SOLUTION-SET INVARIANT MATRICES AND VECTORS IN FUZZY RELATION INEQUALITIES BASED ON MAX-AGGREGATION FUNCTION COMPOSITION

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This paper is dedicated to Professor L. A. Zadeh on the occasion of his 95th birthday and the 50th year of the birth of fuzzy logic

Abstract. Fuzzy relation inequalities based on max-F composition are discussed, where F is a binary aggregation on [0,1]. For a fixed fuzzy relation inequalities system $A \circ^F \mathbf{x} \leq \mathbf{b}$, we characterize all matrices A' For which the solution set of the system $A' \circ^F \mathbf{x} \leq \mathbf{b}$ is the same as the original solution set. Similarly, for a fixed matrix A, the possible perturbations b' of the right-hand side vector b not modifying the original solution set are determined. Several illustrative examples are included to clarify the results of the paper.

1. Introduction

Since the first resolution of fuzzy relation equations was proposed by Sanchez in 1976 [9], fuzzy relation equations (FRE, in short) and inequalities (FRI, in short) and related problems have been widely studied by many researchers. For a general background see [8]. The application of FRE and FRI can be seen in many areas such as fuzzy decision making, fuzzy control, image processing, image and video compression and decompression, image reconstruction, fuzzy modeling, fuzzy diagnosis and especially fuzzy medical diagnosis. So far, the main attention was paid to FREs and FRIs with max-min or max-product compositions. Also, some results of these cases were generalized for max-t composition with additional assumptions on the triangular norm ([5, 10, 11]). To the best of our knowledge, no work has been done in which a fuzzy max-aggregation function is considered in FREs or FRIs.

Some works have been done to investigate the impact of perturbations in the inputs of a system of FREs with max-min or max-product compositions [3, 12]. In this contribution, first, we find the solution set of a system of FRIs in the presence of max-aggregation function composition and then, we aim to investigate the stability of this solution set under possible perturbations in inputs. In this way, we will have two different equivalence relations. Using related partitions, the number of calculations for solving a group of systems of FRIs will decrease.

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The paper is organized as follows. In the next section some basic information concerning aggregation functions is given. Section 3 gives the definition of a max-aggregation function system of fuzzy relation inequalities and its solution set. In Sections 4 and 5 the set of all solution-set invariant matrices and vectors are characterized, respectively. Finally, some concluding remarks are added.

2. **Basic Concepts**

For more details and terminology of this section see [2] and [6].

Definition 2.1. [6] An n-ary aggregation function is a mapping of $n > 1$ arguments that maps the n-dimensional unit cube to the unit interval, $F : [0,1]^n \to [0,1]$, with the following properties

a) $F(\mathbf{0}) = 0$ and $F(\mathbf{1}) = 1$,
b) $\mathbf{x} \leq \mathbf{y}$ implies $F(\mathbf{x}) \leq F(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in [0,1]^n$.

Throughout this paper, we employ 2-ary (or binary) aggregation functions and simply write aggregation functions.

Definition 2.2. [7] A t-norm $T : [0,1]^2 \to [0,1]$ is an aggregation function with neutral element 1 which is symmetric and associative.

Definition 2.3. [4] A function $K : [0,1]^2 \to [0,1]$ is said to be a semicopula if it satisfies the following conditions

a) $K(x,1) = K(1,x) = x$ for all $x \in [0,1]$;
b) $K(x,y) \leq K(x^{'},y^{'})$ for all $x, x^{'}, y, y^{'} \in [0,1], x \leq x^{'}$ and $y \leq y^{'}$.

In other words, a semicopula is an aggregation function with neutral element 1 and consequently, annihilator 0. Important examples of semicopulas are t-norms minimum, product and Lukasiewicz t-norm.

Remark 2.4. [1] Let F be an aggregation function which is left-continuous in the second coordinate. Then,

$$F^{-}(x,y) = \sup\{z \in [0,1] | F(x,z) \leq y\} \tag{1}$$

is a (right) conjugate (adjoint operator Galois connection) related to F. The value $F^{-}(x,y)$ is the maximal solution of the inequality $F(x,z) \leq y$. Note that if F is a t-norm then, F^{-} is also called a residual implication. Moreover, then, $F(x,y) = \inf\{z \in [0,1] | F^{-}(x,z) \geq y\}$.

3. **Problem Formulation**

Let F be any aggregation function which is left-continuous in the second coordinate. Consider the following fuzzy relation inequality system.

$$A \circ^{F} \mathbf{x} \leq \mathbf{b}, \tag{2}$$

where, $A = (a_{ij})_{m\times n}$ is a fixed matrix, $\mathbf{b} = (b_i)_{m\times 1}$ and $\mathbf{x} = (x_j)_{n\times 1}$ are the fixed right hand side and unknown vectors, respectively, such that $a_{ij}, b_i, x_j \in [0,1]$; $i \in$

$I = \{1, 2, \ldots, m\}$ and $j \in J = \{1, 2, \ldots, n\}$. Also, o^F stands for max-aggregation function composition which means that (2) can be rewritten into

$$\vee_{j \in J} F(a_{ij}, x_j) \leq b_i \tag{3}$$

for all $i \in I$. We say (2) is solvable if and only if there is some $\mathbf{x} \in [0,1]^n$ such that (2) is satisfied.

Lemma 3.1. *System (2) is solvable if and only if* $A \circ^F \boldsymbol{0} \leq \boldsymbol{b}$ *where,* $\boldsymbol{0} = (0, 0, \ldots, 0)$.

Proof. It follows from the monotonicity of F. □

Since $\mathbf{x} \in [0,1]^n$, by Lemma 3.1, $\mathbf{0}$ is the unique minimal solution [8] of (2).

From now on, we always suppose the solvability of (2) unless the otherwise is mentioned.

Lemma 3.2. *Let* $\boldsymbol{x} \in [0,1]^n$ *be a solution of (2). Then, for all* $j \in J$,

$$\wedge_{i \in I} F^-(a_{ij}, b_i) \geq x_j. \tag{4}$$

Proof. From (3), we have $F(a_{ij}, x_j) \leq b_i$ for all $i \in I$ and $j \in J$. Therefore, $F^-(a_{ij}, b_i) \geq x_j$ for all $i \in I$ and $j \in J$. So, $\wedge_{i \in I} F^-(a_{ij}, b_i) \geq x_j$ for all $j \in J$. □

Corollary 3.3. *The unique maximal solution of (2) is* $\overline{\boldsymbol{x}} \in [0,1]^n$ *where, for all* $j \in J$,

$$\overline{x}_j = \wedge_{i \in I} F^-(a_{ij}, b_i). \tag{5}$$

Proof. It follows from Lemma 3.2 immediately. □

Corollary 3.4. *The solution set of (2) is* $[\boldsymbol{0}, \overline{\boldsymbol{x}}]$.

Here, we find the solution set of (2) in the presence of some different aggregation functions. For more details see [1].

Example 3.5. a) Let $F = \min$. So, we have

$$min^-(x, y) = \begin{cases} 1 & x \leq y \\ y & x > y \end{cases} \tag{6}$$

which is the Gödel implication.

b) Let F be Lukasiewicz t-norm, $T_L(x, y) = \max(0, x + y - 1)$. We have

$$T_L^-(x, y) = \begin{cases} 1 & x \leq y \\ 1 - x + y & x > y \end{cases} = \min(1, 1 - x + y)$$

which is Lukasiewicz implication.

c) Let F be arithmetic mean, AM. We have

$$AM^-(x, y) = \max(0, \min(1, 2y - x)). \tag{7}$$

d) Let F be product. We have

$$Prod^-(x, y) = \begin{cases} 1 & x \leq y \\ \frac{y}{x} & x > y. \end{cases}$$

Therefore, by (5), one can find the maximal solution of System (2) for different aggregation functions.

4. **Solution-set Invariant Matrices**

Now, let the matrix A and the right hand side vector $\mathbf{b}$ be fixed. We want to determine the amount of perturbation which can be imposed on the matrix A without any changes in the solution set of (2). In other words, we want to determine the set of all matrices which have the same solution set $[\mathbf{0}, \overline{\mathbf{x}}]$ with respect to the given and fixed right hand side vector $\mathbf{b}$. We shall denote this set of all solution-set invariant matrices by $(A, \mathbf{b})_F$.

Lemma 4.1. *$A^{'} \in (A, \boldsymbol{b})_F$ if and only if $\overline{\boldsymbol{x}}$ is the maximal solution of*

$$A^{'} \circ^F \boldsymbol{x} \leq \boldsymbol{b}. \tag{8}$$

Proof. It is obvious, according to the form of solution set. □

For a considered vector $\overline{\mathbf{x}}$ we use the notation $J^{'} = \{j \in J | \overline{x}_j < 1\}$. Now, we are ready to characterize the set $(A, \mathbf{b})_F$.

Theorem 4.2. *$A^{'} \in (A, \boldsymbol{b})_F$ if and only if $F^{-}(a^{'}_{ij}, b_i) \geq \overline{x}_j$ for all $j \in J$ and all $i \in I$ and also, for all $j \in J^{'}$ there exist some $i \in I_j \subseteq I$ such that $F^{-}(a^{'}_{ij}, b_i) = \overline{x}_j$.*

Proof. First, suppose $A^{'} \in (A, \mathbf{b})_F$. Then, by (4), $F^{-}(a^{'}_{ij}, b_i) \geq \overline{x}_j$ for all $i \in I$ and all $j \in J$. Now, let for some $j \in J^{'}$, $I_j = \emptyset$. This means there is no $i \in I_j$ such that $F^{-}(a^{'}_{ij}, b_i) = \overline{x}_j$. Therefore, we have $F^{-}(a^{'}_{ij}, b_i) > \overline{x}_j$ for all $i \in I_j$. So, $\wedge_{i \in I_j} F^{-}(a^{'}_{ij}, b_i) > \overline{x}_j$. Since $A^{'} \in (A, \mathbf{b})_F$, we have $\wedge_{i \in I \setminus I_j} F^{-}(a^{'}_{ij}, b_i) = \overline{x}_j$. Thus, there should be some $i \in I \setminus I_j$ such that $F^{-}(a^{'}_{ij}, b_i) = \overline{x}_j$ which is a contradiction.

Conversely, suppose for all $i \in I$ and all $j \in J$, $F^{-}(a^{'}_{ij}, b_i) \geq \overline{x}_j$. If $j \notin J^{'}$ then, $\overline{x}_j = 1$. So, $F^{-}(a^{'}_{ij}, b_i) \geq 1$ for all $i \in I$ and then, $\wedge_{i \in I} F^{-}(a^{'}_{ij}, b_i) = 1 = \overline{x}_j$. If $j \in J^{'}$ then, $\overline{x}_j < 1$. So, $F^{-}(a^{'}_{ij}, b_i) \geq \overline{x}_j$ for all $i \in I$ and so, $\wedge_{i \in I} F^{-}(a^{'}_{ij}, b_i) \geq \overline{x}_j$. Since by hypothesis, there exist some $i \in I_j$ such that $F^{-}(a^{'}_{ij}, b_i) = \overline{x}_j$ then, $\wedge_{i \in I} F^{-}(a^{'}_{ij}, b_i) = \overline{x}_j$. Hence, $A^{'} \in (A, \mathbf{b})_F$. □

Example 4.3. Consider the following system

$$\begin{bmatrix} 0.5 & 0.4 \\ 1 & 0.8 \end{bmatrix} \circ^{AM} \mathbf{x} \leq \begin{bmatrix} 0.3 \\ 0.8 \end{bmatrix}.$$

We have $\overline{x}_1 = 0.1$ and $\overline{x}_2 = 0.2$. By Theorem 4.2 and Example 3.5, we have $A^{'} \in (A, \mathbf{b})_{AM}$ if and only if $A^{'} = \begin{bmatrix} 0.5 & 0.4 \\ a & a \end{bmatrix}$, where $a \in [0, 1]$.

Now, let K be a continuous semicopula. We have the following corollary.

Corollary 4.4. *$A^{'} \in (A, \boldsymbol{b})_K$ if and only if*

a) For all $j \notin J^{'}$, $a^{'}_{ij} \in [0, b_i]$ for all $i \in I$.

b) For all $j \in J^{'}$, we have $a^{'}_{ij} \in [0, K_{-}(\overline{x}_j, b_i)]$ for all $i \in I$ and there exists some $i \in I$ such that $a^{'}_{ij} = K_{-}(\overline{x}_j, b_i)$. Where, $K_{-}(x, y) = \sup\{z \in [0, 1] | K(z, x) \leq y\}$ is the left conjugate related to F.

Proof. First, let $A^{'} \in (A, \mathbf{b})_K$.

a) Let $j \notin J^{'}$. We have $\wedge_{i\in I} K^{-}(a^{'}_{ij}, b_i) = 1$ that implies $K^{-}(a^{'}_{ij}, b_i) = 1$ for all $i \in I$. So, by (1) and Definition 2.3, $a^{'}_{ij} \leq b_i$ for all $i \in I$. That means $a^{'}_{ij} \in [0, b_i]$ for all $i \in I$.

b) Let $j \in J^{'}$. By Theorem 4.2 we have $K^{-}(a^{'}_{ij}, b_i) \geq \overline{x}_j$ for all $i \in I$ and there exist some $i \in I$ such that $K^{-}(a^{'}_{ij}, b_i) = \overline{x}_j$. But, we have $K^{-}(a^{'}_{ij}, b_i) \geq \overline{x}_j$ if and only if $a^{'}_{ij} \leq K_{-}(\overline{x}_j, b_i)$. Also, $K^{-}(a^{'}_{ij}, b_i) = \overline{x}_j < 1$ if and only if $a^{'}_{ij} = K_{-}(\overline{x}_j, b_i)$ which completes the proof.

The converse part is similar to that of Theorem 4.2. □

Example 4.5. As an example we can consider a semicopula K which is a continuous Archimedean t-norm [7], i.e., it is generated by an additive generator $k : [0, 1] \to [0, \infty]$, k is strictly decreasing, continuous and $k(1) = 0$. Then, for $x \geq y$, it holds $K^{+}(x, y) = k^{-1}(k(y) - k(x))$ with convention $\infty - \infty = 0$, and then Corollary 4.4 (b) reduces to $k(a^{'}_{i,j}) \geq k(b_i) - k(\overline{x}_j)$ for each $j \in J^{'}$ and each i, supposing also the existence of some i for which $k(a^{'}_{i,j}) = k(b_i) - k(\overline{x}_j)$.

For product t-norm we have $k(x) = -\log x$, hence we have $a^{'}_{i,j} \leq b_i.\overline{x}_j$ for each $j \in J^{'}$ and each i, so that for each $j \in J'$, there exist some i such that $a^{'}_{i,j} = b_i.\overline{x}_j$. Where, $\overline{x}$ is derived by Example 3.5.

Corollary 4.6. $A^{'} \in (A, \boldsymbol{b})_{\min}$ *if and only if*

a) For all $j \notin J^{'}$, $a^{'}_{ij} \in [0, b_i]$ *for all* $i \in I$.

b) For all $j \in J^{'}$, *we have*

$$a^{'}_{ij} \in \begin{cases} [0, b_i] & i \notin \overline{I}_j \\ [0, 1] & i \in \overline{I}_j \end{cases}$$

and $a^{'}_{ij} \in (b_i, 1]$ *for some* $i \in I^{'}_j$, *where* $I^{'}_j = \{i | a_{ij} > b_i = \overline{x}_j\}$ *and* $I^{'}_j \subseteq \overline{I}_j = \{i | b_i \geq \overline{x}_j\}$.

Proof. First, let $A^{'} \in (A, \mathbf{b})_{\min}$.

a) Let $j \notin J^{'}$. We have $\wedge_{i\in I} \min^{-}(a^{'}_{ij}, b_i) = 1$ that implies for all $i \in I$, $\min^{-}(a^{'}_{ij}, b_i) = 1$. So, by (6), $a^{'}_{ij} \leq b_i$ for all $i \in I$. That means $a^{'}_{ij} \in [0, b_i]$ for all $i \in I$.

b) Let $j \in J^{'}$. We have $\wedge_{i\in I} \min^{-}(a^{'}_{ij}, b_i) = \overline{x}_j$. Thus, $\min^{-}(a^{'}_{ij}, b_i) \geq \overline{x}_j$ for all $i \in I$ and there exist some $i_. \in I$ such that $\min^{-}(a^{'}_{i_. j}, b_{i_.}) = \overline{x}_j$. But, we have $\min^{-}(a^{'}_{ij}, b_i) \geq \overline{x}_j$ if and only if $a^{'}_{ij} \leq b_i$ or $b_i \geq \overline{x}_j$. So, $a^{'}_{ij} \in [0, b_i]$ if $i \notin \overline{I}_j$ and $a^{'}_{ij} \in [0, 1]$ if $i \in \overline{I}_j$. On the other hand, $\min^{-}(a^{'}_{ij}, b_i) = \overline{x}_j < 1$ if and only if $a^{'}_{ij} > b_i$ for some i and $b_i < 1$. So, $a^{'}_{ij} \in (b_i, 1]$ for some $i \in I^{'}_j$.

The converse part is similar to that of Theorem 4.2. □

Example 4.7. Consider the following system

$$\begin{bmatrix} 0.3 & 0.5 \\ 0.4 & 0.6 \end{bmatrix} \circ^{\min} \mathbf{x} \leq \begin{bmatrix} 0.3 \\ 0.8 \end{bmatrix}. \tag{9}$$

We have $\overline{x}_1 = 1$ and $\overline{x}_2 = 0.3$. Therefore, $I_2^{'} = \{1\}, \overline{I}_2 = \{1, 2\}$. So, $A^{'} \in (A, \mathbf{b})_{\min}$ if and only if $A^{'} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where $a \in [0, 0.3]$, $b \in (0.3, 1]$, $c \in [0, 0.8]$ and $d \in [0, 1]$.

We see how the original matrix A can be perturbed into $A^{'}$, not influencing the solution set $[\mathbf{0}, \overline{\mathbf{x}}]$.

Remark 4.8. We shall denote the set of all matrices which don't have any solution with respect to the given and fixed right hand side vector $\mathbf{b}$ by $(A, \mathbf{b}, \emptyset)_F$. Also, we will use the notation $(\overline{A, \mathbf{b}})_F$ for $(\bigcup_{\mathbf{b} \in \mathbb{B}} (A, \mathbf{b})_F) \cup (\bigcup_{\mathbf{b} \in \mathbb{B}} (A, \mathbf{b}, \emptyset)_F)$ where, $\mathbb{B}$ is the set of all $m \times 1$ vectors with values from $[0, 1]$.

Theorem 4.9. *Let K be a semicopula. Then, $(A, \boldsymbol{b}, \emptyset)_K = \emptyset$.*

Proof. By Lemma 3.1, (2) is not solvable if and only if $K(a_{ij}, 0) > b_i$ for some $i \in I$ and some $j \in J$. So, System (2) is not solvable if and only if $0 > b_i$ for some $i \in I$. Which is not possible because $b_i \geq 0$ for all $i \in I$. So, System (2) in which F is a semicopula is always solvable and $\mathbf{0}$ is it's solution. So, $(A, \mathbf{b}, \emptyset)_K = \emptyset$. □

Theorem 4.10. *$A^{'} \in (A, \boldsymbol{b}, \emptyset)_{AM}$ if and only if $a_{ij}^{'} \in (2b_i, 1]$ for some $i \in I$ and some $j \in J$.*

Proof. It follows directly from Lemma 3.1. □

Example 4.11. Consider the following system

$$A \circ^{AM} \mathbf{x} \leq \begin{bmatrix} 0.3 \\ 0.8 \end{bmatrix}.$$

Then, by Theorem 4.10, each $A^{'} \in (A, \mathbf{b}, \emptyset)_{AM}$ has the following general form

$$\begin{bmatrix} a & c \\ b & b \end{bmatrix} \text{ or } \begin{bmatrix} c & a \\ b & b \end{bmatrix} \text{ or } \begin{bmatrix} a & a \\ b & b \end{bmatrix}$$

where, $a \in (0.6, 1]$, $b \in [0, 1]$ and $c \in [0, 0.6]$.

Now, let $\mathbb{A}$ be the set of all $m \times n$ matrices with values from $[0, 1]$. Let $A_1, A_2 \in \mathbb{A}$, we say $A_1 \sim_{\mathbf{b}} A_2$ if and only if $A_1, A_2 \in (A, \mathbf{b})_F$ or $A_1, A_2 \in (A, \mathbf{b}, \emptyset)_F$ for some matrix $A \in \mathbb{A}$. Clearly, relation $\sim_{\mathbf{b}}$ is an equivalence relation so, $\{(\overline{A, \mathbf{b}})_F\}$ is a partition of $\mathbb{A}$ related to $\sim_{\mathbf{b}}$. Using these partitions, once we solve System (2) we will find the solution of a group of systems and it can decrease the number of calculations.

Example 4.12. Consider the following system

$$A \circ^{\min} \mathbf{x} \leq \begin{bmatrix} 0.3 \\ 0.8 \end{bmatrix}.$$

We have nine different possible maximal solutions based on the elements of matrix A as follows

$$A^1 = \begin{bmatrix} [0,0.3] & (0.3,1] \\ [0,0.8] & [0,1] \end{bmatrix}, \overline{x}^1 = \begin{bmatrix} 1 \\ 0.3 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} (0.3,1] & [0,0.3] \\ [0,1] & [0,0.8] \end{bmatrix}, \overline{x}^2 = \begin{bmatrix} 0.3 \\ 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} [0,0.3] & (0.3,1] \\ (0.8,1] & [0,1] \end{bmatrix}, \overline{x}^3 = \begin{bmatrix} 0.8 \\ 0.3 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} (0.3,1] & [0,0.3] \\ [0,1] & (0.8,1] \end{bmatrix}, \overline{x}^4 = \begin{bmatrix} 0.3 \\ 0.8 \end{bmatrix}$$

$$A^5 = \begin{bmatrix} [0,0.3] & [0,0.3] \\ [0,0.8] & (0.8,1] \end{bmatrix}, \overline{x}^5 = \begin{bmatrix} 1 \\ 0.8 \end{bmatrix}$$

$$A^6 = \begin{bmatrix} [0,0.3] & [0,0.3] \\ (0.8,1] & [0,0.8] \end{bmatrix}, \overline{x}^6 = \begin{bmatrix} 0.8 \\ 1 \end{bmatrix}$$

$$A^7 = \begin{bmatrix} (0.3,1] & (0.3,1] \\ [0,1] & [0,1] \end{bmatrix}, \overline{x}^7 = \begin{bmatrix} 0.3 \\ 0.3 \end{bmatrix}$$

$$A^8 = \begin{bmatrix} [0,0.3] & [0,0.3] \\ (0.8,1] & (0.8,1] \end{bmatrix}, \overline{x}^8 = \begin{bmatrix} 0.8 \\ 0.8 \end{bmatrix}$$

$$A^9 = \begin{bmatrix} [0,0.3] & [0,0.3] \\ [0,0.8] & [0,0.8] \end{bmatrix}, \overline{x}^9 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

5. Solution-set Invariant Vectors

In this section, we analyze another kind of sensitivity in which A is fixed. In this procedure, the amount of perturbation which can be imposed on the right hand side vector $\mathbf{b}$ without any changes in the solution set of (2) is determined. In other words, we want to determine the set of all vectors which has the same solution set with respect to the given and fixed matrix A. We shall denote this set of solution-set invariant vectors by $(\mathbf{b}, A)_F$.

Lemma 5.1. *$\boldsymbol{b}^{'} \in (\boldsymbol{b}, A)_F$ if and only if $\overline{\boldsymbol{x}}$ is the maximal solution of*

$$Ao^F \boldsymbol{x} \leq \boldsymbol{b}^{'}. \tag{10}$$

Proof. It is obvious, according to the form of solution set. □

According to Lemma 5.1, in order to find $(\mathbf{b}, A)_F$, it is enough to determine all vectors which $\overline{\mathbf{x}}$ is the maximal solution of the associated system of fuzzy relation inequalities.

Theorem 5.2. *$\boldsymbol{b}^{'} \in (\boldsymbol{b}, A)_F$ if and only if $b_i^{'} \in [F(a_{ij}, \overline{x}_j), 1]$ for all $j \in J$ and all $i \in I$ and also, for all $j \in J^{'}$ there exist some $i \in I_j \subseteq I$ such that $F^{-}(a_{ij}, b_i^{'}) = \overline{x}_j$.*

Proof. The proof is similar to that of Theorem 4.2. □

Remark 5.3. a) Note that, for some fixed $a, c \in [0,1]$, the equation $F^-(a,b) = c$ need not have any solution b. For example, considering $F = \min$, i.e.,

$$F^-(a,b) = \begin{cases} 1 & a \le b, \\ b & a > b \end{cases}$$

it is obvious that, if $a \in [0,1[$ and $c = a$, $F^-(a,b) = a$ does not hold for any $b \in [0,1]$.

b) Due to the fact that $\overline{x}_j = \wedge_{i\in I} F^-(a_{ij}, b_i)$, it is obvious that for each j there is some i such that $\overline{x}_j = F^-(a_{ij}, b_i^{'})$, where $b_i^{'} = b_i$.

Example 5.4. Consider the following system

$$\begin{bmatrix} 0.3 & 0.5 \\ 0.4 & 0.6 \end{bmatrix} \circ^{\min} \mathbf{x} \le \begin{bmatrix} 0.4 \\ 0.3 \end{bmatrix}.$$

By Theorem 5.2 we have $\mathbf{b}^{'} \in (\mathbf{b}, A)_{\min}$ if and only if $\mathbf{b}^{'} = \begin{bmatrix} a \\ 0.3 \end{bmatrix}$ where, $a \in [0.3, 1]$. Hence, we can modify the original value $b_1 = 0.4$ into any value $b_1^{'} \in [0.3, 1]$ not influencing the solution set.

For the case $F = AM$, we can determine $(\mathbf{b}, A)_F$ more specifically.

Corollary 5.5. *If $\boldsymbol{b}^{'} \in (\boldsymbol{b}, A)_{AM}$ then, $b_i^{'} \in [\vee_{j\in J}\{\frac{1}{2}(\overline{x}_j + a_{ij})\}, 1]$ for all $i \in I$.*

Proof. By (7) and Theorem 5.2, the proof follows immediately. □

Lemma 5.6. *If $\boldsymbol{b}^{'} \in (\boldsymbol{b}, A)_{AM}$ then, $J^{'} = \cup_{i\in I} J_i$ where, $J_i = \{j^{'} \in J^{'} \,|\, \vee_{j\in J} \frac{1}{2}(\overline{x}_j + a_{ij}) = \frac{1}{2}(\overline{x}_{j^{'}} + a_{ij^{'}})\}$ for all $i \in I$.*

Proof. It is clear that $\cup_{i\in I} J_i \subseteq J^{'}$. Let $j^{'} \in J^{'}$ be arbitrary. Since $\overline{x}_{j^{'}} < 1$ then, $\overline{x}_{j^{'}} = \wedge_{i\in I}\{2b_i^{'} - a_{ij^{'}}\}$ and so, there exist $i^{'} \in I$ such that $\overline{x}_{j^{'}} = 2b_{i^{'}}^{'} - a_{i^{'}j^{'}}$. But, by Corollary 5.5, $\frac{1}{2}(\overline{x}_{j^{'}} + a_{i^{'}j^{'}}) = b_{i^{'}}^{'} \ge \vee_{j\in J}\{\frac{1}{2}(\overline{x}_j + a_{i^{'}j})\} \ge \frac{1}{2}(\overline{x}_{j^{'}} + a_{i^{'}j^{'}})$. Thus, $j^{'} \in J_{i^{'}}$. Therefore, $J^{'} \subseteq \cup_{i\in I} J_i$. □

The following theorem determines completely the elements of $(\mathbf{b}, A)_{AM}$.

Theorem 5.7. *$\boldsymbol{b}^{'} \in (\boldsymbol{b}, A)_{AM}$ if and only if for all $i \in I$, $b_i^{'} \in [\vee_{j\in J}\{\frac{1}{2}(\overline{x}_j + a_{ij})\}, 1]$ and for all $j \in \cup_{i\in I} J_i$ there exist $i \in I_j^{'}$ such that $b_i^{'} = \frac{1}{2}(\overline{x}_j + a_{ij})$ where, $I_j^{'} = \{i \in I | j \in J_i\}$.*

Proof. Let $\mathbf{b}^{'} \in (\mathbf{b}, A)_{AM}$. By Corollary 5.5, for all $i \in I$, $b_i^{'} \in [\vee_{j\in J}\{\frac{1}{2}(\overline{x}_j + a_{ij})\}, 1]$. Now, let $j^{'} \in \cup_{i\in I} J_i$ be arbitrary and fixed. By Corollary 5.5, $b_i^{'} \ge \frac{1}{2}(\overline{x}_{j^{'}} + a_{ij^{'}})$ for all $i \in I$. Since by Lemma 5.6, $j^{'} \in J^{'}$ then, $\overline{x}_{j^{'}} < 1$. Hence, $\overline{x}_{j^{'}} = \wedge_{i\in I}\{2b_i^{'} - a_{ij^{'}}\}$ and so, there exist $i^{'} \in I$ such that $\overline{x}_{j^{'}} = 2b_{i^{'}}^{'} - a_{i^{'}j^{'}}$. This implies $b_{i^{'}}^{'} = \frac{1}{2}(\overline{x}_{j^{'}} + a_{i^{'}j^{'}})$. Now, we show that $i^{'} \in I_{j^{'}}^{'}$. By Corollary 5.5, $b_{i^{'}}^{'} \in [\vee_{j\in J}\{\frac{1}{2}(\overline{x}_j + a_{i^{'}j})\}, 1]$. Therefore, $\frac{1}{2}(\overline{x}_{j^{'}} + a_{i^{'}j^{'}}) \le \vee_{j\in J}\{\frac{1}{2}(\overline{x}_j + a_{i^{'}j})\} \le b_{i^{'}}^{'} = \frac{1}{2}(\overline{x}_{j^{'}} + a_{i^{'}j^{'}})$. So, $i^{'} \in I_{j^{'}}^{'}$. Conversely, let for all $i \in I$, $b_i^{'} \in [\vee_{j\in J}\{\frac{1}{2}(\overline{x}_j + a_{ij})\}, 1]$. Fix $j \in J$. We have,

$b_i^{'} \in [\frac{1}{2}(\overline{x}_j + a_{ij}), 1]$ for all $i \in I$. Therefore, $2b_i^{'} - a_{ij} \geq \overline{x}_j$ for all $i \in I$. So, $\wedge_{i \in I}\{2b_i^{'} - a_{ij}\} \geq \overline{x}_j$. If $j \notin J^{'}$ then, $\overline{x}_j = 1$. Hence, $\wedge_{i \in I}\{2b_i^{'} - a_{ij}\} \geq 1$ and so, $\wedge\{1, \wedge_{i \in I}\{2b_i^{'} - a_{ij}\}\} = 1 = \overline{x}_j$. If $j \in J^{'}$ then, $j \in \cup_{i \in I} J_i$ and there exist $i \in I_j^{'}$ such that $b_i^{'} = \frac{1}{2}(\overline{x}_j + a_{ij})$. So, $\wedge_{i \in I}\{2b_i^{'} - a_{ij}\} = \overline{x}_j$ and then, $\wedge\{1, \wedge_{i \in I}\{2b_i^{'} - a_{ij}\}\} = \overline{x}_j$. Therefore, $\mathbf{b}^{'} \in (\mathbf{b}, A)_{AM}$. □

Remark 5.8. We shall denote the set of all vectors which don't have any solution with respect to the given and fixed matrix A by $(\mathbf{b}, A, \emptyset)_F$. Also, we will use notation $(\overline{\mathbf{b}, A})_F = (\bigcup_{A \in \mathbb{A}} (\mathbf{b}, A)_F) \cup (\bigcup_{A \in \mathbb{A}} (\mathbf{b}, A, \emptyset)_F)$.

Theorem 5.9. *Let K be a semicopula. Then, $(\boldsymbol{b}, A, \emptyset)_K = \emptyset$.*

Proof. The proof is similar to the that of Theorem 4.9. □

Theorem 5.10. $\boldsymbol{b}^{'} \in (\boldsymbol{b}, A, \emptyset)_{AM}$ *if and only if* $b_i^{'} \in [0, \frac{1}{2}a_{ij})$ *for some* $i \in I$ *and some* $j \in J$.

Proof. It follows from Lemma 3.1. □

Now, let $\mathbf{b}_1, \mathbf{b}_2 \in \mathbb{B}$, we say $\mathbf{b}_1 \sim_A \mathbf{b}_2$ if and only if $\mathbf{b}_1, \mathbf{b}_2 \in (\mathbf{b}, A)_F$ or $\mathbf{b}_1, \mathbf{b}_2 \in (\mathbf{b}, A, \emptyset)_F$ for some vector $\mathbf{b}$. Clearly, relation $\sim_A$ is an equivalence relation so, $\{(\overline{\mathbf{b}, A})_F\}$ is a partition of $\mathbb{B}$ related to $\sim_A$. Using these partitions, once we solve System (2) we will find the solution of a group of systems and it can decrease the number of calculations.

6. Conclusions

We have studied the max-F composition inequalities system $A \circ^F \mathbf{x} \leq \mathbf{b}$ from the stability of solution set point of view. We have completely characterized the set $(A, \mathbf{b})_F$ of all matrices $A^{'}$ such that $A^{'} \circ^F \mathbf{x} \leq \mathbf{b}$ if and only if $A \circ^F \mathbf{x} \leq \mathbf{b}$. Similarly, for a fixed matrix A we have characterized the set $(\mathbf{b}, A)_F$ of all perturbed vectors $\mathbf{b}^{'}$ such that $A \circ^F \mathbf{x} \leq \mathbf{b}^{'}$ if and only if $A \circ^F \mathbf{x} \leq \mathbf{b}$. Our results show the robustness of max-F composition based fuzzy relation inequalities system. Based on obtained results, we have defined two equivalence relations that can be used for decreasing the number of calculations specially when the dimension of problem is high. For the future, we aim to study the stability of fuzzy relation equations systems and also, other types of fuzzy relation inequalities systems.

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