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ON GENERAL FUZZY RECOGNIZERS

M. HORRY AND M. M. ZAHEDI

Abstract. In this paper, we define the concepts of general fuzzy recognizer, language recognized by a general fuzzy recognizer, the accessible and the coaccessible parts of a general fuzzy recognizer and the reversal of a general fuzzy recognizer. Then we obtain the relationships between them and construct a topology and some hypergroups on a general fuzzy recognizer.

1. **Introduction and Preliminaries**

The concept of fuzzy automata was introduced by Wee in 1967 [16]. Wee and Fu [17] first introduced fuzzy recognizers with a non-fuzzy initial states, non-fuzzy final states and non-fuzzy inputs.
Let Σ be a set. A word in Σ is the product of a finite sequence of elements in Σ. Λ will denote the empty word and Σ^* the set of all words on Σ. The length $\ell(x)$ of the word $x \in \Sigma^*$ is the number of its letters, so $\ell(\Lambda) = 0$. For a nonempty set X, $\tilde{P}(X)$ will denote the set of all fuzzy sets on X and $P(X)$ will denote the set of all subsets on X.
A deterministic finite-state automaton is a five-tuple denoted as $A = (Q, \Sigma, f, T, s)$, where Q is a finite set of states, Σ is a finite set of input symbols, f is the state transition function, from $Q \times \Sigma$ into Q T is a subset of Q of accepting states and $s \in Q$ is the initial state.
A word $x = x_1x_2 \dots x_n \in \Sigma^*$ is said to be accepted by A if there exist states $q_0, q_1, \dots, q_n$ satisfying

(1) $q_0 = s$
(2) $f(q_{i-1}, x_i) = q_i$ for $i = 1, 2, \dots, n$,
(3)$q_n \in T$.

The empty word is accepted by A if and only if $s \in T$.
A nondeterministic finite-state automaton is a five-tuple denoted as $A = (Q, \Sigma, f, T, s)$, where Q is a finite set of states, Σ is a finite set of input symbols, f is the state transition function from $Q \times \Sigma$ into $P(Q)$, T is a subset of Q of accepting states and $s \in Q$ is the initial state.
A fuzzy finite-state automaton (FFA) is a six-tuple $\tilde{F} = (Q, \Sigma, R, Z, \delta, \omega)$, where Q is a finite set of states, Σ is a finite set of input symbols, R is the initial state of

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$\tilde{F}$, Z is a finite set of output symbols, $\delta : Q \times \Sigma \times Q \to [0,1]$ is the fuzzy transition function which is used to map a state (current state) into another state (next state) upon an input symbol, attributing a value in the interval $[0,1]$ and $\omega : Q \to Z$ is the output function. Associated with each fuzzy transition, there is a membership value in $[0,1]$ called the weight of the transition. Transition from state q_i (current state) to state q_j (next state) upon input a_k is denoted by $\delta(q_i, a_k, q_j)$.

We use $\delta(q_i, a_k, q_j)$ to refer both to a transition and its weight in the sense that whenever $\delta(q_i, a_k, q_j)$ is used as a value, it refers to the weight of the transition, and otherwise, specifies the transition itself. The set of all transitions of $\tilde{F}$ will be denoted by Δ. The above definition is generally accepted as a formal definition of an FFA [10], [13], [14].

In 2004, M. Doostfatemeh and S.C. Kremer extended the notion of fuzzy automata and introduced the notion of general fuzzy automata [6].

Santos [15] and Mizumoto [12] studied various fuzzy recognizers (automata) with fuzzy initial states, non-fuzzy final states and non-fuzzy inputs.

Malik, Mordeson and Sen introduced fuzzy recognizers in [11] with fuzzy initial states, fuzzy final states and non-fuzzy inputs.

Kumbhojkar and Chaudhari characterized words recognized by fuzzy recognizer and proved fuzzy recognizability of several crisp sets [9].

In this paper, by using [1],[2],[3],[7] and [9], we extend the notion of fuzzy recognizer and introduce the notion of general fuzzy recognizer and construct a topology and some hypergroups on a general fuzzy recognizer.

Definition 1.1. [6] A general fuzzy automaton (GFA) is an eight-tuple machine $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2)$, where

(i) Q is a finite set of states, $Q = \{q_1, q_2, \ldots, q_n\}$,
(ii) Σ is a finite set of input symbols, $\Sigma = \{a_1, a_2, \ldots, a_m\}$,
(iii) $\tilde{R}$ is the set of fuzzy start states, $\tilde{R} \in \tilde{P}(Q)$,
(iv) Z is a finite set of output symbols, $Z = \{b_1, b_2, \ldots, b_k\}$,
(v) $\omega : Q \to Z$ is the output function,
(vi) $\tilde{\delta} : (Q \times [0,1]) \times \Sigma \times Q \to [0,1]$ is the augmented transition function,
(vii) $F_1 : [0,1] \times [0,1] \to [0,1]$ is the membership assignment function,
(viii) $F_2 : [0,1]^* \to [0,1]$ is called the multi-membership resolution function.

We note that the function $F_1(\mu, \delta)$ has two parameters μ and δ, where μ is the membership value of a predecessor and δ is the weight of a transition. In this definition, the process that takes place upon the transition from state q_i to q_j on input a_k is represented as:

$$\mu^{t+1}(q_j) = \tilde{\delta}((q_i, \mu^t(q_i)), a_k, q_j) = F_1(\mu^t(q_i), \delta(q_i, a_k, q_j)).$$

This means that the membership value (mv) of the state q_j at time $t+1$ is computed by the function F_1 using both the membership value of q_i at time t and the weight of the transition.

The usual options for the function $F_1(\mu, \delta)$ are $\max\{\mu, \delta\}$, $\min\{\mu, \delta\}$ and $(\mu + \delta)/2$.

The multi-membership resolution function resolves the multi-membership active states and assigns a single membership value to them.

Let $Q_{act}(t_i)$ be the fuzzy set of all active states at time t_i, $\forall i \geq 0$. We have $Q_{act}(t_0) = \tilde{R}$ and

$$Q_{act}(t_i) = \{(q, \mu^{t_i}(q)) : \exists q' \in Q_{act}(t_{i-1}), \exists a \in \Sigma, \delta(q', a, q) \in \Delta\}, \forall i \geq 1.$$

Since $Q_{act}(t_i)$ is a fuzzy set, in order to show that a state q belongs to $Q_{act}(t_i)$ and T is a subset of $Q_{act}(t_i)$, we should write: $q \in Domain(Q_{act}(t_i))$ and $T \subset Domain(Q_{act}(t_i))$.

Hereafter, we simply denote them as: $q \in Q_{act}(t_i)$ and $T \subset Q_{act}(t_i)$.

The combination of the operations of functions F_1 and F_2 on a multi-membership state q_j leads to the multi-membership resolution algorithm.

Algorithm 1.2. [6] (Multi-membership resolution) If there are several simultaneous transitions to the active state q_j at time $t+1$, the following algorithm will assign a unified membership value to it:

(1) Each transition weight $\delta(q_i, a_k, q_j)$ together with $\mu^t(q_i)$, will be processed by the membership assignment function F_1, and will produce a membership value. Call this v_i.

$$v_i = \tilde{\delta}((q_i, \mu^t(q_i)), a_k, q_j) = F_1(\mu^t(q_i), \delta(q_i, a_k, q_j)).$$

(2) These membership values are not necessarily equal. Hence, they need to be processed by the multi-membership resolution function F_2.

(3) The result produced by F_2 will be assigned as the instantaneous membership value of the active state q_j,

$$\mu^{t+1}(q_j) = \underset{i=1}{\overset{n}{F_2}}[v_i] = \underset{i=1}{\overset{n}{F_2}}[F_1(\mu^t(q_i), \delta(q_i, a_k, q_j))].$$

where

- n is the number of simultaneous transitions to the active state q_j at time $t+1$.
- $\delta(q_i, a_k, q_j)$ is the weight of a transition from q_i to q_j upon input a_k.
- $\mu^t(q_i)$ is the membership value of q_i at time t.
- $\mu^{t+1}(q_j)$ is the final membership value of q_j at time $t+1$.

Definition 1.3. [18] Let $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}, F_1, F_2)$ be a general fuzzy automaton. We define max-min general fuzzy automata as $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ such that :

$$\tilde{\delta}^* : Q_{act} \times \Sigma^* \times Q \to [0, 1]$$

where $Q_{act} = \{Q_{act}(t_0), Q_{act}(t_1), Q_{act}(t_2), \dots\}$ and for all $i \geq 0$,

$$\tilde{\delta}^*((q, \mu^{t_i}(q)), \Lambda, p) = \begin{cases} 1, & q = p, \\ 0, & otherwise \end{cases}$$

Also, if the input at time t_i be u_i, where $u_i \in \Sigma, \forall 1 \leq i \leq n$, then

$\tilde{\delta}^*((q, \mu^{t_{i-1}}(q)), u_i, p) = \tilde{\delta}((q, \mu^{t_{i-1}}(q)), u_i, p),$

$$\tilde{\delta}^*((q,\mu^{t_{i-1}}(q)),u_iu_{i+1},p)=\bigvee_{q'\in Q_{act}(t_i)}(\tilde{\delta}((q,\mu^{t_{i-1}}(q)),u_i,q')\wedge\tilde{\delta}((q',\mu^{t_i}(q')),u_{i+1},p)),$$

and recursively

$$\tilde{\delta}^*((q,\mu^{t_0}(q)),u_1u_2\dots u_n,p)=\vee\{\tilde{\delta}((q,\mu^{t_0}(q)),u_1,p_1)\wedge\tilde{\delta}((p_1,\mu^{t_1}(p_1)),u_2,p_2)\wedge\dots$$
$$\wedge\tilde{\delta}((p_{n-1},\mu^{t_{n-1}}(p_{n-1})),u_n,p)|p_1\in Q_{act}(t_1),p_2\in Q_{act}(t_2),\dots,p_{n-1}\in Q_{act}(t_{n-1})\}.$$

If $q\in Q_{act}(t_i)$, we should write q belongs to an element of Q_{act}. Hereafter, we simply denote it as: $q\in Q_{act}$.

Definition 1.4. [10] The mapping $\rho:\Sigma^*\longrightarrow\Sigma^*$ is called a reversal of Σ^* if the following conditions hold :

(i) $\rho(\Lambda)=\Lambda$,
(ii) $\rho(a)=a$,
(iii) $\rho(xa)=\rho(a)\rho(x)$,
$\forall a\in\Sigma$ and $x\in\Sigma^*$.

Corollary 1.5. If $\rho:\Sigma^*\longrightarrow\Sigma^*$ be a reversal mapping, then $\rho(\rho(x))=x$.

Proof. IT is streightforward by induction on the length of x.
Let $A\subseteq\Sigma^*$ and $\rho:\Sigma^*\longrightarrow\Sigma^*$ be the reversal mapping. Let

$$A^\rho=\{\rho(x):x\in A\}.$$

□

Definition 1.6. [8] Let X be an arbitrary set. The function $\psi:P(X)\longrightarrow P(X)$ is called a closure operator on X, if for any two elements A and B of $P(X)$, we have

(i) $\psi(\emptyset)=\emptyset$,
(ii) $A\subseteq\psi(A)$,
(iii) $\psi(A\bigcup B)=\psi(A)\bigcup\psi(B)$,
(iv) $\psi(\psi(A))=\psi(A)$.

Definition 1.7. [8] A subset τ of $P(X)$ is called a topology on X if

(i) $\emptyset$, $X\in\tau$,
(ii) If A_1 , $A_2\in\tau$, then $A_1\bigcap A_2\in\tau$,
(iii) If $A_i\in\tau$, $\forall i\in I$, then $\bigcup_{i\in I}A_i\in\tau$.

Definition 1.8. [5] A nonempty set H endowed with a hyperoperation $\circ:H^2\to P^*(H)$ is called a hypergroupoid where $P^*(H)$ is the set of all nonempty subsets of H. The image of the pair $(a,b)\in H^2$ is denoted by $a\circ b$ and is called the hyperproduct of a and b. If A and B are nonempty subsets of H, then $A\circ B=\bigcup\limits_{a\in A,b\in B}a\circ b$.

Definition 1.9. [4] The hypergroupoid $\langle H,\circ\rangle$ is called semihypergroup if the hyperoperation " $\circ$ " is associative. A semihypergroup $\langle H,\circ\rangle$ is called hypergroup if

$$H \circ a = a \circ H = H, \quad \forall a \in H.$$

2. **On General Fuzzy Recognizers**

In this article, we introduce some operations which will simplify proofs of quite a few theorems proved in this paper.

Definition 2.1. Let $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ be a max-min general fuzzy automaton, $\alpha : Q \to [0,1]$, $\tau : Q \to [0,1]$, $x \in \Sigma^*$ and $A \subseteq \Sigma^*$. Define
(i) $\alpha * x : Q \to [0,1]$ by

$$(\alpha * x)(p) = \bigvee\{\alpha(q) \bigwedge \tilde{\delta}^*((q, \mu^t(q)), x, p) : q \in Q_{act}\}$$

$\forall p \in Q$,

(ii) $\alpha * A : Q \to [0,1]$ by

$$\alpha * A = \bigcup_{x \in A} \alpha * x,$$

(iii) $\alpha * x^{-1} : Q_{act} \to [0,1]$ by

$$(\alpha * x^{-1})(p) = \bigvee\{\alpha(q) \bigwedge \tilde{\delta}^*((p, \mu^t(p)), x, q) : q \in Q\}$$

$\forall p \in Q_{act}$,

(vi) $\alpha * A^{-1} : Q_{act} \to [0,1]$ by

$$\alpha * A^{-1} = \bigcup_{x \in A} \alpha * x^{-1},$$

(v)

$$\alpha^{-1} o \tau = \{x \in \Sigma^* : \alpha(q) \bigwedge \tilde{\delta}^*((q, \mu^t(q)), x, p) \bigwedge \tau(p) > 0 \text{ for some } q \in Q_{act}, p \in Q\}.$$

Definition 2.2. A general fuzzy recognizer is a ten-tuple $\tilde{F}_r^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2, \iota, \zeta)$, where
(i) $\tilde{F}^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2)$ is a max-min general fuzzy automaton,
(ii) ι is a fuzzy subset of Q, called the initial fuzzy state,
(iii) ζ is a fuzzy subset of Q, called the fuzzy subset of final states.

Definition 2.3. Let $\tilde{F}_r^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2, \iota, \zeta)$ be a general fuzzy recognizer and $x \in \Sigma^*$. Then x is said to be recognized by $\tilde{F}_r^*$ if

$$\bigvee\{\iota(q) \bigwedge \tilde{\delta}^*((q, \mu^t(q)), x, p) \bigwedge \zeta(p) : q \in Q_{act}, p \in Q\} > 0.$$

Definition 2.4. Let $\tilde{F}_r^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2, \iota, \zeta)$ be a general fuzzy recognizer and

$$L(\tilde{F}_r^*) = \{x \in \Sigma^* : x \text{ is recognized by } \tilde{F}_r^*\}.$$

$L(\tilde{F}_r^*)$ is called the language recognized by the general fuzzy recognizer $\tilde{F}_r^*$.

Theorem 2.5. Let $\tilde{F}_r^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2, \iota, \zeta)$ be a general fuzzy recognizer. Then

$$L(\tilde{F}_r^*) = \{x \in \Sigma^* : \iota(q) \bigwedge \tilde{\delta}^*((q, \mu^t(q)), x, p) \bigwedge \zeta(p) > 0 \text{ for some } q \in Q_{act}, p \in Q\}.$$

Proof. The proof is obvious. □

Theorem 2.6. Let $\tilde{F}_r^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2, \iota, \zeta)$ be a general fuzzy recognizer. We define the equivalence relation R on Σ^* by

$$xRy \iff \tilde{\delta}^*((q, \mu^t(q)), x, p) = \tilde{\delta}^*((q, \mu^t(q)), y, p) : \exists\, q \in Q_{act}, p \in Q.$$

Let $B(x) = \{y \in \Sigma^* : xRy\}$, $B(D) = \bigcup_{x \in D} B(x)$. We define

$$\begin{aligned} B : P(\Sigma^*) &\longrightarrow P(\Sigma^*) \\ D &\longrightarrow B(D). \end{aligned}$$

Then B is a closure operator on Σ^*.

Proof. (i) $B(\emptyset) = \emptyset$.

(ii) Let $x \in D$. Since xRx, $x \in B(x) \subseteq B(D)$. Thus, $D \subseteq B(D)$.

(iii)$B(C \bigcup D) = \bigcup_{x \in C \bigcup D} B(x) = (\bigcup_{x \in C} B(x)) \bigcup (\bigcup_{x \in D} B(x)) = B(C) \bigcup B(D)$.

(iv) By (ii), we have $B(D) \subseteq B(B(D))$. Conversely, let $x \in B(B(D))$. Then there exists $y \in B(D)$ such that $x \in B(y)$. Thus $y \in B(x_1)$, for some $x_1 \in D$. Consequently, xRy and yRx_1. Since R is an equivalence relation, xRx_1. Thus $x \in B(x_1) \subseteq B(D)$. Therefore $B(B(D)) \subseteq B(D)$. □

Theorem 2.7. Let $\tilde{F}_r^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2, \iota, \zeta)$ be a general fuzzy recognizer. Then $\xi = \{D^C : D \subseteq \Sigma^*, B(D) = D\}$ is a topology on Σ^*.

Proof. (i) Since $B(\emptyset) = \emptyset$, then $\Sigma^* = (\emptyset)^C \in \xi$.

(ii) Since B is a closure operator on Σ^*, $\Sigma^* \subseteq B(\Sigma^*)$. On the other hand, since $B(\Sigma^*) \in P(\Sigma^*)$, $B(\Sigma^*) \subseteq \Sigma^*$. Thus, $B(\Sigma^*) = \Sigma^*$. Therefore, we conclude that $\emptyset = (\Sigma^*)^C \in \xi$.

(iii) Let D_1^C and D_2^C belong to ξ. Then $B(D_1) = D_1$ and $B(D_2) = D_2$. Thus, we have $B(D_1 \bigcup D_2) = B(D_1) \bigcup B(D_2) = D_1 \bigcup D_2$. That is $D_1^C \bigcap D_2^C = (D_1 \bigcup D_2)^C \in \xi$.

(iv) Let $D_i^C \in \xi$, $\forall i \in I$. Then $B(D_i) = D_i$, $\forall i \in I$. Since B is a closure operator on Σ^*, $\bigcap_{i \in I} D_i \subseteq B(\bigcap_{i \in I} D_i)$. On the other hand, since $D_i \bigcup (\bigcap_{i \in I} D_i) = D_i$, so we get that $B(D_i) \bigcup (B(\bigcap_{i \in I} D_i)) = B(D_i)$. Then $B(\bigcap_{i \in I} D_i) \subseteq B(D_i) = D_i$. Thus $B(\bigcap_{i \in I} D_i) \subseteq \bigcap_{i \in I} D_i$. Hence, $B(\bigcap_{i \in I} D_i) = \bigcap_{i \in I} D_i$. That is, $\bigcup_{i \in I} D_i^C = (\bigcap_{i \in I} D_i)^C \in \xi$. □

Theorem 2.8. Let $\tilde{F}_r^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2, \iota, \zeta)$ be a general fuzzy recognizer, $x \in \Sigma^*$ and $A = L(\tilde{F}_r^*)$. Then

(i) $A = \iota^{-1} o \zeta$.

(ii) $x^{-1}A = (\iota * x)^{-1} o\zeta$, where $x^{-1}A = \{y \in \Sigma^* : xy \in A\}$.

Proof. (i) The proof is not complicated.

(ii) Let $y \in (\iota * x)^{-1} o\zeta$. Then $(\iota * x)(q) \bigwedge \tilde{\delta}^*((q, \mu^t(q)), y, p) \bigwedge \zeta(p) > 0$ for some $q \in Q_{act}$, $p \in Q$. This implies that $(\iota * x)(q) > 0$ and $\tilde{\delta}^*((q, \mu^t(q)), y, p) \bigwedge \zeta(p) > 0$. Now $(\iota * x)(q) = \bigvee\{\iota(r) \bigwedge \tilde{\delta}^*((r, \mu^t(r)), x, q) : r \in Q_{act}\} > 0$ and so

$$\iota(r) \bigwedge \tilde{\delta}^*((r, \mu^t(r)), x, q) > 0$$

for some $r \in Q_{act}$. Thus we have $\iota(r) \bigwedge \tilde{\delta}^*((r, \mu^t(r)), x, q) \bigwedge \tilde{\delta}^*((q, \mu^t(q)), y, p) \bigwedge \zeta(p) > 0$ for some $r, q \in Q_{act}$, $p \in Q$. This implies that $\iota(r) \bigwedge \tilde{\delta}^*((r, \mu^t(r)), xy, p) \bigwedge \zeta(p) > 0$ for some $r \in Q_{act}$, $p \in Q$. Thus $xy \in A$ and so $y \in x^{-1}A$.
Conversely, Let $y \in x^{-1}A$. Then $xy \in A$ and so there exist $r \in Q_{act}$, $p \in Q$ such that $\iota(r) \bigwedge \tilde{\delta}^*((r, \mu^t(r)), xy, p) \bigwedge \zeta(p) > 0$. This implies that $\tilde{\delta}^*((r, \mu^t(r)), xy, p) > 0$. Now $\tilde{\delta}^*((r, \mu^t(r)), xy, p) = \bigvee_{q \in Q_{act}}\{\tilde{\delta}^*((r, \mu^t(r)), x, q) \bigwedge \tilde{\delta}^*((q, \mu^t(q)), y, p)\} > 0$. Thus there exists $q \in Q_{act}$ such that $\tilde{\delta}^*((r, \mu^t(r)), x, q) \bigwedge \tilde{\delta}^*((q, \mu^t(q)), y, p) > 0$. So we have $\tilde{\delta}^*((r, \mu^t(r)), x, q) > 0$ and $\tilde{\delta}^*((q, \mu^t(q)), y, p) > 0$. Now $(\iota * x)(q) = \bigvee\{\iota(s) \bigwedge \tilde{\delta}^*((s, \mu^t(s)), x, q) : s \in Q_{act}\} \geq \iota(r) \bigwedge \tilde{\delta}^*((r, \mu^t(r)), x, q) > 0$. Hence $(\iota * x)(q) \bigwedge \tilde{\delta}^*((q, \mu^t(q)), y, p) \bigwedge \zeta(p) > 0$ for some $q \in Q_{act}$, $p \in Q$. Thus $y \in (\iota * x)^{-1} o\zeta$. □

Theorem 2.9. Let $\tilde{F}_r^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2, \iota, \zeta)$ be a general fuzzy recognizer, $x \in \Sigma^*$ and $A = L(\tilde{F}_r^*)$. Then

(i) $x \in A$ if and only if there exists $q \in Q_{act}$ such that $[\iota \bigcap (\zeta * x^{-1})](q) > 0$.

(ii) $x \in A$ if and only if there exists $p \in Q$ such that $[(\iota * x) \bigcap \zeta](p) > 0$.

Proof. (i) Let $x \in A$. Then there exist $q \in Q_{act}$ and $p \in Q$ such that $\iota(q) \bigwedge \tilde{\delta}^*((q, \mu^t(q)), x, p) \bigwedge \zeta(p) > 0$. So there exist $q \in Q_{act}$ and $p \in Q$ such that $\iota(q) > 0$ and $\tilde{\delta}^*((q, \mu^t(q)), x, p) \bigwedge \zeta(p) > 0$. Thus there exists $q \in Q_{act}$ such that $\iota(q) > 0$ and $\bigvee\{\tilde{\delta}^*((q, \mu^t(q)), x, p) \bigwedge \zeta(p) : p \in Q\} > 0$. Therefore there exists $q \in Q_{act}$ such that $\iota(q) > 0$ and $(\zeta * x^{-1})(q) > 0$. Consequently, there exists $q \in Q_{act}$ such that $[\iota \bigcap (\zeta * x^{-1})](q) > 0$. Similarly, we can prove the converse of this part.

(ii) Let $x \in A$. So there exist $q \in Q_{act}$ and $p \in Q$ such that $\iota(q) \bigwedge \tilde{\delta}^*((q, \mu^t(q)), x, p) \bigwedge \zeta(p) > 0$. Then there exist $q \in Q_{act}$ and $p \in Q$ such that $\iota(q) \bigwedge \tilde{\delta}^*((q, \mu^t(q)), x, p) > 0$ and $\zeta(p) > 0$. Thus there exists $p \in Q$ such that $\bigvee\{\iota(q) \bigwedge \tilde{\delta}^*((q, \mu^t(q)), x, p) : q \in Q_{act}\} > 0$ and $\zeta(p) > 0$. Therefore there exists $p \in Q$ such that $(\iota * x)(p) > 0$ and $\zeta(p) > 0$. Consequently, there exists $p \in Q$ such that $[(\iota * x) \bigcap \zeta](p) > 0$. Similarly, we can show the converse of this part. □

Definition 2.10. Let $\tilde{F}_r^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2, \iota, \zeta)$ be a general fuzzy recognizer and $S_1 = \{q \in Q_{act} : (\iota * \Sigma^*)(q) > 0\}$. If $Q = S_1$, then $\tilde{F}_r^*$ is called accessible. Consider $\tilde{\delta}^{*a} = \tilde{\delta}^*|_{\bar{S_1} \times \Sigma^* \times S_1}$, $\iota^a = \iota|_{S_1}$, $\zeta^a = \zeta|_{S_1}$ and $\tilde{F}_r^{*a} = (S_1, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^{*a}, F_1, F_2, \iota^a, \zeta^a)$, where $\bar{S_1} = \{(q, \mu^t(q)) : q \in S_1\}$. Then $\tilde{F}_r^{*a}$ is a general fuzzy recognizer.

Definition 2.11. Let $\tilde{F}_r^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2, \iota, \zeta)$ be a general fuzzy recognizer. Then $\tilde{F}_r^{*a}$ is called the accessible part of $\tilde{F}_r^*$.

Theorem 2.12. Let $\tilde{F}_r^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2, \iota, \zeta)$ be a general fuzzy recognizer. Then $L(\tilde{F}_r^*) = L(\tilde{F}_r^{*a})$.

Proof. By Theorem 2.6 (i), it is sufficient to show that $\iota^{-1} o\zeta = (\iota^a)^{-1} o\zeta^a$. Let $x \in \iota^{-1} o\zeta$. Then $\iota(q) \bigwedge \tilde{\delta}^*((q, \mu^t(q)), x, p) \bigwedge \zeta(p) > 0$ for some $q \in Q_{act}$, $p \in Q$. Since $\iota(q) \bigwedge \tilde{\delta}^*((q, \mu^t(q)), x, p) > 0$, we have $(\iota * \Sigma^*)(p) > 0$. So $p \in S_1$. Also, since $\iota(q) > 0$ and $\tilde{\delta}^*((q, \mu^t(q)), \Lambda, q) = 1 > 0$, we have $(\iota * \Sigma^*)(q) > 0$. So $q \in S_1$. Hence $\iota^a(q) \bigwedge \tilde{\delta}^{*a}((q, \mu^t(q)), x, p) \bigwedge \zeta^a(p) > 0$ and thus $x \in (\iota^a)^{-1} o\zeta^a$. Therefore, $\iota^{-1} o\zeta \subseteq (\iota^a)^{-1} o\zeta^a$.

Now, let $x \in (\iota^a)^{-1} o\zeta^a$. Then $\iota^a(q) \bigwedge \tilde{\delta}^{*a}((q, \mu^t(q)), x, p) \bigwedge \zeta^a(p) > 0$ for some $q, p \in S_1$. Therefore $\iota^a(q) > 0$, $\tilde{\delta}^{*a}((q, \mu^t(q)), x, p) > 0$ and $\zeta^a(p) > 0$. Thus $\iota(q) = \iota^a(q)$, $\zeta(p) = \zeta^a(p)$ and $\tilde{\delta}^*((q, \mu^t(q)), x, p) = \tilde{\delta}^{*a}((q, \mu^t(q)), x, p)$. Consequently, $\iota(q) \bigwedge \tilde{\delta}^*((q, \mu^t(q)), x, p) \bigwedge \zeta(p) > 0$. Hence $x \in \iota^{-1} o\zeta$. So $(\iota^a)^{-1} o\zeta^a \subseteq \iota^{-1} o\zeta$. □

Theorem 2.13. Let $\tilde{F}_r^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2, \iota, \zeta)$ be a general fuzzy recognizer and $S_1 = \{q \in Q_{act} : (\iota * \Sigma^*)(q) > 0\}$. We define on S_1 the following hyperoperation :

$$p \circ p = \{r \in S_1 : (\iota * \Sigma^*)(p) \geq (\iota * \Sigma^*)(r)\},$$

and

$$p \circ q = (p \circ p) \cup (q \circ q), \text{ where } p \neq q.$$

Then $\langle S_1, \circ \rangle$ is a commutative hypergroup.

Proof. We first show that the hyperoperation " $\circ$ " is associative. We have

$$\begin{aligned}
(p \circ q) \circ s &= [\{r_1 \in S_1 : (\iota * \Sigma^*)(p) \geq (\iota * \Sigma^*)(r_1)\} \cup \{r_2 \in S_1 : (\iota * \Sigma^*)(q) \geq \\
&\quad (\iota * \Sigma^*)(r_2)\}] \circ s \\
&= \{r_1' \in S_1 : (\iota * \Sigma^*)(r_1) \geq (\iota * \Sigma^*)(r_1'), (\iota * \Sigma^*)(p) \geq (\iota * \Sigma^*)(r_1)\} \\
&\cup \{r_2' \in S_1 : (\iota * \Sigma^*)(r_2) \geq (\iota * \Sigma^*)(r_2'), (\iota * \Sigma^*)(q) \geq (\iota * \Sigma^*)(r_2)\} \\
&\cup \{r_3 \in S_1 : (\iota * \Sigma^*)(s) \geq (\iota * \Sigma^*)(r_3)\} \\
&\subseteq \{r_1' \in S_1 : (\iota * \Sigma^*)(p) \geq (\iota * \Sigma^*)(r_1')\} \cup \{r_2' \in S_1 : (\iota * \Sigma^*)(q) \geq \\
&\quad (\iota * \Sigma^*)(r_2')\} \cup \{r_3 \in S_1 : (\iota * \Sigma^*)(s) \geq (\iota * \Sigma^*)(r_3)\}.
\end{aligned}$$

Let

$$\begin{aligned}
T = \{r_1 \in S_1 : (\iota * \Sigma^*)(p) \geq (\iota * \Sigma^*)(r_1)\} &\cup \{r_2 \in S_1 : (\iota * \Sigma^*)(q) \geq (\iota * \Sigma^*)(r_2)\} \\
&\cup \{r_3 \in S_1 : (\iota * \Sigma^*)(s) \geq (\iota * \Sigma^*)(r_3)\}.
\end{aligned}$$

Thus $(p \circ q) \circ s \subseteq T$. Now, let $r \in \{r_1 \in S_1 : (\iota * \Sigma^*)(p) \geq (\iota * \Sigma^*)(r_1)\}$. Then $(\iota * \Sigma^*)(p) \geq (\iota * \Sigma^*)(r)$. Hence $r \in p \circ p \subseteq (p \circ q) \circ s$. So $(p \circ q) \circ s \supseteq T$. Therefore

$$(p \circ q) \circ s = T.$$

Similarly,

$$p \circ (q \circ s) = T.$$

Therefore the hyperoperation " $\circ$ " is associative. We claim that

$$S_1 \circ q = q \circ S_1 = S_1, \quad \forall q \in S_1.$$

It is clear that $S_1 \circ q \subseteq S_1$. For the reverse inclusion, let $p \in S_1$. Since $(\iota * \Sigma^*)(p) \geq (\iota * \Sigma^*)(p)$, hence

$$p \in \{r_1 \in S_1 : (\iota*\Sigma^*)(p) \geq (\iota*\Sigma^*)(r_1)\} \cup \{r_2 \in S_1 : (\iota*\Sigma^*)(q) \geq (\iota*\Sigma^*)(r_2)\} = p \circ q.$$

Thus $p \in S_1 \circ q$. So $S_1 \subseteq S_1 \circ q$. □

Definition 2.14. Let $\tilde{F}_r^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2, \iota, \zeta)$ be a general fuzzy recognizer. $\tilde{F}_r^*$ is called complete if for all $q \in Q_{act}$, $x \in \Sigma^*$, there exists $p \in Q$ such that $\tilde{\delta}^*((q, \mu^t(q)), x, p) > 0$.

Theorem 2.15. Let $\tilde{F}_r^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2, \iota, \zeta)$ be a complete general fuzzy recognizer, then so is $\tilde{F}_r^{*a}$.

Proof. Let $(q, x) \in S_1 \times \Sigma^*$. Then $(q, x) \in Q_{act} \times \Sigma^*$. Since $\tilde{F}_r^*$ is complete, there exists $p \in Q$ such that $\tilde{\delta}^*((q, \mu^t(q)), x, p) > 0$. Now $q \in S_1$, then $(\iota * \Sigma^*)(q) > 0$. So there exist $p_0 \in Q_{act}$ and $x_0 \in \Sigma^*$ such that $\iota(p_0) \bigwedge \tilde{\delta}^*((p_0, \mu^t(p_0)), x_0, q) > 0$. This implies that $\iota(p_0) > 0$ and $\tilde{\delta}^*((p_0, \mu^t(p_0)), x_0, q) > 0$. Hence $\tilde{\delta}^*((p_0, \mu^t(p_0)), x_0, q) \bigwedge \tilde{\delta}^*((q, \mu^t(q)), x, p) > 0$. Now $\tilde{\delta}^*((p_0, \mu^t(p_0)), x_0x, p) = \bigvee_{r \in Q_{act}} \{\tilde{\delta}^*((p_0, \mu^t(p_0)), x_0, r) \bigwedge \tilde{\delta}^*((r, \mu^t(r)), x, p)\} > 0$. Hence $\iota(p_0) \bigwedge \tilde{\delta}^*((p_0, \mu^t(p_0)), x_0x, p) > 0$. Therefore, $(\iota * \Sigma^*)(p) > 0$ and so $p \in S_1$. Since $\tilde{\delta}^{*a}((q, \mu^t(q)), x, p) = \tilde{\delta}^*((q, \mu^t(q)), x, p)$, $\tilde{F}_r^{*a}$ is complete. □

Definition 2.16. Let $\tilde{F}_r^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2, \iota, \zeta)$ be a general fuzzy recognizer and $S_2 = \{q \in Q_{act} : (\zeta * (\Sigma^*)^{-1})(q) > 0\}$. If $Q = S_2$, then $\tilde{F}_r^*$ is called coaccessible.
Consider $\tilde{\delta}^{*b} = \tilde{\delta}^*|_{\bar{S}_2 \times \Sigma^* \times S_2}$, $\iota^b = \iota|_{S_2}$, $\zeta^b = \zeta|_{S_2}$ and $\tilde{F}_r^{*b} = (S_2, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^{*b}, F_1, F_2, \iota^b, \zeta^b)$, where $\bar{S}_2 = \{(q, \mu^t(q)) : q \in S_2\}$. Then $\tilde{F}_r^{*b}$ is a general fuzzy recognizer.

Definition 2.17. Let $\tilde{F}_r^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2, \iota, \zeta)$ be a general fuzzy recognizer. Then $\tilde{F}_r^{*b}$ is called the coaccessible part of $\tilde{F}_r^*$.

Theorem 2.18. Let $\tilde{F}_r^* = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}^*, F_1, F_2, \iota, \zeta)$ be a general fuzzy recognizer. Then $L(\tilde{F}_r^*) = L(\tilde{F}_r^{*b})$.

Proof. By Theorem 2.8 (i), it is sufficient to show that $\iota^{-1} o \zeta = (\iota^b)^{-1} o \zeta^b$. Let $x \in \iota^{-1} o \zeta$. Then $\iota(q) \bigwedge \tilde{\delta}^*((q, \mu^t(q)), x, p) \bigwedge \zeta(p) > 0$ for some $q \in Q_{act}$, $p \in Q$. Since $\tilde{\delta}^*((q, \mu^t(q)), x, p) \bigwedge \zeta(p) > 0$, we have $(\zeta * (\Sigma^*)^{-1})(q) > 0$. So $q \in S_2$. Also, since $\zeta(p) > 0$ and $\tilde{\delta}^*((p, \mu^t(p)), \Lambda, p) = 1 > 0$, $(\zeta * (\Sigma^*)^{-1})(p) > 0$. So $p \in S_2$. Hence $\iota^b(q) \bigwedge \tilde{\delta}^{*b}((q, \mu^t(q)), x, p) \bigwedge \zeta^b(p) > 0$ and thus $x \in (\iota^b)^{-1} o \zeta^b$. Therefore

$\iota^{-1}o\zeta \subseteq (\iota^b)^{-1}o\zeta^b$.

Now, let $x \in (\iota^b)^{-1}o\zeta^b$. Then $\iota^b(q) \bigwedge \tilde{\delta}^{*b}((q,\mu^t(q)),x,p) \bigwedge \zeta^b(p) > 0$ for some $q,p \in S_2$. Therefore $\iota^b(q) > 0$, $\tilde{\delta}^{*b}((q,\mu^t(q)),x,p) > 0$ and $\zeta^b(p) > 0$. Thus $\iota(q) = \iota^b(q)$, $\zeta(p) = \zeta^b(p)$ and $\tilde{\delta}^*((q,\mu^t(q)),x,p) = \tilde{\delta}^{*b}((q,\mu^t(q)),x,p)$. Consequently, $\iota(q) \bigwedge \tilde{\delta}^*((q,\mu^t(q)),x,p) \bigwedge \zeta(p) > 0$. Hence $x \in \iota^{-1}o\zeta$. So $(\iota^b)^{-1}o\zeta^b \subseteq \iota^{-1}o\zeta$. □

Theorem 2.19. Let $\tilde{F}_r^* = (Q,\Sigma,\tilde{R},Z,\omega,\tilde{\delta}^*,F_1,F_2,\iota,\zeta)$ be a general fuzzy recognizer and $S_2 = \{q \in Q_{act} : (\zeta * (\Sigma^*)^{-1})(q) > 0\}$. We define on S_2 the following hyperoperation :

$$p \circ p = \{r \in S_2 : (\zeta * (\Sigma^*)^{-1})(p) \geq (\zeta * (\Sigma^*)^{-1})(r)\},$$

and

$$p \circ q = (p \circ p) \cup (q \circ q), \text{ where } p \neq q.$$

Then $\langle S_2, \circ \rangle$ is a commutative hypergroup.

Proof. The proof is similar to Theorem 2.13, by using suitable modification. □

Definition 2.20. Let $\tilde{F}_r^* = (Q,\Sigma,\tilde{R},Z,\omega,\tilde{\delta}^*,F_1,F_2,\iota,\zeta)$ be a general fuzzy recognizer. Define $\tilde{\delta}^{*\rho}((p,\mu^s(p)),x,q) = \tilde{\delta}^*((q,\mu^t(q)),\rho(x),p)$, $\forall p,q \in Q_{act}$, $\forall x \in \Sigma^*$. Then the general fuzzy recognizer $\tilde{F}_r^{*\rho} = (Q,\Sigma,\tilde{R},Z,\omega,\tilde{\delta}^{*\rho},F_1,F_2,\zeta,\iota)$ is called the reversal of $\tilde{F}_r^*$.

Theorem 2.21. Let $\tilde{F}_r^* = (Q,\Sigma,\tilde{R},Z,\omega,\tilde{\delta}^*,F_1,F_2,\iota,\zeta)$ be a general fuzzy recognizer. If $Q = Q_{act}$, then $L(\tilde{F}_r^{*\rho}) = [L(\tilde{F}_r^*)]^\rho$.

Proof. By Theorem 2.8 (i), it is sufficient to show that $\zeta^{-1}o\iota = [\iota^{-1}o\zeta]^\rho$. Let $\rho(x) \in [\iota^{-1}o\zeta]^\rho$. Then $x \in \iota^{-1}o\zeta$. This implies that $\iota(q) \bigwedge \tilde{\delta}^*((q,\mu^t(q)),x,p) \bigwedge \zeta(p) > 0$ for some $q \in Q_{act}$, $p \in Q$. Since $Q = Q_{act}$, so $\zeta(p) \bigwedge \tilde{\delta}^{*\rho}((p,\mu^s(p)),\rho(x),q) \bigwedge \iota(q) > 0$ for some $p \in Q_{act}$, $q \in Q$. Therefore $\rho(x) \in \zeta^{-1}o\iota$. Consequently, $[\iota^{-1}o\zeta]^\rho \subseteq \zeta^{-1}o\iota$. Similarly, we can prove that $\zeta^{-1}o\iota \subseteq [\iota^{-1}o\zeta]^\rho$. □

Theorem 2.22. Let $\tilde{F}_r^* = (Q,\Sigma,\tilde{R},Z,\omega,\tilde{\delta}^*,F_1,F_2,\iota,\zeta)$ be a general fuzzy recognizer and $Q = Q_{act}$. Then $\tilde{F}_r^*$ is coaccessible if and only if $\tilde{F}_r^{*\rho}$ is accessible.

Proof. Let $\tilde{F}_r^*$ be coaccessible. By Definition 2.16, we have $Q = \{q \in Q_{act} : (\zeta * (\Sigma^*)^{-1})(q) > 0\}$. Then for all $q \in Q$, there exist $x \in \Sigma^*$ and $p \in Q$ such that $\zeta(p) \bigwedge \tilde{\delta}^*((q,\mu^t(q)),x,p) > 0$. Since $Q = Q_{act}$, for all $q \in Q$, there exist x^ρ namely $\rho(x) \in \Sigma^*$ and $p \in Q_{act}$ such that $\zeta(p) \bigwedge \tilde{\delta}^{*\rho}((p,\mu^s(p)),\rho(x),q) > 0$. So $Q = \{q \in Q_{act} : (\zeta * \Sigma^*)(q) > 0\}$. Therefore $\tilde{F}_r^{*\rho} = (Q,\Sigma,\tilde{R},Z,\omega,\tilde{\delta}^{*\rho},F_1,F_2,\zeta,\iota)$ is accessible. Similarly, we can verify the converse of this theorem. □

References

[1] S. Bozapalidis and O. L. Bozapalidoy, *On the recognizability of fuzzy languages I*, Fuzzy Sets and Systems, **157** (2006), 2394-2402.

[2] S. Bozapalidis and O. L. Bozapalidoy, *On the recognizability of fuzzy languages II*, Fuzzy Sets and Systems, **159(1)** (2008), 107-113.

[3] S. Bozapalidis and O. L. Bozapalidoy, *Fuzzy tree language recognizability*, Fuzzy Sets and Systems, **161(5)** (2010), 716-734.
[4] P. Corsini, *Prolegomena of hypergroup theory*, Aviani Editore, 1993.
[5] P. Corsini and V. Leoreanu, *Applications of hyperstructure theory*, Kluwer Academic Publishers, Advances in Mathematics, 2003.
[6] M. Doostfatemeh and S. C. Kremer, *New directions in fuzzy automata*, International Journal of Approximate Reasoning, **38** (2005), 175-214.
[7] M. Horry and M. M. Zahedi, *Hypergroups and general fuzzy automata*, Iranian Journal of Fuzzy Systems, **6(2)** (2009), 61-74.
[8] K. Kuratowski, *Topology*, Academic Presss, 1966.
[9] H. V. Kumbhojkar and S. R. Chaudhari, *Fuzzy recognizers and recognizable sets*, Fuzzy Sets and Systems, **131(3)** (2002), 381-392.
[10] J. N. Mordeson and D. S. Malik, *Fuzzy automata and languages, theory and applications*, Chapman and Hall/CRC, London/Boca Raton, FL, 2002.
[11] D. S. Malik, J. N. Mordeson and M. K. Sen, *On subsystems of fuzzy finite state machines*, Fuzzy Sets and Systems, **68** (1994), 83-92.
[12] M. Mizumoto, J. Tanaka and K. Tanaka, *Some consideration on fuzzy automata*, J. Compute. Systems Sci., **3** (1969), 409-422.
[13] W. Omlin, K. K. Giles and K. K. Thornber, *Equivalence in knowledge representation: automata, rnns, and dynamic fuzzy systems*, Proc. IEEE, **87(9)** (1999), 1623-1640.
[14] W. Omlin, K. K. Thornber and K. K. Giles, *Fuzzy finite-state automata can be deterministically encoded into recurrent neural networks*, IEEE Trans. Fuzzy Syst., **5(1)** (1998), 76-89.
[15] E. S. Santos, *Realization of fuzzy languages by probabilistic, max-prod and maximin automata*, Information Sciences, **8** (1975), 39-53.
[16] W. G. Wee, *On generalization of adaptive algorithm and application of the fuzzy sets concept to pattern classif ication*, Ph.D. dissertation Purdue University, IN, 1967.
[17] W. G. Wee and K. S. Fu, *A formulation of fuzzy automata and its applicationset as a modele of learning systems*, IEEE Trans. Systems Sci. Cybernet., **5** (1969), 215-223.
[18] M. M. Zahedi, M. Horry and K. Abolpor, *Bifuzzy (General) topology on max-min general fuzzy automata*, Advanced in Fuzzy Mathematics, **3(1)** (2008), 51-68.

M. Horry*, Shahid Chamran University of Kerman, Kerman, Iran
E-mail address: mohhorry@chamran-edu.ir

M. M. Zahedi, Department of Mathematics, Shahid Bahonar University of Kerman, Kerman, Iran
E-mail address: zahedi_mm@mail.uk.ac.ir

*Corresponding author