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SOME RESULTS ON L-COMPLETE LATTICES

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Abstract. The paper deals with special types of L-ordered sets, L-fuzzy complete lattices, and fuzzy directed complete posets. First, a theorem for constructing monotone maps is proved, a characterization for monotone maps on an L-fuzzy complete lattice is obtained, and it's proved that if f is a monotone map on an L-fuzzy complete lattice $(P; e)$, then the least fixpoint of f is meet of a special element of L^P. A relation between L-fuzzy complete lattices and fixpoints is found and fuzzy versions of monotonicity, rolling, fusion and exchange rules on L-complete lattices are stated. Finally, we investigate the set of all monotone maps on a fuzzy directed complete posets, $DCPO$s, and find a condition which under the set of all fixpoints of a monotone map on a fuzzy $DCPO$ is a fuzzy $DCPO$.

1. Introduction

Fixed point theory serves as an essential tool for various branches of mathematical analysis and its applications in many areas. For example, in theoretical computer science, least fixed points of monotone functions are used to define program semantics. There are three main approaches to fixed point theory. The first one is the metric approach in which one makes use of the metric properties of the underlying spaces and self-maps. A primary example of this approach is Banach's Contraction Mapping Theorem. The second approach is the topological one in which one utilizes the topological properties of the underlying spaces and continuity of self-maps. A primary example of this approach is Brouwer's Fixed Point Theorem. Finally, the third approach is the order-theoretic one.

Recently, based on complete Heyting algebras and on the fuzzy L-order relation, Zhang, Xie and Fan [24] have defined and studied L-fuzzy complete lattices, which are generalizations of traditional complete lattices. They discussed their properties, showed that they coincide with complete and co-complete categories enriched over the frame L [18], and they proved the Tarski Fixed-Point Theorem for an L-fuzzy complete lattice.

Using complete Heyting algebras, Fan and Zhang [11, 23] studied quantitative domains through fuzzy set theory. Their approach first defines a fuzzy partial order, specifically a degree function on a non-empty set. Then they define and study fuzzy directed subsets and (continuous) fuzzy directed complete posets ($DCPO$s for short). Moreover, Yao [21] and Yao and Shi [22] pursued an investigation on

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quantitative domains via fuzzy sets. They defined the notions of fuzzy Scott topology on fuzzy $DCPO$s, Scott convergence and topological convergence for stratified L-filters and studied them. They showed that the category of fuzzy $DCPO$s with fuzzy Scott continuous maps is Cartesian-closed. In [19], Yao introduced an L-frame by an L-ordered set equipped with some further conditions. It is a complete L-ordered set with the meet operation having a right fuzzy adjoint. He established an adjunction between the category of stratified L-topological spaces and the category of L-locales, the opposite category of this kind of L-frames. Borzooei et al. [6] applied an L-relation on a group, introduced an L-ordered group and investigated its properties.

In the mid-1950's Tarski [16] published an interesting result: Every complete lattice has the fixed point property, that is, every order-preserving mapping has a fixpoint. Davis [8] proved the converse: Every lattice with the fixed point property is complete. Tarski's Fixpoint Theorem generalized to CPO (an abbreviation for a $\vee$-complete poset with a bottom element), i.e. if $f: P \to P$ is an order-preserving map and P is a CPO, then the set of fixpoints of f, $Fix(f)$, is a CPO and so $Fix(f)$ has a least element. But, what is the relationship between the least element of $Fix(f)$ and other points of P? The main purpose of this paper is to answer this question.

The present paper is organized as follows. In Section 2, we list some preliminary notions and results that will be used in the paper. In Section 3, we consider L-fuzzy complete lattices. We show that if f is a monotone map on an L-fuzzy complete lattice $(P; e)$, then the least and greatest fixpoint of f are meet and join of some special L-subsets of P. Also, we show that every L-complete lattice is, up to isomorphism, an L-complete lattice of fixpoints and we propose fuzzy versions of monotonicity, rolling, fusion and exchange rules on L-complete lattices. In Section 4, we define the concept of a t-fixpoint and we prove that if $(P; e)$ is a fuzzy $DCPO$, then H_P, the set of monotone maps on $(P; e)$, is a fuzzy $DCPO$. Finally, we investigate this fuzzy $DCPO$.

2. **Preliminaries**

We start with some notions from [7, 11]. A non-empty subset D of a poset $(P; \leq)$ is called *directed* if, for each pair of elements $x, y \in D$, there exists $z \in D$ such that $x, y \leq z$. We say that a poset $(P; \leq)$ is a *pre-CPO* or a $DCPO$ (an abbreviation for a directed complete partially ordered set) if, for each directed subset D of P, the join of D, $\bigvee D$ (the least upper bound of D in L) exists. A $DCPO$ $(P; \leq)$ is called a CPO (an abbreviation for a complete partially ordered set) if P has a bottom element.

Let $(L; \vee, \wedge, 0, 1)$ be a bounded lattice. For $a, b \in L$, we say that $c \in L$ is a *relative pseudocomplement* of a with respect to b if c is the largest element with $a \wedge c \leq b$ and we denote it by $a \to b$. A lattice $(L; \vee, \wedge)$ is said to be a *Heyting algebra* if the relative pseudocomplement $a \to b$ exists for all elements $a, b \in L$. The relative pseudocomplement is well-known in fuzzy set theory as *residual implication* or *residuum.* A *frame* is a complete lattice $(L; \vee, \wedge)$ satisfying the infinite distributive law $a \wedge \bigvee S = \bigvee_{s \in S}(a \wedge s)$ for every $a \in L$ and $S \subseteq L$. It is well known that L is a frame if and only if it is a complete Heyting algebra.

In fact, if $(L; \vee, \wedge)$ is a frame, then for each $a, b \in L$, the relative pseudocomplement of a with respect to b, is the element $a \to b := \bigvee\{x \in L | \ a \wedge x \leq b\}$ and with this definition $\to$ is a binary operation on L. In the following, we list some important properties of complete Heyting algebras, for more details relevant to frames and Heyting algebras, we refer to [13] and [3, Section 7]:

(i) $(x \wedge y) \to z = x \to (y \to z)$;
(ii) $x \to (\bigwedge Y) = \bigwedge_{y \in Y}(x \to y)$;
(iii) $(\bigvee Y) \to z = \bigwedge_{y \in Y}(y \to z)$.

From now on, in this paper, $(L; \vee, \wedge, 0, 1)$ or simply L always denotes a frame and L^X denotes the set of all maps from a set X into L.

Definition 2.1. [1, 2, 4, 9, 10] Let P be a set and $e : P \times P \to L$ be a map. The pair $(P; e)$ is called an L- *fuzzy ordered set* (L-ordered set for short) if, for all $x, y, z \in P$ we have

(E1) $e(x, x) = 1$;
(E2) $e(x, y) \wedge e(y, z) \leq e(x, z)$;
(E3) $e(x, y) = e(y, x) = 1$ implies $x = y$.

Proposition 2.2. [21, Prop. 3.7] *Let $(X; e)$ be an L-ordered set. Then for each $x, y \in X$,*

$$e(x, y) = \bigwedge_{z \in X} \big(e(z, x) \to e(z, y)\big) = \bigwedge_{z \in X} \big(e(y, z) \to e(x, z)\big).$$

In an L-ordered set $(P; e)$, the map e is called an *L-fuzzy order relation* (L-order relation for short) on P. If $(P; \leq)$ is a classical poset, then $(P; \chi_{\leq})$ is an L-ordered set, where $\chi_{\leq}$ is the characteristic function of $\leq$. We usually denote this L-ordered set by $(P; e_{\leq})$. Moreover, for each L-ordered set $(P; e)$, the set $\leq_e = \{(x, y) \in P \times P | \ e(x, y) = 1\}$ is a crisp partial order on P and $(P; \leq_e)$ (if there is no ambiguity we write $(P; \leq)$) is a poset. Assume that $(P; e)$ is an L-ordered set and $\phi \in L^P$. Define $\downarrow \phi \in L^P$ and $\uparrow \phi \in L^P$, see [12, 23], as follows:

$$\downarrow \phi(x) = \bigvee_{x' \in P} (\phi(x') \wedge e(x, x')), \qquad \uparrow \phi(x) = \bigvee_{x' \in P} (\phi(x') \wedge e(x', x)), \quad \forall x \in P.$$

Definition 2.3. [19, 21] A map $f : (P; e_P) \to (Q; e_Q)$ between two L-ordered sets is called *monotone* if for all $x, y \in P$, $e_P(x, y) \leq e_Q(f(x), f(y))$.

A monotone map $f : (P; e_P) \to (Q; e_Q)$ is called an *L-order isomorphism* if f is one-to-one, onto and $e_P(x, y) = e_Q(f(x), f(y))$ for all $x, y \in P$.

Definition 2.4. [20, 21] Let $(P; e_P)$ and $(Q; e_Q)$ be two L-ordered sets and $f : P \to Q$ and $g : Q \to P$ be two monotone maps. The pair (f, g) is called a *fuzzy Galois connection* between P and Q if $e_Q(f(x), y) = e_P(x, g(y))$ for all $x \in P$ and $y \in Q$, where f is called the *fuzzy left adjoint* of g, and dually, g is called the *fuzzy right adjoint* of f.

Theorem 2.5. [20, Thm. 3.2] *A pair (f, g) is a fuzzy Galois connection on (X, e_X) and (Y, e_Y) if and only if both f and g are monotone and (f, g) is a (crisp) Galois connection on $(X, \leq_{e_X})$ and $(Y, \leq_{e_Y})$.*

Definition 2.6. [21, 20] Let $(P; e)$ be an L-ordered set and $S \in L^P$. An element x_0 is called the *join* (respectively, the *meet*) of S, in symbols $x_0 = \sqcup S$ (respectively, $x_0 = \sqcap S$) if, for all $x \in P$,

(J1) $S(x) \leq e(x, x_0)$ (respectively, (M1), $S(x) \leq e(x_0, x)$);
(J2) $\bigwedge_{y \in P}(S(y) \to e(y, x)) \leq e(x_0, x)$ (respectively, (M2), $\bigwedge_{y \in P}(S(y) \to e(x, y)) \leq e(x, x_0)$).

If the join and the meet of S exist, then they are unique (see [23]).

Theorem 2.7. [23, Thm. 2.2] *Let $(P; e)$ be an L-ordered set, $x_0 \in P$, and $S \in L^P$. Then*

(i) *$x_0 = \sqcup S$ if and only if $e(x_0, x) = \bigwedge_{y \in P}(S(y) \to e(y, x))$ for all $x \in P$;*
(ii) *$x_0 = \sqcap S$ if and only if $e(x, x_0) = \bigwedge_{y \in P}(S(y) \to e(x, y))$ for all $x \in P$.*

Definition 2.8. Let $(P; e)$ be an L-ordered set. For all $a \in P$, $\downarrow a : P \to L$ and $\uparrow a : P \to L$ are defined by $\downarrow a(x) = e(x, a)$ and $\uparrow a(x) = e(a, x)$, respectively. It can be easily shown that $\sqcup \downarrow a = a$ and $\sqcap \uparrow a = a$ for all $a \in P$ ([21, Prop 3.16]).

In [24], an L-fuzzy complete lattice was introduced: An L-ordered set $(P; e)$ is called an *L-fuzzy complete lattice* (or an *L-complete lattice*, for short) if, for all $S \in L^P$, $\sqcap S$ and $\sqcup S$ exist. If $(P; e)$ is an L-complete lattice, then $(P; \leq_e)$ is a complete lattice, where $\vee S = \sqcup \chi_S$ and $\wedge S = \sqcap \chi_S$ for any $S \subseteq P$. We must note that an L-fuzzy complete lattice was called a complete L-ordered set, in [19, Definition 2.5].

Theorem 2.9. [5, 17, 24] *Let X be a non-empty set. Then $(L^X; \tilde{e})$ is an L-complete lattice, where $\tilde{e}(f, g) = \bigwedge_{x \in X}(f(x) \to g(x))$ for all $f, g \in L^X$.*

Suppose that X and Y are two sets. For each mapping $f : X \to Y$, we have two maps $f^{\to} : L^X \to L^Y$ and $f^{\leftarrow} : L^Y \to L^X$ defined by

$$(\forall y \in Y)(\forall A \in L^X) \quad \Big(f^{\to}(A)(y) = \bigvee\{A(x)|\ x \in X,\ f(x) = y\}\Big)$$
$$(\forall x \in X)(\forall B \in L^Y) \quad f^{\leftarrow}(B)(x) = B(f(x)).$$

For simplicity, for any $A \in L^X$, we use $f(A)$ instead of $f^{\to}(A)$.

Definition 2.10. [21] Let $(P; e)$ be an L-ordered set. An element $x \in P$ is called a *maximal (or minimal)* element of $A \in L^P$, in symbols $x = \max A$ (or $x = \min A$), if $A(x) = 1$ and for all $y \in P$, $A(y) \leq e(y, x)$ (or $A(y) \leq e(x, y)$). It is easy to see that if A has a maximal (or minimal) element, then it is unique.

Definition 2.11. [14, 21] Let $(X; e)$ be an L-ordered set. An element $D \in L^X$ is called a *fuzzy directed subset* of $(X; e)$ if

(FD1) $\bigvee_{x \in X} D(x) = 1$;
(FD2) for all $x, y \in X$, $D(x) \wedge D(y) \leq \bigvee_{z \in X} \big(D(z) \wedge e(x, z) \wedge e(y, z)\big)$.

An L-ordered set $(X;e)$ is called a *fuzzy DCPO* if every fuzzy directed subset of $(X;e)$ has a join.

3. Fixpoints on L-complete Lattices

In this section, monotone maps on L-ordered set play an important role. Thus, in Theorem 3.3, we propose a procedure for constructing monotone maps. Then we show that every L-complete lattice is, up to isomorphism, an L-complete lattice of fixpoints. We find a relation between least and greatest fixpoints of a monotone map f on an L-complete lattice $(P;e)$ with join and meet of some special elements of L^P. Finally, we present a fuzzy version of monotonicity, rolling, fusion and exchange rules on L-complete lattices.

Lemma 3.1. *Let $(L;\vee,\wedge)$ be a frame and $a_i,b_i\in L$ for all $i\in I$. Then*

$$\bigwedge_{i\in I}(a_i\to b_i)\leq(\bigvee_{i\in I}a_i)\to(\bigvee_{i\in I}b_i).$$

Proof. The proof is straightforward. □

In Theorem 3.3 and Corollary 3.5, we will use the following proposition that has been stated by W. Yao [21, Prop. 3.16].

Proposition 3.2. [21] *Let $(P;e)$ be an L-ordered set, $S\in L^P$. Then $a=\max S$ (resp. $a'=\min S$) if and only if $S(a)=1$ and $a=\sqcup S$ (resp. $S(a')=1$ and $a'=\sqcap S$).*

Theorem 3.3. *Let $(P;e)$ be an L-ordered set, $(Q;e')$ be an L-complete lattice and $f:P\to Q$ be a map. For each $x\in P$, we define $S_x:Q\to L$, by $S_x(y)=f_L^{\to}(\downarrow x)(y)$, for all $y\in Q$. Let $F:P\to Q$ be defined by $F(a)=\sqcup S_a$ for all $a\in P$. Then F is monotone. Moreover, f is monotone if and only if $f=F$.*

Proof. First we show that F is monotone, that is $e(a,b)\leq e'(F(a),F(b))$ for all $a,b\in P$. Put $a,b\in P$. Set $u_a=\sqcup S_a$ and $u_b=\sqcup S_b$. By Theorem 2.7, for all $x\in Q$, we obtain

$$e'(u_a,x)=\bigwedge_{y\in Q}\big(S_a(y)\to e'(y,x)\big),\quad e'(u_b,x)=\bigwedge_{y\in Q}\big(S_b(y)\to e'(y,x)\big).$$

Hence,

$$e'(u_a,u_b)=\bigwedge_{y\in Q}\big(S_a(y)\to e'(y,u_b)\big)\tag{1}$$

$$S_b(y)\leq e'(y,u_b)\text{ for all }y\in Q.\tag{2}$$

From (2), it follows that $S_a(y)\to S_b(y)\leq S_a(y)\to e'(y,u_b)$ for all $y\in Q$ and so by (1), $\bigwedge_{y\in Q}\big(S_a(y)\to S_b(y)\big)\leq\bigwedge_{y\in Q}\big(S_a(y)\to e'(y,u_b)\big)=e'(u_a,u_b)$. Moreover, we have

$$\begin{aligned}
\bigwedge_{y\in Q}\big(S_a(y)\to S_b(y)\big) &= \bigwedge_{y\in f(P)}\big(S_a(y)\to S_b(y)\big),\ \text{since } S_a(y)=0, \text{ for all } y\in Q-f(P)\\
&= \bigwedge_{y\in P}\big(S_a(f(y))\to S_b(f(y))\big)\\
&= \bigwedge_{y\in P}\Big(\big(\bigvee_{f(z)=f(y)} e(z,a)\big)\to\big(\bigvee_{f(w)=f(y)} e(w,a)\big)\Big)\\
&\geq \bigwedge_{y\in P}\Big(\bigwedge_{f(z)=f(y)} (e(z,a)\to e(z,b))\Big),\ \text{by Lemma 3.1}\\
&= \bigwedge_{z\in P}\big(e(z,a)\to e(z,b)\big)\\
&= e(a,b),\ \text{by Proposition 2.2}
\end{aligned}$$

so $F:(P;e)\to(Q;e')$ is monotone. Now, we show that, if f is monotone, then $f=F$. Suppose that $f:(P;e)\to(Q;e')$ is monotone. Put $a\in P$. For all $y\in Q$, $S_a(y)=\bigvee_{\{z\in P|\ f(z)=y\}} e(z,a)$, so $S_a(f(a))=1$. Since f is monotone, $f(z)=y$ implies that $e(z,a)\leq e'(f(z),f(a))=e'(y,f(a))$, hence $S_a(y)=\bigvee_{\{z\in P|\ f(z)=y\}} e(z,a)\leq e'(y,f(a))$. By Proposition 3.2, $\sqcup S_a=f(a)$ and so $F(a)=f(a)$ for all $a\in P$. □

Suppose that $(P;e)$ is an L-ordered set and $f:P\to P$ is a map. Define three maps $S_f:P\to L$, $T_f:P\to L$ and $M_f:P\to L$ by $S_f(x)=e(f(x),x)$, $T_f(x)=e(x,f(x))$ and $M_f(x)=S_f(x)\wedge T_f(x)$ for all $x\in P$. Moreover, by $Fix(f)$ we denote the set of all fixpoints of f, that is, $Fix(f)=\{x\in P|\ f(x)=x\}$ and every point $x\in Fix(f)$ is said to be a *fixpoint* of f.

Consider the assumptions of Theorem 3.3. Let $Hom(P,Q)$ be the set of all monotone maps from P to Q. Clearly, $(Q^P;\varepsilon')$ and $(Hom(P,Q);\varepsilon')$ are L-ordered sets, where $\varepsilon'(\alpha,\beta)=\bigwedge_{x\in P} e'(\alpha(x),\beta(x))$ for all $\alpha,\beta\in Q^P$. It can be easily seen that, the map, $\phi:Q^P\to Q^P$, sending f to F (see the notations in Theorem 3.3) is a monotone map and $Fix(\phi)=Hom(P,Q)$.

Theorem 3.4. *Let $(P;e)$ be an L-complete lattice and $f:(P;e)\to(P;e)$ be a monotone map. Then $\sqcap S_f$ and $\sqcup T_f$ are fixpoints of f. In addition, $\sqcap S_f$ is the least fixpoint and $\sqcup T_f$ is the greatest fixpoint of f.*

Proof. Let $\sqcup T_f=a$ and $\sqcap S_f=b$. Then $e(a,x)=\bigwedge_{y\in P}\big(T_f(y)\to e(y,x)\big)$ and $e(x,b)=\bigwedge_{y\in P}\big(S_f(y)\to e(x,y)\big)$ for all $x\in P$ and so $T_f(y)\leq e(y,a)$ for all $x\in P$. Since f is a monotone map, then $e(y,a)\leq e(f(y),f(a))$ and hence $T_f(y)\leq e(y,f(y))\wedge e(f(y),f(a))\leq e(y,f(a))$ for all $y\in P$. Thus by Theorem 2.7, $e(a,f(a))=\bigwedge_{y\in P}\big(T_f(y)\to e(y,f(a)\big)=1$. Also, $1=e(a,f(a))\leq e(f(a),f(f(a)))=T_f(f(a))$, so $f(a)\leq a$. Therefore, $f(a)=a$ and a is a fixpoint of f. Now, let u be another fixpoint of f, then $1=e(u,f(u))=T_f(u)$, hence, $u\leq a$, whence a is the greatest fixpoint of f. By a similar way, we can show that b is the least fixpoint of f. □

Corollary 3.5. *Let $(P;e)$ be an L-complete lattice and $f:(P;e)\to(P;e)$ be a monotone map. Then* $\max T_f$ *and* $\min S_f$ *exist and* $\max T_f=\sqcup T_f=\sqcup M_f$ *and* $\min S_f=\sqcap S_f=\sqcap M_f$.

Proof. By Theorem 3.4 and Proposition 3.2, it can be easily obtained that $\max T_f=\sqcup T_f$ and $\min S_f=\sqcap S_f$. Let $a=\min S_f$ and $b=\max T_f$. By Theorem 3.4, $M_f(a)=1=M_f(b)$. Also, for all $y\in P$, we have $M_f(y)\le S_f(y)\le e(y,a)$ and $M_f(y)\le T_f(y)\le e(y,a)$, so by definition, $\min M_f=a$ and $\max M_f=b$. Now, from [21, Prop. 3.16] we conclude that $\sqcup M_f=b$ and $\sqcap M_f=a$. □

Example 3.6. Let X be a set and $f:X\to X$ be a map. By Theorem 2.9, $(L^X;\tilde{e})$ is an L-complete lattice. Put $A,B\in L^X$. By Lemma 3.1, for each $y\in X$, $\bigwedge\{A(x)\to B(x)|\ x\in X,\ f(x)=y\}\le(\bigvee\{A(x)|\ x\in X,\ f(x)=y\})\to(\bigvee\{B(x)|\ x\in X,\ f(x)=y\})=f^{\to}(A)(y)\to f^{\to}(B)(y)$. It follows that

$$\begin{aligned}\tilde{e}(A,B) &= \bigwedge_{x\in X}(A(x)\to B(x))=\bigwedge_{y\in Imf}\ \bigwedge_{\{x\in X|\ f(x)=y\}}(A(x)\to B(x))\\ &\le \bigwedge_{y\in Imf} f^{\to}(A)(y)\to f^{\to}(B)(y)\\ &= \bigwedge_{y\in X} f^{\to}(A)(y)\to f^{\to}(B)(y)=\tilde{e}(f(A),f(B)).\end{aligned}$$

So, $f^{\to}$ is a monotone map on $(L^X;\tilde{e})$. On the other hand,

$$Fix(f^{\to})=\{A\in L^X|\ \forall y\in X,\ \bigvee_{\{f(x)=y\}}A(x)=A(y)\}$$

Let $S_{f^{\to}}:L^X\to L$ and $T_{f^{\to}}:L^X\to L$ be defined by $S_{f^{\to}}(A)=\tilde{e}(f^{\to}(A),A)$ and $T_{f^{\to}}(A)=\tilde{e}(A,f^{\to}(A))$, for all $A\in L^X$. Consider the maps $\phi_0,\phi_1\in L^X$ define by

$$\phi_0(x)=\bigvee_{\phi\in L^X}(T_{f^{\to}}(\phi)\wedge\phi(x)),\quad \phi_1(x)=\bigwedge_{\phi\in L^X}(S_{f^{\to}}(\phi)\to\phi(x)),\quad \forall x\in X$$

Then by the proof of [24, Thm. 2.29], $\sqcup T_{f^{\to}}=\phi_0$ and $\sqcap S_{f^{\to}}=\phi_1$ and so by Theorem 3.4, ϕ_0 and ϕ_1 are greatest and least fixpoints of $\tilde{f}$, respectively.

By [24, Thm. 2.29], we know that, if $(X;e)$ is an L-complete lattice and $f:X\to X$ is a monotone map, then $Fix(f)$ is an L-complete lattice. In the next theorem, we will show that each L-complete lattice is of this form. That is, any L-complete lattice is L-isomorphic to $Fix(f)$ for some suitable monotone map f on a suitable L-complete lattice.

Theorem 3.7. *Let $(P;e)$ be an L-complete lattice. Define $f:(L^P,\tilde{e})\to(L^P,\tilde{e})$ by $f(S)=\downarrow\sqcup S$ for each $S\in L^P$. Then f is monotone and there exists an L-order isomorphism between $(Fix(f);\tilde{e})$ and $(P;e)$.*

Proof. By [24, Thm. 2.29], $(L^P;\tilde{e})$ is an L-complete lattice. First, we show that f is monotone (clearly, f is well defined). Let $S,T\in L^P$ and $x\in P$. Then by Theorem 2.7,

$$e(\sqcup S, x) = \bigwedge_{y\in P} \big(S(y) \to e(y,x)\big), \quad e(\sqcup T, x) = \bigwedge_{y\in P} \big(T(y) \to e(y,x)\big) \tag{3}$$

and so

$$\begin{aligned}\tilde{e}(\downarrow \sqcup S, \downarrow \sqcup T) &= \bigwedge_{y\in P} \big((\downarrow \sqcup S)(y) \to (\downarrow \sqcup T)(y)\big) = \bigwedge_{y\in P} \big(e(y,\sqcup S) \to e(y,\sqcup T)\big) \\ &= e(\sqcup S, \sqcup T), \text{ by Proposition 2.2} \\ &\geq \bigwedge_{y\in P} \big(S(y) \to e(y,\sqcup T)\big), \text{ by (3).}\end{aligned}$$

Also, by (3), for all $y \in P$, $T(y) \leq e(y, \sqcup T)$, so $S(y) \to T(y) \leq S(y) \to e(y, \sqcup T)$ for all $y \in P$ which implies that $\tilde{e}(S,T) = \bigwedge_{y\in P} \big(S(y) \to T(y)\big) \leq \bigwedge_{y\in P} \big(S(y) \to e(y, \sqcup T)\big)$. By summing up the above results, it follows that $\tilde{e}(S,T) \leq \tilde{e}(\downarrow \sqcup S, \downarrow \sqcup T)$. That is, $f : (L^P, \tilde{e}) \to (L^P, \tilde{e})$ is monotone. Define $\alpha : P \to Fix(f)$ by $\alpha(x) = \downarrow x$ for all $x \in P$. We know that $(Fix(f); \tilde{e})$ is an L-complete lattice. Clearly, α is one-to-one. Put $S \in Fix(f)$. Then $\downarrow \sqcup S = f(S) = S$, so $S \in Im(\alpha)$. That is, α is onto. Moreover, by Proposition 2.2, for all $a, b \in P$, $\tilde{e}(\alpha(a), \alpha(b)) = \tilde{e}(\downarrow a, \downarrow b) = \bigwedge_{y\in P} \big(e(y,a) \to e(y,b)\big) = e(a,b)$. Therefore, α is an L-order isomorphism. □

We know that if $(P; \leq)$ is a complete lattice and $f, g : P \to P$ are two order-preserving maps such that $f(x) \leq g(x)$ for all $x \in P$, then $\mu_f \leq \mu_g$, where μ_f and μ_g are the least fixpoints of f and g, respectively (see [7, Section 8]). In the next theorem, we generalize this result for L-complete lattices.

Theorem 3.8. (Monotonicity rule.) *Let $(P; e)$ be an L-complete lattice and $f, g : (P; e) \to (P; e)$ be monotone maps such that $\sqcap S_f = a$ and $\sqcap S_g = b$. Then $\bigwedge_{y\in P} e(f(y), g(y)) \leq e(a,b)$.*

Proof. Since $\sqcap S_f = a$ and $\sqcap S_g = b$, then for each $x \in P$, we have

$$e(x,a) = \bigwedge_{y\in P} \big(e(f(y),y) \to e(x,y)\big) \tag{4}$$

$$e(x,b) = \bigwedge_{y\in P} \big(e(g(y),y) \to e(x,y)\big) \tag{5}$$

By (5), $e(a,b) = \bigwedge_{y\in P} \big(e(g(y),y) \to e(a,y)\big)$. Also, by (4), $e(f(y),y) \leq e(a,y)$ for all $y \in P$, so $e(g(y),y) \to e(f(y),y) \leq e(g(y),y) \to e(a,y)$ for all $y \in P$ and hence, $e(a,b) = \bigwedge_{y\in P} \big(e(g(y),y) \to e(a,y)\big) \geq \bigwedge_{y\in P} \big(e(g(y),y) \to e(f(y),y)\big)$. Now, we claim that $e(g(y),y) \to e(f(y),y) \geq e(f(y),g(y))$ for all $y \in P$. In order to show that our claim is true, it suffices to prove that $e(f(y),g(y)) \wedge e(g(y),y) \leq e(f(y),y)$, which clearly hold by (E3). Hence, our claim is true and so $e(a,b) = \bigwedge_{y\in P} \Big(e(g(y),y) \to e(f(y),y)\Big) \geq \bigwedge_{y\in P} e(f(y),g(y))$. □

Theorem 3.9. (Rolling rule.) *Let $(P; e)$ and $(Q; e')$ be L-complete lattices and $f : (P; e) \to (Q; e')$, $g : (Q; e') \to (P; e)$ be monotone maps. Then the following hold:*

(i) $g(\sqcap S_{f\circ g}) = \sqcap S_{g\circ f}$.

(ii) $\sqcap(g(S_{f\circ g})) = g(\sqcap S_{g\circ f})$.

Proof. (i) The proof of this part follows from Theorem 3.4, and the Rolling Rule from [7, 8.29].

(ii) Let $\sqcap(g(S_{f\circ g})) = b$ and $a = \sqcap S_{g\circ f}$. Then for each $x \in P$,

$$\begin{aligned} e(x,b) = \bigwedge_{y\in P}\big(g(S_{f\circ g})(y) \to e(x,y)\big) &= \bigwedge_{y\in P}\big(\big(\bigvee_{\{z\in Q|\ g(z)=y\}} S_{f\circ g}(z)\big) \to e(x,y)\big) \\ &= \bigwedge_{y\in P}\ \bigwedge_{\{z\in Q|\ g(z)=y\}} \big(S_{f\circ g}(z) \to e(x,y)\big) \\ &= \bigwedge_{y\in Q} \big(S_{f\circ g}(y) \to e(x,g(y))\big). \qquad (6) \end{aligned}$$

$$e(x,a) = \bigwedge_{y\in P} \big(S_{g\circ f}(y) \to e(x,y)\big) = \bigwedge_{y\in P} \big(e(g\circ f(y),y) \to e(x,y)\big). \qquad (7)$$

Since g is monotone, by (6) and (7), for each $x \in Q$, $e(g(x),b) = \bigwedge_{y\in Q}\big(S_{f\circ g}(y) \to e(g(x),g(y))\big) \geq \bigwedge_{y\in Q}\big(S_{f\circ g}(y) \to e(x,y)\big) = e(x,\sqcap S_{f\circ g})$. Hence, $e(g(\sqcap S_{f\circ g}),b) \geq e(\sqcap_{f\circ g},\sqcap_{f\circ g}) = 1$ and so by (i), $a = \sqcap S_{g\circ f} = g(\sqcap S_{f\circ g}) \leq b$. Moreover, by (6), $e(f\circ g(y),y) = S_{f\circ g}(y) \leq e(b,g(y))$ for all $y \in Q$, so $e\big(f\circ g(f(a)),f(a)\big) \leq e(b,g(f(a)))$. By Theorem 3.4, we have $g(f(a)) = a$ (since $a = \sqcap S_{gof}$) and $f\circ g(f(a)) = f(a)$. It follows that $1 = e(f(a),f(a)) = e\big(f\circ g(f(a)),f(a)\big) = e(b,a)$. Therefore, $a = b$, and so by (i), the proof of this part is completed. □

Theorem 3.10. (Fusion rule.) *Let $(P;e)$ and $(Q;e')$ be two L-complete lattices and let $f : P \to Q$ possess a right adjoint $f' : Q \to P$. Let $g : P \to P$ and $h : Q \to Q$ be monotone. Then*

(i) $\bigwedge_{y\in P} e'(f\circ g(y), h\circ f(y)) \leq e'(f(\sqcap S_g),\sqcap S_h)$.

(ii) $e'(h\circ f(\sqcap S_g), f\circ g(\sqcap S_g)) \leq e'(\sqcap S_h, f(\sqcap S_g))$.

Proof. (i) By Theorem 2.5, (f,f') is a fuzzy Galois connection between P and Q, hence

$$\begin{aligned} 1 = e(f'(\sqcap S_h),f'(\sqcap S_h)) &= e'(f(f'(\sqcap S_h)),\sqcap S_h) \\ &\Rightarrow 1 = e'\big(h(f(f'(\sqcap S_h))),h(\sqcap S_h)\big), \text{ since } h \text{ is monotone} \\ &\Rightarrow 1 = e'\big(h(f(f'(\sqcap S_h))),\sqcap S_h\big), \text{ by Theorem 3.4.} \end{aligned}$$

It follows that

$$\begin{aligned} &e'(f\circ g(f'(\sqcap S_h)),h\big(f(f'(\sqcap S_h))\big)) \\ &\quad= e'(h\big(f(f'(\sqcap S_h))\big),\sqcap S_h) \wedge e'(f\circ g(f'(\sqcap S_h)),h\big(f(f'(\sqcap S_h))\big)) \\ &\quad\leq e'(f\circ g(f'(\sqcap S_h)),\sqcap S_h) = e(g(f'(\sqcap S_h)),f'(\sqcap S_h)). \end{aligned}$$

By Theorem 2.7, for each $y \in P$, $e(g(y),y) = S_g(y) \leq e(\sqcap S_g,y)$, so

$$e(g(f'(\sqcap S_h)),f'(\sqcap S_h)) \leq e(\sqcap S_g,f'(\sqcap S_h)) = e'(f(\sqcap S_g),\sqcap S_h).$$

Therefore, $\bigwedge_{y\in P} e'(f\circ g(y),h\circ f(y)) \leq e'(f(\sqcap S_g),\sqcap S_h)$.

(ii) By Theorem 2.7(ii), we know that $S_h(y) \leq e'(\sqcap S_h, y)$ for all $y \in Q$, and so by Theorem 3.4, $e'(\sqcap S_h, f(\sqcap S_g)) \geq S_h(f(\sqcap S_g)) = e'(h(f(\sqcap S_g)), f(\sqcap S_g)) = e'(h(f(\sqcap S_g)), f(g(\sqcap S_g)))$. □

Note that, in Theorem 3.10, we showed that $e'(f \circ g(f'(\sqcap S_h)), h\big(f(f'(\sqcap S_h))\big)) \leq e'(f(\sqcap S_g), \sqcap S_h)$.

Corollary 3.11. (Exchange rule.) *Let $(P; e)$ and $(Q; e')$ be L-complete lattices and $f, g : P \to Q$ and $h : Q \to P$ be monotone maps. If f possesses a right adjoint $f' : Q \to P$, then*

$$e'\Big(f \circ h \circ g(f'(\sqcap S_{g\circ h})), g \circ h \circ f(f'(\sqcap S_{g\circ h}))\Big) \leq e'(\sqcap S_{f\circ h}, \sqcap S_{g\circ h}).$$

Proof. Let $h' : Q \to Q$ and $g' : P \to P$ be defined by $h' = g \circ h$ and $g' = h \circ g$. By Theorem 3.10,

$$e'\Big(f \circ g'(f'(\sqcap S_{h'})), h' \circ f(\sqcap S_{h'})\Big) \leq e'(f(\sqcap S_{g'}), \sqcap S_{h'}) = e'(f(\sqcap S_{h\circ g}), \sqcap S_{g\circ h})$$

and by Theorem 3.9(i), $\sqcap S_{h\circ g} = h(\sqcap S_{g\circ h})$, so that

$$e'(f(\sqcap S_{h\circ g}), \sqcap S_{g\circ h}) = e'(f(h(\sqcap S_{g\circ h})), \sqcap S_{g\circ h}) = S_{f\circ h}(\sqcap S_{g\circ h}).$$

Also, by Theorem 2.7, for each $y \in Q$, $S_{f\circ h}(y) \leq e'(\sqcap S_{f\circ h}, y)$, so

$$e'\Big(f \circ g'(f'(\sqcap S_{h'})), h' \circ f(\sqcap S_{h'})\Big) \leq e'(\sqcap S_{f\circ h}, \sqcap S_{g\circ h}).$$

□

4. **Fuzzy DCPOs**

In this section, we define the concept of a t-fixpoint and prove that if $(P; e)$ is a fuzzy $DCPO$, then the set of monotone maps on $(P; e)$ is a fuzzy $DCPO$. This will serve us in order to find some of the t-fixpoints of f. Finally, we find conditions under which we can find the least fixpoint of f.

Theorem 4.1. *Let $(P; e)$ be a fuzzy $DCPO$ and H_P be the set of all monotone maps on $(P; e)$. Then $(H_P; \overline{e})$ is a fuzzy $DCPO$, where $\overline{e}(f, g) = \bigwedge_{x\in P} e(f(x), g(x))$ for all $f, g \in H_P$.*

Proof. It is easy to see that $(H_P; \overline{e})$ is an L-ordered set. Let $S : H_P \to L$ be a fuzzy directed subset of $(H_P; \overline{e})$. Then $\bigvee_{f\in H_P} S(f) = 1$ and for each $f, g \in H_P$,

$$S(f) \wedge S(g) \leq \bigvee_{\gamma\in H_P} \Big(S(\gamma) \wedge \overline{e}(f, \gamma) \wedge \overline{e}(g, \gamma)\Big). \tag{8}$$

First, we show that $\sqcup S$ exists. That is, there exists a map $\alpha_0 : P \to P$ such that

$$\overline{e}(\alpha_0, f) = \bigwedge_{\gamma\in H_P} \Big(S(\gamma) \to \overline{e}(\gamma, f)\Big) \quad \text{for all } f \in H_P,$$

which is equivalent to

$$\begin{aligned}
\overline{e}(\alpha_0, f) &= \bigwedge_{\gamma\in H_P} \Big(S(\gamma) \to (\bigwedge_{y\in P} e\big(\gamma(y), f(y)\big))\Big) \\
&= \bigwedge_{\gamma\in H_P} \bigwedge_{y\in P} \Big(S(\gamma) \to e\big(\gamma(y), f(y)\big)\Big) \\
&= \bigwedge_{y\in P} \bigwedge_{\gamma\in H_P} \Big(S(\gamma) \to e(\gamma(y), f(y))\Big).
\end{aligned}$$

So, it suffices to show that

$$\bigwedge_{y\in P} e(\alpha_0(y), f(y)) = \bigwedge_{y\in P}\bigwedge_{\gamma\in H_P} \Big(S(\gamma)\rightarrow e(\gamma(y), f(y))\Big) \text{ for all } f\in H_P\ . \qquad (9)$$

Put $f \in H$. We claim that, for all $y \in P$, there exists an element $u_y \in P$ such that

$$e(u_y, f(y)) = \bigwedge_{\gamma\in H_P} \Big(S(\gamma)\rightarrow e(\gamma(y), f(y))\Big).$$

Let $x \in P$ and $X := \{\gamma(x)|\gamma \in H_P\}$. Define the map $T_x : P \to L$ by

$$T_x(y) = \begin{cases} 0 & y\in P-X \\ \bigvee\{S(h)|\ h\in H_P,\ h(x)=y\} & y\in X. \end{cases}$$

Clearly, T_x is a well-defined map. In the following, we show that T_x is a fuzzy directed subset on $(P; e)$.

(i)

$$\begin{aligned} \bigvee_{u\in P} T_x(u) &= \bigvee_{u\in X} T_x(u) = \bigvee_{\gamma\in H_p} T_x(\gamma(x)) \\ &= \bigvee_{\gamma\in H_P}\ \bigvee_{\{h\in H_P|\ h(x)=\gamma(x)\}} S(h) \end{aligned}$$

Since $S : H_P \to L$ is a fuzzy directed set on $(H_P; \overline{e})$, $\bigvee_{h\in H_P} S(h) = 1$. Therefore, $\bigvee_{u\in P} T_x(u) = 1$.

(ii) Let $u, v \in P$. If $u \in P-X$ or $v \in P-X$, then by definition, $T_x(u)\wedge T_x(v) = 0$ and so $T_x(u) \wedge T_x(v) \le \bigvee_{z\in P}\big(T_x(z) \wedge e(u, z) \wedge e(v, z)\big)$. Otherwise, $u = \gamma_1(x)$ and $v = \gamma_2(x)$ for some $\gamma_1, \gamma_2 \in H_P$. It follows that

$$T_x(u)\wedge T_x(v) = \bigvee_{\{h\in H_P|\ h(x)=\gamma_1(x)\}}\ \bigvee_{\{k\in H_P|\ k(x)=\gamma_2(x)\}} (S(h)\wedge S(k)). \qquad (10)$$

Also, we have

$$\begin{aligned} \bigvee_{z\in P}\big(T_x(z)\wedge e(u,z)\wedge e(v,z)\big) &= \bigvee_{z\in X}\big(T_x(z)\wedge e(u,z)\wedge e(v,z)\big) \\ &= \bigvee_{\gamma\in H_P}\big(T_x(\gamma(x))\wedge e(u,\gamma(x))\wedge e(v,\gamma(x))\big) \\ &= \bigvee_{\gamma\in H_P}\big(\big(\bigvee_{\{h\in H_P|\ h(x)=\gamma(x)\}} S(h)\big)\wedge e(u,\gamma(x))\wedge e(v,\gamma(x))\big) \\ &= \bigvee_{\gamma\in H_P}\ \bigvee_{\{h\in H_P|\ h(x)=\gamma(x)\}}\big(S(h)\wedge e(u,\gamma(x))\wedge e(v,\gamma(x))\big) \\ &= \bigvee_{\gamma\in H_P}\big(S(\gamma)\wedge e(u,\gamma(x))\wedge e(v,\gamma(x))\big) \\ &= \bigvee_{\gamma\in H_P}\big(S(\gamma)\wedge e(\gamma_1(x),\gamma(x))\wedge e(\gamma_2(x),\gamma(x))\big) \\ &\ge \bigvee_{\gamma\in H_P}\big(S(\gamma)\wedge \overline{e}(\gamma_1,\gamma)\wedge \overline{e}(\gamma_2,\gamma)\big) \end{aligned}$$

so by (8), $\bigvee_{z\in P}\big(T_x(z) \wedge e(u, z) \wedge e(v, z)\big) \ge S(\gamma_1) \wedge S(\gamma_2)$. Since γ_1 and γ_2 are arbitrary elements of H_P such that $\gamma_1(x) = u$ and $\gamma_2(x) = v$, then $\bigvee_{z\in P}\big(T_x(z)\wedge e(u,z)\wedge e(v,z)\big) \geq S(h)\wedge S(k)$ for all $k,h\in H_P$ such that $h(x)=u$ and $k(x)=v$, thus by (10),

$$\begin{aligned}\bigvee_{z\in P}\big(T_x(z)\wedge e(u,z)\wedge e(v,z)\big) &\geq \bigvee_{\{h\in H_P\mid\ h(x)=\gamma_1(x)\}}\ \bigvee_{\{k\in H_P\mid\ k(x)=\gamma_2(x)\}}(S(h)\wedge S(k))\\ &= T_x(u)\wedge T_x(v)\end{aligned}$$

(i) and (ii) imply that T_x is a fuzzy directed set on $(P;e)$ for all $x\in P$, whence by the assumption, $\sqcup T_x$ exists for all $x\in P$. Let $u_x := \sqcup T_x$ for all $x\in P$. Then for each $x\in P$, we have

$$e(u_x,z) = \bigwedge_{t\in P}\big(T_x(t)\to e(t,z)\big),\qquad \text{for all } z\in P. \tag{11}$$

Define a map $\alpha_0 : P\to L$ by $\alpha_0(x) = \sqcup T_x$ for all $x\in P$. By (11), for each $f\in H_P$ and $y,z\in P$,

$$\begin{aligned}e(\alpha_0(y),z) &= e(u_y,z) = \bigwedge_{t\in P}\big(T_y(t)\to e(t,z)\big)\\ &= \bigwedge_{\gamma\in H_P}\big(T_y(\gamma(y))\to e(\gamma(y),z)\big),\ \text{by definition of } T_y\\ &= \bigwedge_{\gamma\in H_P}\Big(\Big(\bigvee_{\{h\in H_P\mid\ h(y)=\gamma(y)\}} S(h)\Big)\to e(\gamma(y),z)\Big)\\ &= \bigwedge_{\gamma\in H_P}\ \bigwedge_{\{h\in H_P\mid\ h(y)=\gamma(y)\}}(S(h)\to e(\gamma(y),z))\\ &= \bigwedge_{\gamma\in H_P}(S(\gamma)\to e(\gamma(y),z)).\end{aligned} \tag{12}$$

It follows that $\bigwedge_{y\in P} e(\alpha_0(y),f(y)) = \bigwedge_{y\in P}\bigwedge_{\gamma\in H_P}\big(S(\gamma)\to e(\gamma(y),f(y))\big)$ and so (9) holds. Hence $\alpha_0 = \sqcup S$. Now, we show that $\alpha_0 : P\to P$ is a monotone map. Let $x,y\in P$. Also, by (12), for all $z\in P$,

$$e(\alpha_0(x),z) = \bigwedge_{\gamma\in H_P}(S(\gamma)\to e(\gamma(x),z)),\quad e(\alpha_0(y),z) = \bigwedge_{\gamma\in H_P}(S(\gamma)\to e(\gamma(y),z)).$$

so,

$$S(\gamma)\leq e(\gamma(y),\alpha_0(y)) \tag{13}$$

$$e(\alpha_0(x),\alpha_0(y)) = \bigwedge_{\gamma\in H_P}(S(\gamma)\to e(\gamma(x),\alpha_0(y))) \tag{14}$$

Hence, by (14),

$$\begin{aligned}e(x,y)\leq e(\alpha_0(x),\alpha_0(y)) &\Leftrightarrow e(x,y)\leq S(\gamma)\to e(\gamma(x),\alpha_0(y)),\ \text{for all } \gamma\in H_P\\ &\Leftrightarrow e(x,y)\wedge S(\gamma)\leq e(\gamma(x),\alpha_0(y)),\ \text{for all } \gamma\in H_P\\ &\Leftrightarrow S(\gamma)\leq e(x,y)\to e(\gamma(x),\alpha_0(y)),\ \text{for all } \gamma\in H_P\ .\end{aligned}$$

Since for each $\gamma\in H_P$, $e(x,y)\leq e(\gamma(x),\gamma(y))$, by Proposition 2.2, we have

$$e(\gamma(y),\alpha_0(y))\leq e(\gamma(x),\gamma(y))\to e(\gamma(x),\alpha_0(y))\leq e(x,y)\to e(\gamma(x),\alpha_0(y)).$$

Thus by (13), $S(\gamma) \leq e(x,y) \rightarrow e(\gamma(x), \alpha_0(y))$. Therefore, α_0 is monotone, and so it belongs to H_P. $\square$

Theorem 4.2. *Let $(P;e)$ be a fuzzy DCPO, H_P be the set of all monotone maps on $(P;e)$ and $S : H_P \rightarrow L$ be defined by $S(\alpha) = \overline{e}(Id_P, \alpha)$ for all $\alpha \in H_P$ (Id_P is the identity map on P). Then $\sqcup S$ exists and belongs to H_P. Moreover, $S(\alpha) \leq \overline{e}(\alpha \circ \sqcup S, \sqcup S) \wedge \overline{e}(\sqcup S, \alpha \circ \sqcup S)$ for all $\alpha \in H_P$.*

Proof. First, we show that S is a fuzzy directed subset of $(H_P, \overline{e})$. Since $Id_P \in H$ and $S(Id_P) = \overline{e}(Id_P, Id_P) = 1$, then $\bigvee_{\gamma \in H_P} S(\gamma) = 1$. Now, we show that $S(f) \wedge S(g) \leq \bigvee_{\alpha \in H_P} \big(S(\alpha) \wedge \overline{e}(f,\alpha) \wedge \overline{e}(g,\alpha)\big)$ for all $f, g \in H_P$. Put $f, g \in H_P$.

$$S(f) \wedge S(g) = \big(\bigwedge_{x \in P} e(x, f(x))\big) \wedge \big(\bigwedge_{x \in P} e(x, g(x))\big) = \bigwedge_{x \in P} \big(e(x, f(x)) \wedge e(x, g(x))\big). \tag{15}$$

$$\begin{aligned} &\bigvee_{\alpha \in H_P} \big(S(\alpha) \wedge \overline{e}(f,\alpha) \wedge \overline{e}(g,\alpha)\big) \\ &\quad = \bigvee_{\alpha \in H_P} \Big(\big(\bigwedge_{x \in P} e(x, \alpha(x))\big) \wedge \big(\bigwedge_{x \in P} e(f(x), \alpha(x))\big) \wedge \big(\bigwedge_{x \in P} e(g(x), \alpha(x))\big) \Big) \\ &\quad = \bigvee_{\alpha \in H_P} \Big(\big(\bigwedge_{x \in P} e(x, \alpha(x)) \wedge e(f(x), \alpha(x))\big) \wedge \big(\bigwedge_{x \in P} e(g(x), \alpha(x))\big) \Big). \end{aligned} \tag{16}$$

Clearly, $f \circ g \in H_P$ and for each $x \in P$, we have

(i) $e(x, f \circ g(x)) \geq e(x, f(x)) \wedge e(f(x), f \circ g(x)) \geq e(x, f(x)) \wedge e(x, g(x))$ (since f is monotone).

(ii) $e(f(x), f \circ g(x)) \geq e(x, g(x))$, (since f is monotone),

which imply that $\bigwedge_{x \in P} \big(e(x, f \circ g(x)) \wedge e(f(x), f \circ g(x))\big) \geq \bigwedge_{x \in P} \big(e(x, f(x)) \wedge e(x, g(x))\big)$. Also,

$$\bigwedge_{x \in P} e(g(x), f \circ g(x)) \geq \bigwedge_{x \in P} e(x, f(x)), \qquad \text{since } Im(g) \subseteq P$$

so, we have

$$\begin{aligned} &\bigwedge_{x \in P} \Big(e(x, f \circ g(x)) \wedge e(f(x), f \circ g(x))\Big) \wedge \bigwedge_{x \in P} e(g(x), f \circ g(x)) \\ &\qquad \geq \bigwedge_{x \in P} \big(e(x, f(x)) \wedge e(x, g(x))\big) \wedge \bigwedge_{x \in P} e(x, f(x)) \\ &\qquad = \bigwedge_{x \in P} \big(e(x, f(x)) \wedge e(x, g(x))\big). \end{aligned}$$

From $f \circ g \in H_P$, (15) and (16), it follows that

$$\begin{aligned} S(f) \wedge S(g) &= \bigwedge_{x \in P} \big(e(x, f(x)) \wedge e(x, g(x))\big) \\ &\leq \bigvee_{\alpha \in H_P} \Big(\big(\bigwedge_{x \in P} e(x, \alpha(x)) \wedge e(f(x), \alpha(x))\big) \wedge \big(\bigwedge_{x \in P} e(g(x), \alpha(x))\big) \Big) \\ &= \bigvee_{\alpha \in H_P} \big(S(\alpha) \wedge \overline{e}(f,\alpha) \wedge \overline{e}(g,\alpha)\big). \end{aligned}$$

Therefore, S is a fuzzy directed subset of $(H_P, \overline{e})$. By Theorem 4.1, there exists $\beta \in H_P$ such that $\beta = \sqcup S$, hence by Theorem 2.7(i), for each $f \in H_P$, $\overline{e}(\beta, f) = \bigwedge_{\alpha \in H_P} \big(S(\alpha) \to \overline{e}(\alpha, f)\big)$, whence $S(\alpha \circ \beta) \le \overline{e}(\alpha \circ \beta, \beta)$. From (E2), it can be easily obtained that $S(\alpha \circ \beta) = \overline{e}(Id_P, \alpha \circ \beta) \ge \overline{e}(Id_P, \alpha) \wedge \overline{e}(\alpha, \alpha \circ \beta) \ge \overline{e}(Id_P, \alpha) \wedge \overline{e}(Id_P, \beta) = S(\alpha) \wedge S(\beta) = S(\alpha)$ (since $S(Id_P) = 1$, then by Definition 2.6, $1 = S(Id_P) \le \overline{e}(Id_P, \beta) = S(\beta)$. Thus, $S(\alpha) \wedge S(\beta) \le \overline{e}(\alpha \circ \beta, \beta)$. On the other hand, $\overline{e}(\beta, \alpha \circ \beta) \ge \overline{e}(Id_P, \alpha) = S(\alpha)$ (since $Im(\beta) \subseteq P$). By summing up the above results, we get that $S(\alpha) \le \overline{e}(\beta, \alpha \circ \beta) \wedge \overline{e}(\alpha \circ \beta, \beta)$. □

Definition 4.3. Let $(P; e)$ be an L-ordered set, $t \in L$ and $f : (P; e) \to (P; e)$ be monotone. An element $x \in P$ is called a *t-fixpoint* of f if $t \le e(x, f(x)) \wedge e(f(x), x)$. Obviously, the concepts of a 1-fixpoint and a fixpoint are the same.

Corollary 4.4. *Consider the assumptions of Theorem* 4.2. *Then*

(i) *For each $x \in P$ and each $f \in H_P$, $\beta(x)$ is a t-fixpoint of f, where $t = S(f)$.*
(ii) *For each $f \in H_P$, $S(f) \le \overline{e}(f, \beta \circ F)$.*
(iii) *For each $f \in H_P$, there exists $u \in P$ such that $S(f) \le e(f(u), u) \wedge e(u, f(u))$.*
(iv) *If $f \in H_P$ such that $S(f) = 1$, then $f \circ \beta = f$. That is, $\beta(x)$ is a fixpoint for f for all $x \in P$.*

Proof. (i) The proof is a straightforward consequence of Theorem 4.2.

(ii) Let $f \in H_P$. Since $\beta = \sqcup S$, then

$$\begin{aligned} S(f) &= \overline{e}(Id_P, f) \le \overline{e}(\beta, \beta \circ f), \text{ since } \beta \text{ is monotone} \\ &= \bigwedge_{\alpha \in P} \big(S(\alpha) \to \overline{e}(\alpha, \beta \circ f)\big), \text{ by Theorem 2.7(i).} \end{aligned}$$

So, $S(f) \le S(f) \to \overline{e}(f, \beta \circ f)$, which implies that $S(f) \le \overline{e}(f, \beta \circ f)$ (since L is a frame, $a \le a \to b$ implies that $a = a \wedge (a \to b) = a \wedge b$, so $a \le b$).

(iii) By (i), for each $x \in P$, the element $u = \beta(x)$ satisfies the condition $S(f) \le e(f(u), u) \wedge e(u, f(u))$.

(iv) Let $f \in H_P$ such that $S(f) = 1$. Then by Theorem 2.7(i), $1 = S(f) \le \overline{e}(f, \beta)$, hence $f(x) \le \beta(x)$ for all $x \in P$, which implies that $S(\beta) = \bigwedge_{x \in P} e(x, \beta(x)) \ge \bigwedge_{x \in P} e(x, f(x)) = S(f) = 1$. So, by (i), $1 = S(f) \wedge S(\beta) \le \overline{e}(f \circ \beta, \beta) \wedge \overline{e}(\beta, f \circ \beta)$. That is, $f \circ \beta = \beta$. □

We know that if $(P; \le)$ is a CPO and $f : P \to P$ is an order-preserving map, then $Fix(f)$ has a least element. Moreover, by Theorem 3.4, if $(P; e)$ is an L-complete lattice, then $\sqcap S_f$ exists and is the least fixpoint of f. In the sequel, we attempt to find conditions for a monotone map on a fuzzy $DCPO$ $(P; e)$ under which $\sqcap S_f$ exists.

Definition 4.5. Let $(P; e)$ be an L-ordered set, $S \in L^P$ and $X \subseteq P$. An element $b \in X$ is called a *join of S in X* and is denoted by $\bigsqcup_X S = b$ if, for each $x \in X$, $e(b, x) = \bigwedge_{y \in X} \big(S(y) \to e(y, x)\big)$. In a similar way, we can define the notion of a meet of S in X, $\bigsqcap_X S$.

Remark 4.6. Let $(P;e)$ be an L-ordered set, $S \in L^P$ such that $a = \sqcup S$ and $X \subseteq P$. Suppose that b is a join of S in X. Then $b \in X$ and for each $x \in X$, $e(b,x) = \bigwedge_{y\in X}\big(S(y) \to e(y,x)\big) \geq \bigwedge_{y\in P}\big(S(y) \to e(y,x)\big) = e(a,x)$ and so $1 = e(b,b) = e(a,b)$. Thus, $a \leq_e b$. By a similar way, we can show that, if $a' = \sqcap S$ and b' is a meet of S in X, then $b' \leq_e a'$.

In a special case, if X is a subset of an L-ordered set $(P;e)$, $S \in L^P$, $b = \bigsqcup_X S$ ($b' = \bigsqcap_X S$) and $a = \sqcup S$ ($a' = \sqcap S$) such that $Supp(S) := \{x \in P|\ S(x) \neq 0\}$ is a subset of X, then for all $x \in X$, we have

$$e(a,x) = \bigwedge_{y\in P}\big(S(y) \to e(y,x)\big) = \bigwedge_{y\in X}\big(S(y) \to e(y,x)\big) = e(b,x).$$

So, $e(a,b) = e(b,b) = 1$. That is, $a \leq_e b$. A similar proof shows that $b' \leq_e a'$.

Definition 4.7. Let X be a non-empty subset of an L-ordered set $(P;e)$ and $a \in P$. An element $b \in X$ is called a *strong L-cover* for a in X if $e(a,x) = e(b,x)$ for all $x \in X$. We must note that, from $b \in X$ it follows that $e(a,b) = e(b,b) = 1$. Also, if a has a strong L-cover in X, then it is unique. Indeed, if $b, b' \in X$ are strong L-covers for a in X, then by definition, $1 = e(b',b') = e(a,b') = e(b,b')$. Similarly, $e(b',b) = 1$, so $b = b'$.

Let $(P,\leq)$ be a poset and b be a cover for a in P. Then $a < b$ and for each $x \in P$, if $a < x \leq b$, then $x = b$. On the other hand, if $X = \{x \in P|\ a < x \leq b\}$, then b is a cover for a if and only if $e_{\leq}(a,x) = 1 = e_{\leq}(b,x)$, for all $x \in X$. That is, b is a strong $\{0,1\}$-cover for a in X.

Proposition 4.8. *Let X be a non-empty subset of an L-ordered set $(P;e)$, $S \in L^P$ $a = \sqcup S$ such that $Supp(S) \subseteq X$. Then b is a strong L-cover for a in X if and only if $b = \bigsqcup_X S$.*

Proof. Let $b = \bigsqcup_X S$. Then for all $x \in P$,

$$\begin{aligned} e(b,x) &= \bigwedge_{y\in X}\big(S(y) \to e(y,x)\big) = \bigwedge_{y\in P}\big(S(y) \to e(y,x)\big), \text{ since } Supp(S) \subseteq X \\ &= e(a,x), \text{ since } \sqcup S = a. \end{aligned}$$

It follows that b is a strong L-cover for a in X. The proof of the converse is similar. □

Theorem 4.9. *Let $(P;e)$ be a fuzzy DCPO with zero (that is, there is $0 \in P$ such that $e(0,x) = 1$ for all $x \in P$), $f \in H_P$, $Y = \{a \in P|\ e(a,f(a)) = 1\}$ and $M = Fix(f)$. If each element of Y has a strong L-cover in M, then $(M;e)$ is a fuzzy sub DCPO of $(P;e)$.*

Proof. Clearly, $(P;\leq_e)$ is a CPO, so by [7, Thm. 8.22], f has a fixpoint (or M has a least element). Assume that each element of Y has a strong L-cover. Put a fuzzy directed subset S of $(M;e)$. Define $\overline{S} : P \to L$, by $\overline{S}(x) = S(x)$ for all $x \in M$ and $\overline{S}(x) = 0$ for all $x \in P - M$. It can be easily shown that $\overline{S}$ is a fuzzy directed subset of $(P;e)$. It follows that $\sqcup\overline{S}$ exists. Let $a = \sqcup\overline{S}$. Then for each $x \in P$,

$$e(a,x) = \bigwedge_{y\in P} \big(\overline{S}(y) \to e(y,x)\big) = \bigwedge_{y\in M} \big(S(y) \to e(y,x)\big) \tag{17}$$

and so $1 = e(a,a) = \bigwedge_{y\in M} \big(S(y) \to e(y,a)\big) \le \bigwedge_{y\in M} \big(S(y) \to e(f(y),f(a))\big) = \bigwedge_{y\in M} \big(S(y) \to e(y,f(a))\big) = e(a,f(a))$, which implies that $a \in Y$. Hence by the assumption, a has a strong L-cover in M, say, b whence $e(a,x) = e(b,x)$ for all $x \in M$. From (17) it follows that $e(b,x) = e(a,x) = \bigwedge_{y\in M} \big(S(y) \to e(y,x)\big)$. Therefore, $b = \bigsqcup_M S$ and so $(M;e)$ is a fuzzy $DCPO$ with zero. □

Consider the assumption in the proof of Theorem 4.9, if S has a join in M, then there is no proper element of M between $\bigsqcup_M S$ and $\sqcup\overline{S}$. Clearly, $\sqcup\overline{S} \le_e \bigsqcup_M S$. If c is an element of M such that $\sqcup\overline{S} \le_e c \le_e \bigsqcup_M S$, then by (17), $1 = e(\sqcup\overline{S}, c) = \bigwedge_{y\in M}(S(y) \to e(y,c)) = e(\bigsqcup_M S, c)$, so $c = a$. It follows that $\bigsqcup_M S$ is a (ordinary) cover for $\sqcup\overline{S}$ in the poset $(P, \le_e)$. This note can be useful when we want to find the join of S in $Fix(f)$. It suffices to search it between the elements of $Fix(f)$ which are (ordinary) cover of $\sqcup\overline{S}$.

Now, let $(P;e)$ be a fuzzy $DCPO$ with zero, $f \in H_P$ and $Y = \{a \in P \mid e(a,f(a)) = 1\}$. We note that, if $Fix(f)$ is a fuzzy sub $DCPO$ of $(P;e)$, then clearly, for each fuzzy directed subset S of $(M;e)$, $\sqcup\overline{S}$ has a strong L-cover in $Fix(f)$ (which is $\bigsqcup_{Fix(f)} S$).

Example 4.10. Let $(P;\le)$ be a CPO. It can be easily shown that the $\{0,1\}$-ordered set $(P;e_\le)$ is a fuzzy $DCPO$ with zero. Put a monotone map $f : (P;e_\le) \to (P;e_\le)$. Since f is monotone, then for each $a \in P$ satisfying the condition $a \le f(a)$, we have $A = \{x \in P \mid a \le x\}$ is a CPO and $f : A \to A$ is a monotone map, so by [7, Thm. 8.22] f has a least fixpoint on A, b say. Let x be another fixpoint of f.

(1) If $a \le x$, then $b \le x$, so $e_\le(a,x) = e_\le(b,x) = 1$.

(2) If $a \not\le x$, then $b \not\le x$, so $e_\le(a,x) = e_\le(b,x) = 0$. Hence, b is a strong L-cover for a in $Fix(f)$, whence $(P;e_\le)$ satisfies the conditions of Theorem 4.9.

We finish this section with a corollary can be obtained from the proof of Theorem 4.9. We recall that by [22], a monotone map $f : (X;e_X) \to (Y;e_Y)$ between two fuzzy $DCPO$s is called *fuzzy Scott continuous* if f preserves the join of fuzzy directed sets, that is $f(\sqcup D) = \sqcup f^{\to}(D)$, for all fuzzy directed subset D of (X, e_X). Note that by [15, Thm. 3.11], any L-order isomorphism between fuzzy $DCPO$s is fuzzy Scott continuous For more details about fuzzy Scott continuous maps on $DCPO$s we refer to [15, 22].

Corollary 4.11. *Let $(P;e)$ be a fuzzy DCPO with zero, $f \in H_P$, $Y = \{a \in P \mid e(a,f(a)) = 1\}$ and $M = Fix(f)$. If f is fuzzy Scott continuous, then $(M;e)$ is a fuzzy sub DCPO of $(P;e)$.*

Proof. Let S be a fuzzy directed subset of $(M;e)$ and $\overline{S}$ be the map in the proof of Theorem 4.9. Similar to the proof of Theorem 4.9, $\sqcup\overline{S}$ exists and $e(\sqcup\overline{S}, f(\sqcup\overline{S})) = 1$. On the other hand, by the assumption, $f(\sqcup\overline{S}) = \sqcup f^{\to}(\overline{S})$ and so for each $x \in P$, we have

$$\begin{aligned} e(f(\sqcup\overline{S}), x) &= \bigwedge_{y\in P}\Big(\bigvee_{z\in f^{-1}(y)} \overline{S}(z) \to e(y,x)\Big) = \bigwedge_{y\in P}\bigwedge_{z\in f^{-1}(y)} (\overline{S}(z) \to e(y,x)) \\ &= \bigwedge_{z\in M} (S(z) \to e(f(z),x)),\ \text{by definition of } \overline{S} \\ &= \bigwedge_{z\in M} (S(z) \to e(z,x)),\ \text{since } f(z)=z, \text{ for all } z\in M \\ &= \bigwedge_{z\in P} (\overline{S}(z) \to e(z,x)) = e(\sqcup\overline{S}, x). \end{aligned}$$

It follows that $e(f(\sqcup\overline{S}), \sqcup\overline{S}) = e(\sqcup\overline{S}, \sqcup\overline{S}) = 1$. That is $\sqcup\overline{S} = f(\sqcup\overline{S})$ and so $\sqcup\overline{S} \in M$. Therefore, $(M; e)$ is a fuzzy sub $DCPO$ of $(P; e)$. □

5. **Conclusions**

The main purpose of the paper was to generalize some theorems about fixpoints of complete lattices and CPOs for L-fuzzy complete lattices and fuzzy CPOs. Since monotone maps are important objects in the third section, first monotone maps on L-complete lattices were characterized. By fixpoint theorem, we know that each monotone map $f : P \to P$ on a complete lattice P has the least and greatest fixpoints, but we have no information about the relation between other elements of P. In Theorem 3.4, we found a relation for them. In Theorem 3.7, we proved the converse of [24, Thm. 2.29]. It is a generalization of an important theorem in lattice theory "Any complete lattice is a lattice of fixpoints of an order-preserving map". In Theorem 3.8–Corollary 3.11, we generalized Monotonicity rule, Rolling rule, Fusion rule and Exchange rule for L-complete lattices. The fourth section began with two theorems which said that the set of all monotone maps on a fuzzy DCPO $(P; e)$ was a DCPO and $S(\alpha) \leq \overline{e}(\alpha \circ \sqcup S, \sqcup S) \wedge \overline{e}(\sqcup S, \alpha \circ \sqcup S)$, where $S(\alpha) = \overline{e}(Id_P, \alpha)$ for all monotone map α on P. Then we found a condition under which the set of fixpoints of a monotone map f of a CPO is a fuzzy sub DCPO of $(P; e)$.

The presented results have obtained new ways for application of fixed point theory. We note that the presented results can also be formulated in the language of residuated lattices.

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