Iranian Journal of Fuzzy Systems **Vol. 8, No. 3,** (2011) pp. 113-124

FUZZY GOULD INTEGRABILITY ON ATOMS

A. CRISTIANA GAVRILUŢ

Abstract. In this paper we study the relationships existing between total measurability in variation and Gould type fuzzy integrability (introduced and studied in [21]), giving a special interest on their behaviour on atoms and on finite unions of disjoint atoms.

We also establish that any continuous real valued function defined on a compact metric space is totally measurable in the variation of a regular finitely purely atomic multisubmeasure and it is also Gould integrable with respect to regular finitely purely atomic multisubmeasures.

1. Introduction

In the last decades, people began intensively to study the non-additive case, and, recently, the set-valued case, due to their numerous practical applications in many fields, such as, decision theory, artificial intelligence, statistics, sociology, biology, theory of games, economic mathematics. A real interest was given to atoms, pseudo-atoms, (finitely) purely atomicity and non-atomicity for set (multi)functions. We mention here the contributions of Chiţescu [4, 5] (on finitely purely atomic measures), Dobrakov [6], Drewnowski [7] (on submeasures), Pap [18], Jiang, Suzuki [19], Suzuki [22], Wu, Bu [23] (various problems concerning different types of fuzzy (non-additive) set functions).

In the set valued case, many researches were made (see, for instance, Abbas, Imdad and Gopal [1], Alimohammady, Ekici, Jafari and Roohi [2], Altun [3]) and different generalizations of well-known results from the classical (fuzzy) measure theory were obtained.

In the last years, a new type of integral (called the Gould integral) was intensively studied for different types of set (multi)functions: vector valued measures (Gould [16]), multimeasures (Precupanu, Croitoru [20]), multisubmeasures (Gavriluţ [8, 9]), submeasures (Gavriluţ, Petcu [14, 15]), monotone set multifunctions (called fuzzy multimeasures) (Precupanu, Gavriluţ, Croitoru [21]).

In this paper we continue the study of Precupanu, Gavriluţ and Croitoru [21], giving a special interest on atoms, finitely purely atomicity and also on a special type of measurability, called total measurability in variation, which seems to be a useful tool in the set valued subadditive case.

Received: March 2010; Revised: September 2010; Accepted: November 2010

Key words and phrases: Fuzzy Gould integral, Totally measurability (in variation), (Multi) (sub) measure, Atom, Finitely purely atomic, Regularity.

2. **Terminology and Basic Results**

We introduce notations, definitions and results used throughout the paper. Let T be an abstract nonvoid space, $\mathcal{P}(T)$ the family of subsets of T, $\mathcal{C}$ a ring of subsets of T, X a real normed space, $\mathcal{P}_0(X)$ the family of all nonvoid subsets of X, $\mathcal{P}_f(X)$ the family of all nonvoid, closed subsets of X, $\mathcal{P}_{bf}(X)$ the family of all nonvoid, closed, bounded subsets of $X, \mathcal{P}_{kc}(X)$ the family of all nonvoid, compact, convex subsets of X and h the Hausdorff pseudometric on $\mathcal{P}_f(X)$, which becomes a metric on $\mathcal{P}_{bf}(X)$ [17]. If X is a Banach space, then $\mathcal{P}_{bf}(X)$ is a complete metric space [17].

It is known that $h(M,N) = \max\{e(M,N), e(N,M)\}$, where $e(M,N) = \sup\limits_{x\in M} d(x,N)$, for every $M,N \in \mathcal{P}_f(X)$ and

$d(x,N)$ is the distance from x to N induced by the norm of X.

e is called *the excess of M over N*.

We define $|M| = h(M,\{0\})$, for every $M \in \mathcal{P}_f(X)$, where 0 is the origin of X.

On $\mathcal{P}_0(X)$ we consider the Minkowski addition " $\stackrel{\bullet}{+}$ " [17], defined by:

$$M \stackrel{\bullet}{+} N = \overline{M+N}, \text{ for every } M,N \in \mathcal{P}_0(X),$$

where $M+N = \{x+y/x \in M, y \in N\}$ and $\overline{M+N}$ is the closure of $M+N$ with respect to the topology induced by the norm of X.

If $n \in \mathbb{N}^*$, then by $i = \overline{1,n}$, we mean $i \in \{1,2,...,n\}$.

As we shall see, the Minkowski addition will be used in the definition of multi(sub)measures taking values in the family of nonvoid, closed subsets of X, because the classical addition of two closed sets is not, generally, a closed set, too.

Definition 2.1. Let $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ be a set multifunction, with $\mu(\emptyset) = \{0\}$.

I) μ is said to be:

i) *monotone* if $\mu(A) \subseteq \mu(B)$, for every $A,B \in \mathcal{C}$, with $A \subseteq B$;

ii) a *multisubmeasure* [8-13] if it is monotone and

$\mu(A \cup B) \subseteq \mu(A) \stackrel{\bullet}{+} \mu(B)$, for every $A,B \in \mathcal{C}$, with $A \cap B = \emptyset$
(or, equivalently, for every $A,B \in \mathcal{C}$);

iii) a *multimeasure* if $\mu(A\cup B) = \mu(A) \stackrel{\bullet}{+} \mu(B)$, for every $A,B \in \mathcal{C}$, with $A\cap B = \emptyset$.

iv) *null-additive* if $\mu(A \cup B) = \mu(A)$, for every $A,B \in \mathcal{C}$, with $\mu(B) = \{0\}$.

II) [7] A set $A \in \mathcal{C}$ is said to be an *atom* of μ if $\mu(A) \supsetneq \{0\}$ and for every $B \in \mathcal{C}$, with $B \subset A$, we have $\mu(B) = \{0\}$ or $\mu(A \backslash B) = \{0\}$.

III) μ is said to be *finitely purely atomic* if $T = \bigcup\limits_{i=1}^{p} A_i$, where $A_i \in \mathcal{C}$, $i = \overline{1,p}$ are pairwise disjoint atoms of μ (evidently, here, $\mathcal{C}$ has to be an algebra).

Remark 2.2. I) Suppose $m : \mathcal{C} \to \mathbb{R}_+$, with $m(\emptyset) = 0$ is an arbitrary set function.

By the aid of the set multifunction $\mu : \mathcal{C} \to \mathcal{P}_{kc}(\mathbb{R}_+)$ defined for every $A \in \mathcal{C}$ by $\mu(A) = [0, m(A)]$ (called *the induced set multifunction*), and by the above definitions, we immediately obtain the classical corresponding notions for set functions: monotonicity, submeasure, finitely additive (null-additive, respectively) set function, atom, finitely purely atomic set function. Also, one can easily transfer in this manner the results of this paper back to set functions.

II) Any multisubmeasure is null-additive. The converse is not generally valid.

We consider the extended real valued set function $|\mu| : \mathcal{C} \to \overline{\mathbb{R}}_+$ defined for every $A \in \mathcal{C}$ by $|\mu|(A) = |\mu(A)|$.

If T is a locally compact Hausdorff space, we denote by $\mathcal{B}_0$ the Baire δ-ring generated by the G_δ-compact subsets of T (that is, compact sets which are countable intersections of open sets) and by $\mathcal{B}$ the Borel δ-ring generated by the compact subsets of T. If, moreover, T is compact, then $\mathcal{B}$ is an algebra.

We recall a type of regularity that we have defined and studied in [9], [11] and [12] for different types of set multifunctions with respect to the Hausdorff topology induced by the Hausdorff pseudometric h. Suppose T is a locally compact, Hausdorff space.

Definition 2.3. Let $\mu : \mathcal{C} \longrightarrow \mathcal{P}_f(X)$ be a set multifunction, with $\mu(\emptyset) = \{0\}$.

I) μ is said to be *regular* on $A \in \mathcal{C}$ (or A is said to be *regular* with respect to μ) if for every $\varepsilon > 0$, there exists a compact set $K \subset A$, $K \in \mathcal{C}$ such that $|\mu(B)| < \varepsilon$, for every $B \in \mathcal{C}$, $B \subset A\backslash K$.

II) μ is said to be *regular* on $\mathcal{C}$ if μ is regular on every $A \in \mathcal{C}$.

The ring $\mathcal{C}$ may be considered, for instance, $\mathcal{B}_0$ or $\mathcal{B}$.

If μ is monotone, then it is regular on $A \in \mathcal{C}$ if and only if for every $\varepsilon > 0$, there exists a compact set $K \subset A$, $K \in \mathcal{C}$ such that $|\mu(A\backslash K)| < \varepsilon$.

One can easily check the following result which generalizes the corresponding one obtained in [10] for multisubmeasures:

Theorem 2.4. *Let T be a locally compact, Hausdorff space and $\mu : \mathcal{B} \to \mathcal{P}_f(X)$ monotone, null-additive and regular. If $A \in \mathcal{B}$ is an atom of μ, then $\exists! a \in A$ so that $\mu(A\backslash\{a\}) = \{0\}$.*

Remark 2.5. If T is a compact, Hausdorff space, $\mu : \mathcal{B} \to \mathcal{P}_f(X)$ is a finitely purely atomic regular multisubmeasure and $A_i \in \mathcal{B}$, $i = \overline{1,p}$ are pairwise disjoint atoms of μ so that $T = \bigcup\limits_{i=1}^{p} A_i$, then, by Theorem 2.4, $\exists! \{a_1, \ldots, a_p\} \subset T$ so that $\mu(A_i\backslash\{a_i\}) = \{0\}$, for every $i = \overline{1,p}$. Then

$$\begin{aligned} \{0\} \quad &\subseteq \quad \mu(T\backslash\{a_1, \ldots, a_p\}) \subseteq \mu(A_1\backslash\{a_1\}) \stackrel{\bullet}{+} \\ \stackrel{\bullet}{+} \ldots \stackrel{\bullet}{+} \mu(A_p\backslash\{a_p\}) \quad &= \quad \{0\}, \end{aligned}$$

so, $\mu(T\backslash\{a_1, \ldots, a_p\}) = \{0\}$ and this implies $\exists! \{a_1, \ldots, a_p\} \subset T$ such that $\mu(T) = \mu(\{a_1, \ldots, a_p\})$, that is, the measure of the entire space T is precisely known.

In the sequel, $\mathcal{C}$ is a ring of subsets of an abstract space T.

We consider the following extended real valued set functions associated with a set multifunction $\mu : \mathcal{C} \longrightarrow \mathcal{P}_f(X)$, with $\mu(\emptyset) = 0$:

i) *the variation* $\overline{\mu}$ of μ defined by $\overline{\mu}(A) = \sup\{\sum\limits_{i=1}^{n} |\mu(A_i|)\}$, for every $A \subset T$, where the supremum is extended over all finite families of pairwise disjoint sets $\{A_i\}_{i=\overline{1,n}} \subset \mathcal{C}$, with $A_i \subset A$, for every $i = \overline{1,n}$.
μ is said to be *of finite variation* on $\mathcal{C}$ if $\overline{\mu}(A) < \infty$, for every $A \in \mathcal{C}$.

ii) If $\mathcal{C}$ is an algebra,
$\widetilde{\mu}$ defined by
$\widetilde{\mu}(A) = \inf\{\overline{\mu}(B); A \subseteq B, B \in \mathcal{C}\}$, for every $A \subseteq T$.

Remark 2.6. I) i) If μ is of finite variation on $\mathcal{C}$, then $\mu : \mathcal{C} \longrightarrow \mathcal{P}_{bf}(X)$.

ii) $\overline{\mu}$ is monotone and super-additive on $\mathcal{P}(T)$.

II) [10, 11] If μ is a multisubmeasure, then:
i) $|\mu|$ is a submeasure in the sense of Drewnowski [7].

ii) $\overline{\mu}$ is finitely additive on $\mathcal{C}$ and $\overline{\mu}(A) \geq |\mu(A)|$.

iii) $\overline{\mu}(A) = |\mu(A)|$, for every atom $A \in \mathcal{C}$ of μ.

Therefore, if $\mathcal{C}$ is an algebra and $\mu : \mathcal{C} \longrightarrow \mathcal{P}_f(X)$ is a finitely purely atomic multisubmeasure, then $\overline{\mu}(T) = \sum\limits_{i=1}^{p} |\mu(A_i|$, where $A_i \in \mathcal{C}$, $i = \overline{1,p}$ are pairwise disjoint atoms of μ and $T = \bigcup\limits_{i=1}^{p} A_i$.

III) If $\mathcal{C}$ is an algebra, then:
i) $\widetilde{\mu}(A) = \overline{\mu}(A)$, for every $A \in \mathcal{C}$;
ii) If μ is a multisubmeasure, then $\widetilde{\mu}$ is a submeasure on $\mathcal{P}(T)$.

In what follows, let $\mathcal{A}$ be an algebra of subsets of the abstract space T and X be a Banach space.

Definition 2.7. I) A *partition* of T is a finite family $P = \{A_i\}_{i=\overline{1,n}} \subset \mathcal{A}$ such that $A_i \cap A_j = \emptyset, i \neq j$ and $\bigcup\limits_{i=1}^{n} A_i = T$.

II) Let $P = \{A_i\}_{i=\overline{1,n}}$ and $P' = \{B_j\}_{j=\overline{1,m}}$ be two partitions of T. P' is said to be *finer than* P, denoted $P \leq P'$ (or $P' \geq P$), if for every $j = \overline{1,m}$, there exists $i_j = \overline{1,n}$ so that $B_j \subseteq A_{i_j}$.

We denote by $\mathcal{P}$ the class of all partitions of T and if $A \in \mathcal{A}$ is fixed, by $\mathcal{P}_A$, the class of all partitions of A. Suppose $\mu : \mathcal{A} \longrightarrow \mathcal{P}_f(X)$, with $\mu(\emptyset) = \{0\}$ is a set multifunction.

In what follows, $f : T \to \mathbb{R}$ is a real valued, bounded function.

Definition 2.8. [12, 21] I) f is said to be *$\overline{\mu}$-totally-measurable (totally measurable in the variation of μ) on* $(T, \mathcal{A}, \mu)$ if for every $\varepsilon > 0$ there exists a partition $P_\varepsilon = \{A_i\}_{i=\overline{0,n}}$ of T such that:

a) $\overline{\mu}(A_0) < \varepsilon$ and
b) $\sup\limits_{t,s\in A_i} |f(t) - f(s)| < \varepsilon$, for every $i = \overline{1,n}$.

II) f is said to be *$\overline{\mu}$-totally-measurable on* $B \in \mathcal{A}$ if the restriction $f|_B$ of f to B is $\overline{\mu}$-totally measurable on $(B, \mathcal{A}_B, \mu_B)$, where $\mathcal{A}_B = \{A \cap B; A \in \mathcal{A}\}$ and $\mu_B = \mu|_{\mathcal{A}_B}$.

Remark 2.9. If f is $\overline{\mu}$-totally-measurable on T, then f is $\overline{\mu}$-totally-measurable on every $A \in \mathcal{A}$. Moreover,

Proposition 2.10. *Let us consider $A \in \mathcal{A}$ and $\{A_i\}_{i=\overline{1,p}} \subset \mathcal{A}$ so that $A = \bigcup\limits_{i=1}^{p} A_i$.*

If $\mu : \mathcal{A} \longrightarrow \mathcal{P}_f(X)$ is a (multi)(sub)measure, then f is $\overline{\mu}$-totally-measurable on A if and only if it is $\overline{\mu}$-totally-measurable on every $A_i, i = \overline{1,p}$.

Proof. According to the previous remark, the *Only if part* immediately follows.

For the *If part*, one can suppose without loos of generality that $p = 2$. We denote $A_1 = M$ and $A_2 = N$.

Suppose first that $M \cap N = \emptyset$.

By the $\overline{\mu}$-totally-measurability of f on M and N, there are $P_\varepsilon^M = \{M_i\}_{i=\overline{0,n}} \in \mathcal{P}_M$ and $P_\varepsilon^N = \{N_j\}_{i=\overline{0,q}} \in \mathcal{P}_N$ satisfying the conditions of totally measurability in variation.

Since μ is a (multi)(sub)measure, $\overline{\mu}$ is additive on $\mathcal{A}$, so $P_\varepsilon^{M\cup N} = \{M_0 \cup N_0, M_1, ..., M_n, N_1, ..., N_q\} \in \mathcal{P}_{M\cup N}$ also satisfies the conditions of totally measurability in variation.

Consequently, f is $\overline{\mu}$-totally-measurable on $M \cup N$.

If $M \cap N \neq \emptyset$, since $M \cup N = (M\backslash N) \cup N$ and $\overline{\mu}$-totally-measurability is hereditary, the statement easily follows. $\square$

The reader is refereed to [8, 9, 12] for other properties of totally measurability in variation.

Also, it is easy to observe that f is $\overline{\mu}$-totally-measurable if and only if f is $\overline{\lambda\mu}$-totally-measurable, for every $\lambda \in \mathbb{R}$.

In the following, we establish that, in some conditions, continuity implies totally-measurability in variation:

Theorem 2.11. *i)* [12] *Suppose T is a compact, metric space, $f : T \to \mathbb{R}$ is continuous on T and $\mu : \mathcal{B} \to \mathcal{P}_f(X)$ is regular, null-additive and monotone.*

If μ has atoms, then f is $\overline{\mu}$-totally-measurable on every atom $A_0 \in \mathcal{B}$ of μ.

ii) If, moreover, μ is a finitely purely atomic multisubmeasure, then f is $\overline{\mu}$-totally-measurable on T.

Proof. ii) If μ is a finitely purely atomic multisubmeasure, then, according to Proposition 2.10 and Remark 2.2-II), f is $\overline{\mu}$-totally-measurable on T. $\square$

In what follows, without any special assumptions, $\boldsymbol{\mu : \mathcal{A} \to \mathcal{P}_f(X)}$ *will be* **a monotone set multifunction, with $\boldsymbol{\mu(\emptyset) = 0}$**, *of finite variation. As before,* $\boldsymbol{f : T \to} \mathbb{R}$ *will be* **a real valued, bounded function**, *T being an abstract space.*

$\sigma(P)$ denotes $\sum\limits_{i=1}^{n} f(t_i)\mu(A_i)$, for every partition $P = \{A_i\}_{i=\overline{1,n}}$ of T and every $t_i \in A_i, i = \overline{1,n}$.

Definition 2.12. [21] I) The function f is said to be *μ-integrable on T* if the net $(\sigma(P))_{P\in(\mathcal{P},\leq)}$ is convergent in $\mathcal{P}_f(X)$, where $\mathcal{P}$, the set of all partitions of T, is ordered by the relation " $\leq$ " given in Definition 2.7-II).

If $(\sigma(P))_{P\in(\mathcal{P},\leq)}$ is convergent, then its limit is called *the integral of f on T with respect to μ*, denoted by $\int_T f d\mu$.

II) If $B \in \mathcal{A}, f$ is said to be *μ-integrable on B* if the restriction $f|_B$ of f to B is μ-integrable on $(B, \mathcal{A}_B, \mu_B)$.

Remark 2.13. f is μ-integrable on T if and only if there is $I \in \mathcal{P}_f(X)$ such that for every $\varepsilon > 0$, there exists a partition P_ε of T, so that for every other partition of T, $P = \{A_i\}_{i=\overline{1,n}}$, with $P \geq P_\varepsilon$ and every choice of points $t_i \in A_i, i = \overline{1,n}$, we have $h(\sigma(P), I) < \varepsilon$.

Theorem 2.14. *Suppose $\mu : \mathcal{A} \to \mathcal{P}_f(X)$ is monotone and null-additive. If $A \in \mathcal{A}$ is an atom of μ and if f is bounded and $\overline{\mu}$-totally-measurable on A, then f is μ-integrable on A.*

Proof. First, we observe that, if A is an atom of μ and if $\{A_i\}_{i=\overline{1,n}} \in \mathcal{P}_A$, then, there exists only one set, for instance, without loss of generality, A_1, so that $\mu(A_1) \supsetneqq \{0\}$ and $\mu(A_2) = ... = \mu(A_n) = \{0\}$.

Let $A \in \mathcal{A}$ be an atom of μ.

Since f is $\overline{\mu}$-totally-measurable on A, then for every $\varepsilon > 0$ there exists a partition $P_\varepsilon = \{A_i\}_{i=\overline{0,n}}$ of A such that:

$$\begin{cases} i)\ \overline{\mu}(A_0) < \frac{\varepsilon}{2M} \text{ (where } M = \sup\limits_{t\in T}|f(t)| > 0, \text{ - otherwise,} \\ f \equiv 0 \text{ on } T \text{ and the proof finishes) and} \\ ii)\ \sup\limits_{t,s\in A_i} |f(t) - f(s)| < \frac{\varepsilon}{\overline{\mu}(T)} \text{ (where } \overline{\mu}(T) > 0, \text{ - otherwise,} \\ \overline{\mu}(T) = 0, \text{ so } \mu(A) = \{0\} \text{ , for every } A \in \mathcal{A}, \text{ and,} \\ \text{consequently, there are no atoms } A \in \mathcal{A} \text{ of } \mu), \\ \text{for every } i = \overline{1,n}. \end{cases}$$

Let $\{B_j\}_{j=\overline{1,k}}$, $\{C_p\}_{p=\overline{1,s}} \in \mathcal{P}_A$ be two arbitrary partitions which are finer than P_ε and consider $s_j \in B_j, j = \overline{1,k}, \theta_p \in C_p, p = \overline{1,s}$.

We prove that

$$h(\overset{\bullet}{\sum_{j=1}^{k}} f(s_j)\mu(B_j), \overset{\bullet}{\sum_{p=1}^{s}} f(\theta_p)\mu(C_p)) < \varepsilon.$$

We have two cases:

I. $\mu(A_0) \supsetneq \{0\}$. Then $\mu(A_1) = ... = \mu(A_n) = \{0\}$.

Suppose, without loss of generality that $\mu(B_1) \supsetneq \{0\}$, $\mu(C_1) \supsetneq \{0\}$ and $\mu(B_2) = ... = \mu(B_k) = \{0\}, \mu(C_2) = ... = \mu(C_s) = \{0\}$. Then $B_1 \subset A_0$ and $C_1 \subset A_0$. Consequently,

$$\begin{aligned} h(\overset{\bullet}{\sum_{j=1}^{k}} f(s_j)\mu(B_j), \overset{\bullet}{\sum_{p=1}^{s}} f(\theta_p)\mu(C_p)) &= h(f(s_1)\mu(B_1), f(\theta_1)\mu(C_1)) \leq \\ &\leq |f(s_1)||\mu(B_1)| + |f(\theta_1)||\mu(C_1)| \leq \\ &\leq 2M\overline{\mu}(A_0) < \varepsilon. \end{aligned}$$

II. $\mu(A_0) = \{0\}$. Then, without loss of generality, $\mu(A_1) \supsetneq \{0\}$ and $\mu(A_i) = \{0\}$, for every $i = \overline{2, n}$. Suppose that $\mu(B_1) \supsetneq \{0\}$, $\mu(C_1) \supsetneq \{0\}$ and $\mu(B_2) = ... = \mu(B_k) = \{0\}, \mu(C_2) = ... = \mu(C_s) = \{0\}$. Then $B_1 \subset A_1$ and $C_1 \subset A_1$, and, therefore,

$$h(\overset{\bullet}{\sum_{j=1}^{k}} f(s_j)\mu(B_j), \overset{\bullet}{\sum_{p=1}^{s}} f(\theta_p)\mu(C_p)) = h(f(s_1)\mu(B_1), f(\theta_1)\mu(C_1)).$$

Since A is an atom of μ and $\mu(B_1) \supsetneq \{0\}$, then $\mu(A \backslash B_1) = \{0\}$, so $\mu(C_1 \backslash B_1) = \{0\}$. By the null-additivity of μ, we get $\mu(C_1) = \mu(B_1)$. Then

$$\begin{aligned} h(\overset{\bullet}{\sum_{j=1}^{k}} f(s_j)\mu(B_j), \overset{\bullet}{\sum_{p=1}^{s}} f(\theta_p)\mu(C_p)) &= h(f(s_1)\mu(B_1), f(\theta_1)\mu(C_1)) = \\ &= h(f(s_1)\mu(B_1), f(\theta_1)\mu(B_1)). \end{aligned}$$

Because, generally, $h(\alpha M, \beta M) \leq |\alpha - \beta||M|$, for every $\alpha, \beta \in \mathbb{R}$ and every $M \in \mathcal{P}_f(X)$, we have

$$h(\overset{\bullet}{\sum_{j=1}^{k}} f(s_j)\mu(B_j), \overset{\bullet}{\sum_{p=1}^{s}} f(\theta_p)\mu(C_p)) \leq |\mu(B_1)||f(s_1) - f(\theta_1)| \leq \overline{\mu}(T)\frac{\varepsilon}{\overline{\mu}(T)} = \varepsilon.$$

Therefore, $(\sigma(P))_{P \in \mathcal{P}_A}$ is a Cauchy net in the complete metric space $(\mathcal{P}_{bf}(X), h)$, and so f is μ-integrable on A. □

By the properties of fuzzy Gould type integrability [21] and by the above theorem, we get:

Corollary 2.15. *If $\mu : \mathcal{A} \to \mathcal{P}_f(X)$ is finitely purely atomic, monotone and null-additive, then every bounded function $f : T \to \mathbb{R}$ which is $\overline{\mu}$-totally-measurable on atoms, is μ-integrable on T.*

If, moreover, $\mu : \mathcal{A} \to \mathcal{P}_{kc}(X)$, then f is μ-integrable on every set $A \in \mathcal{A}$.

Also, by [21], Theorem 2.11 and Theorem 2.14, we obtain:

Corollary 2.16. *If T is a compact metric space, $f : T \to \mathbb{R}$ is continuous on T and $\mu : \mathcal{B} \to \mathcal{P}_f(X)$ is finitely purely atomic, monotone, null-additive and regular, then f is μ-integrable on T.*

If, moreover, $\mu : \mathcal{B} \to \mathcal{P}_{kc}(X)$, then f is μ-integrable on every $A \in \mathcal{B}$.

In the end of this section, we give several examples:

Example 2.17. i) If $m_1, m_2 : \mathcal{C} \to \mathbb{R}_+$, m_1 is a finitely additive set function and m_2 is a submeasure (finitely additive set function, respectively), then the set multifunction $\mu : \mathcal{C} \to \mathcal{P}_f(\mathbb{R}_+)$, defined for every $A \in \mathcal{C}$ by $\mu(A) = [-m_1(A), m_2(A)]$, is a multisubmeasure (multimeasure, respectively).

ii) If $T = \{t, s\}$ and $\mathcal{C} = \mathcal{P}(T)$, then the set multifunction $\mu : \mathcal{C} \to \mathcal{P}_f(\mathbb{R}_+)$, defined by $\mu(T) = [0, 4], \mu(\emptyset) = \{0\}$ and $\mu(\{t\}) = \mu(\{s\}) = [0, 1]$ is null-additive and it is not a multisubmeasure/multimeasure.

iii) If $\mathcal{C}$ is a ring of subsets of $T, m : \mathcal{C} \to \mathbb{R}_+$ is finitely additive, then $\mu : \mathcal{C} \to \mathcal{P}_f(\mathbb{R}_+)$, defined for every $A \in \mathcal{C}$ by $\mu(A) = \begin{cases} [-m(A), m(A)], \text{ if } m(A) \leq 1 \\ [-m(A), 1], \text{ if } m(A) > 1 \end{cases}$

is a multisubmeasure.

iv) If $\mathcal{C}$ is a ring of subsets of $T, m : \mathcal{C} \to \mathbb{R}_+$ is a set function with $m(\emptyset) = \{0\}$ and $\mu : \mathcal{C} \to \mathcal{P}_{kc}(\mathbb{R}_+)$ is the induced set multifunction defined for every $A \in \mathcal{C}$ by $\mu(A) = [0, m(A)]$, then A is an atom of μ if and only if A is an atom of m.

v) If T is a countable set and $\mathcal{C} = \{A \subset T; A \text{ is finite or } cA \text{ is finite}\}$, then for the multisubmeasure $\mu : \mathcal{C} \to \mathcal{P}_f(\mathbb{R}_+)$, defined for every $A \in \mathcal{C}$ by $\mu(A) = \begin{cases} \{0\}, \text{ if } A \text{ is finite} \\ \{0, 1\}, \text{ if } cA \text{ is finite,} \end{cases}$

every $A \in \mathcal{C}$ where cA is finite is an atom of μ.

3. Gould Integrability on Atoms

In this section we point out several relationships existing between Gould integrability and total measurability in variation on atoms. We calculate the corresponding integrals on atoms and on finite unions of disjoint atoms.

Unless stated otherwise, let $\mathcal{A}$ be an algebra of subsets of an abstract space T, X be a Banach space, $\mu : \mathcal{A} \to \mathcal{P}_f(X)$ be a monotone set multifunction of finite variation, with $\mu(\emptyset) = \{0\}$ and $f : T \to \mathbb{R}$ be a bounded function.

Theorem 3.1. *Suppose T is a compact Hausdorff space and $\mu : \mathcal{B} \to \mathcal{P}_f(X)$ is monotone, null-additive and regular.*

If μ has atoms and if f is μ-integrable on an atom $A_0 \in \mathcal{B}$ of μ, then $\exists! a_0 \in A_0$ so that $\mu(A_0 \backslash \{a_0\}) = \{0\}$ and, in this case,

$$\int_{A_0} f d\mu = f(a_0)\mu(A_0).$$

Proof. By the μ-integrability of f on A_0, for every $\varepsilon > 0$, there is $P_\varepsilon = \{B_j\}_{j=\overline{1,k}}$ $\in \mathcal{P}_{A_0}$ so that for every $t_j \in B_j, j = \overline{1,k}$, we have $h(\int_{A_0} f d\mu, \sum\limits_{j=1}^{k} f(t_j)\mu(B_j)) < \varepsilon$.

Since $A_0 \in \mathcal{B}$ is an atom of μ, we may suppose without loss of generality that $\mu(B_1) = \mu(A_0)$ and $\mu(B_2) = ... = \mu(B_k) = \{0\}$.

By Theorem 2.3, $\exists! a_0 \in A_0$ so that $\mu(A_0 \backslash \{a_0\}) = \{0\}$. By the null-additivity, we also have $\mu(A_0) = \mu(\{a_0\})$.

If $a_0 \notin B_1$, then suppose, for instance, that $a_0 \in B_2$. In this case, by the monotonicity of μ and since $\mu(B_2) = \{0\}$, we obtain $\mu(\{a_0\}) = \{0\}$, so $\mu(A_0) = \{0\}$, which is a contradiction. Consequently, $a_0 \in B_1$.

Let then, particularly, $t_1 = a_0$. We get $h(\int_{A_0} f d\mu, f(a_0)\mu(A_0)) < \varepsilon$, for every $\varepsilon > 0$, so finally $\int_{A_0} f d\mu = f(a_0)\mu(A_0)$. □

Corollary 3.2. (A Lebesgue Type Theorem) *Suppose T is a compact Hausdorff space and $\mu : \mathcal{B} \to \mathcal{P}_f(X)$ is monotone, null-additive and regular.*

If for every $n \in \mathbb{N}$, $f_n, f : T \to \mathbb{R}$ are μ-integrable on an atom $A_0 \in \mathcal{B}$ of μ and if $(f_n)_n$ pointwise converges to f, then

$$\lim_{n\to\infty} \int_{A_0} f_n d\mu = \int_{A_0} f d\mu \text{ (with respect to } h).$$

Proof. By Theorem 3.1, $\exists! a_0 \in A_0$ so that $\mu(A_0 \backslash \{a_0\}) = \{0\}$ and, in this case, for every $n \in \mathbb{N}$, $\int_{A_0} f_n d\mu = f_n(a_0)\mu(A_0)$ and $\int_{A_0} f d\mu = f(a_0)\mu(A_0)$.

Since in fact $\mu : \mathcal{B} \to \mathcal{P}_{bf}(X)$, we have $\lim\limits_{n\to\infty} f_n(a_0)\mu(A_0) = f(a_0)\mu(A_0)$, so the conclusion holds. □

By Theorem 3.1 and since fuzzy Gould type integral [21] is finitely additive and hereditary on $\mathcal{A}$ when μ is $\mathcal{P}_{kc}(X)$-valued, we obtain the following:

Corollary 3.3. *If T is a compact Hausdorff space, $\mu : \mathcal{B} \to \mathcal{P}_{kc}(X)$ is finitely purely atomic, monotone, null-additive (with $T = \bigcup\limits_{i=1}^{p} A_i$, where $A_i \in \mathcal{B}$, $i = \overline{1,p}$ are pairwise disjoint atoms of μ) and regular and if $f : T \to \mathbb{R}$ is μ-integrable on T, then for every $i = \overline{1,p}$, $\exists! a_i \in A_i$ so that $\mu(A_i \backslash \{a_i\}) = \{0\}$ and, in this case,*

$$\int_T f d\mu = f(a_1)\mu(A_1) + ... + f(a_p)\mu(A_p).$$

Proposition 3.4. *Suppose $\mu : \mathcal{A} \to \mathcal{P}_{kc}(X)$ is a multisubmeasure of finite variation, with $\mu(\emptyset) = \{0\}$ and $f : T \to \mathbb{R}$ is μ-integrable on T.*

Let $M : \mathcal{A} \to \mathcal{P}_{kc}(X)$ be defined for every $A \in \mathcal{A}$ by $M(A) = \int_A f d\mu$ (by [21], M is a monotone multimeasure).

If f is $\overline{\mu}$-totally-measurable, then f is $\overline{M}$-totally-measurable.

Proof. The statement easily follows since $|M(A)| \leq K\overline{\mu}(A)$, for every $A \in \mathcal{A}$, where $K = \sup\limits_{t\in T}|f(t)|$. Therefore, if $\{B_j\}_{j=\overline{1,l}} \in \mathcal{P}_A$ is arbitrary, we have:

$$\sum_{j=1}^{l}|M(B_j)| \leq K\sum_{j=1}^{l}\overline{\mu}(B_j) = K\overline{\mu}(A),$$

whence $\overline{M}(A) \leq K\overline{\mu}(A)$, for every $A \in \mathcal{A}$, so the conclusion follows. □

In the sequel, we shall discuss total measurability in variation and Gould integrability for monotone set functions $m : \mathcal{A} \to \mathbb{R}_+$:

Remark 3.5. By the definitions, one can easily obtain the following results:

Let $m : \mathcal{A} \to \mathbb{R}_+$, with $m(\emptyset) = 0$ be an arbitrary monotone set function of finite variation and $f : T \to \mathbb{R}$ be a bounded function.

Consider the induced monotone set multifunction $\mu : \mathcal{A} \to \mathcal{P}_f(\mathbb{R})$, of finite variation, with $\mu(\emptyset) = \{0\}$, defined by $\mu(A) = [0, m(A)]$, for every $A \in \mathcal{A}$.

We observe that $\overline{\mu}(A) = \overline{m}(A)$.

The definition of $\overline{m}$-total-measurability is then the following:

f is said to be:

i) *$\overline{m}$-totally-measurable on* $(T, \mathcal{A}, \mu)$ if for every $\varepsilon > 0$ there exists a partition $P_\varepsilon = \{A_i\}_{i=\overline{0,n}}$ of T such that:

a) $\overline{m}(A_0) < \varepsilon$ and

b) $\sup\limits_{t,s\in A_i} |f(t) - f(s)| < \varepsilon$, for every $i = \overline{1,n}$.

ii) *$\overline{m}$-totally-measurable on* $B \in \mathcal{A}$ if the restriction $f|_B$ of f to B is $\overline{m}$-totally measurable on $(B, \mathcal{A}_B, m_B)$, where $\mathcal{A}_B = \{A \cap B; A \in \mathcal{A}\}$ and $m_B = m|_{\mathcal{A}_B}$.

We remark that f is $\overline{m}$-totally-measurable if and only if f is $\overline{\mu}$-totally-measurable.

We also observe that f is m-integrable on T if and only if there is $I \in \mathbb{R}$ such that for every $\varepsilon > 0$, there exists a partition P_ε of T, so that for every other partition of T, $P = \{A_i\}_{i=\overline{1,n}}$, with $P \geq P_\varepsilon$ and every choice of points $t_i \in A_i, i = \overline{1,n}$, we have $|\sigma(P) - I| = |\sum\limits_{i=1}^{n} f(t_i)m(A_i) - I| < \varepsilon$. Here, $I = \int_T fdm$.

One can easily check that on every $A \in \mathcal{A}$, f is m-integrable if and only if it is μ-integrable and in this case,

$$\int_A fd\mu = [0, \int_A fdm].$$

One may immediately derive by [21] the properties of fuzzy Gould type integrability with respect to a monotone set function.

It would be interesting to study the relationships among this types of integrability and other fuzzy integrals.

In what follows, we investigate the relationships between total measurability in variation and Gould integrability for different types of set functions:

Theorem 3.6. *Suppose $m : \mathcal{A} \to \mathbb{R}_+$ is a monotone, null-additive set function of finite variation and $f : T \to \mathbb{R}$ is a bounded function. If $A \in \mathcal{A}$ is an atom of m and if f is m-integrable on A, then f is $\overline{m}$-totally-measurable on A.*

Proof. Because f is m-integrable on A, then for every $\varepsilon > 0$, there exists $P_\varepsilon = \{A_i\}_{i=\overline{1,n}} \in \mathcal{P}_A$ so that for every other $\{B_j\}_{j=\overline{1,p}} \geq P_\varepsilon$ and every $t_j, s_j \in B_j$, we have

$$|\sum_{j=1}^{p} f(t_j)m(B_j) - \sum_{j=1}^{p} f(s_j)m(B_j)| < \varepsilon.$$

On the other hand, since $A \in \mathcal{A}$ is an atom of m, then we may suppose, without loss of generality that $m(A \backslash B_1) = 0, m(B_2) = m(B_3) = ... = m(B_p) = 0$.

Consequently, for every $\varepsilon > 0$ and every $t, s \in B_1, |f(t) - f(s)|m(A) < \varepsilon$, so $\sup\limits_{t,s \in B_1} |f(t) - f(s)| \leq \frac{\varepsilon}{m(A)}$.

Also, $\overline{m}(A \backslash B_1) = 0$ because $m(A \backslash B_1) = 0$ and this means that f is $\overline{m}$-totally-measurable on A. □

By Remark 3.5, Theorem 3.6 and Theorem 2.14, the last one applied for the induced set multifunction, we get:

Corollary 3.7. *Suppose $m : \mathcal{A} \to \mathbb{R}_+$ is monotone, null-additive and of finite variation. If $f : T \to \mathbb{R}$ is a bounded function, then on every atom $A \in \mathcal{A}$ of m, f is $\overline{m}$-totally-measurable if and only if f is m-integrable.*

Let us recall that for submeasures, total-measurability in variation and Gould integrability are equivalent on any subset of $\mathcal{A}$, not only on atoms, as we obtained here for monotone, null-additive set functions, which are more general than submeasures.

Remark 3.8. [14] If $m : \mathcal{A} \to \mathbb{R}_+$ is a submeasure of finite variation and if $f : T \to \mathbb{R}$ is a bounded function, then, on any $A \in \mathcal{A}$, f is $\overline{m}$-totally-measurable if and only if f is m-integrable and, moreover, $\int_A f dm = \int_A f d\overline{m}$.

4. **Concluding Remarks**

In this paper, we study total measurability in variation, Gould integrability and we calculate the corresponding integrals on atoms and on finite unions of pairwise disjoint atoms. We also prove that, under some conditions, regular finitely purely atomic multisubmeasures provide remarkable properties such as, total measurability in variation and Gould type integrability. It is our intention to continue the study of total measurability in variation and to establish relationships with other types of measurability. Also, it would be interesting to compare the Gould type integral with other types of well-known fuzzy integrals, for instance, with the Choquet one.

Acknowledgements. The author is very grateful to the referees for their valuable suggestions in improving the paper.

References

[1] M. Abbas, M. Imdad and D. Gopal, *$\varepsilon-$weak contractions in fuzzy metric spaces*, Iranian Journal of Fuzzy Systems, in print.

[2] M. Alimohammady, E. Ekici, S. Jafari and M. Roohi, *On fuzzy upper and lower contra-continuous multifunctions*, Iranian Journal of Fuzzy Systems, in print.

[3] I. Altun, *Some fixed point theorems for single and multivalued mappings on ordered non-archimedean fuzzy metric spaces*, Iranian Journal of Fuzzy Systems, **7(1)** (2010), 91-96.

[4] I. Chiţescu, *Finitely purely atomic measures and L_p-spaces*, An. Univ. Bucureşti Şt. Natur., **24** (1975), 23-29.

[5] I. Chiţescu, *Finitely purely atomic measures: coincidence and rigidity properties*, Rend. Circ. Mat. Palermo(2), **50(3)** (2001), 455-476.

[6] I. Dobrakov, *On submeasures, I*, Dissertationes Math., **112** (1974), 5-35.

[7] L. Drewnowski, *Topological rings of sets, continuous set functions, Integration*, I, II, III, Bull. Acad. Polon. Sci. Sér. Math. Astron. Phys., **20** (1972), 269-286.

[8] A. Gavriluţ, *A Gould type integral with respect to a multisubmeasure*, Math. Slovaca, **58(1)** (2008), 1-20.

[9] A. Gavriluţ, *On some properties of the Gould type integral with respect to a multisubmeasure*, An. Şt. Univ. Iaşi, **52(1)** (2006), 177-194.

[10] A. Gavriluţ, *Non-atomicity and the Darboux property for fuzzy and non-fuzzy Borel/Baire multivalued set functions*, Fuzzy Sets and Systems, **160** (2009), 1308-1317.

[11] A. Gavriluţ, *Regularity and autocontinuity of set multifunctions*, Fuzzy Sets and Systems, **160** (2009), 681-693.

[12] A. Gavriluţ, *A Lusin type theorem for regular monotone uniformly autocontinuous set multifunctions*, Fuzzy Sets and Systems, **161** (2010), 2909-2918.

[13] A. Gavriluţ and A. Croitoru, *Non-atomicity for fuzzy and non-fuzzy multivalued set functions*, Fuzzy Sets and Systems, **160** (2009), 2106-2116.

[14] A. Gavriluţ and A. Petcu, *A Gould type integral with respect to a submeasure*, An. Şt. Univ. Iaşi, Tomul LIII, **2** (2007), 351-368.

[15] A. Gavriluţ and A. Petcu, *Some properties of the Gould type integral with respect to a submeasure*, Bul. Inst. Pol. Iaşi, LIII (LVII), **5** (2007), 121-131.

[16] G. G. Gould, *On integration of vector-valued measures*, Proc. London Math. Soc., **15** (1965), 193-225.

[17] S. Hu and N. S. Papageorgiou, *Handbook of multivalued analysis*, Kluwer Acad. Publ., Dordrecht, **I** (1997).

[18] E. Pap, *Null-additive set functions*, Kluwer Academic Publishers, Dordrecht-Boston-London, 1995.

[19] Q. Jiang and H. Suzuki, *Fuzzy measures on metric spaces*, Fuzzy Sets and Systems, **83** (1996), 99-106.

[20] A. Precupanu and A. Croitoru, *A Gould type integral with respect to a multimeasure, I*, An. Şt. Univ. Iaşi, **48** (2002), 165-200.

[21] A. Precupanu, A. Gavriluţ and A. Croitoru, *A fuzzy Gould type integral*, Fuzzy Sets and Systems, **161** (2010), 661-680.

[22] H. Suzuki, *Atoms of fuzzy measures and fuzzy integrals*, Fuzzy Sets and Systems, **41** (1991), 329-342.

[23] C. Wu and S. Bo, *Pseudo-atoms of fuzzy and non-fuzzy measures*, Fuzzy Sets and Systems, **158** (2007), 1258-1272.

Alina Cristiana Gavriluţ, Faculty of Mathematics, Al. I. Cuza University, Iaşi, Romania

E-mail address: gavrilut@uaic.ro