GRADATION OF CONTINUITY IN FUZZY TOPOLOGICAL SPACES

R. THAKUR, K. K. MONDAL AND S. K. SAMANTA

ABSTRACT. In this paper we introduce a definition of gradation of continuity in graded fuzzy topological spaces and study its various characteristic properties. The impact of the grade of continuity of mappings over the N-compactness grade is examined. Concept of gradation is also introduced in openness, closedness, homeomorphic properties of mappings and T_2 separation axiom. Effect of the grades interrelated with N-compactness, closedness, T_2 separation and homeomorphism of mappings are studied.

1. Introduction

After the introduction of fuzzy topology by C. L. Chang [4] many authors worked with this concept leading to the introduction of various types of fuzzy topologies in the last four decades. Accordingly studies with different notions of continuity of mappings over fuzzy topological spaces have been an interesting field of investigation. We would like to mention in brief about some of the works which are relevant to our work. But these are mostly related to "crisp continuity" weaker or stronger forms, suitably matching in the context of fuzzy topology. As continuity is such a property of transformation which enables it to preserve some spatial characteristics while transforming one space to another, it is a natural curiosity to study how does the "fuzziness of continuity" passes the information of spatial characteristics under transformation. In this connection mention may be made of M. Demirci [6], who has introduced a concept of continuity degree using grade of membership relation of crisp point and a fuzzy subset via neighborhood operator. Recently in 2007 A. M. Zahran et al [27] defines degree of continuity using M. S. Ying [24] type fuzzifying topology involving crisp subsets. Studies on continuity like structures over other types of fuzzy topological spaces have been made by many authors. Recently, Cong-hua Yan and Jin-xuan Fang [23] has worked on L-fuzzy Bilinear Operator and its Continuity.

The notions of the sets and functions in fuzzy topological spaces are highly developed and are used in many practical and engineering problems, computational topology for geometric design, computer-aided geometric design, engineering design

Received: November 2009; Revised: May 2010; Accepted: August 2010

Key words and phrases: Fuzzy belongingness, Fuzzy quasi-coincidence, Gradation of openness, Gradation of neighborhoodness, Fuzzy topological spaces, Fuzzy filters, Graded continuity.

Subject classification no. 54A40, 03E72

The present work is supported by Council of Scientific and Industrial Research(CSIR), Pusa, New Delhi, India [File No. 202(0003)/EMR-I dated 27/04/2006].

research and mathematical sciences. The notion of continuity is an important concept in general topology and fuzzy topology as well as in almost all branches of mathematics and quantum physics. The importance of fuzzy topology in studying quantum physics especially related to both string theory and ϵ^{∞} theory has been shown by Ms. El. Naschie [8, 9, 11, 12]. The topology of quantum space-time is shadowed closely by the Mobius geometry of quasi-Fuschian and Kleinian groups and that is the cause behind the phenomena of high-energy particle physics [8, 12]. Recently continuity of functions in the topological space has been investigated by mathematicians and quantum physicists [3, 7, 9, 18] from the different points of views.

In this paper we give a definition of gradation of continuity in graded fuzzy topological spaces and investigate the extent of preservation of some graded topological properties under the influence of this graded continuity. Similarly we also introduce definition of gradation of mapping openness and gradation of homeomorphism and also show the properties of graded T_2 space and graded N-compactness under gradation of continuity.

2. **Notation and Preliminaries**

In this paper X denotes a nonempty set, I denotes the unit closed interval [0,1], I_0 denotes the set $I - \{0\}$, I^X denotes the set of all fuzzy subsets of X whereas $Pt(I^X)$ denotes the set of all fuzzy points of X. $1_X, 0_X$ denote the constant fuzzy subsets defined by $1_X(x) = 1,\ \forall x \in X$ and $0_X(x) = 0,\ \forall x \in X$ respectively. A' denotes the complement of A (i.e. $A'(x) = 1 - A(x)$).

Definition 2.1. [5] A function $\tau : I^X \to I$ is called a gradation of openness (GO) on X if it satisfies the following conditions:

(O1) : $\tau(\tilde{0}) = \tau(\tilde{1}) = 1$,
(O2) : $\tau(A_1 \wedge A_2) \geq \tau(A_1) \wedge \tau(A_2)$, for $A_1, A_2 \in I^X$,
(O3) : $\tau\Big(\vee_{i\in\Delta} A_i \Big) \geq \wedge_{i\in\Delta}\tau(A_i)$ for any $\{A_i\}_{i\in\Delta} \subseteq I^X$.

The pair (X, τ) is called a fuzzy topological space (fts).

Definition 2.2. [5] A function $\mathcal{F} : I^X \to I$ is called a gradation of closedness (GC) on X if it satisfies the following conditions:

(C1) : $\mathcal{F}(\tilde{0}) = \mathcal{F}(\tilde{1}) = 1$,
(C2) : $\mathcal{F}(A_1 \vee A_2) \geq \mathcal{F}(A_1) \wedge \mathcal{F}(A_2)$, for $A_1, A_2 \in I^X$,
(C3) : $\mathcal{F}\Big(\wedge_{i\in\Delta} A_i \Big) \geq \wedge_{i\in\Delta}\mathcal{F}(A_i)$ for any $\{A_i\}_{i\in\Delta} \subseteq I^X$.

The pair $(X, \mathcal{F})$ is called a fuzzy co-topological space.

Definition 2.3. [2] Let $\mathcal{G} : I^X \to I$ be a mapping satisfying

(GF1) : $\mathcal{G}(\tilde{0}) = 0;\ \mathcal{G}(\tilde{1}) = 1$,
(GF2) : $\mathcal{G}(A_1 \wedge A_2) \geq \mathcal{G}(A_1) \wedge \mathcal{G}(A_2);\ \forall\ A_1, A_2 \in I^X$,
(GF3) : $\mathcal{G}(B) \geq \mathcal{G}(A)$ if $A \subseteq B;\ A, B \in I^X$,

then $\mathcal{G}$ is said to be a generalized filter (g-filter) on I^X.

The following (*) marked definitions and results can be deduced directly from [15, 16] and [26] by choosing $L = I$, the unit closed interval [0,1].

Definition 2.4. * Let $(X, \mathcal{F})$ be a fuzzy co-topological space with $\mathcal{F}$ as a GC on X. For each $r \in I_0$ and for each $A \in I^X$ we define $cl(A, r) = \cap\{D \in I^X\ ;\ A \subseteq D\ ;\ D \in \mathcal{F}_r\}$, where $\mathcal{F}_r = \{C \in I^X\ ;\ \mathcal{F}(C) \geq r\}$. '$cl$' is said to be fuzzy closure operator in $(X, \mathcal{F})$.

Definition 2.5. * Let (X, τ) be a fts with τ as a GO on X. For each $r \in I_0$ and for each $A \in I^X$ we define $int(A, r) = \vee\{D \in I^X\ ;\ A \supseteq D\ ;\ D \in \tau_r\}$, where $\tau_r = \{U \in I^X\ ;\ \tau(U) \geq r\}$. '$int$' is said to be fuzzy interior operator in (X, τ).

Definition 2.6. * Let (X, τ) be a fuzzy topological space with τ as a GO on X and let $Q : Pt(I^X) \times I^X \to I$ be a mapping defined by $Q(p_x, A) = \vee\{\tau(U)\ ;\ p_x\hat{q}U \subseteq A\}$. Then Q is said to be a gradation of q-neighborhoodness on X, where $p_x\hat{q}U$ holds if $U(x) + p > 1$.

Proposition 2.7. * *Let Q be a gradation of q-neighborhoodness in an fts (X, τ). Then*

(QN1) : $Q(p_x, \tilde{1}) = 1, Q(p_x, \tilde{0}) = 0,\ \forall\ p_x \in Pt(I^X)$,

(QN2) : $Q(p_x, A) \leq Q(p_x, B)$ *if* $A, B \in I^X,\ A \subseteq B$ *and* $Q(p_x, A) \leq Q(s_x, A),\ \forall A \in I^X$ *if* $p_x \leq s_x$,

(QN3) : $Q(p_x, A \wedge B) = Q(p_x, A) \wedge Q(p_x, B),\ \forall\ p_x \in Pt(I^X)\ \&\ A, B \in I^X$,

(QN4) : $Q(p_x, A) > k$ $\Rightarrow\ \exists B_p \in I^X$ *s.t* $p_x\hat{q}B_p \subseteq A\ \&\ \wedge\{Q(r_y, B_p);\ r_y \in Pt(I^X);\ r_y\hat{q}B_p\} > k$.

Proposition 2.8. * *Let $Q : Pt(I^X) \times I^X \to I$ be a mapping satisfying (QN1) - (QN3) of Proposition 2.7. Let $\bar{\tau} : I^X \to I$ be defined by $\bar{\tau}(A) = \wedge\{Q(p_x, A);\ p_x \in Pt(I^X)\ \&\ p_x\hat{q}A\}$. Then $(X, \bar{\tau})$ forms an fts. If further the condition (QN4) of Proposition 2.7 is satisfied by Q then the mapping $\bar{Q} : Pt(I^X) \times I^X \to I$ defined by $\bar{Q}(p_x, A) = \vee\{\bar{\tau}(U)\ ;\ p_x\hat{q}U \subseteq A\}$ is identical with Q.*

Proposition 2.9. * *Let Q be a gradation of q-neighborhoodness in an fts (X, τ) and $\bar{\tau} : I^X \to I$ be defined by $\bar{\tau}(A) = \wedge\{Q(p_x, A);\ p_x \in Pt(I^X)\ \&\ p_x\hat{q}A\}$ then $\bar{\tau}$ is a GO on X and $\bar{\tau} = \tau$.*

Definition 2.10. * Let (X, τ) be an fts and $e \in Pt(I^X)$. The q-neighborhood system of the fuzzy point e with respect to the Chang fuzzy topology τ_r, denoted by $\tilde{Q}_r(e)$, is defined by $\tilde{Q}_r(e) = \{U \in I^X;\ \exists\ V \in \tau_r$ satisfying $e\hat{q}V \subseteq U\}$.

Definition 2.11. [22] Let X be a nonempty crisp set. A fuzzy filter on I^X is a non-empty family $\mathcal{G}$ of fuzzy subsets of X such that

i) $\tilde{0} \notin \mathcal{G}$,
ii) $\mathcal{G}$ is closed under finite intersection,
iii) if $B \in \mathcal{G}\ \&\ B \subset A$ then $A \in \mathcal{G}\ \forall A, B \in I^X$.

Definition 2.12. [16] Let X be a nonempty crisp set. A nonempty subfamily $\mathcal{A} \subseteq I^X$ is called a filter base on I^X, if $\tilde{0} \notin \mathcal{A}$ and $\mathcal{A}$ is closed under finite intersection. For a fuzzy filter base $\mathcal{A}$ on I^X, define the fuzzy filter generated by $\mathcal{A}$ by $\uparrow \mathcal{A} =$

$\{U \in I^X;\ \exists V \in \mathcal{A}$ satisfying $V \subseteq U\}$.
Let $f : X \to Y$ be any mapping then for any fuzzy filter $\mathcal{G}$ on I^X the set $f[\mathcal{G}] = \{f(A);\ A \in \mathcal{G}\}$ forms a filter base on I^Y. The fuzzy filter generated by $f[\mathcal{G}]$ is called the image fuzzy filter of $\mathcal{G}$ and it is denoted by $\uparrow f[\mathcal{G}]$.

Definition 2.13. [26] Let (X, δ) be a Chang fuzzy topological space (Cfts), $A \in I^X, \Phi \subset I^X, \alpha \in I$. Then Φ is called an α-Q-cover of A, denoted by $\vee\Phi\hat{q}A(\alpha)$, if for every $\alpha_x \in A$, there exists $U \in \Phi$ such that $\alpha_x\hat{q}U$. Φ is called an open α-Q-cover of A, if $\Phi \subset \delta$ and Φ is an α-Q-cover of A. $\Phi_0 \subset I^X$ is called a *sub*-α-Q-cover of Φ, if $\Phi_0 \subset \Phi$ and Φ_0 is also an α-Q-cover of A. Φ is called an α^--Q-cover of A, denoted by $\vee\Phi\hat{\hat{q}}A(\alpha)$, if there exists $\gamma < \alpha$ such that Φ is a γ-Q-cover of A.

Definition 2.14. [26] Let $((X, \delta)$ be an Cfts, $A \in I^X$. A is called N-compact, if for every $\alpha \in I_0$, every open α-Q-cover of A has a finite subfamily which is an α^--Q-cover of A. (X, δ) is called N-compact, if $\tilde{1}_X$ is N-compact.

Definition 2.15. [26] Let (X, τ) be a Cfts and $\alpha \in I_0$. Then (X, τ) is called α-T_2, if for every distinguished points $x, y \in X$ there exists $U \in \mathcal{Q}(\alpha_x)(= \{U \in I^X : \exists P \in \tau$ satisfying $\alpha_x\hat{q}P \subseteq U\}), V \in \mathcal{Q}(\alpha_y)$ such that $U \wedge V = \underline{0}$.
(X, τ) is called *level*-T_2, if (X, τ) is α-T_2 for every $\alpha \in I_0$.

Theorem 2.16. [26] *Let (X, δ) be a stratified (i.e. containing all constant fuzzy subsets $\bar{\alpha}$ $\forall\alpha \in I$) level-T_2 Cfts and A be N-compact subset in (X, δ). Then A is closed in (X, δ).*

Theorem 2.17. [26] *Let (X, τ) be a fuzzy topological space in the sense of Chang. Also let $A \in I^X$ be N-compact, $B \in I^X$ be closed. Then $A \wedge B$ is N-compact.*

3. Gradation of Continuity

By using Lukasiewicz's technique, we define gradation of continuity of a mapping over fuzzy topological spaces and studies some characteristic properties of such gradation. A notion of gradation on N-compactness in a fuzzy topological space is introduced and combined effect of grades of continuity and N-compactness is analyzed.

Definition 3.1. Let (X, τ) and (Y, δ) be any two fts and $f : (X, \tau) \to (Y, \delta)$ be a mapping. Then the gradation of continuity of f is defined by $cont(f) = \wedge_{V \in I^Y}\{1 - \delta(V) + \tau(f^{-1}(V))\}$.

Proposition 3.2. *Let (X, τ) and (Y, δ) be any two fts, and $f : (X, \tau) \to (Y, \delta)$ be a mapping with τ and δ as gradations of openness on X and Y respectively. Then $cont(f) = \wedge_{V \in I^Y}\{1 - \bar{\mathcal{F}}(V) + \mathcal{F}(f^{-1}(V))\}$, where $\mathcal{F}$ and $\bar{\mathcal{F}}$ are the gradations of closedness on X and Y respectively, defined by $\mathcal{F}(A) = \tau(A'),\ \forall\ A \in I^X$ and $\bar{\mathcal{F}}(B) = \delta(B'),\ \forall B \in I^Y$.*

Proof. We have $cont(f) = \wedge_{V \in I^Y}\{1 - \delta(V) + \tau(f^{-1}(V))\}$
$= \wedge_{V \in I^Y}\{1 - \delta(V') + \tau(f^{-1}(V'))\}$
$= \wedge_{V \in I^Y}\{1 - \bar{\mathcal{F}}(V) + \tau([f^{-1}(V)]')\}$ [as $f^{-1}(V') = [(f^{-1}(V))']$
$= \wedge_{V \in I^Y}\{1 - \bar{\mathcal{F}}(V) + \mathcal{F}(f^{-1}(V))\}$. □

Proposition 3.3. . *Let (X,τ) and (Y,δ) be any two fts, and $f:(X,\tau)\to(Y,\delta)$ be a mapping. Then for each $l\in I$, the following statements are equivalent:*

(a) $cont(f)\geq l$.
(b) $f^{-1}(\delta_{1-r})\subseteq\tau_{l-r},\forall r\in I$, *with* $r<l$.
(c) $f^{-1}(\bar{\mathcal{F}}_{1-r})\subseteq\mathcal{F}_{l-r},\forall r\in I$, *with* $r<l$.
(d) $f^{-1}(\underline{\tilde{Q}}_{1-r}(f(e)))\subseteq\tilde{Q}_{l-r}(e)$, $\forall r\in I$, *with* $r<l$ *and* $\forall\ e\in Pt(I^X)$, *where* $\tilde{Q}_{l-r}(e)$ *and* $\underline{\tilde{Q}}_{1-r}(f(e))$ *be the q-neighborhood systems of* e *and* $f(e)$ *with respect to the Chang fuzzy topologies* τ_{l-r} *and* δ_{1-r} *respectively, defined by* $\tilde{Q}_{l-r}(e)=\{U\in I^X;\ \exists V\in\tau_{l-r}\ s.t.\ e\hat{q}V\subseteq U\}$ *and* $\underline{\tilde{Q}}_{1-r}(f(e))=\{U\in I^Y;\ \exists V\in\delta_{1-r}$ *such that* $f(e)\hat{q}V\subseteq U\}$.
(e) $f(cl(A,l-r))\subseteq cl(f(A),1-r),\ \forall\ A\in I^X$ *and* $\forall r\in I$ *with* $r<l$.
(f) $f^{-1}(int(A,1-r))\subseteq int(f^{-1}(A),l-r),\ \forall\ A\in I^Y$ *and* $\forall r\in I$ *with* $r<l$.
(g) $\bigwedge_{(e,V)\in Pt(I^X)\times I^Y}\left\{1-\underline{Q}(f(e),V)+Q(e,f^{-1}(V))\right\}\geq l$, *where* Q *and* $\underline{Q}$ *are the gradations of q-neighborhoodness in* (X,τ) *and* (Y,δ) *respectively.*

Proof. $(a)\Rightarrow(b)$:
We have $\wedge_{V\in I^Y}\{1-\delta(V)+\tau(f^{-1}(V))\}\geq l$.
$\Rightarrow 1-\delta(V)+\tau(f^{-1}(V))\geq l,\ \forall\ V\in I^Y$,
$\Rightarrow 1-\delta(V)\geq l-\tau(f^{-1}(V)),\ \forall\ V\in I^Y$.
So, $1-\delta(V)\leq r\Rightarrow l-\tau(f^{-1}(V))\leq r,\ \forall V\in I^Y$ for $r<l$.
i.e. $\delta(V)\geq 1-r\Rightarrow\tau(f^{-1}(V))\geq l-r\ \forall V\in I^Y$ for $r<l$.
i.e. $V\in\delta_{1-r}\Rightarrow f^{-1}(V)\in\tau_{l-r},\forall V\in I^Y$ with $r<l$.
Hence $(a)\Rightarrow(b)$ holds.
$(b)\Rightarrow(a)$:
Let (b) hold.
If possible let $\wedge_{V\in I^Y}\{1-\delta(V)+\tau(f^{-1}(V))\}<l$.
Then $\exists\ W\in I^Y$ such that $\{1-\delta(W)+\tau(f^{-1}(W))\}<l$,
$\Rightarrow 1-\delta(W)<l-\tau(f^{-1}(W))\}$.
Then $\exists\ r\in I_0$ such that $1-\delta(W)<r<l-\tau(f^{-1}(W))$.
So $\delta(W)>1-r$ and $\tau(f^{-1}(W))<l-r$ which implies $W\in\delta_{1-r}$ but $f^{-1}(W)\notin\tau_{l-r}$ for some $r<l$, which is a contradiction.
Hence $cont(f)\geq l$. So $(b)\Rightarrow(a)$.
$(b)\Leftrightarrow(c)$:
This results directly follows from the Proposition 3.2.
$(b)\Rightarrow(d)$:
Let (b) hold.
Let $A\in f^{-1}(\underline{\tilde{Q}}_{1-r}(f(e)))$ where $r(<l)\in I$. Then $\exists U\in\underline{\tilde{Q}}_{1-r}(f(e))$ such that $A=f^{-1}(U)$.
Now $U\in\underline{\tilde{Q}}_{1-r}(f(e))\Rightarrow\exists V\in\delta_{1-r}$ such that $f(e)\hat{q}V\subseteq U$.
$\Rightarrow f^{-1}(V)\in\tau_{l-r}$ (by (b)) and $e\hat{q}f^{-1}(V)\subseteq f^{-1}(U)$.
$\Rightarrow A=f^{-1}(U)\in\tilde{Q}_{l-r}(e)$,
$\Rightarrow f^{-1}(\underline{\tilde{Q}}_{1-r}(f(e)))\subseteq\tilde{Q}_{l-r}(e)$. So the proof follows.

$(d) \Rightarrow (b)$:
Let $A \in f^{-1}(\delta_{1-r})$ where $r \in I$ with $r < l$. Then $\exists V \in \delta_{1-r}$ such that $A = f^{-1}(V)$.
If $A = \tilde{0}_X$ then obviously $A \in \tau_{l-r}$.
So let $A = f^{-1}(V) \neq \tilde{0}_X$.
Take any $\alpha_x \in Pt(I^X)$ such that $\alpha_x \hat{q} f^{-1}(V)$. Then $\alpha_{f(x)} \hat{q} V$ where $V \in \delta_{1-r}$,
$\Rightarrow V \in \underset{\sim}{\tilde{Q}}_{1-r}(f(\alpha_x))$,
$\Rightarrow f^{-1}(V) \in \tilde{Q}_{l-r}(\alpha_x)$ (by (d)).
So $f^{-1}(V) \in \tilde{Q}_{l-r}(\alpha_x)$ $\forall \alpha_x \hat{q} f^{-1}(V)$. Thus for each $\alpha_x \hat{q} f^{-1}(V)$, $\exists W_{\alpha_x} \in \tau_{l-r}$ such that $\alpha_x \hat{q} W_{\alpha_x} \subseteq f^{-1}(V)$.
So $f^{-1}(V) \in \tau_{l-r}$ i.e. $A \in \tau_{l-r}$.
Hence $f^{-1}(\delta_{1-r}) \subseteq \tau_{l-r}$ and this is true for all $r < l$, so the proof follows.

$(b) \Rightarrow (e)$:
Let (b) hold.
Let $\mathcal{A} = \{D \in I^X; D \in \mathcal{F}_{l-r}$ & $A \subseteq D\}$ where $\mathcal{F}_{l-r} = \{C \in I^X; \mathcal{F}(C) \geq l - r\}$ and $\mathcal{B} = \{D \in I^Y; D \in \bar{\mathcal{F}}_{1-r}$ & $f(A) \subseteq D\}$ where $\bar{\mathcal{F}}_{1-r} = \{C \in I^Y; \bar{\mathcal{F}}(C) \geq 1 - r\}$.
So $cl(A, l-r) = \wedge\mathcal{A}$ and $cl(f(A), 1-r) = \wedge\mathcal{B}$.
Let $D \in \mathcal{B}$ i.e. $D \in \bar{\mathcal{F}}_{1-r}$ & $f(A) \subseteq D$,
$\Rightarrow f^{-1}(D) \in \mathcal{F}_{l-r}$ (by (c)) & $f^{-1}f(A) \subseteq f^{-1}(D)$,
$\Rightarrow f^{-1}(D) \in \mathcal{F}_{l-r}$ & $A \subseteq f^{-1}(D)$ (since $A \subseteq f^{-1}f(A)$),
$\Rightarrow f^{-1}(D) \in \mathcal{A}$.
Hence $\mathcal{A} \supseteq f^{-1}(\mathcal{B})$.
Clearly $\bar{\mathcal{F}}_{1-r}$ is closed under arbitrary intersection. So $cl(f(A), 1-r) \in \mathcal{B}$. Hence $f^{-1}(cl(f(A), 1-r)) \in \mathcal{A}$,
$\Rightarrow cl(A, l-r) \subseteq f^{-1}(cl(f(A), 1-r))$,
$\Rightarrow f(cl(A, l-r)) \subseteq f(f^{-1}(cl(f(A), 1-r))) \subseteq cl(f(A), 1-r)$. Hence (e).

$(e) \Rightarrow (a)$:
Let (e) hold.
If possible let $cont(f) < l$ i.e. $\wedge_{V \in I^Y}\{1 - \delta(V) + \tau(f^{-1}(V))\} < l$.
Then $\exists V \in I^Y$ such that $1 - \delta(V) + \tau(f^{-1}(V)) < l$,
$\Rightarrow l - \tau(f^{-1}(V)) > 1 - \delta(V)$,
$\Rightarrow \exists r \in I_0$ such that $l - \tau(f^{-1}(V)) > r > 1 - \delta(V)$,
$\Rightarrow \tau(f^{-1}(V)) < l - r$ and $\delta(V) > 1 - r$.
Let $V^c = W$.
Then we have $\mathcal{F}(f^{-1}(W)) < l-r$, and $\bar{\mathcal{F}}(W) > 1-r$ [since $f^{-1}(V^c) = \{f^{-1}(V)\}^c$],
$\Rightarrow cl(f^{-1}(W), l-r) \supset (\neq) f^{-1}(W)$ and $cl(W, 1-r) = W$.
i.e $\exists \gamma_x \in cl(f^{-1}(W), l-r)$ but $\gamma_x \notin f^{-1}(W)$,
$\Rightarrow \gamma_{y(=f(x))} \in f(cl(f^{-1}(W), l-r))$ but $\gamma_y \notin W \cap f(1_X)(= ff^{-1}(W))$.
i.e. $f(cl(f^{-1}(W), l-r)) \supset (\neq) f(f^{-1}(W)) = W \cap f(1_X) = cl(W, 1-r) \cap f(1_X)$
$\subseteq cl(ff^{-1}(W), 1-r) \cap f(1_X)$.
i.e. $f(cl(f^{-1}(W), l-r)) \supset (\neq) cl(ff^{-1}(W), 1-r) \cap f(1_X)$,
$\Rightarrow \exists \gamma_x \in f(1_X)$ such that $\gamma_x \in f(cl(f^{-1}(W), l-r))$ but $\gamma_x \notin cl(ff^{-1}(W), 1-r) \cap f(1_X)$,
$\Rightarrow \exists \gamma_x \in f(1_X)$ such that $\gamma_x \in f(cl(f^{-1}(W), l-r))$ but $\gamma_x \notin cl(ff^{-1}(W), 1-r)$, which contradicts (e). Hence $cont(f) \geq l$.

$(a) \Rightarrow (f)$:

Suppose (a) hold and if possible let $f^{-1}(int(A, 1-r) \not\subseteq int(f^{-1}(A), l-r)$ for some $A \in I^Y$.

Then there exists $\alpha \in I_0$ and $x \in X$ such that $f^{-1}(int(A, 1-r))(x) > \alpha > int(f^{-1}(A), l-r)(x)$,

$\Rightarrow \alpha_x \in f^{-1}(int(A, 1-r))$ and

$$\alpha_x \notin int(f^{-1}(A), l-r), \tag{1}$$

Now $\alpha_x \in f^{-1}(int(A, 1-r))$,

$\Rightarrow \alpha_y \in int(A, 1-r)$ where $y = f(x)$.

Again $int(A, 1-r) \in \delta_{1-r}$. So taking $W = int(A, 1-r)$ we have

$$\delta(W) \geq 1-r \textit{ and } \alpha_x \in f^{-1}(W) \subset f^{-1}(A). \tag{2}$$

From (1) we have $\alpha_x \notin int(f^{-1}(A), l-r) = \cup\{U \in \tau_{l-r} : f^{-1}(A) \supseteq U\}$,

which shows that $\forall\, U \in I^X$ with $\alpha_x \in U \subseteq f^{-1}(A) \Rightarrow \tau(U) < l-r$.

This condition holds for $U = f^{-1}(W)$,

Hence we have,

$$\tau(f^{-1}(W)) < l-r. \tag{3}$$

From (2) and (3) we have $l - \tau(f^{-1}(W)) > r \geq 1 - \delta(W)$,

$\Rightarrow 1 - \delta(W) + \tau(f^{-1}(W)) < l \Rightarrow cont(f) < l$, which is a contradiction.

Hence $f^{-1}(int(A, 1-r)) \subseteq int(f^{-1}(A), l-r)$.

$(f) \Rightarrow (a)$:

Suppose (f) hold.

Then we have to show that $\wedge_{V \in I^Y}\{1 - \delta(V) + \tau(f^{-1}(V))\} \geq l$.

If possible let $\wedge_{V \in I^Y}\{1 - \delta(V) + \tau(f^{-1}(V))\} < l$.

Then $\exists V \in I^Y$ such that $1 - \delta(V) + \tau(f^{-1}(V)) < l$,

$\Rightarrow 1 - \delta(V) < r < l - \tau(f^{-1}(V))$ for some $r \in I_0$,

$\Rightarrow \delta(V) > 1 - r$ and $\tau(f^{-1}(V)) < l - r$,

$\Rightarrow int(V, 1-r) = V$ and $int(f^{-1}(V), l-r) \subset (\neq) f^{-1}(V)$.

So, $int(f^{-1}(V), l-r) \subset (\neq) f^{-1}(V) = f^{-1}(int(V, 1-r))$, which contradicts our assumption.

Hence $cont(f) \geq l$.

$(b) \Rightarrow (g)$:

Suppose (b) hold and if possible let $\displaystyle\bigwedge_{(e,V) \in Pt(I^X) \times I^Y} \Big\{1 - \underline{Q}(f(e), V) + Q(e, f^{-1}(V))\Big\} < l$.

Then there exists $(e^0, V^0) \in Pt(I^X) \times I^Y$ such that $1 - \underline{Q}(f(e^0), V^0) + Q(e^0, f^{-1}(V^0)) < l$.

i.e. $1 - \underline{Q}(f(e^0), V^0) < l - Q(e^0, f^{-1}(V^0))$,

$\Rightarrow\ \exists r \in I_0$ such that $1 - \underline{Q}(f(e^0), V^0) < r < l - Q(e^0, f^{-1}(V^0)$

i.e. $\exists r \in I_0$ such that $\underline{Q}(f(e^0), V^0) > 1 - r$ and $Q(e^0, f^{-1}(V^0)) < l - r$.

Now $\underline{Q}(f(e^0), V^0) > 1 - r$,

$\Rightarrow \vee\{\delta(U); f(e^0)\hat{q}U \subset V^0\} > 1 - r$,

$\Rightarrow \exists U^0 \in I^Y$ such that $f(e^0)\hat{q}U^0 \subset V^0$ and $\delta(U^0) > 1 - r$.
$\Rightarrow \exists U^0 \in I^Y$ such that $e^0\hat{q}f^{-1}(U^0) \subseteq f^{-1}(V^0)$ and $\tau(f^{-1}(U^0)) \geq l - r$[by (b)]. (4)

Again $Q(e^0, f^{-1}(V^0)) < l - r$,
$\Rightarrow \vee\{\tau(U); e^0\hat{q}U \subset f^{-1}(V^0)\} < l - r$,
$\Rightarrow \forall U \in I^X$ satisfying $e^0\hat{q}U \subset f^{-1}(V^0)$, we have $\tau(U) < l - r$, which contradicts the condition (4). Hence $(b) \Rightarrow (g)$.

$(g) \Rightarrow (b)$:
Suppose (g) hold and if possible let $\exists\ r \in I_0$ such that $r < l$ and $f^{-1}(\delta_{1-r}) \nsubseteq \tau_{l-r}$.
Then $\exists\ V \in \delta_{1-r}$ but $f^{-1}(V) \notin \tau_{l-r}$,
$\Rightarrow \delta(V) \geq 1 - r$ but $\tau(f^{-1}(V)) < l - r$.
Now $\tau(f^{-1}(V)) < l - r$,
$\Rightarrow \wedge\{Q(e, f^{-1}(V)); e \in Pt(I^X)\ \&\ e\hat{q}f^{-1}(V)\} < l - r$.(from Proposition 2.9) So $\exists\ e^0 \in Pt(I^X)$ such that $e^0\hat{q}f^{-1}(V)$ and $Q(e^0, f^{-1}(V)) < l - r$. (5)

Again $\delta(V) > 1 - r$,
$\Rightarrow \wedge\{\underline{Q}(e, V); e \in Pt(I^Y)\ \&\ e\hat{q}V\} \geq 1 - r$,
$\Rightarrow \underline{Q}(e, V) \geq 1 - r\ \forall\ e \in Pt(I^Y)$ with $e\hat{q}V$. As $e^0\hat{q}f^{-1}(V) \Rightarrow f(e^0)\hat{q}V$, so $\underline{Q}(f(e^0), V) \geq 1 - r$,
$\Rightarrow 1 - \underline{Q}(f(e^0), V) \leq r$. (6)

From (5) and (6) we have $1 - \underline{Q}(f(e^0), V) + Q(e^0, f^{-1}(V)) < r + l - r = l$, which is contradictory to (g). Hence $(g) \Rightarrow (b)$. □

Definition 3.4. Let (X, τ) be an fts, $G \subseteq I^X$ be a fuzzy filter on $X, e \in Pt(I^X)$. Then e is called a cluster point of G with grade l, denoted by $\mathcal{G}\infty^l e$ iff $l' = \wedge\{r \in I_0 : U \cap A \neq \tilde{0},\ \forall\ U \in \tilde{Q}_r(e)$ and $\forall A \in \mathcal{G}\}$.
And e is called a limit point of $\mathcal{G}$ with grade l, denoted by $\mathcal{G} \rightarrow^l e$, iff $l' = \wedge\{r \in I_0 : \tilde{Q}_r(e) \subseteq \mathcal{G}\}$.

Proposition 3.5. *Let (X, τ) and (Y, δ) be any two fuzzy topological spaces and $f : (X, \tau) \rightarrow (Y, \delta)$. Then the following two statements are equivalent:*

(a) *$cont(f) \geq l$ for some $l \in I_0$.*
(b) *For any fuzzy filter base $\mathcal{A}$ in (X, τ) and $e \in Pt(I^X), \uparrow \mathcal{A} \rightarrow^m e \Rightarrow f[\mathcal{A}] \rightarrow^k f(e)$ for some $m, k \in I_0$ with $k \geq l + m - 1($ provided $l + m > 1)$.*

Proof. $(a) \Rightarrow (b)$: Suppose (a) hold. Let $\tilde{Q}_r(e)$ and $\underline{\tilde{Q}}_r(f(e))$ be q-neighborhood systems of e and $f(e)$ with respect to the Chang fuzzy topologies τ_r and δ_r respectively. Let $\mathcal{A}$ be a fuzzy filter base in (X, τ) s.t. $\uparrow \mathcal{A} \rightarrow^m e$ for some $e \in Pt(I^X)$ and $f : (X, \tau) \rightarrow (Y, \delta)$ be a mapping with $cont(f) \geq l$, where $l + m > 1$.
As $\uparrow \mathcal{A} \rightarrow^m e$ means $m' = \wedge\{s \in I_0; \tilde{Q}_s(e) \subseteq\uparrow \mathcal{A}\}$.
Also, $l + m > 1 \Rightarrow m' < l$. So, $\exists s \in I_0$ such that $s < l$ and $\tilde{Q}_s(e) \subseteq\uparrow \mathcal{A}$.
Let $r = l - s$, then $\tilde{Q}_{l-r}(e) \subseteq\uparrow \mathcal{A}$.
Then $V \in \underline{\tilde{Q}}_{1-r}(f(e)) \Rightarrow f^{-1}(V) \in \tilde{Q}_{l-r}(e)$(by Proposition 3.3(d)) $\Rightarrow f^{-1}(V) \in\uparrow \mathcal{A}$ [as $\tilde{Q}_{l-r}(e) \subseteq\uparrow \mathcal{A}$].
$\Rightarrow \exists A \in \mathcal{A}$ such that $f^{-1}(V) \supseteq A$.

$\Rightarrow f(A) \subseteq ff^{-1}(V) \subseteq V \Rightarrow V \in f[\uparrow \mathcal{A}]$.
So $\underline{\tilde{Q}}_{1-r}(f(e)) \subseteq\uparrow f[\mathcal{A}]$.
i.e. if $\tilde{Q}_s(e) \subseteq\uparrow \mathcal{A}$ then $\underline{\tilde{Q}}_{1-l+s}(f(e)) \subseteq\uparrow f[\uparrow \mathcal{A}]$,
i.e. $\uparrow \mathcal{A} \to e$ with respect to τ_s,
$\Rightarrow\uparrow f[\uparrow \mathcal{A}] \to f(e)$ with respect to δ_{1-l+s}.
Let $\Delta_1 = \{t \in I_0; \uparrow \mathcal{A} \to e$ with respect to $\tau_t\}$ and $\Delta_2 = \{t \in I_0; \uparrow f[\uparrow \mathcal{A}] \to f(e)$ with respect to $\delta_t\}$ thus $s \in \Delta_1 \Rightarrow 1 - l + s \in \Delta_2$.
Let $m' = \wedge\{t \in I_0; t \in \Delta_1\}$ and $k' = \wedge\{t \in I_0; t \in \Delta_2\}$.
We shall now show that $k' \leq 1 - l + m'$. i.e. $k \geq l + m - 1$.
If possible let $k' > 1 - l + m' \Rightarrow m' < l + k' - 1$,
$\Rightarrow$ there exists $p \in \Delta_1$ such that $m' \leq p < l + k' - 1$.
Now $p < l + k' - 1 \Rightarrow k' > 1 - l + p \Rightarrow 1 - l + p \notin \Delta_2$, which is a contradiction.
Hence if $\uparrow \mathcal{A} \to^m e$ then $\uparrow f[\uparrow \mathcal{A}] \to^k f(e)$ where $k \geq l + m - 1$.

$(b) \Rightarrow (a)$: Suppose (b) hold. If possible let $\wedge_{V \in I^Y}\{1 - \delta(V) + \tau(f^{-1}(V))\} < l$.
Then there exists $W \in I^Y$ such that $1 - \delta(W) + \tau(f^{-1}(W)) < l$,
$\Rightarrow \tau(f^{-1}(W)) < \delta(W) + l - 1 \Rightarrow \exists k_1, k_2 \in I_0$ such that $\tau(f^{-1}(W)) < k_1 < k_2 < \delta(W) - l'$.
Now $\tau(f^{-1}(W)) < k_1$,
$\Rightarrow \wedge\{Q(e, f^{-1}(W)); e \in Pt(I^X)$ & $e\hat{q}f^{-1}(W)\} < k_1$, [by Proposition 2.9]
$\Rightarrow \exists e^0 \in Pt(I^X)$ such that $e^0\hat{q}f^{-1}(W)$ and $Q(e^0, f^{-1}(W)) < k_1$.
Again $Q(e^0, f^{-1}(W)) < k_1 \Rightarrow \vee\{\tau(U); e_0\hat{q}U \subseteq f^{-1}(W)\} < k_1$.
So $\forall\, U \in I^X, \tau(U) \geq k_1$ and $e^0\hat{q}U \Rightarrow U \nsubseteq f^{-1}(W)$,
$\Rightarrow f^{-1}(W) \notin \tilde{Q}_{k_1}(e^0)$,

$$\Rightarrow W \notin f(\tilde{Q}_{k_1}(e^0)), \text{ for some } e^0 \in Pt(I^X). \tag{7}$$

Again $\delta(W) > k_2 + l'$ and $f(e^0)\hat{q}W$,(since $e^0\hat{q}f^{-1}(W)$)

$$\Rightarrow W \in \underline{\tilde{Q}}_{k_2+l'}(f(e^0)). \tag{8}$$

By (7) and (8) we have $f(\tilde{Q}_{k_1}(e^0) \not\supset \underline{\tilde{Q}}_{k_2+l'}(f(e^0))$.
This means if $\uparrow f[\uparrow \tilde{Q}_{k_1}(e^o)] \to^k f(e^0)$ then $k' \geq k_2 + l'$.
Again $\tilde{Q}_{k_1}(e^0)$ is a fuzzy filter and by definition $\tilde{Q}_{k_1}(e^0) \to^m e^0$ then $m' \leq k_1$.
Hence $k' \geq k_2 + l' > k_1 + l' \geq m' + l'$
$\Rightarrow k < (m' + l')' = (2 - m - l)' = l + m - 1$, which is a contradiction. Hence the result. □

Definition 3.6. In a fts (X, τ) a fuzzy subset A is said to be N-compact of grade m, if $m' = \wedge\{r \in I_0; A$ is N-compact in $(X, \tau_r)\}$, symbolically we say A is N^m-compact.

Proposition 3.7. *Let (X, τ) and (Y, δ) be any two fuzzy topological spaces and $f : (X, \tau) \to (Y, \delta)$ be a mapping with gradation of continuity l. If A be any fuzzy subset of X with m as gradation of N-compactness such that $l + m > 1$, then $f(A)$ is N-compact with gradation $\geq l + m - 1$, i.e. if A is N^m-compact then f(A) is $N^{(\geq l+m-1)}$- compact.*

Proof. Let A be N^m-compact and if possible let f(A) be N^t-compact where $t < l + m - 1$.
Then we have $t' = \wedge\{r \in I_0; f(A)$ is N-compact in $(Y, \delta_r)\}$,
$\Rightarrow (l + m - 1)' < \wedge\{r \in I_0; f(A)$ is N-compact in $(Y, \delta_r)\}$,
$\Rightarrow \exists s \in I_0$ such that $(l + m - 1)' = 2 - (l + m) < s < \wedge\{r \in I_0; f(A)$ is N-compact in $(Y, \delta_r)\}$. So, $f(A)$ is not N-compact in (Y, δ_s).
Let $k = 1 - s$. Then $s = 1 - k$ and f(A) is not N-compact in (Y, δ_{1-k}).
So there exists a $\alpha \in I_0$ and $\exists\alpha$-Q-cover $\psi(\subseteq \delta_{1-k})$ of f(A) such that no finite sub-collection of ψ is an α^--Q-cover of $f(A)$.
Let $\phi = \{f^{-1}(V); V \in \psi\}$. Then by Proposition 3.2 (b) $\phi \subseteq \tau_{l-k}$. Then clearly ϕ is an α-Q-cover of A. Now we show that no finite sub-collection of ϕ is an α^--Q-cover of A.
If possible let $\phi_0 = \{f^{-1}(V_i), i = 1, 2, 3........n\}$ be a finite sub-collection of ϕ which is an α^--Q-cover of A.
Then $\exists\gamma < \alpha$ such that $\vee\phi_0\hat{q}A(\gamma)$.
Take λ such that $\gamma < \lambda < \alpha$. Now for every $\lambda_y \in f(A)$, by definition we have $\lambda \leq f(A)(y) = \vee\{A(x) : x \in X, y = f(x)\}$. Since $\gamma < \lambda$, $\exists\ x \in X$ such that $f(x) = y, x_\gamma \in A$. Since $\vee\phi_0\hat{q}A(\gamma)$, $\exists\ i \leq n$ such that $x_\gamma\hat{q}f^{-1}(V_i)$. That is to say, $\gamma > (f^{-1}(V_i))'(x) = f^{-1}(V_i')(x) = V_i'(f(x)) = V_i'(y)$.
By $\gamma < \lambda$, we have $\lambda_y\hat{q}V_i$. So $\psi_0 = \{V_i : i = 1, 2, 3, ...n\}$ is an open λ-Q-cover of $f(A)$ and hence an α^--Q-cover of f(A) which is a contradiction.
Hence A is not N-compact with respect to τ_{l-k}. i.e. $m' \geq l - k \Rightarrow 1 - m \geq l - k$,
$\Rightarrow 1 - m \geq l - 1 + s$,
$\Rightarrow s \leq 2 - l - m$, which is a contradiction. Hence the result follows. □

4. Gradation of Mapping Openness

In this section concept of gradations are introduced in openness, closedness and homeomorphic properties of mappings and their properties are studied. Problem of defining initial fuzzy topology through a mapping f under this graded situation is considered and it is shown that corresponding to a preassigned value $l \in (0, 1]$ there exists an initial fuzzy topology with respect to which the grade of continuity of f is l. Apart from this a gradation in the T_2 separation axiom is also introduced and the famous result that every continuous function from a compact space to a T_2-space is homeomorphism, has been fuzzified under this graded form.

Definition 4.1. Let $(X, \tau), (Y, \delta)$ be any two fuzzy topological spaces with τ and δ as gradations of openness on X and Y respectively. We define the gradation of mapping openness of the mapping $f : X \to Y$ by $O(f) = \wedge_{U \in I^X}\{min\{1, 1 - \tau(U) + \delta(f(U))\}\} = \wedge_{U \in I^X}\{1 - \tau(U) + \delta(f(U))\}$.

Proposition 4.2. *Let (X, τ) and (Y, δ) be any two fts, and $f : (X, \tau) \to (Y, \delta)$ be a mapping. Then for each $l \in I$, the following statements are equivalent:*

(a) $O(f) \geq l$.
(b) $f(\tau_{1-r}) \subseteq \delta_{l-r}, \forall r \in I$, *with* $r < l$.
(c) $f(int(A, 1 - r)) \subseteq int(f(A), l - r),\ \forall\ r < l$ *and* $\forall\ A \in I^X$.

(d) $f(\tilde{Q}_{1-r}(p_x)) \subseteq \underline{\tilde{Q}}_{l-r}(f(p_x)),\ \forall r < l\ \&\ \forall p_x \in Pt(I^X)$, *where* $\tilde{Q}_{1-r}(p_x)$ *and* $\underline{\tilde{Q}}_{l-r}(f(p_x))$ *are the q-neighborhood systems of* p_x *and* $f(p_x)$ *with respect to Chang fuzzy topologies* τ_{1-r} *and* δ_{l-r} *respectively.*

Proof. $(a) \Leftrightarrow (b)$:
In fact $l \leq \wedge_{V\in I^X}\{1-\tau(V)+\delta(f(V))\}$,
$\Leftrightarrow 1-\tau(V)+\delta(f(V)) \geq l,\ \forall\ V \in I^X$,
$\Leftrightarrow 1-\tau(V) \geq l-\delta(f(V)),\ \forall\ V \in I^X$,
$\Leftrightarrow 1-\tau(V) \leq r$ implies $l-\delta(f(V)) \leq r,\ \forall\ V \in I^Y,\ \forall\ r \in [0,l)$.
$\Leftrightarrow \tau(V) \geq 1-r$ implies $\delta(f(V)) \geq l-r,\ \forall\ V \in I^X$ and $\forall r \in [0,l)$,
$\Leftrightarrow V \in \tau_{1-r}$ implies $f(V) \in \delta_{l-r},\ \forall\ V \in I^X$ and $\forall r \in [0,l)$.

$(b) \Rightarrow (c)$:
Suppose (b) holds.
We have to show that $f(int(A,1-r)) \subseteq int(f(A),l-r), \forall\ r < l$ and $\forall\ A \in I^X$.
If possible, let $f(int(A,1-r)) \nsubseteq int(f(A),l-r)$ for some $r < l$ and for some $A \in I^X$.
Then $\exists\ y \in Y$ such that $f(int(A,1-r))(y) > \alpha > int(f(A),l-r)(y)$.
Now $f(int(A,1-r))(y) > \alpha$,
$\Rightarrow sup_{x\in f^{-1}(y)}\{int(A,1-r)(x)\} > \alpha$,
$\Rightarrow \exists\ x \in f^{-1}(y)$ such that $int(A,1-r)(x) > \alpha$.
i.e. $\exists\ x \in f^{-1}(y)$ such that $sup\{U(x); U \in \tau_{1-r}$ and $A \supset U\} > \alpha$,
$\Rightarrow \exists\ U_0 \in \tau_{1-r}$ such that $U_0(x) > \alpha$ and $U_0 \subseteq A$.
i.e. $\exists U_0 \in \tau_{1-r}$ and $\alpha_x \in U_0 \subseteq A$ for some $x \in f^{-1}(y)$. (9)

Again $\alpha > int(f(A),l-r)(y)$,
$\Rightarrow \alpha > sup\{V(y); V \in \delta_{l-r}$ and $f(A) \supseteq V\}$,
$\Rightarrow V(y) < \alpha, \forall\ V \in \delta_{l-r}$ with $V \subseteq f(A)$ (10)

From (9), we have $U_0 \in \tau_{1-r}$. So by (b), $f(U_0) \in \delta_{l-r}$. Again $\alpha_x \in U_0 \subset A$ implies $\alpha_y \in f(U_0) \subseteq f(A)$, which is contradictory to (10). Hence $f(int(A,1-r)) \subseteq int(f(A),l-r),\ \forall r < l$ and $\forall\ A \in I^X$.

$(c) \Rightarrow (a)$:
Assume (c) holds. If possible, let (a) does not hold.
Then $l > \wedge_{V\in I^X}\{1-\tau(U)+\delta(f(U))\}$,
$\Rightarrow \exists\ W \in I^X$ such that $1-\tau(W)+\delta(f(W)) < l$,
$\Rightarrow 1-\tau(W) < l-\delta(f(W))$,
$\Rightarrow \exists\ \alpha \in I_0$ such that $1-\tau(W) < \alpha < l-\delta(f(W))$,
$\Rightarrow \tau(W) > 1-\alpha$ and $\delta(f(W)) < l-\alpha$,
$\Rightarrow W \in \tau_{1-\alpha}$ and $f(W) \notin \delta_{l-\alpha}$,
$\Rightarrow f(int(W,1-\alpha)) = f(W)$, but $int(f(W),l-\alpha) \subset (\neq) f(W)$,
$\Rightarrow int(f(W),l-\alpha) \subset (\neq) f(int(W,1-\alpha))$, which contradicts (c). Hence $(c) \Rightarrow (a)$.

$(b) \Rightarrow (d)$:
Suppose (b) holds. Choose any $r < l$ and any $p_x \in Pt(I^X)$. Let $U \in \tilde{Q}_{1-r}(p_x)$.
Then $\exists V \in \tau_{1-r}$ such that $p_x \hat{q} V \subseteq U$ and hence $f(p_x)\hat{q} f(V) \subseteq f(U)$.
Now from (b), $f(V) \in \delta_{l-r}$, so$f(U) \in \underline{\tilde{Q}}_{l-r}(f(p_x))$.
Hence $f(\tilde{Q}_{1-r}(p_x)) \subseteq \underline{\tilde{Q}}_{l-r}(f(p_x)),\ \forall r < l\ \&\ \forall p_x \in Pt(I^X)$.

$(d) \Rightarrow (b)$:
Suppose (d) holds and let $V \in \tau_{1-r}$ with $r < l$.
If $V = \tilde{0}_X$, then obviously $f(V) = \tilde{0}_Y \in \delta_{l-r}$. Let $V \neq \tilde{0}_X$, then $f(V) \neq \tilde{0}_Y$.
So $\exists y_0 \in Y$ such that $f(V)(y_0) > 0$ i.e $(f(V))'(y_0) < 1$,
$\Rightarrow \exists \alpha \in I_0$ such that $(f(V))'(y_0) < \alpha < 1$. i.e. $\alpha_{y_0} \hat{q} f(V)$.
Now for each $\alpha_y \hat{q} f(V)$, we have $\alpha > (f(V))'(y)$ i.e. $\alpha' < f(V)(y)$,
$\Rightarrow \alpha' < \bigvee\{V(\xi); \xi \in f^{-1}(y)\}$,
$\Rightarrow \exists x \in f^{-1}(y)$ such that $\alpha' < V(x)$,
$\Rightarrow \alpha_x \hat{q} V$.
So, $\alpha_x \hat{q} V$ and $V \in \tau_{1-r} \Rightarrow V \in \tilde{Q}_{1-r}(\alpha_x)$.
Then by (d), $f(V) \in \underline{\tilde{Q}}_{l-r}(f(\alpha_x))$,
$\Rightarrow \exists W \in \delta_{l-r}$ such that $\alpha_y \hat{q} W \subseteq f(V)$. This shows that for each $\alpha_y \hat{q} f(V), \exists W \in \delta_{l-r}$ such that $\alpha_y \hat{q} W \subseteq f(V)$, and this is true $\forall y$ with $f(V)(y) > 0$ and for all α satisfying $(f(V))'(y) < \alpha < 1$. So $f(V) = \vee_{\alpha_y \hat{q} f(V)} W \Rightarrow f(V) \in \delta_{l-r}$.
Hence $f(\tau_{1-r}) \subseteq \delta_{l-r}$, $\forall r < l$. □

Definition 4.3. Let (X, τ) be a fuzzy topological space with τ as gradation of openness then (X, τ) is said to be T_2 space with grade l if $l = \vee\{r \in I; (X, \tau_r)$ is a T_2 Cfts$\}$. Symbolically this is written as $T_2^{(l)}$.

Proposition 4.4. *Let (X, τ) be an fts which is N^l-compact and A be a fuzzy subset such that $\mathcal{F}_\tau(A) = k > l'$. Then A is $N^{(\geq l)}$-compact, where $\mathcal{F}_\tau(A) = \tau(A')$.*

Proof. We have $l' = \wedge\{r \in I_0; (X, \tau_r)$ is N-compact $\}$.
Let $r_0 \in I_0$ such that $l' < r_0 < k$.
Then $r_0 > \wedge\{r \in I_0; (X, \tau_r)$ is N-compact$\}$,
$\Rightarrow \exists\ s \in I_0$ such that $r_0 > s$ and (X, τ_s) is N-compact.
As $r_0 > s \Rightarrow \tau_{r_0} \subseteq \tau_s$.
So (X, τ) is N^l-compact $\Rightarrow$ (X, τ_{r_0}) is N-compact. Also $\mathcal{F}_\tau(A) \geq r_0$ means A is closed in N-compact space (X, τ_{r_0}).
So, A is N-compact in (X, τ_{r_0})(by Theorem 2.17). i.e. for any r_0 with $l' < r_0 < k$, we have A is N-compact in (X, τ_{r_0}). (11)

Now let $a' = \wedge\{r \in I_0; A$ is N-compact in $(X, \tau_r)\}$.
i.e. a is the gradation of compactness of A.
So $a' \leq l'$ i.e. A is $N^{(\geq l)}$ compact. Hence the proof. □

Proposition 4.5. *Let (X, τ) be an fts such that $\tau(\bar{\alpha}) = 1\ \forall \alpha \in [0, 1]$. Let (X, τ) be ${T_2}^{(k)}$ where $k > 0$ and A be a fuzzy subset of X which is $N^{(l)}$ compact, where $l > k'$. Then $\mathcal{F}_\tau(A) \geq k$.*

Proof. We have $k = \vee\{r \in I_0; (X, \tau_r)$ is a T_2 Chang fts$\}$.
$l' = \wedge\{r \in I_0; A$ is N-compact in $(X, \tau_r)\}$.
Let $r_0 \in I_0$ be such that $l' < r_0 < k$. Then $\exists s_O \in I_0$ such that $r_0 < s_0$ and (X, τ_{s_0}) is T_2 Cfts. Since $r_0 < s_0$ implies $\tau_{r_0} \supset \tau s_0$. So, (X, τ_{r_0}) is T_2 Cfts. Again we have $\tau(\bar{\alpha}) = 1\ \forall\ \alpha \in I$.
So, (X, τ_{r_0}) is stratified *level*-T_2 Chang fuzzy topological space. On the contrary

$l' < r \Rightarrow \wedge\{r \in I_0; A$ is N-compact in $(X, \tau_r)\} < r_0$,
$\Rightarrow \exists\ t_0 \in I_0$ such that $l' < t_0 < r_0$ and A is N-compact in (X, τ_{t_0}) and since $\tau_{t_0} \supseteq \tau_{r_0}$. So, A is N-compact in (X, τ_{r_0}). So A is closed in (X, τ_{r_0})(by Theorem 2.17). This holds for every r_0 satisfying $l' < r_0 < k$. Hence $b = \vee\{r \in I_0; A$ is closed in $(X, \tau_r)\} \geq k$. i.e. $\mathcal{F}_\tau(A) \geq k$. j □

Proposition 4.6. *Let X be a non-empty set, (Y, δ) be a fuzzy topological space and $f : X \to (Y, \delta)$ be an onto mapping and $l \in (0, 1)$. Let $\tau : I^X \to I$, be defined by*

$$\tau(A) = \begin{cases} 1 & \text{if } A = \tilde{0}_X \text{ or } \tilde{1}_X \\ l + \delta(V) - 1 & \text{if } A \text{ can be expressed as } A = f^{-1}(V); \text{ for some } V \in I^Y \\ & \text{such that } \delta(V) \geq 1 - l\} \\ 0 & \text{otherwise.} \end{cases}$$

Then τ is the smallest gradation of openness on X to make $cont(f) = l$.

Proof. O1. Follows from the definition of τ.

O2. Let $A_1, A_2 \in I^X$:

If at least one of A_1, A_2 is $\tilde{0}_X$ or $\tilde{1}_X$ then the case is obvious.
Now let A_1, A_2 be two proper fuzzy subsets of X. Then if at least one of $\tau(A_1), \tau(A_2)$ is 0 then the case also holds trivially.
So let A_1, A_2 be two proper fuzzy subsets which can be expressed as $A_1 = f^{-1}(V_1)$ & $A_2 = f^{-1}(V_2)$ for some $V_1, V_2 \in I^Y$ with $\delta(V_1), \delta(V_2) \geq 1 - l$.
So, $A_1 \cap A_2 = f^{-1}(V_1) \cap f^{-1}(V_2) = f^{-1}(V_1 \cap V_2)$ and $\delta(V_1 \cap V_2) \geq \delta(V_1) \wedge \delta(V_2) \geq 1 - l$.
So, $\tau(A_1 \cap A_2) = l + \delta(V_1 \cap V_2) - 1$
$\geq l + \delta(V_1) \wedge \delta(V_2) - 1 = (l + \delta(V_1) - 1) \wedge (l + \delta(V_2) - 1) = \tau(A_1) \wedge \tau(A_2)$

O3. Let $\{A_i; i \in \Lambda\}$ be a family of fuzzy sets in X. If either of the following two cases
i) one of $\tau(A_i) = 0$, ii) $\tau(\cup_{i\in\Lambda} A_i) = 1$ holds then O3 holds trivially.
So, let us consider the case when $\cup_{i\in\Lambda} A_i \neq \tilde{1}_X$ and $\tau(A_i) \neq 0,\ \forall\ i \in \Lambda$.
So, for each $i \in \Lambda, A_i$ can be expressed as $A_i = f^{-1}(V_i)$ for some $V_i \in I^Y$ such that $\delta(V_i) \geq 1 - l$,
$\Rightarrow \cup_{i\in\Lambda} A_i = \cup_{i\in\Lambda} f^{-1}(V_i) = f^{-1}(\cup_{i\in\Lambda} V_i)$. Also $\delta(\cup_{i\in\Lambda} V_i) \geq \wedge_{i\in\Lambda}\delta(V_i) \geq 1 - l$.
Hence $\tau(\cup_{i\in\Lambda} A_i) = l + \delta(\cup_{i\in\Lambda} V_i) - 1$
$\geq l + \wedge_{i\in\Lambda}\delta(V_i) - 1$
$= \wedge_{i\in\Lambda}(l + \delta(V_i) - 1)$
$= \wedge_{i\in\Lambda}\tau(A_i)$

To show that $cont(f) = l$ consider the following cases:

1. Let $V = \tilde{0}_Y$ or $\tilde{1}_Y$ then $1 - \delta(V) + \tau(f^{-1}(V)) = 1$.
2. If $V \in I^Y$ be such that $\delta(V) < 1-l$ then $1-\delta(V)+\tau(f^{-1}(V)) > 1-1+l+0 = l$.
3. If $V(\neq \tilde{0}_Y, \tilde{1}_Y) \in I^Y$ be such that $\delta(V) \geq 1 - l$ then $1 - \delta(V) + \tau(f^{-1}(V)) = l$.

These three cases show that $cont(f) = l$.

Then τ is a gradation of openness (GO) on X such that $cont(f) = l$.
Lastly we have to show that τ is the smallest gradation of openness to make $cont(f) = l$. Let σ be any gradation of openness on X such that $cont(f) = l$ then
i) obviously, $\sigma(\tilde{0}_X) = \sigma(\tilde{1}_X) = 1$.
ii) For $A = f^{-1}(V)$ such that $\delta(V) \geq 1 - l, 1 - \delta(V) + \sigma(f^{-1}(V)) \geq l$,(since $cont(f) = l$ w.r.t. σ and δ)
$\Rightarrow \sigma(A) \geq l + \delta(V) - 1$.
iii) For $A = f^{-1}(V)$ such that $\delta(V) < 1 - l$, then obviously $\sigma(A) \geq \tau(A) = 0$. These three cases show the result. □

Example 4.7. Let X be a non-empty set and I be the unit closed interval [0, 1]. Let $\tau : I^X \to I$ be defined by

$$\tau(A) = \begin{cases} 1 & \text{if } A = \tilde{0}_X \text{ or } \tilde{1}_X \\ \frac{1}{n} & \text{if } A = \frac{\tilde{1}}{n}, (n \in \mathbb{N}\text{-set of natural Nos.}) \\ 0 & \text{otherwise.} \end{cases}$$

$$\delta(A) = \begin{cases} 1 & \text{if } A = \tilde{0}_X \text{ or } \tilde{1}_X \\ 0.2 + \frac{1}{n} & \text{if } A = \frac{\tilde{1}}{n}, (n \geq 2 \text{ and } n \in \mathbb{N} \text{ -set of natural Nos.}) \\ 0 & \text{otherwise.} \end{cases}$$

Then clearly τ and δ are gradations of openness on X.
Let $f : (X, \tau) \to (X, \delta)$ be the identity mapping . Then as we have $\tau_{0.6} = \{\tilde{0}, \tilde{1}\}$ and $\delta_{0.6} = \{\tilde{0}, \tilde{1}, \frac{\tilde{1}}{2}\}$, so, $f^{-1}(\delta_{0.6}) \not\subseteq \tau_{0.6}$.
Hence f is not a gradation preserving (gp) mapping.
Now to find the gradation of continuity of f we see that $cont(f) = \wedge_{V \in I^X}\{1 - \delta(V) + \tau(f^{-1}(V))\}$
$= min\{1 - 0.2 - \frac{1}{n} + \frac{1}{n}\}$
$= 0.8$

Example 4.8. Let $\mathcal{U}$ and $\mathcal{T}$ be the usual and lower limit topology on **R** respectively(**R**- the set of real numbers).
Let $\tau : I^R \to I$ be defined by
$\tau(\mu_A) = 1$ if $A \in \mathcal{U}$
$= 0$ otherwise,
where μ_A denotes the characteristic function of A.
Let $\delta : I^R \to I$ be defined by
$\delta(\mu_A) = 1$ if $A \in \mathcal{U}$
$= 0.5$ if $A \in \mathcal{T} - U$
$= 0$ otherwise.

Then τ and δ are gradations of openness (GO) on R and the identity map $f : (R, \tau) \to (R, \delta)$ is not a gp-map.
However $cont(f) = \wedge_{U \in R}\{1 - \delta(V) + \tau(f^{-1}(V))\} = 0.5$

Definition 4.9. Let $(X, \tau), (Y, \delta)$ be any two fuzzy topological spaces with τ, δ as gradations of openness on X,Y respectively. We define the gradation of mapping closedness of the mapping $f : X \to Y$ by $C(f) = \wedge_{U \in I^X}\{1 - \tilde{\tau}(U) + \tilde{\delta}(f(U))\}$, where $\tilde{\tau}$ and $\tilde{\delta}$ are the gradations of closedness on X and Y respectively.

Definition 4.10. Let $(X, \tau), (Y, \delta)$ be any two fuzzy topological spaces with τ and δ as gradations of openness on X and Y respectively and $f : X \to Y$ be a mapping. The pair $(cont(f), O(f))$ is called the homeomorphism grade of f and is denoted by $\mathcal{H}(f)$

Proposition 4.11. *Let $(X, \tau), (Y, \delta)$ be any two fuzzy topological spaces with τ, δ as gradations of openness on X, Y, respectively. Then a bijective mapping $f : X \to Y$ has a gradation of mapping closedness l iff f has a gradation of mapping openness l, where $\tilde{\tau}$ and $\tilde{\delta}$ are the gradations of closedness on X and Y respectively.*

Proof. We have
$C(f) = \wedge_{V \in I^X}\{1 - \tilde{\tau}(V) + \tilde{\delta}(f(V))\}$
$= \wedge_{V' \in I^X}\{1 - \tilde{\tau}(V') + \tilde{\delta}(f(V'))\}$,
$= \wedge_{V' \in I^X}\{1 - \tau(V) + \tilde{\delta}(f(V))'\}$ (since f is bijective) ,
$= \wedge_{V' \in I^X}\{1 - \tau(V) + \delta(f(V))\}$,
$= \wedge_{V \in I^X}\{1 - \tau(V) + \delta(f(V))\}, = O(f)$. □

Proposition 4.12. *Let (X, τ) and (Y, δ) be any two fuzzy topological spaces with τ and δ as gradations of openness respectively and $f : (X, \tau) \to (Y, \delta)$ be a mapping with gradation of mapping closedness $\geq l$ iff for any $U \in \tilde{\tau}_{1-r} \Rightarrow f(U) \in \tilde{\delta}_{l-r}, \ \forall\, r < l$. i.e. $f(\tilde{\tau}_{1-r}) \subseteq \tilde{\delta}_{l-r}, \ \forall\, r < l$.*

Proposition 4.13. *Let $(X, \tau), (Y, \delta)$ be any two fuzzy topological spaces with τ, δ as gradations of openness on X, Y respectively. Now if (X, τ) is N^1-compact and (Y, δ) is T_2 with grade 1, where $\delta(\bar{\alpha}) = 1\ \forall \alpha \in [0, 1]$ and $f : X \to Y$ is a bijective mapping with grade of continuity l. Then $\mathcal{H}(f) = (l, m)$, where $m \geq l$.*

Proof. Now there are two cases:

1. Let A be any fuzzy set having gradation of closedness r where $l + r > 1$. Let $0 < s < r$. Then (X, τ_s) is N-compact and A is closed in (X, τ_s). So by Theorem 2.17 A is N-compact in (X, τ_s). Since A is N-compact in (X, τ_s) for every s satisfying $0 < s < r$. Hence A is $N^{(1)}$-compact. So by Proposition 3.7 $f(A)$ is $N^{(\geq l)}$-compact. i.e. $f(A)$ is closed with grade $\geq l$(by Proposition 4.5). i.e. $1 - \mathcal{F}(A) + \bar{\mathcal{F}}(f(A)) \geq 1 - r + l \geq l$.

2. $l + r \leq 1$ then $1 - \mathcal{F}(A) + \bar{\mathcal{F}}(f(A)) \geq 1 - r \geq l$. i.e. in both cases $1 - \mathcal{F}(A) + \bar{\mathcal{F}}(f(A)) \geq l$. Now since A is arbitrary this relation holds for any $A \in I^X$.
i.e. f has a gradation of mapping closedness $\geq l$. i.e. f has a gradation of mapping openness $\geq l$.i.e. $\mathcal{H}(f) = (l, m)$, where $m \geq l$. □

5. **Application**

With topological spaces defined for design and manufacturing representation, it is possible to investigate the mapping from the design space to the manufacturing space. One of the most important properties of such a mapping is its continuity, the crisp version of which might be failed because of the manufacturing process

[25]. But under the fuzzy approach, introduced in this paper, where a gradation of continuity is used and also topologies of domain and codomain spaces are fuzzified, a theoretical background can be developed with soft reasoning which will be very useful in this field.

Acknowledgements. The authors are grateful to the referees for their valuable comments. The authors are also thankful to the managing editor for his valuable comments to prepare the paper in the present form.

References

[1] M. E. Abd El-Monsef, S. N. ElDeep, F. M. Zeyada and I. M. Hanafy, *On fuzzy δ-continuity and fuzzy δ-closed graph*, Periodica Mathematica Hungarica, **26(1)** (1993), 43-53.

[2] M. H. Burton, M. Muraleetharan and J. Gutierrez Garcia, *Generalized filters 1*, Fuzzy Sets and Systems, **106** (1999), 275-284.

[3] M. Caldas, S. Jafari, T. Noiri and M. Simoes, *A new generalization of contra-continuity via Levine's g-closed sets*, Chaos, Solitons & Fractals, **32(4)** (2007), 1597-1603.

[4] C. L. Chang, *Fuzzy topological spaces*, J. Math. Anal. Appl., **24** (1968), 182-190.

[5] K. C. Chattopadhyay, R. N. Hazra and S. K. Samanta, *Gradation of openness : fuzzy topology*, Fuzzy Sets and Systems, **49** (1992), 237-242.

[6] M. Demirci, *On the convergence structure of L-topological spaces and the graded continuity in L-topological spaces, part I-II*, International Conference on Applicable and General Topology, August: 12-18, 2001.

[7] E. Ekici, *Generalization of weakly clopen ansd strongly θ-b-continuous functions*, Chaos, Solitons & Fractals, **38(1)** (2008), 79-88.

[8] M. S. El. Naschie, *Superstring, knots and non-commutative geometry in space*, Int. J Theor Phys, **37(12)** (1998), 2935-51.

[9] M. S. El. Naschie, *Topics in the mathematical physics of E-infinity theory*, Chaos, Solitons & Fractals, **30** (2006), 656-63.

[10] M. S. El. Naschie, *Quantum gravity, Cliford algebras, fuzzy set theory and fundamental constants of nature*, Chaos, Solitons & Fractals, **20** (2004), 437-50.

[11] M. S. El. Naschie, *Topics of mathematical physics of E-infinity theory*, Chaos, Solitons & Fractals, **30** (2006), 656-63.

[12] M. S. El. Naschie, *A review of E-infinity theory and mass spectrum of high energy particle physics*, Chaos, Solitons & Fractals, **19** (2004), 209-36.

[13] F. Jinming, *MV-quasi-coincident neighborhood systems in I-topology*, Preprint.

[14] T. Kubiak, *On fuzzy topologies*, Ph. D. Thesis, A. Mickiewiez, Poznan, 1985.

[15] K. K. Mondal and S. K. Samanta, *Fuzzy convergence theory -I*, Journal of the Korea Society of Mathematical Education Series B: PAM, **12(1)** (2005), 75-91.

[16] K. K. Mondal and S. K. Samanta, *Fuzzy convergence theory -II*, Journal of the Korea Society of Mathematical Education Series B: PAM, **12(2)** (2005), 105-124.

[17] K. K. Mondal and S. K. Samanta, *Fuzzy belongingness, fuzzy quasi-coincidence and convergence of generalized fuzzy filters*, The Journal of Fuzzy Mathematics, **15(4)** (2007).

[18] T. Noiri and V. Popa, *Faintly m-continuous functions*, Chaos, Solitons & Fractals, **19** (2004), 1147-59.

[19] P. Pao-Ming and L. Ying Ming, *Fuzzy topology I, neighborhood structure of a fuzzy point and Moor-Smith convergence*, J. Math. Anal. Appl., **76** (1980), 571-599.

[20] D. W. Rosen and T. J. Peters, *The role of toplogy in engineering design research*, Res Eng Des, **2** (1996), 81-98

[21] A. P. Sostak, *On a fuzzy topological structure*, Supp. Rend. Circ. Math. Palermo (Ser. II) II, (1985), 89-103.

[22] G. Werner, *The general fuzzy filter approach to fuzzy topology, I*, Fuzzy Sets and Systems, **76** (1995), 205-224.

[23] C. H. Yan and J. X. Fang, *L-fuzzy bilinear operator and its continuity*, Iranian Journal of Fuzzy Systems, **4(1)** (2007), 65-73.
[24] M. S. Ying, *A new approach for fuzzy topology (I)*, Fuzzy Sets and Systems, **39** (1991), 302-321.
[25] M. S. Ying, *On the method of neighborhood systems in fuzzy topology*, Fuzzy Sets and Systems, **68** (1994), 227-238.
[26] L. Ying-Ming and L. Mao-Kang, *Fuzzy topology*, World Scientific.
[27] A. M. Zahran, O. R. Sayed and A. K. Mousa, *Completely continuous functions and R-map in fuzzifying topological space*, Fuzzy Sets and Systems, **158** (2007), 409-23.

Ramkrishna Thakur, Department of Mathematics, Visva-Bharati, Santiniketan-731235, West Bengal, India
E-mail address: thakurram06@gmail.com

S. K. Samanta*, Department of Mathematics, Visva-Bharati, Santiniketan-731235, West Bengal, India
E-mail address: syamal_123@yahoo.co.in, syamal.samanta@visva-bharati.ac.in

K. K. Mondal, Department of Mathematics, Kurseong College, Kurseong-734203, West Bengal, India
E-mail address: krishnakmondal@yahoo.co.in

*Corresponding author