilde{A}_{j_i}(\mathbf{x}_0) \to \widetilde{B}_j(y)\right) = \bigvee_{i \in I_2} \widetilde{A}_{j_i}(\mathbf{x}_0) \to \widetilde{B}_j(y) = \widetilde{A}_{j_2}(\mathbf{x}_0) \to \widetilde{B}_j(y)$, respectively. Without loss of generality, we set $j_1 = j$ and $j_2 = j+1$. Therefore, equation (18) can be rewritten as follows:

$$\widetilde{B}'(y) = \left(\widetilde{A}_j(\mathbf{x}_0) \to \widetilde{B}_j(y)\right) \wedge \left(\widetilde{A}_{j+1}(\mathbf{x}_0) \to \widetilde{B}_{j+1}(y)\right). \tag{19}$$

4. **Approximation Capability of Fuzzy Systems Based on Interval-valued R- and S-implications**

Generally speaking, the centroid of average as a result of defuzzification is not appropriate to the interval-valued fuzzy systems based on interval-valued implications. This reason is that the rule R_j should be invalid for the output if $\widetilde{A}_j(\mathbf{x}) = 0_{L^I}$. However, $\widetilde{B}'(y)$ in equation (8) (or (11)) is not zero on the support of $\widetilde{B}_j$ (that is, the set $\{y \in V | \widetilde{B}_j(y) > 0_{L^I}\}$) because the action of interval-valued implications. this implies that some unrelated factors will be considered if centroid of average defuzzifier is used. In order to obtain a good approximation capability, we thus will use averaging of maximum defuzzifier.

Let $Y = \{y | \widetilde{B}(y) = \max_{y \in V} \widetilde{B}(y)\}$, then the control output y_0 is arbitrary one chosen point of Y (such as the middle point of Y, etc.). Hence, for any input $\mathbf{x} = \mathbf{x}_0$, we can obtain a mapping $y_0 = f(\mathbf{x}_0)$, which is called the interval-valued fuzzy system function in this paper. Generally speaking, the expression of the interval-valued system function $y = f(\mathbf{x})$ cannot be obtained directly. Therefore, we only discuss the fuzzy systems based on interval-valued R-implications induced by continuous interval-valued t-norms satisfying the residuation principle and several specific interval-valued S-implications mentioned above for the rest of this paper.

Lemma 4.1. *Let $\to_R$ be an interval-valued R-implication generated by continuous interval-valued t-norm satisfying the residuation principle and $pr_2\mathcal{T}(x, [y_2, y_2]) = pr_2\mathcal{T}(x, [0, y_2])$ for $x \in D$, $y_2 \in [0, 1]$. Then, for any input $\mathbf{x}_0$ the system function of interval-valued fuzzy systems based on $\to_R$ is*

$$y_0 = f(\mathbf{x}_0) = (B_j^l)^{-1}(z_1)(or\ (B_j^u)^{-1}(z_2)), \tag{20}$$

where y_0 satisfies the condition $B_j^l(y_0) + B_{j+1}^u(y_0) = 1$ (or $B_j^u(y_0) + B_{j+1}^l(y_0) = 1$).

Proof. For any input $\mathbf{x} = \mathbf{x}_0$, we have $\widetilde{B}'(y) = \Big(\widetilde{A}_j(\mathbf{x}_0) \to \widetilde{B}_j(y)\Big) \wedge \Big(\widetilde{A}_{j+1}(\mathbf{x}_0) \to \widetilde{B}_{j+1}(y)\Big)$. Therefore, $\widetilde{B}'(y)$ gets its maximum value when the following equation hold:

$$\widetilde{A}_j(\mathbf{x}_0) \to \widetilde{B}_j(y) = \widetilde{A}_{j+1}(\mathbf{x}_0) \to \widetilde{B}_{j+1}(y). \tag{21}$$

Let $Y_1 = \{y \in V | \widetilde{A}_j(\mathbf{x}_0) \le \widetilde{B}_j(y)$ and $\widetilde{A}_{j+1}(\mathbf{x}_0) \le \widetilde{B}_{j+1}(y)\}$. We consider the following two cases:

i. $Y_1 \ne \emptyset$. This implies that $A^l_j(\mathbf{x}_0) + A^u_{j+1}(\mathbf{x}_0) \le B^l_j(y) + B^u_{j+1}(y) = 1$ and $A^u_j(\mathbf{x}_0) + A^l_{j+1}(\mathbf{x}_0) \le B^u_j(y) + B^l_{j+1}(y) = 1$. Therefore, there exist z_1 and z_2 such that $A^l_j(\mathbf{x}_0) \le z_1 \le 1 - A^u_{j+1}(\mathbf{x}_0)$, $A^u_j(\mathbf{x}_0) \le z_2 \le 1 - A^l_{j+1}(\mathbf{x}_0)$, respectively. And then $z_1 \le z_2$ holds. Let $z = [z_1, z_2] = \widetilde{B}_j(y)$. Since $B^l_j(y)$ and $B^u_{j+1}(y)$ are pseudo-trapeziod-shaped respectively, there exists one point $y_0 = (B^l_j)^{-1}(z_1)$(or $(B^u_j)^{-1}(z_2)$. Obviously, $y_0 \in Y_1$. And then it satisfies the condition $B^l_j(y_0) + B^u_{j+1}(y_0) = 1$ (or $B^u_j(y_0) + B^l_{j+1}(y_0) = 1$).

ii. $Y_1 = \emptyset$. Since continuous $\mathcal{T}$ satisfies the residuation principle, we have $\mathcal{T}(\widetilde{A}_j(\mathbf{x}_0), \widetilde{A}_j(\mathbf{x}_0) \to \widetilde{B}_j(y)) = \widetilde{B}_j(y), \mathcal{T}(\widetilde{A}_{j+1}(\mathbf{x}_0), \widetilde{A}_{j+1}(\mathbf{x}_0) \to \widetilde{B}_{j+1}(y)) = \widetilde{B}_{j+1}(y)$. Further, we can obtain the following equation:

$$\mathcal{T}(\widetilde{A}_{j+1}(\mathbf{x}_0), \widetilde{B}_j(y)) = \mathcal{T}(\widetilde{A}_j(\mathbf{x}_0), \widetilde{B}_{j+1}(y)). \tag{22}$$

By Lemma 2.13, there exists two continuous t-norm T_1, T_2 on $[0,1]$ and a real number $a \in [0,1]$ such that the following equations hold:

$$\begin{cases} T_1(A^l_{j+1}(\mathbf{x}_0), B^l_j(y)) = T_1(A^l_j(\mathbf{x}_0), B^l_{j+1}(y)) \\ \max(T_2(a, T_2(A^u_{j+1}(\mathbf{x}_0), B^u_j(y))), T_2(A^l_{j+1}(\mathbf{x}_0), B^l_j(y)), T_2(A^u_{j+1}(\mathbf{x}_0), B^u_j(y))) \\ = \max(T_2(a, T_2(A^u_j(\mathbf{x}_0), B^u_{j+1}(y))), T_2(A^l_j(\mathbf{x}_0), B^l_{j+1}(y)), T_2(A^u_j(\mathbf{x}_0), B^u_{j+1}(y))) \end{cases} \tag{23}$$

Let $g_1(z_1, z_2) = T_1(A^l_{j+1}(\mathbf{x}_0), z_1) - T_1(A^l_j(\mathbf{x}_0), 1 - z_2)$. Obviously, $g_1(z_1, z_2)$ is continuous and non-decreasing for the first and second variables. It is no difficult to see that $g_1(0,0) = T(A^l_{j+1}(\mathbf{x}_0), 0) - T(A^l_j(\mathbf{x}_0), 1) = -A^l_j(\mathbf{x}_0) < 0$ and $g_1(1,1) = T(A^l_{j+1}(\mathbf{x}_0), 1) - T(A^l_j(\mathbf{x}_0), 0) = A^l_{j+1}(\mathbf{x}_0) > 0$. Thus, there exists $(z^0_1, z^0_2) \in L^I$ such that $g_1(z^0_1, z^0_2) = 0$ by the intermediate value theorem. Similarly, let $g_2(z_1, z_2) = \max(T_2(a, T_2(A^u_{j+1}(\mathbf{x}_0), z_2)), T_2(A^l_{j+1}(\mathbf{x}_0), z_2),\ T_2(A^u_{j+1}(\mathbf{x}_0), z_2)) - \max(T_2(a, T_2(A^u_j(\mathbf{x}_0), 1 - z_1)), T_2(A^l_j(\mathbf{x}_0), 1 - z_1), T_2(A^u_j(\mathbf{x}_0), 1 - z_1))$. It is easy to find that $g_2(z_1, z_2)$ is continuous and non-decreasing for the first and second variables. Moreover, we have $g_2(0,0) = \max(T_2(a, T_2(A^u_{j+1}(\mathbf{x}_0), 0)), T_2(A^l_{j+1}(\mathbf{x}_0), 0),$ $T_2(A^u_{j+1}(\mathbf{x}_0),\ 0)) - \max(T_2(a, T_2(A^u_j(\mathbf{x}_0), 1)), T_2(A^l_j(\mathbf{x}_0), 1), T_2(A^u_j(\mathbf{x}_0), 1)) = -A^u_j$ $(\mathbf{x}_0) < 0$ and $g_2(1,1) = \max(T_2(a, T_2(A^u_{j+1}(\mathbf{x}_0), 1)), T_2(A^l_{j+1}(\mathbf{x}_0), 1), T_2(A^u_{j+1}(\mathbf{x}_0),$ $1)) - \max(T_2(a, T_2(A^u_j(\mathbf{x}_0), 0)), T_2(\ A^l_j(\mathbf{x}_0), 0), T_2(A^u_j(\mathbf{x}_0), 0)) = -A^l_{j+1}(\mathbf{x}_0) > 0$. Therefore, there exists $(z^{0'}_1, z^{0'}_2) \in L^I$ such that $g_2(z^{0'}_1, z^{0'}_2) = 0$ by the intermediate value theorem. This implies that equation (23) can be rewritten as

$$\begin{cases} g_1(z_1, z_2) = 0 \\ g_2(z_1, z_2) = 0 \end{cases}, \tag{24}$$

where $0 \leq z_1 \leq z_2 \leq 1$. Select any (z_1, z_2) as a solution of equation (24). And then set $[z_1, z_2] = \widetilde{B}_j(y)$. Since $B_j^l(y)$ and $B_{j+1}^u(y)$ are pseudo-trapeziod-shaped respectively, there exists one unique point $y_0 = (B_j^l)^{-1}(z_1)$(or $(B_j^u)^{-1}(z_2)$). Obviously, y_0 satisfies the condition $B_j^l(y_0) + B_{j+1}^u(y_0) = 1$ (or $B_j^u(y_0) + B_{j+1}^l(y_0) = 1$). $\square$

As a special example, we now show the expression of interval-valued fuzzy systems based on interval-valued R-implications generated by Łukasiewicz t-norm $\mathcal{T}_L$ according to Lemma 4.1.

Corollary 4.2. *If $\rightarrow_R$ is the interval-valued R-implication generated by Łukasiewicz t-norm $\mathcal{T}_L$, then the interval-valued fuzzy system has the following form:*

$$y_0 = f(\mathbf{x}_0) = (B_j^l)^{-1}\left(\frac{1 + 2A_j^l(\mathbf{x}_0) - A_j^u(\mathbf{x}_0) - A_{j+1}^l(\mathbf{x}_0)}{2}\right) \tag{25}$$

or

$$(B_j^u)^{-1}\left(\frac{1 + A_j^u(\mathbf{x}_0) - A_{j+1}^l(\mathbf{x}_0)}{2}\right), \tag{25'}$$

or

$$y_0 = f(\mathbf{x}_0) = (B_j^l)^{-1}\left(\frac{1 + A_j^l(\mathbf{x}_0) - A_{j+1}^u(\mathbf{x}_0)}{2}\right) \tag{26}$$

or

$$(B_j^u)^{-1}\left(\frac{1 + A_j^l(\mathbf{x}_0) + A_{j+1}^u(\mathbf{x}_0) - 2A_{j+1}^l(\mathbf{x}_0)}{2}\right). \tag{26'}$$

Proof. According to Lemma 4.1, we have the following equation:

$$\mathcal{T}_L(\widetilde{A}_{j+1}(\mathbf{x}_0), \widetilde{B}_j(y)) = \mathcal{T}_L(\widetilde{A}_j(\mathbf{x}_0), \widetilde{B}_{j+1}(y)).$$

This implies that the following system of equations holds:

$$\begin{cases} (A_{j+1}^l(\mathbf{x}_0) + B_j^l(y) - 1) \vee 0 = (A_j^l(\mathbf{x}_0) + B_{j+1}^l(y) - 1) \vee 0 \\ (A_{j+1}^l(\mathbf{x}_0) + B_j^u(y) - 1) \vee (A_{j+1}^u(\mathbf{x}_0) + B_j^l(y) - 1) \vee 0) \\ \qquad = (A_j^l(\mathbf{x}_0) + B_{j+1}^u(y) - 1) \vee (A_j^u(\mathbf{x}_0) + B_{j+1}^l(y) - 1) \vee 0 \end{cases}. \tag{27}$$

The two solutions of this system of equations are $z_1 = \frac{1}{2}(1 + 2A_j^l(\mathbf{x}_0) - A_j^u(\mathbf{x}_0) - A_{j+1}^l(\mathbf{x}_0))$, $z_2 = \frac{1}{2}(1 + A_j^u(\mathbf{x}_0) - A_{j+1}^l(\mathbf{x}_0))$ and $z_1' = \frac{1}{2}(1 + A_j^l(\mathbf{x}_0) - A_{j+1}^u(\mathbf{x}_0))$, $z_2' = \frac{1}{2}(1 + A_j^l(\mathbf{x}_0) + A_{j+1}^u(\mathbf{x}_0) - 2A_{j+1}^l(\mathbf{x}_0))$, respectively. Obviously, $z_1 \leq z_2$ and $z_1' \leq z_2'$. If we take $[z_1, z_2] = \widetilde{B}_j(y)$, then $y_0 = (B_j^l)^{-1}\left(\frac{1+2A_j^l(\mathbf{x}_0)-A_j^u(\mathbf{x}_0)-A_{j+1}^l(\mathbf{x}_0)}{2}\right)$ or $(B_j^u)^{-1}\left(\frac{1+A_j^u(\mathbf{x}_0)-A_{j+1}^l(\mathbf{x}_0)}{2}\right)$.

If we take $[z_1', z_2'] = \widetilde{B}_j(y)$, then $y_0 = (B_j^l)^{-1}\left(\frac{1+A_j^l(\mathbf{x}_0)-A_{j+1}^u(\mathbf{x}_0)}{2}\right)$ or $(B_j^u)^{-1}\left(\frac{1+A_j^l(\mathbf{x}_0)+A_{j+1}^u(\mathbf{x}_0)-2A_{j+1}^l(\mathbf{x}_0)}{2}\right)$. $\square$

Lemma 4.3. *Let $\to_S$ be an interval-valued S-implication generated by continuous representable interval-valued s-norm and involutive negation $\mathcal{N}$. Then, for any input $\mathbf{x}_0$ the system function of interval-valued fuzzy systems based on $\to_S$ is*

$$y_0 = f(\mathbf{x}_0) = (B_j^l)^{-1}(z_1)(\mathit{or}\ (B_j^u)^{-1}(z_2)), \tag{28}$$

where y_0 satisfies condition $B_j^l(y_0) + B_{j+1}^u(y_0) = 1$ (or $B_j^u(y_0) + B_{j+1}^l(y_0) = 1$).

Proof. For any input $\mathbf{x} = \mathbf{x}_0$, Equation (19) can be rewritten as follows

$$\widetilde{B}'(y) = \mathcal{S}(\mathcal{N}(\widetilde{A}_j(\mathbf{x}_0)), \widetilde{B}_j(y)) \wedge \mathcal{S}(\mathcal{N}(\widetilde{A}_{j+1}(\mathbf{x}_0)), \widetilde{B}_{j+1}(y)). \tag{29}$$

Without loss of generality, we assume that $[y_{j-1}, y_{j+1}]$ and $[y_j, y_{j+2}]$ are the support set of $\widetilde{B}_j(y)$ and $\widetilde{B}_{j+1}(y)$, respectively. We can assert that the point y_0 that $\widetilde{B}'(y)$ gets its maximum value lies in $[y_j, y_{j+1}]$. Otherwise, we first assume that $y_0 \notin [y_{j-1}, y_{j+2}]$, then $\widetilde{B}'(y_0) = \mathcal{N}(\widetilde{A}_j(\mathbf{x}_0)) \wedge \mathcal{N}(\widetilde{A}_{j+1}(\mathbf{x}_0))$. However, there exist some $y \in [y_{j-1}, y_{j+2}]$ such that $\mathcal{S}(\mathcal{N}(\widetilde{A}_j(\mathbf{x}_0)), \widetilde{B}_j(y)) > \mathcal{S}(\mathcal{N}(\widetilde{A}_j(\mathbf{x}_0)), 0_{L^I}) = \mathcal{N}(\widetilde{A}_j(\mathbf{x}_0))$ and $\mathcal{S}(\mathcal{N}(\widetilde{A}_{j+1}(\mathbf{x}_0)), \widetilde{B}_{j+1}(y)) > \mathcal{S}(\mathcal{N}(\widetilde{A}_{j+1}(\mathbf{x}_0)), 0_{L^I}) = \mathcal{N}(\widetilde{A}_{j+1}(\mathbf{x}_0))$ hold. And then we have $\widetilde{B}'(y) > \mathcal{N}(\widetilde{A}_j(\mathbf{x}_0)) \wedge \mathcal{N}(\widetilde{A}_{j+1}(\mathbf{x}_0))$. This is a contradiction.

Further, it is no difficult to see that $\widetilde{B}'(y)$ gets its maximum value when the following equation holds:

$$\mathcal{S}(\mathcal{N}(\widetilde{A}_j(\mathbf{x}_0)), \widetilde{B}_j(y)) = \mathcal{S}(\mathcal{N}(\widetilde{A}_{j+1}(\mathbf{x}_0)), \widetilde{B}_{j+1}(y)). \tag{30}$$

Since $\mathcal{S}$ is representable and $\mathcal{N}$ is involutive, there exists s-norms S_1, S_2 and negations N_1, N_2 on $[0,1]$ such that the following system of equations holds:

$$\begin{cases} S_1(N_1(A_j^u(\mathbf{x}_0)), B_j^l(y)) = S_1(N_1(A_{j+1}^u(\mathbf{x}_0)), B_{j+1}^l(y)) \\ S_2(N_2(A_j^l(\mathbf{x}_0)), B_j^u(y)) = S_2(N_2(A_{j+1}^l(\mathbf{x}_0)), B_{j+1}^u(y)) \end{cases} \tag{31}$$

Let $h_1(z_1, z_2) = S_1(N_1(A_j^u(\mathbf{x}_0)), z_1) - S_1(N_1(A_{j+1}^u(\mathbf{x}_0)), 1-z_2)$. Obviously, $h_1(z_1, z_2)$ is continuous and non-decreasing for the first and second variables. It is no difficult to see that $h_1(0,0) = S_1(N_1(A_j^u(\mathbf{x}_0)), 0) - S_1(N_1(A_{j+1}^u(\mathbf{x}_0)), 1) = N_1(A_j^u(\mathbf{x}_0)) - 1 \le 0$ and $h_1(1,1) = S_1(N_1(A_j^u(\mathbf{x}_0)), 1) - S_1(N_1(A_{j+1}^u(\mathbf{x}_0)), 0) = 1 - N_1(A_{j+1}^u(\mathbf{x}_0)) \ge 0$. Thus, there exists $(z_1^0, z_2^0) \in L^I$ such that $h_1(z_1^0, z_2^0) = 0$ by the intermediate value theorem. Similarly, let $h_2(z_1, z_2) = S_2(N_2(A_j^l(\mathbf{x}_0)), z_2) - S_2(N_2(A_{j+1}^l(\mathbf{x}_0)), 1 - z_1)$. It is easy to find that $h_2(z_1, z_2)$ is continuous and non-decreasing for the first and second variables. Moreover, we can obtain that $h_2(0,0) = S_2(N_2(A_j^l(\mathbf{x}_0)), 0) - S_2(N_2(A_{j+1}^l(\mathbf{x}_0)), 1) = N_2(A_j^l(\mathbf{x}_0)) - 1 \le 0$ and $h_2(1,1) = S_2(N_2(A_j^l(\mathbf{x}_0)), 1) - S_2(N_2(A_{j+1}^l(\mathbf{x}_0)), 0) = 1 - N_2(A_{j+1}^l(\mathbf{x}_0)) \ge 0$. Therefore, there exists $(z_1^{0'}, z_2^{0'}) \in L^I$ such that $h_2(z_1^{0'}, z_2^{0'}) = 0$ by the intermediate value theorem. This implies that equation (31) can be rewritten as

$$\begin{cases} h_1(z_1, z_2) = 0 \\ h_2(z_1, z_2) = 0 \end{cases}, \tag{32}$$

where $0 \le z_1 \le z_2 \le 1$. Choose any (z_1, z_2) as a solution of equation (32) and take $[z_1, z_2] = \widetilde{B}_j(y)$. Since $B_j^l(y)$ and $B_{j+1}^u(y)$ are pseudo-trapeziod-shaped respectively, there exists one unique point $y_0 = (B_j^l)^{-1}(z_1)$(or $(B_j^u)^{-1}(z_2)$. Obviously, y_0 satisfies the condition $B_j^l(y_0) + B_{j+1}^u(y_0) = 1$ (or $B_j^u(y_0) + B_{j+1}^l(y_0) = 1$). □

The following fact reveals that interval-valued fuzzy systems based on interval-valued R-and S-implications are universal approximators.

Theorem 4.4. *Let $U \subseteq \mathbf{R}^m$ be a unlimited domain and $\forall \varepsilon > 0$. For any continuous function $g : U \to \mathbf{R}$ such that $\lim_{||\mathbf{x}|| \to \infty} g(\mathbf{x})$ exists for any $\mathbf{x} \in U$, there exist some interval-valued fuzzy systems $y = f(\mathbf{x})$ based on R- and S-implications such that*

$$\max_{\mathbf{x} \in U} |f(\mathbf{x}) - g(\mathbf{x})| < \varepsilon. \tag{33}$$

Proof. In order to simply and better showing the proof, we shall only consider the case when $m = 2$. The proof for other cases is similar. Without loss of generality, we assume $U = [a_1, +\infty) \times [a_2, +\infty)$ and $\lim_{||\mathbf{x}|| \to +\infty} g(\mathbf{x}) = A$. It is not difficult to prove that g is uniformly continuous on $[a_1, +\infty) \times [a_2, +\infty)$. And then there exists a closed interval $[c, d]$ such that $g(U) \subseteq [c, d]$. Since $\lim_{||\mathbf{x}|| \to +\infty} g(\mathbf{x}) = A$, for any $\varepsilon > 0$ we can find two positive numbers $b_1 (> a_1)$ and $b_2 (> a_2)$ such that $|g(x_1, x_2) - A| < \frac{\varepsilon}{2}$ if $x_1 > b_1$ and $x_2 > b_2$. For any $\varepsilon > 0$, the continuity of g implies that there exists $\delta > 0$ such that $|g(x_1, x_2) - g(x_1', x_2')| < \frac{\varepsilon}{2}$ if $|x_1 - x_1'| < \delta$ and $|x_2 - x_2'| < \delta$ for any $x_1, x_1' \in [a_1, +\infty), x_2, x_2' \in [a_2, +\infty)$. With the δ defined above, natural numbers n_1 and n_2 such that $\frac{b_1 - a_1}{n_1} < \frac{\delta}{2}$ and $\frac{b_2 - a_2}{n_2} < \delta$, respectively. Choose $x_i^1 \in [a_1, b_1] (i = 1, 2, \cdots, n_1)$ and $x_j^2 \in [a_2, b_2] (j = 1, 2, \cdots, n_2)$ such that $\max(|x_{i+1}^1 - x_i^1|, |x_{j+1}^2 - x_j^2|) < \delta$.

Following the idea presented in [26], let $y_k = g(x_i^1, x_j^2) (k = 1, 2, \cdots, n_1 n_2)$. Take the distinct y_k's and sort them ascending. In other words, $y_k = y'_{\sigma(k)}$, where σ is a permutation from $\mathbb{N}_{n_1 n_2}$ to $\mathbb{N}_l$ and $y'_{\sigma(k)}$ are in ascending order. Although $g(x_i^1, x_j^2) = y_k = y'_{\sigma(k)}$, $(x_i^1, x_j^2), (x_{i+1}^1, x_{j+1}^2) \in U$ may not be mapped to $y'_{\sigma(k)}, y'_{\sigma(k)+1}$ or $y'_{\sigma(k)}, y'_{\sigma(k)-1}$. Therefore, we need to refine every sub-interval $[x_i^1, x_{i+1}^1] \times [x_j^2, x_{j+1}^2]$ according to the following three cases.

Case 1: If $y'_{\sigma(k+1)} = y'_{\sigma(k)-1}$ or $y'_{\sigma(k)+1}$ then we do nothing.

Case 2: If $y'_{\sigma(k+1)} = y'_{\sigma(k)+p}$ with $p \geq 2$. In this case, we can find a point $(x^1, x^2) \in [x_i^1, x_{i+1}^1] \times [x_j^2, x_{j+1}^2]$ such that $y' = g(x^1, x^2)$ for any $y' \in \{y'_{\sigma(k)+1}, y'_{\sigma(k)+2}, \cdots, y'_{\sigma(k)+p-1}\}$ by the continuity of g. Further, we split $[x_i^1, x_{i+1}^1]$ and $[x_j^2, x_{j+1}^2]$ into $[x_i^1, x_{i_1}^1]$, $[x_{i_1}^1, x_{i_2}^1], \cdots, [x_{i_{p-1}}^1, x_{i+1}^1]$ and $[x_j^2, x_{j_1}^2]$, $[x_{j_1}^2, x_{j_2}^2], \cdots, [x_{j_{p-1}}^2, x_{j+1}^2]$, respectively.

Case 3: If $y'_{\sigma(k+1)} = y'_{\sigma(k)-p}$ with $p \geq 2$. Similar to Case 2, $[x_i^1, x_{i+1}^1]$ and $[x_j^2, x_{j+1}^2]$ are split into $[x_i^1, x_{i_1}^1]$, $[x_{i_1}^1, x_{i_2}^1], \cdots, [x_{i_{p-1}}^1, x_{i+1}^1]$ and $[x_j^2, x_{j_1}^2]$, $[x_{j_1}^2, x_{j_2}^2], \cdots, [x_{j_{p-1}}^2, x_{j+1}^2]$, respectively.

After achieved the above procedure, we rename the points $x_i^1 (i = 1, 2, \cdots, n_1)$, $x_{i_s}^1$ and $x_j^2 (j = 1, 2, \cdots, n_2)$, $x_{j_s}^2$ $(s = 1, 2, \cdots, p-1)$ as $x_i^1 (i = 1, 2, \cdots, n_1')$ and x_j^2 $(j = 1, 2, \cdots, n_2')$, respectively.

Next, we construct interval-valued fuzzy sets on U and $[c, d]$ with the (x_i^1, x_j^2)'s and y_k's mentioned above as follows. Two group of complete and consistent interval-valued fuzzy sets $\{\widetilde{A}_i^1\}_{i=1}^{n_1'+1}$ on $[a_1, +\infty)$ and $\{\widetilde{A}_j^2\}_{j=1}^{n_2'+1}$ on $[a_2, +\infty)$ are respectively constructed such that $\widetilde{A}_1^1 = ((A_1^1)^l, (A_1^1)^u) = ((0, x_1^1, x_2^1; 1), (0, x_1^1, x_2^1, x_3^1; 1))$, $\widetilde{A}_i^1 = ((A_i^1)^l, (A_i^1)^u) = ((x_{i-1}^1, x_i^1, x_{i+2}^1, x_{i+3}^1; 1), (x_i^1, x_{i+1}^1, x_{i+1}^1, x_{i+2}^1; 1))$ for each $1 < i \leq n_1' - 1$, $\widetilde{A}_{n_1'}^1 = ((A_{n_1'}^1)^l, (A_{n_1'}^1)^u) = ((x_{n_1'-2}^1, x_{n_1'-1}^1, b_1 + \delta_1, b_1 + 2\delta_1; 1), (x_{n_1'-1}^1, b_1, b_1 + \delta_1, b_1 + 2\delta_1; 1))$, $\widetilde{A}_{n_1'+1}^1 = ((A_{n_1'+1}^1)^l, (A_{n_1'+1}^1)^u) = ((b_1, b_1 + \delta_1, +\infty; 1), (b_1, b_1 + 2\delta_1, +\infty; 1))$ and $\widetilde{A}_1^2 = ((A_1^2)^l, (A_1^2)^u) = ((0, x_1^2, x_2^2; 1), (0, x_1^2, x_2^2, x_3^2; 1))$, $\widetilde{A}_j^2 = ((A_j^2)^l, (A_j^2)^u) = ((x_{j-1}^2, x_j^2, x_{j+2}^2, x_{j+3}^2; 1), (x_j^2, x_{j+1}^2, x_{j+1}^2, x_{j+2}^2; 1))$ for each $1 < j \leq n_2' - 1$, $\widetilde{A}_{n_2'}^2 = ((A_{n_2'}^2)^l, (A_{n_2'}^2)^u) = ((x_{n_2'-2}^2, x_{n_2'-1}^2, b_2 + \delta_2, b_2 + 2\delta_2; 1), (x_{n_2'-1}^2, b_2, b_2 + \delta_2, b_2 + 2\delta_2; 1))$, $\widetilde{A}_{n_2'+1}^2 = ((A_{n_2'+1}^2)^l, (A_{n_2'+1}^2)^u) = ((b_2, b_2 + \delta_2, +\infty; 1), (b_2, b_2 + 2\delta_2, +\infty; 1))$, respectively. A group of complete and consistent interval-valued fuzzy sets $\{\widetilde{B}_l\}_{l=1}^{k+1}$ on $[c, d]$ such that $\widetilde{B}_1 = (B_1^l, B_1^u) = ((0, y_1, y_2; 1), (0, y_1, y_2, y_3; 1))$, $\widetilde{B}_j = (B_j^l, B_j^u) = ((y_{j-1}, y_j, y_{j+2}, y_{j+3}; 1), (y_j,\ y_{j+1}, y_{j+1}, y_{j+2}; 1))$ for each $1 < j \leq k - 1$, $\widetilde{B}_k = (B_k^l, B_k^u) = ((y_{k-2}, y_k, y_k, y_k; 1), (y_{k-1}, y_k, y_k,\ y_k; 1))$, $\widetilde{B}_{k+1} = \begin{cases} 1_{L^I} & y = A \\ 0_{L^I} & \text{otherwise} \end{cases}$.

Then, interval-valued fuzzy rules can be designed as follows:

R_{ij} : IF x_1 is $\widetilde{A}_i^1$ AND x_2 is $\widetilde{A}_j^2$ THEN y is $\widetilde{B}_l$ $(i = 1, 2, \cdots, n_1'; j = 1, 2, \cdots, n_2')$,

$\mathrm{R}_{n_1'+1, n_2'+1}$: IF x_1 is $\widetilde{A}_{n_1'+1}^1$ or x_2 is $\widetilde{A}_{n_2'+1}^2$ THEN y is $\widetilde{B}_{k+1}$,

where $\widetilde{B}_l$ is chosen such that $y_l = g(x_i^1, x_j^2)$ if x_i^1 and x_j^2 belong to $\mathrm{Ker}\widetilde{A_i^1}$ and $\mathrm{Ker}\widetilde{A_j^2}$, respectively. It is easy to prove that if the rules R_{ij} are fired, then the membership functions of the consequent part of these rules have at most two adjacent different elements $\widetilde{B}_j, \widetilde{B}_{j+1}$.

Thus, we have constructed the interval-valued fuzzy system. Moreover, for any given input $\mathbf{x}_0 \in U$ either there exists $[x_{i_0}^1, x_{i_0+1}^1] \times [x_{j_0}^1, x_{j_0+1}^1]$ for some $i_0 \leq n_1' - 1$ and $j_0 \leq n_1' - 1$ such that $\mathbf{x}_0 \in [x_{i_0}^1, x_{i_0+1}^1] \times [x_{j_0}^1, x_{j_0+1}^1]$ or $\mathbf{x}_0 \in [b_1, +\infty) \times [b_2, +\infty)$. According to our construction, we have $\widetilde{B}'(y) = \Big(\widetilde{A}_i(\mathbf{x}_0) \to \widetilde{B}_i(y)\Big) \wedge \Big(\widetilde{A}_{i+1}(\mathbf{x}_0) \to \widetilde{B}_{i+1}(y)\Big)$. Let us consider the following three cases:

i. $\mathbf{x}_0 \in [x_{i_0}^1, x_{i_0+1}^1] \times [x_{j_0}^1, x_{j_0+1}^1]$. This case implies that $y_0 = (B_i^l)^{-1}(z_1)$ (or $(B_i^u)^{-1}(z_2)$) is in $[y_{j-1}, y_j]$ or $[y_j, y_{j+1}]$ by Lemma 4.1. Without of loss generality, we assume that $y_0 \in [y_j, y_{j+1}]$. Therefore, $\max\{|y_0 - y_j|, |y_0 - y_{j+1}|\} < \frac{\varepsilon}{2}$. Thus, $|f(\mathbf{x}_0) - g(\mathbf{x}_0)| = |y_0 - g(\mathbf{x}_0)| \leq \min\{|y_0 - y_j| + |y_j - g(\mathbf{x}_0)|, |y_0 - y_{j+1}| + |y_{j+1} - g(\mathbf{x}_0)|\} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

ii. $\mathbf{x}_0 \in [b_1, +\infty) \times [b_2, +\infty)$. This case implies that $y_0 = A$ by Lemma 4.1. So, $|f(\mathbf{x}_0) - g(\mathbf{x}_0)| = |y_0 - g(\mathbf{x}_0)| = |A - g(\mathbf{x}_0)|\} < \varepsilon$.

Hence, $\max\limits_{\mathbf{x} \in U} |f(\mathbf{x}) - g(\mathbf{x})| = \max(\max\limits_{\mathbf{x} \in [a_1, b_1] \times [a_2, b_2]} |f(\mathbf{x}) - g(\mathbf{x})|, \max\limits_{\mathbf{x} \in [b_1, +\infty) \times [b_2, +\infty)} |f(\mathbf{x}) - g(\mathbf{x})|) < \varepsilon$. □

5. The Sufficient Condition for Interval-valued Fuzzy Systems as Universal Approximators

This section investigates the sufficient condition for interval-valued fuzzy systems based on R- and S-implications as universal approximators in order to answer the question: "Given a real continuous function, how should the membership functions of interval-valued fuzzy sets be designed, and how many interval-valued fuzzy sets are needed for each input and output variable in order to guarantee the desired approximation accuracy?"

As argument in Theorem 4.4, it is sufficient to approximate a real continuous function defined on a compact subsets of $\mathbf{R}^m$ by an interval-valued fuzzy system. Therefore, we only consider the real continuous functions defined on $[a,b]^m$ in the section.

Definition 5.1. For a normal interval-valued fuzzy set $\widetilde{A}$ on U, x_0 is called the central point of $\widetilde{A}$ if x_0 is the central point of $\text{Ker}\widetilde{A}$.

Definition 5.2. A group of normal interval-valued fuzzy sets $\widetilde{A}_i$ $(i = 1, 2, \cdots, n)$ defined on U is said to be overlapping if $c_{i-1} \leq \inf(\text{Supp}(\widetilde{A}_i)) \leq \sup(\text{Supp}(\widetilde{A}_{i-1})) \leq c_i$ holds for any $i = 2, 3, \cdots, n$, where c_i is the central point of the fuzzy set $\widetilde{A}_i$.

Let a group of normal interval-valued fuzzy sets $\widetilde{A}_i$ $(i = 1, 2, \cdots, n)$ on U be overlapping. It can be easily verified that for any $i \neq j$, $i, j = 1, 2, \cdots, n$, $\widetilde{A}_i(c_j) = 0_{L^I}$ holds, and for any $x \in U$, at most two adjacent fuzzy sets $\widetilde{A}_j$ and $\widetilde{A}_{j+1}$ can have memberships of x greater than 0_{L^I} and must have $c_j \leq x \leq c_{j+1} (j \in \{1, 2, \cdots, n-1\})$.

Definition 5.3. Let the central point of the i-th interval-valued fuzzy set $\widetilde{A}_i$ on U be chosen at c_i $(i = 1, 2, \cdots, n)$, where $c_1 < c_2 < \cdots < c_n$. For each variable $x \in U$, the distance of the fuzzy partition is defined as

$$D_k = c_k - c_{k-1} \ (k = 1, 2, \cdots, n+1). \tag{34}$$

Based on the above argument, the maximum distance of fuzzy partition is defined as

$$D_{\max} = \max_k D_k \ (k = 1, 2, \cdots, n+1). \tag{35}$$

Definition 5.4. A group of normal interval-valued fuzzy sets on U are said to be equally distributed if $D_k = D_{\max}$ $(k = 1, 2, \cdots, n+1)$.

Lemma 5.5. *Suppose that n_0 overlapping and equally distributed interval-valued fuzzy sets are assigned to each output variable of interval-valued fuzzy systems based on R- or S-implications, and n_i' overlapping and equally distributed interval-valued fuzzy sets are assigned to their each input variable. Then, for an arbitrary polynomial $P(\mathbf{x})$ defined on $[a,b]^m$ and the approximation error bound $\varepsilon > 0$, there exist interval-valued fuzzy systems $f(\mathbf{x})$ based on R- or S- implications such that $\max\limits_{\mathbf{x}\in[a,b]^m} |P(\mathbf{x}) - f(\mathbf{x})| < \varepsilon$ holds whenever $n_0 > \frac{d-c}{2\varepsilon} - 1$, $n_i' > \frac{n_0(b-a)}{2\varepsilon} \max\limits_{\mathbf{x}\in[a,b]^m} |\frac{\partial P}{\partial x_i}(\mathbf{x})| - 1 (i = 1, 2, \cdots, m)$.*

Proof. Since $P(\mathbf{x})$ is continuous on $[a,b]^m$, its image set is a closed interval, says $[c,d]$. For any $\mathbf{x} \in [a,b]^m$, we can construct an interval-valued fuzzy system $y = f(\mathbf{x})$ based on R- (or S-) implication as follows. First, a group of normal interval-valued fuzzy sets $\{\widetilde{A}^i_{k_i}\}_{i=1}^m$ is constructed such that they are a Ruspini Partition of $[a, b]$, and the central points are chosen at $c_{k_i}^{'i}$, respectively. Since these fuzzy sets assigned to each input variable are overlapping and equally distributed, $|c_{k_i}^{'i} - c_{k_i-1}^{'i}| = \frac{b-a}{n_i+1}$ $(k_i = 1, 2, \cdots, n_i + 1)$ hold. Next, a group of normal fuzzy sets $\{\widetilde{B}_k\}_{k=1}^{n}$ is constructed such that they are a Ruspini Partition of $[c, d]$, and the central points are chosen at $c_k = P(c_{k_1}^{'1}, \cdots, c_{k_m}^{'m})$, respectively. Then, the fuzzy rules can be designed as follows:

$$\mathrm{R}_{i_1 i_2 \cdots i_m}: \text{IF } x_1 \text{ is } \widetilde{A}_{i_1}^{1} \text{ and } x_2 \text{ is } \widetilde{A}_{i_2}^{2} \text{ and } \cdots \text{ and } x_m \text{ is } \widetilde{A}_{i_m}^{m} \text{ THEN } y \text{ is } \widetilde{B}_k$$

with $i_1 = 1, 2, \cdots, n_1'$, $\cdots$, $i_m = 1, 2, \cdots, n_m'$ and $k = 1, 2, \cdots, n$.

It is easily proven that only fuzzy rule $\mathrm{R}_{i_1 i_2 \cdots i_m}$ is fired for any $c_i' = (c_{i_1}^{'1}, c_{i_2}^{'2}, \cdots, c_{i_m}^{'m})$, and that $f(c_i') = c_k$ holds. Using Taylor's formula, yields $P(\mathbf{x}) = c_k + r_1(\mathbf{x})$, where the Lagrange remainder $r_1(x)$ is expressed as $r_1(\mathbf{x}) = \frac{\partial P}{\partial \mathbf{x}}|_{\mathbf{x}=\eta}(\mathbf{x} - \mathbf{x}')$, η is located in the minimum super sphere containing $\mathbf{x}$ and $(c_{k_1}^{'1}, \cdots, c_{k_m}^{'m})$, and $\frac{\partial P}{\partial \mathbf{x}}$ is the Jacobian determinant of P. Let $|c_{k+1} - c_k| < \frac{\varepsilon}{2}$ and $\sum\limits_{i=1}^{m} \max\limits_{\mathbf{x}\in[a,b]^m} |\frac{\partial P}{\partial x_i}(\mathbf{x})| \frac{1}{n_i'+1} < \frac{\varepsilon}{2}$. Since $|f(\mathbf{x}) - c_k| \le |c_{k+1} - c_k| < \frac{\varepsilon}{2}$, we have $n_0 > \frac{d-c}{2\epsilon} - 1$.

Moreover, $|r_1(\mathbf{x})| = |\frac{\partial P}{\partial \mathbf{x}}|_{\mathbf{x}=\eta}(\mathbf{x} - \mathbf{x}')| \le \sum\limits_{i=1}^{m} \max\limits_{\mathbf{x}\in[a,b]^m} |\frac{\partial P}{\partial x_i}(\mathbf{x})| \frac{b-a}{n_i'+1} < \frac{\varepsilon}{2}$; furthermore, if we let $\max\limits_{\mathbf{x}\in[a,b]^m} |\frac{\partial P}{\partial x_i}(\mathbf{x})| \frac{1}{n_i'+1} < \frac{\varepsilon}{2n_0(b-a)}$, then $n_i' > \frac{n_0(b-a)}{2\varepsilon} \max\limits_{\mathbf{x}\in[a,b]^m} |\frac{\partial P}{\partial x_i}(\mathbf{x})| - 1$ holds. Hence, $|f(\mathbf{x}) - P(\mathbf{x})| = |f(\mathbf{x}) - c_k - r_1(\mathbf{x})| \le |f(\mathbf{x}) - c_k| + |r_1(\mathbf{x})|$; furthermore, $\max\limits_{\mathbf{x}\in[a,b]^m} |f(\mathbf{x}) - P(\mathbf{x})| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon$. □

Theorem 5.6. *Suppose that n_0 overlapping and equally distributed interval-valued fuzzy sets are assigned to each output variable of an interval-valued fuzzy system based on R- or S-implication, and n_i' overlapping and equally distributed interval-valued fuzzy sets are assigned to each input variable of the interval-valued fuzzy system. Then, for an arbitrary real continuous function $g(\mathbf{x}) \in [a, b]^m$ and approximation error bound $\varepsilon > 0$, there exists an interval-valued fuzzy system $y = f(\mathbf{x})$ based on R- (or S-implication) such that $\max\limits_{\mathbf{x}\in[a,b]^m} |f(\mathbf{x}) - g(\mathbf{x})| < \varepsilon$ holds when $n_0 > \frac{d-c}{2(\varepsilon-\varepsilon_1)} - 1$, $n_i' > \frac{n_0(b-a)}{2(\varepsilon-\varepsilon_1)} \max\limits_{\mathbf{x}\in[a,b]^m} |\frac{\partial P}{\partial x_i}(\mathbf{x})| - 1$, where $i = 1, 2, \cdots, m$, $0 < \varepsilon_1 < \varepsilon$ and $\max\limits_{\mathbf{x}\in[a,b]^m} |f(\mathbf{x}) - g(\mathbf{x})| < \varepsilon_1$.*

Proof. By the Weierstrass Theorem, there always exists a polynomial $P(\mathbf{x})$ such that $\max\limits_{\mathbf{x}\in[a,b]^m} |g(\mathbf{x}) - P(\mathbf{x})| < \varepsilon_1$ holds. According to Lemma 5.5, there exists an interval-valued fuzzy system based on *R*- (or *S*-implication), which can reassume $\max\limits_{\mathbf{x}\in[a,b]^m} |P(\mathbf{x}) - g(\mathbf{x})| < \varepsilon - \varepsilon_1$ if $n_0 > \frac{d-c}{2(\varepsilon-\varepsilon_1)} - 1$ holds. This implies that $\max\limits_{\mathbf{x}\in[a,b]^m} |g(\mathbf{x}) - f(\mathbf{x})| \le \max\limits_{\mathbf{x}\in[a,b]^m} |g(\mathbf{x}) - P(\mathbf{x})| + \max\limits_{\mathbf{x}\in[a,b]^m} |P(\mathbf{x}) - g(\mathbf{x})| < \varepsilon$. Hence, $n_i' > \frac{n_0}{2(\varepsilon-\varepsilon_1)} \max\limits_{\mathbf{x}\in[a,b]^m} |\frac{\partial P}{\partial x_i}(\mathbf{x})| - 1$ holds. □

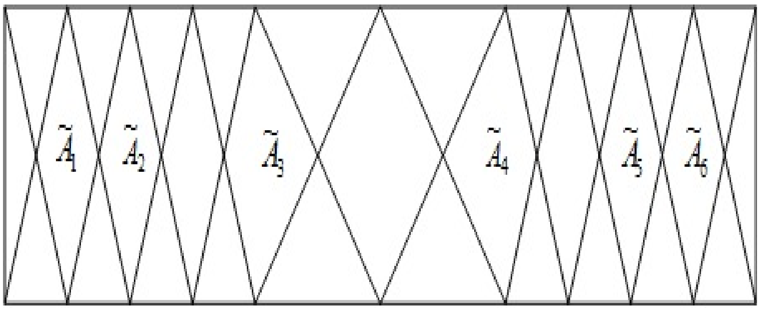

Figure 2. The Membership Functions of Input Variable in Example 6.1

6. Examples

Example 6.1. Design a fuzzy system based on interval-valued R-implications to approximate the continuous function $g(x) = \begin{cases} 1 & x = 0 \\ \frac{\sin x}{x} & \text{otherwise} \end{cases}$ defined over $[-3, 3]$ with an accuracy of $\varepsilon = 0.2$.

The proposed method for design the fuzzy systems based on interval-valued R-implications is now presented as follows:

Step 1: Construct the input and output interval-valued fuzzy membership functions. The image set of $g(x)$ on $[-3, 3]$ is $[0.047, 1]$. By Theorem 4.4, the output discourse is divided into $\lceil \frac{1-0.047}{0.2} \rceil + 1 = 5$ parts. Thus, we construct the membership functions of consequence as $\widetilde{B}_1, \widetilde{B}_2, \widetilde{B}_3$ which are depicted as Figure 2.

Let $\delta = \frac{2}{3}$. Then the input discourse is divided into $\lceil (3-(-3)) \times \frac{3}{2} \rceil + 1 = 10$ parts, implying that the membership functions of the antecedent consist of $\widetilde{A}_1, \widetilde{A}_2, \cdots, \widetilde{A}_6$ (Figure 3).

Step 2: Construct the interval-valued fuzzy rules base. The interval-valued fuzzy rules base consists of:

R_1: IF x is $\widetilde{A}_1$ OR $\widetilde{A}_6$, THEN y is $\widetilde{B}_1$;
R_2: IF x is $\widetilde{A}_2$ OR $\widetilde{A}_5$, THEN y is $\widetilde{B}_2$;
R_3: IF x is $\widetilde{A}_3$ OR $\widetilde{A}_4$, THEN y is $\widetilde{B}_3$.

Step 3: Choose fuzzifier in equation (10) (or equation (12)) and implication operator in equation (19). Here, we choose R-implication generated by interval-valued Łukasiewicz t-norm as implication operator in equation (19).

Step 4: Choose defuzzifier in equation (19) and compute the output by equation (20) and equation (26). Here, we use averaging of maximum defuzzifier and compute the output by equation (19).

Finally, we obtain the interval-valued fuzzy system $y = f(x)$ depicted in Figure 4. The approximation error is $\varepsilon = \max\limits_{x \in [-3,3]} |f(x) - g(x)| = 0.0554 < 0.2$. Figure 4 depicts also the comparison of the interval-valued fuzzy system function $y = f(x)$ and the origin function $y = g(x)$. In this example, we use only 3 rules to

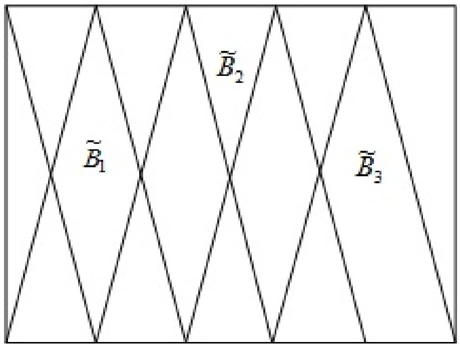

FIGURE 3. The Membership Functions of Output Variable in Example 6.1

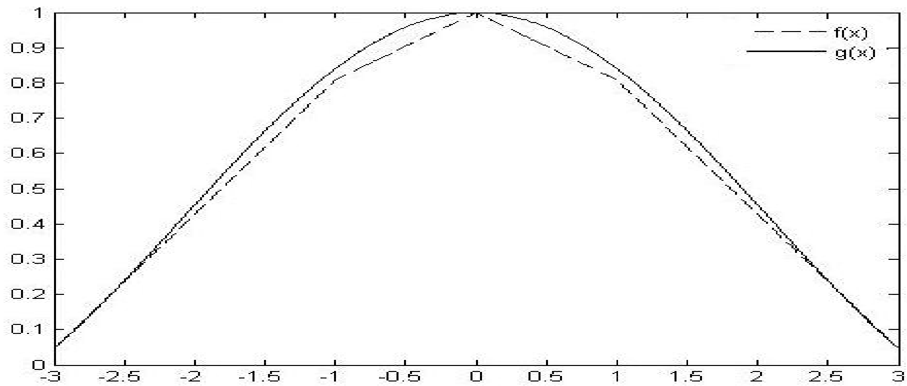

FIGURE 4. Comparison of The Interval-valued Fuzzy System $y = f(x)$ and The Origin Function $y = g(x)$

approximate $g(x)$ with an accuracy $\varepsilon = 0.2$, while Ying in [37] used 207 rules to approximate the same function.

Example 6.2. Design a fuzzy system based on interval-valued R-implications to approximate the polynomial $P(x_1, x_2) = 0.52 + 0.1x_1 + 0.38x_2 - 0.06x_1x_2$ defined over $[-1, 1] \times [-1, 1]$ with an accuracy of $\varepsilon = 0.1$.

Since the image set of $P(x_1, x_2)$ is $[-0.02, 0.94]$, by Theorem 5.6 we obtain $n_0 = \lceil \frac{0.94+0.02}{0.1} \rceil + 1 = 10$.

Thus, the membership functions of consequence consist of $\widetilde{B}_1, \widetilde{B}_2, \cdots, \widetilde{B}_6$. Let $\delta_1 = 0.5$ and $\delta_2 = 0.14$. Then $n_1 = \lceil \frac{1-(-1)}{0.5} \rceil + 1 = 5$ and $n_2 = \lceil \frac{1-(-1)}{0.14} \rceil + 1 = 15$ hold. Thus, the membership functions of the input variables x_1 and x_2 are $\widetilde{A}^1_1, \widetilde{A}^1_2, \widetilde{A}^1_3, \widetilde{A}^1_4$ and $\widetilde{A}^2_1, \widetilde{A}^2_2, \cdots, \widetilde{A}^2_9$ respectively. Table 1 presents the fuzzy rules.

$i\backslash j$	1	2	3	4	5	6	7	8	9
1	1	1	2	3	3	3	4	4	5
2	2	2	2	3	3	3	4	4	5
3	2	2	3	3	3	4	4	4	5
4	2	3	3	3	4	4	4	5	6

TABLE 1. The Interval-valued Fuzzy Rules for Example 6.2

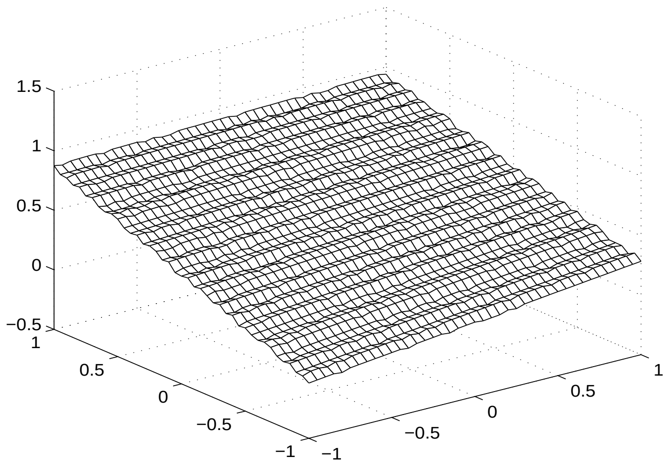

FIGURE 5. Graphical Illustration of the Interval-valued Fuzzy System $f(x_1, x_2)$ in Example 6.2.

Let $x \to_R y = [\frac{y_1}{x_1} \wedge \frac{y_2}{x_2} \wedge 1, \frac{y_2}{x_2} \wedge 1]$ and the product operator, the interval-valued system $f(x_1, x_2)$ is depicted in Figure 5. The approximation error is $\epsilon = 0.078 < 0.1$.

In this example, we use only 36 rules to approximate $P(x_1, x_2) = 0.52 + 0.1x_1 + 0.38x_2 - 0.06x_1x_2$ defined over $[-1, 1] \times [-1, 1]$ with accuracy $\varepsilon = 0.1$ while Ying in [37] used 225 rules to approximate the same function.

7. **Conclusions**

In this paper, we constructively proved that the fuzzy systems based on interval-valued R- and S-implications have the ability to uniformly approximate any multivariate continuous function defined on a compact set to arbitrary accuracy. A formula was presented to compute the lower upper bounds on the number of fuzzy sets to achieve pre-specified approximation accuracy for an arbitrary multivariate continuous function. Finally, a general approach to construct fuzzy systems based on interval-valued R- and S-implications was also represented. This method avoids to calculate the centroid of an interval-valued fuzzy set, implying the type-reduction problem of traditional interval-valued fuzzy systems can be effectively resolved. These results may lay a theoretical foundation for the fuzzy systems development.

In the future, we wish to investigate the necessary condition for the interval-valued fuzzy systems based on R- and S- implications as universal approximators with minimal system configurations.

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