

## EXPECTED PAYOFF OF TRADING STRATEGIES INVOLVING EUROPEAN OPTIONS FOR FUZZY FINANCIAL MARKET

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ABSTRACT. Uncertainty inherent in the financial market was usually considered to be random. However, randomness is only one special type of uncertainty and appropriate when describing objective information. For describing subjective information it is preferred to assume that uncertainty is fuzzy. This paper defines the expected payoff of trading strategies in a fuzzy financial market within the framework of credibility theory. In addition, a computable integral form is obtained for expected payoff of each strategy.

### 1. Introduction

Uncertainty inherent in the financial market was usually considered to be random and it was assumed that stock prices follow some stochastic process. In other words, for each fixed time the stock price was considered to be a random variable. In fact, as early as 1900, Bachelier supposed that stock prices follow Brownian motion. However, Brownian motion is clearly inadequate as a market model since it may predict negative stock prices and in 1965 Samuelson [21] assumed that stock prices follow a geometric Brownian motion which inherits the very irregular paths of Brownian motion. Black and Scholes [2] and, independently, Metron [16] employed the geometric Brownian motion to construct a theory for determining the price of stock options in 1973. This is the famous Black-Scholes option pricing formula which represents a milestone in the development of mathematical finance. Since then, numerous researches surrounding the subject have been carried out.

However, sometimes, such as for a new security market, there is a lack of historical data or maybe the emergence of financial crisis leads to the failure of previous data. In these cases, the subjective experience of experts will help better estimate parameters such as security prices. In other words, subjective uncertainty, i.e., fuzziness is very important in real life. In fact, randomness is only one type of uncertainty. The idea of fuzziness was initially introduced by Zadeh [28]; a fuzzy set is determined by its membership function which can be suggested by experienced experts. Fuzzy set theory has been widely applied in financial areas such as [3] [5] [6] [17] [19] [22] [23] [24] [25] [26] [27]. Liu [13] founded fuzzy calculus to deal with dynamic phenomena, and applied it to finance by assuming that stock prices follow a geometric Liu process. Based on this assumption, Liu's stock model was formulated to describe fuzzy financial market in which uncertainty is dealt with as

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Received: May 2010; Revised: October 2010; Accepted: December 2010

*Key words and phrases:* Credibility measure, Liu process, Expected value, Fuzzy process.

fuzziness. Qin and Li [18] first deduced an option pricing formula for European options for Liu's stock model. Further development may be founded in Qin and Li [20].

In practice, an investor wants to eliminate risk or gain profit through trading strategies using options and the underlying stock. In the face of different market forecasts, flexible trading strategies should be constructed to reduce risk. Consequently, we consider the commonly used covered and spread strategies, when the options involved in the strategies are assumed to be European. In general, the profit of each strategy is calculated as the final payoff minus the initial cost of buying the stock or option. Naturally, the expected payoff of each strategy is important for an investor at the beginning of an investment. In this paper, we investigate the expected payoff of trading strategies for a fuzzy financial market in detail. Furthermore, we also give the integral forms of expected payoffs based on credibility theory.

The rest of this paper is organized as follows. Section 2 recalls the basic results of fuzzy variables and Liu's stock model used in the rest of this paper. Section 3 considers the expected payoff of covered strategies, and Section 4 investigates the expected payoff of alternative of spread strategies. Conclusions and future research directions are given in the final section.

## 2. Preliminaries

In this section, we introduce several basic concepts and properties of fuzzy variables, and discuss Liu's stock model which is a fuzzy counterpart of the Black-Scholes stock model.

**2.1. Fuzzy Variable.** Since the concept of fuzzy set was initiated by Zadeh in 1965, it has been widely applied in many areas. In order to measure a fuzzy event, Liu and Liu [14] proposed a concept of credibility measure in 2002. In addition, Li and Liu [7] proved a sufficient and necessary condition for a credibility measure. Credibility theory was founded by Liu [10] and refined by Liu [11] as a branch of mathematics for studying the behavior of fuzzy phenomena. For detailed expositions on credibility theory, the readers may consult Liu [12].

Let  $\xi$  be a fuzzy variable with membership function  $\mu$ . Then for any set  $B$  of  $\mathfrak{R}$ ,  $\xi \in B$  is a fuzzy event whose credibility measure, defined by Liu and Liu [14], is

$$\text{Cr}\{\xi \in B\} = \frac{1}{2} \left( \sup_{x \in B} \mu(x) + 1 - \sup_{x \in B^c} \mu(x) \right) \quad (1)$$

This is also called the *credibility inversion theorem*. Once the membership function of a fuzzy variable  $\xi$  is given, the credibility measure of fuzzy event  $\xi \in B$  may be obtained using the above formula.

Let  $\xi$  and  $\eta$  be two fuzzy variables. If  $\text{Cr}\{\xi \in B\} = \text{Cr}\{\eta \in B\}$  for any set  $B$  of  $\mathfrak{R}$ , then  $\xi$  and  $\eta$  are said to be indentially distributed [10]. The independence of fuzzy variables has been discussed by several scholars and this paper uses the following definition:

**Definition 2.1.** [15] The fuzzy variables  $\xi_1, \xi_2, \dots, \xi_n$  are said to be independent if

$$\text{Cr} \left\{ \bigcap_{i=1}^n \{\xi_i \in B_i\} \right\} = \min_{1 \leq i \leq n} \text{Cr}\{\xi_i \in B_i\} \quad (2)$$

for any sets  $B_1, B_2, \dots, B_n$  of  $\mathfrak{R}$ .

In order to rank fuzzy variables, the definition of expected value of a fuzzy variable is as follows [14]:

$$E[\xi] = \int_0^{+\infty} \text{Cr}\{\xi \geq r\}dr - \int_{-\infty}^0 \text{Cr}\{\xi \leq r\}dr \quad (3)$$

provided that at least one of the two integrals is finite. If  $\xi$  has a finite expected value, then its variance is defined as  $V[\xi] = E[(\xi - E[\xi])^2]$ .

Next, we introduce two special types of fuzzy variables. A fuzzy variable  $\xi$  is normally distributed if it has the following membership function,

$$\mu(x) = 2 \left( 1 + \exp \left( \frac{\pi|x - et|}{\sqrt{6}\sigma t} \right) \right)^{-1}, \quad -\infty < x < +\infty.$$

It follows from the definitions of expected value and variance that  $E[\xi] = e$  and  $V[\xi] = \sigma^2$  [8]. A lognormally distributed fuzzy variable was introduced by Li and Qin [9] using the lognormal membership function

$$\mu(x) = 2 \left( 1 + \exp \left( \frac{\pi|\ln x - e|}{\sqrt{6}\sigma} \right) \right)^{-1}, \quad x > 0. \quad (4)$$

This fuzzy variable is often encountered in fuzzy calculus and will be used in what follows.

**2.2. Liu's Stock Model.** Let  $T$  be an index set. If, for each  $t \in T$ ,  $X_t$  is a fuzzy variable, then  $\{X_t, t \in T\}$  is called a fuzzy process. In what follows, we use  $X_t$  to denote a fuzzy process. In particular, a fuzzy process  $X_t$  is said [13] to have independent increments if  $X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \dots, X_{t_k} - X_{t_{k-1}}$  are independent fuzzy variables for  $t_0 < t_1 < \dots < t_k$ , and to have stationary increments if, for any given  $t > 0$ , the  $X_{s+t} - X_s$  are identically distributed fuzzy variables for all  $s > 0$ .

**Definition 2.2.** [13] A fuzzy process  $C_t$  is said to be a Liu process if

- (i)  $C_0 = 0$ ,
- (ii)  $C_t$  has stationary and independent increments,
- (iii) every increment  $C_{t+s} - C_s$  is a normally distributed fuzzy variable with expected value  $et$  and variance  $\sigma^2 t^2$ .

A Liu process is said to be standard if  $e = 0$  and  $\sigma = 1$ . If  $C_t$  is a Liu process, then the fuzzy process  $X_t = \exp(et + \sigma C_t)$  is called a geometric Liu process.

The assumption that the stock price follows a geometric Liu process was suggested by Liu [13] in 2008. Based on this assumption, Liu's stock model was formulated to describe a fuzzy financial market. In Liu's stock model, the bond price  $X_t$  and the stock price  $Y_t$  follow

$$\begin{cases} X_t = X_0 \exp(rt) \\ Y_t = Y_0 \exp(et + \sigma C_t) \end{cases} \quad (5)$$

where  $r$  is the riskless interest rate,  $e$  is the stock drift,  $\sigma$  is the stock diffusion, and  $C_t$  is a standard Liu process. For each fixed time  $t$ , it is easy to see that  $Y_t$  is a lognormally distributed fuzzy variable (See Li and Qin [9]). In addition, if  $C_t$  is replaced by Brownian motion  $B_t$ , then  $Y_t$  is just a geometric Brownian motion, and the model becomes the Black-Scholes stock model [2]. Thus, Liu's stock model is a fuzzy counterpart of the Black-Scholes stock model.

### 3. Covered Strategies

A European call option gives the holder the right, but not the obligation, to buy a stock at a specified time  $T$  and for a specified price  $K$ . Similarly, a European put option gives the holder the right to sell a stock at time  $T$  for a specified price  $K$ . The time  $T$  is called the expiration date and the price  $K$  is called the strike price. In this paper, we will assume that the underlying asset is a stock whose price follows a geometric Liu process, and the option used in each strategy is European.

Covered strategies involve taking a position on a single option of a stock and the stock itself. There are four types of covered strategies: a covered call, a reversed covered call, a covered put and a reversed covered put. In this section, the expected payoff of each strategy for Liu's stock model is given in an integral form. An investor will determine which strategy to choose based on his/her observations and judgments for financial market.

#### Covered Call

A covered call is a strategy in which an investor holds a long position in a stock plus a short position in a call option. When there is a sharp rise in the stock price, the long stock position protects the investor from the payoff on the short call option. Suppose that  $K$  is the strike price of the call option. If  $Y_T$  is the final price of the underlying stock, then the payoff of the covered call at time  $T$  is

$$C_1(T, Y_T) = Y_T - (Y_T - K)^+.$$

It is easy to see that  $C_1(T, Y_T)$  is a fuzzy variable.

**Definition 3.1.** The expected payoff of a covered call is defined as the expected value of  $C_1(T, Y_T)$ .

**Theorem 3.2.** *The expected payoff of a covered call for Liu's stock model is  $\int_0^K \text{Cr}\{Y_T \geq r\} dr$ .*

*Proof.* We note that  $Y_T - (Y_T - K)^+ = \min(Y_T, K) \geq 0$ . Since  $Y_T$  is nonnegative, we get

$$\begin{aligned} E[C_1(T, Y_T)] &= \int_0^{+\infty} \text{Cr}\{\min(Y_T, K) \geq r\} dr \\ &= \int_0^K \text{Cr}\{\min(Y_T, K) \geq r\} dr \\ &= \int_0^K \text{Cr}\{Y_T \geq r\} dr. \end{aligned}$$

The proof is complete.  $\square$

Since  $Y_T$  is a lognormally distributed fuzzy variable with membership function

$$\mu_{Y_T}(x) = 2 \left( 1 + \exp \left( \frac{\pi |\ln x - eT|}{\sqrt{6}\sigma T} \right) \right)^{-1}, \quad x > 0,$$

the expression for  $\text{Cr}\{Y_T \geq r\}$  can be obtained by the credibility inversion theorem. For simplicity, in what follows we write  $\alpha = \pi/(\sqrt{6}\sigma T)$  and  $\beta = (\pi e)/(\sqrt{6}\sigma)$ . Thus

$$E[C_1(T, Y_T)] = \int_0^K \frac{\exp(\beta - \alpha \ln r)}{1 + \exp(\beta - \alpha \ln r)} dr. \quad (6)$$

For given values of the strike price  $K$  and expiration date  $T$ , we compare the size of  $K$  with  $\exp(eT)$  based on the value of stock drift  $e$ . Then we choose equation (6) to compute  $E[C_1(T, Y_T)]$  by computing the integral expression. In practice, these can be calculated by using numerical integration methods such as ‘quad’, ‘quadl’ or other Matlab functions.

### Reversed Covered Call

A reversed covered call is a strategy in which an investor holds a short position in a stock plus a long position in a call option. The payoff of a reversed covered call is

$$C_2(T, Y_T) = (Y_T - K)^+ - Y_T.$$

**Definition 3.3.** The expected payoff of a reversed covered call is defined as the expected value of  $C_2(T, Y_T)$ .

**Theorem 3.4.** *Expected payoff of a reversed covered call for Liu’s stock model is  $-\int_0^K \text{Cr}\{Y_T \geq r\} dr$ .*

*Proof.* Note that  $C_2(T, Y_T) = -C_1(T, Y_T)$ . By the nonnegativity of  $C_1(T, Y_T)$ , we have

$$\begin{aligned} E[C_2(T, Y_T)] &= - \int_{-\infty}^0 \text{Cr}\{-C_1(T, Y_T) \leq r\} dr \\ &= - \int_0^{\infty} \text{Cr}\{C_1(T, Y_T) \geq r\} dr \\ &= - \int_0^K \text{Cr}\{Y_T \geq r\} dr. \end{aligned}$$

The proof is complete.  $\square$

Theorem 2 implies that  $E[C_2(T, Y_T)] = -E[C_1(T, Y_T)]$ . Thus, the computable forms of  $E[C_2(T, Y_T)]$  can be obtained by equation (6).

### Covered Put

A covered put is a strategy which involves holding a long position in a put option and the underlying stock itself. It is sometimes referred to as a protective put, since, if the stock price drops, then the investor will sell the stock at the strike price. Suppose that  $K$  is the strike price of the put option. Then the payoff of a covered put at time  $T$  is

$$C_3(T, Y_T) = Y_T + (K - Y_T)^+. \quad (7)$$

**Definition 3.5.** The expected payoff of a covered put is defined as the expected value of  $C_3(T, Y_T)$ .

**Theorem 3.6.** *The expected payoff of a covered put for Liu's stock model is  $K + \int_K^{+\infty} \text{Cr}\{Y_T \geq r\}dr$ .*

*Proof.* We note that  $Y_T + (K - Y_T)^+ = \max(Y_T, K) \geq 0$ . Then by equation (3) we have

$$\begin{aligned} E[C_3(T, Y_T)] &= \int_0^{+\infty} \text{Cr}\{\max(Y_T, K) \geq r\}dr \\ &= \int_0^K \text{Cr}\{\max(Y_T, K) \geq r\}dr + \int_K^{+\infty} \text{Cr}\{\max(Y_T, K) \geq r\}dr \\ &= \int_0^K 1dr + \int_K^{+\infty} \text{Cr}\{Y_T \geq r\}dr \\ &= K + \int_K^{+\infty} \text{Cr}\{Y_T \geq r\}dr. \end{aligned}$$

The proof is complete.  $\square$

The integral form of  $E[C_3(T, Y_T)]$  may be obtained by using credibility inversion theorem as follows:

$$E[C_3(T, Y_T)] = K + \int_K^{+\infty} \frac{1}{1 + \exp(\alpha \ln r - \beta)} dr. \quad (8)$$

The value of  $K$  will determine which expression to choose to obtain the required result.

### Reversed Covered Put

A reversed covered put is a strategy which involves holding a short position in a put option and the underlying stock itself. It is the reverse of a covered put. The payoff of a reversed covered put at time  $T$  is

$$C_4(T, Y_T) = -Y_T - (K - Y_T)^+. \quad (9)$$

**Definition 3.7.** The expected payoff of a reversed covered put is defined as the expected value of  $C_4(T, Y_T)$ .

**Theorem 3.8.** *Expected payoff of a reversed covered put for Liu's stock model is  $-K - \int_K^{+\infty} \text{Cr}\{Y_T \geq r\}dr$ .*

*Proof.* Since  $C_4(T, Y_T) = -C_3(T, Y_T)$ , we have

$$\begin{aligned} E[C_4(T, Y_T)] &= - \int_{-\infty}^0 \text{Cr}\{-C_3(T, Y_T) \leq r\}dr \\ &= - \int_0^{+\infty} \text{Cr}\{C_3(T, Y_T) \geq r\}dr \\ &= -K - \int_K^{+\infty} \text{Cr}\{Y_T \geq r\}dr. \end{aligned}$$

The proof is complete. □

We note that  $E[C_4(T, Y_T)] = -E[C_3(T, Y_T)]$ . Thus, the computable expressions of  $E[C_4(T, Y_T)]$  can be obtained by equation (8).

#### 4. Spread Strategies

Spread strategies involve taking a position in two or more options of the same type with different strike prices. There are a number of types of spread strategies in practice. In this section, we study the expected payoffs of three popular spread strategies including bull spread, bear spread and butterfly spread.

##### Bull Spread

A bull spread strategy consists of two types. One is created using call options and the other is created using put options. A bull spread using call options involve buying a call option on a stock with strike price  $K_1$  and selling a call option on the same stock with a higher strike price  $K_2$ . Both call options have the same expiration date. Suppose that  $T$  is the expiration date of the call options. The payoff of the strategy is

$$B_1(T, Y_T) = (Y_T - K_1)^+ - (Y_T - K_2)^+. \tag{10}$$

For a better understanding, the payoff  $B_1(T, Y_T)$  is illustrated in Figure 1 where  $Y_T$  is a deterministic function of  $T$ .

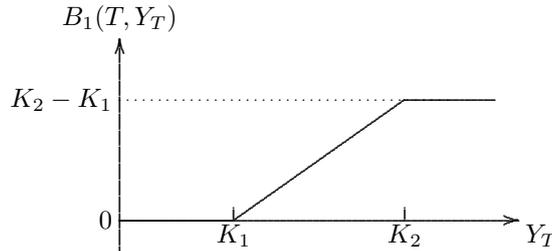


FIGURE 1. The Payoff  $B_1(T, Y_T)$  of Bull Spread Created Using Call Options

**Definition 4.1.** The expected payoff of a bull spread using call options is defined as the expected value of  $B_1(T, Y_T)$ .

**Theorem 4.2.** The expected payoff of a bull spread using call options for Liu's stock model is  $\int_{K_1}^{K_2} \text{Cr}\{Y_T \geq r\} dr$ .

*Proof.* Since  $K_1 < K_2$ , it is easy to prove that  $0 \leq (Y_T - K_1)^+ - (Y_T - K_2)^+ \leq K_2 - K_1$ . It follows from equation (3) that

$$\begin{aligned} E[B_1(T, Y_T)] &= \int_0^{+\infty} \text{Cr}\{(Y_T - K_1)^+ - (Y_T - K_2)^+ \geq r\} dr \\ &= \int_0^{K_2 - K_1} \text{Cr}\{(Y_T - K_1)^+ - (Y_T - K_2)^+ \geq r\} dr. \end{aligned}$$

For any  $0 < r \leq K_2 - K_1$ , we have

$$\begin{aligned} &\{(Y_T - K_1)^+ - (Y_T - K_2)^+ \geq r\} \\ &= \{Y_T \leq K_1 \mid r \leq 0\} \cup \{K_1 < Y_T < K_2 \mid Y_T - K_1 \geq r\} \cup \{Y_T \geq K_2 \mid K_2 - K_1 \geq r\} \\ &= \{K_1 + r \leq Y_T < K_2\} \cup \{Y_T \geq K_2\} \\ &= \{Y_T \geq K_1 + r\}. \end{aligned}$$

Therefore

$$E[B_1(T, Y_T)] = \int_0^{K_2 - K_1} \text{Cr}\{Y_T \geq K_1 + r\} dr = \int_{K_1}^{K_2} \text{Cr}\{Y_T \geq r\} dr.$$

The proof is complete.  $\square$

For given values of  $K_1$  and  $K_2$ , we may obtain the computable forms of  $E[B_1(T, Y_T)]$ . More specifically, we have

$$E[B_1(T, Y_T)] = \int_{K_1}^{K_2} \frac{1}{1 + \exp(\alpha \ln r - \beta)} dr. \quad (11)$$

A bull spread using put options involves buying a put option with a low strike price  $K_1$  and selling a put option with high strike price  $K_2$ . Both put options have the same expiration date  $T$ . The total payoff of a bull spread using put options at time  $T$  is

$$B_2(T, Y_T) = (K_1 - Y_T)^+ - (K_2 - Y_T)^+. \quad (12)$$

**Definition 4.3.** The expected payoff of a bull spread using put options is defined as the expected value of  $B_2(T, Y_T)$ .

**Theorem 4.4.** The expected payoff of a bull spread using put options for Liu's stock model is  $-\int_{K_1}^{K_2} \text{Cr}\{Y_T \leq r\} dr$ .

*Proof.* Since  $K_1 - K_2 \leq B_2(T, Y_T) \leq 0$ , we get

$$E[B_2(T, Y_T)] = - \int_{-\infty}^0 \text{Cr}\{B_2(T, Y_T) \leq r\} dr = - \int_{K_1 - K_2}^0 \text{Cr}\{B_2(T, Y_T) \leq r\} dr.$$

For  $K_1 - K_2 \leq r < 0$ , we have

$$\{B_2(T, Y_T) \leq r\} = \{(K_1 - Y_T)^+ - (K_2 - Y_T)^+ \leq r\} = \{Y_T \leq K_2 + r\}.$$

Therefore,

$$E[B_2(T, Y_T)] = - \int_{K_1 - K_2}^0 \text{Cr}\{Y_T \leq K_2 + r\}dr = - \int_{K_1}^{K_2} \text{Cr}\{Y_T \leq r\}dr.$$

The proof is complete. □

For given values of  $K_1$  and  $K_2$ , we have the following computational form,

$$E[B_2(T, Y_T)] = - \int_{K_1}^{K_2} \frac{\exp(\alpha \ln r - \beta)}{1 + \exp(\alpha \ln r - \beta)} dr. \tag{13}$$

### Bear Spread

Like a bull spread strategy, a bear spread strategy also includes two types, one using call options and the other using put options. A bear spread using call options is a strategy in which an investor sells a call option with strike price  $K_1$  and buys a call option with a higher strike price  $K_2$ . The payoff of this is

$$B_3(T, Y_T) = -(Y_T - K_1)^+ + (Y_T - K_2)^+. \tag{14}$$

A graphical representation of  $B_3(T, Y_T)$  is given in Figure 2 when  $Y_T$  is a deterministic function of  $T$ .

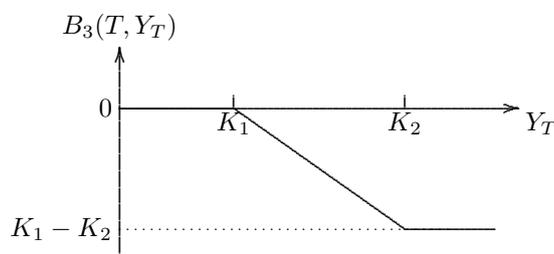


FIGURE 2. The Payoff  $B_3(T, Y_T)$  of Bear Spread Created Using Call Options

**Definition 4.5.** The expected payoff of a bear spread using call options is defined as the expected value of  $B_3(T, Y_T)$ .

**Theorem 4.6.** The expected payoff of a bear spread using call options for Liu's stock model is  $-\int_{K_1}^{K_2} \text{Cr}\{Y_T \geq r\}dr$ .

*Proof.* Since  $K_1 < K_2$ , we have  $-(K_2 - K_1) \leq -(Y_T - K_1)^+ + (Y_T - K_2)^+ \leq 0$ . It follows from the definition of expected value of fuzzy variable that

$$\begin{aligned} E[B_3(T, Y_T)] &= - \int_{-\infty}^0 \text{Cr}\{B_3(T, Y_T) \leq r\} dr \\ &= - \int_{-\infty}^0 \text{Cr}\{-B_3(T, Y_T) \geq -r\} dr \\ &= - \int_0^{+\infty} \text{Cr}\{B_1(T, Y_T) \geq r\} dr \\ &= -E[B_1(T, Y_T)] = - \int_{K_1}^{K_2} \text{Cr}\{Y_T \geq r\} dr. \end{aligned}$$

The proof is complete.  $\square$

Again, like a bull spread using put options, a bear spread using put options is a strategy in which an investor sells a put option with a low strike price  $K_1$  and buys a put option with a higher strike price  $K_2$ . The payoff of a bear spread using put options at time  $T$  is

$$B_4(T, Y_T) = -(K_1 - Y_T)^+ + (K_2 - Y_T)^+. \quad (15)$$

**Definition 4.7.** The expected payoff of a bear spread using put options is defined as the expected value of  $B_4(T, Y_T)$ .

**Theorem 4.8.** *The expected payoff of a bear spread using put options for Liu's stock model is  $\int_{K_1}^{K_2} \text{Cr}\{Y_T \leq r\} dr$ .*

*Proof.* We note that  $B_4(T, Y_T) = -B_2(T, Y_T)$ . Hence

$$\begin{aligned} E[B_4(T, Y_T)] &= \int_0^{+\infty} \text{Cr}\{-B_2(T, Y_T) \geq r\} dr \\ &= \int_{-\infty}^0 \text{Cr}\{B_2(T, Y_T) \leq r\} dr \\ &= - \int_{K_1 - K_2}^0 \text{Cr}\{B_2(T, Y_T) \leq r\} dr \\ &= \int_{K_1}^{K_2} \text{Cr}\{Y_T \leq r\} dr. \end{aligned}$$

The proof is complete.  $\square$

Note that  $E[B_3(T, Y_T)] = -E[B_1(T, Y_T)]$  and  $E[B_4(T, Y_T)] = -E[B_2(T, Y_T)]$ . Thus, the computable expressions of  $E[B_3(T, Y_T)]$  and  $E[B_4(T, Y_T)]$  are immediately obtained.

### Butterfly Spread

A butterfly spread involves buying or selling options with three different strike prices. A butterfly spread using call options involves buying two call options with

strike price  $K_1$  and  $K_2$  respectively and selling two call options with strike price  $(K_1 + K_2)/2$ . Suppose that  $K_1 < K_2$ . The payoff of a butterfly spread strategy using call options at time  $T$  is

$$B_5(T, Y_T) = (Y_T - K_1)^+ + (Y_T - K_2)^+ - 2 \left( Y_T - \frac{K_1 + K_2}{2} \right)^+.$$

A butterfly spread can also be created using put options. It involves buying a put option with a low strike price  $K_1$ , buying a put option with a high strike price  $K_2$  and selling two put options with a strike price  $(K_1 + K_2)/2$ . The payoff of this strategy at time  $T$  is

$$(K_1 - Y_T)^+ + (K_2 - Y_T)^+ - 2 \left( \frac{K_1 + K_2}{2} - Y_T \right)^+$$

which may be proved to be equal to  $B_5(T, Y_T)$ . A graphical representation of  $B_5(T, Y_T)$  is shown in Figure 3.

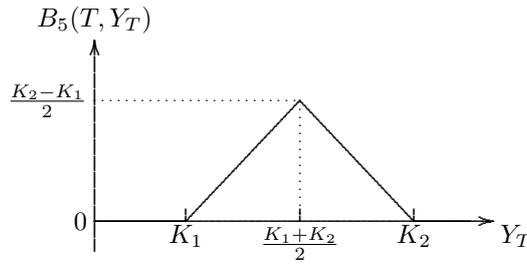


FIGURE 3. The Payoff  $B_5(T, Y_T)$  of Butterfly Spread

**Definition 4.9.** The expected payoff of a butterfly spread using call options is defined as the expected value of  $B_5(T, Y_T)$ .

**Theorem 4.10.** The expected payoff of a bull spread using call options for Liu's stock model is

$$\int_0^{\frac{K_2 - K_1}{2}} \text{Cr}\{K_1 + r \leq Y_T \leq K_2 - r\} dr. \tag{16}$$

*Proof.* First note that  $B_5(T, Y_T)$  is nonnegative. Since the maximum value of  $B_5(T, Y_T)$  is  $(K_2 - K_1)/2$ , we have

$$E[B_5(T, Y_T)] = \int_0^{\frac{K_2 - K_1}{2}} \text{Cr}\{B_5(T, Y_T) \geq r\} dr.$$

For any  $0 \leq r \leq (K_2 - K_1)/2$ ,

$$\begin{aligned}
& \{B_5(T, Y_T) \geq r\} \\
&= \left\{ (K_1 - Y_T)^+ + (K_2 - Y_T)^+ - 2 \left( \frac{K_1 + K_2}{2} - Y_T \right)^+ \geq r \right\} \\
&= \left\{ K_1 \leq Y_T \leq \frac{K_1 + K_2}{2} \mid Y_T - K_1 \geq r \right\} \cup \\
&\quad \left\{ \frac{K_1 + K_2}{2} \leq Y_T \leq K_2 \mid K_2 - Y_T \geq r \right\} \\
&= \{K_1 + r \leq Y_T \leq K_2 - r\}.
\end{aligned}$$

Hence,

$$E[B_5(T, Y_T)] = \int_0^{\frac{K_2 - K_1}{2}} \text{Cr}\{K_1 + r \leq Y_T \leq K_2 - r\} dr.$$

The proof is complete.  $\square$

In particular, if  $K_1 \geq \exp(eT)$ , then

$$E[B_5(T, Y_T)] = - \int_{K_1}^{\frac{K_1 + K_2}{2}} \frac{1}{1 + \exp(\alpha \ln r - \beta)} dr, \quad (17)$$

and if  $K_2 \leq \exp(eT)$ , then

$$E[B_5(T, Y_T)] = - \int_{\frac{K_1 + K_2}{2}}^{K_2} \frac{1}{1 + \exp(\beta - \alpha \ln r)} dr. \quad (18)$$

## 5. Conclusions

In this paper, we have considered the expected payoff of trading strategies for a fuzzy financial market. These strategies include covered strategies involving taking a position in a single option on a stock and the stock itself, and spread strategies involving taking a position in two or more options of the same type. After the proof of each theorem, we have given the integral form of each expected payoff. These integrals are easy to compute by numerical integration techniques.

We have only considered covered strategies and spread strategies using European options. Thus, future work may extend the contribution of the paper in two directions. In one direction, American options, Asian options and other options may be considered for these trading strategies. In another direction, more strategies may be considered such as calendar spread, straddle, strip and strap, strangle and so on.

**Acknowledgements.** This work was partially supported by National Natural Science Foundation of China Grant Nos. 71001003 and 71101007, and Specialized Research Fund for the Doctoral Program of Higher Education of China Grant No. 20101102120022.

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