Fuzzy Ordered Sets and Duality for Finite Fuzzy Distributive Lattices

A. Amroune and B. Davvaz

Abstract. The starting point of this paper is given by Priestley’s papers, where a theory of representation of distributive lattices is presented. The purpose of this paper is to develop a representation theory of fuzzy distributive lattices in the finite case. In this way, some results of Priestley’s papers are extended. In the main theorem, we show that the category of finite fuzzy Priestley spaces is equivalent to the dual of the category of finite fuzzy distributive lattices. Several examples are also presented.

1. Introduction

The study of fuzzy relations was started by Zadeh [17] in 1971. In that celebrated paper the author introduced the concept of fuzzy relation, defined the notion of equivalence, and gave the concept of fuzzy orderings. The concept of fuzzy order was introduced by generalizing the notion of reflexivity, antisymmetry and transitivity, there by facilitating the derivation of known results in various areas and stimulating the discovery of new ones. Fuzzy orderings have broad utility. They can be applied, for example, when expressing our preferences with a set of alternatives.

Since then many notions and results from the theory of ordered sets have been extended to the fuzzy ordered sets. In [16], Venugopalan introduced a definition of fuzzy ordered set (foset) \((P, \mu)\) and presented an example on the set of positive integers. He extended this concept to obtain a fuzzy lattice in which he defined a (fuzzy) relation as a generalization of equivalence. The notion of a multichain in a fuzzy ordered set is defined in [1]. In [14], Šešelja and Tepavčević presented a survey on representations of ordered structures by fuzzy sets. An order relation and a ranking method for type-2 fuzzy values are proposed in [10]. See also [3, 6, 8, 9, 13].

In a series of papers, Priestley [11, 12] gave a theory of representation of distributive lattices. In this paper, we extend some results of [11, 12], more precisely we give a representation theory of fuzzy distributive lattices in the finite case.

This paper is organized as follows: In the next section, basic definitions and notions are presented. In the third section, we give and prove the main result using a definition of fuzzy ordering admitting the minimum \(t\)-norm. The result can be generalized to any \(t\)-norm as introduced in [3]. Using the previous result of Priestley [12], this extension is obtained in a natural way.

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2. Preliminaries

There are two types of relations which often arise in mathematics: order relations and equivalence relations. An order relation is a generalization of both set inclusion and the order relation on the real line. In this section, we recall some definitions and concepts that we shall need in the sequel.

Let $X$ be a non-empty set. A fuzzy set $R$ on $X \times X$ (i.e., $X \times X \rightarrow [0,1]$ mapping) is called a fuzzy binary relation on $X$. A fuzzy binary relation $R$ on $X$ is called

1. reflexive, if $R(x, x) = 1$, for all $x \in X$,
2. antisymmetric, if $R(x, y) \land R(y, x) = 0$ whenever $x \neq y$, for all $x, y \in X$,
3. transitive, if $R(x, y) \land R(y, z) \leq R(x, z)$, for all $x, y, z \in X$.

A reflexive and transitive fuzzy relation is called a fuzzy preordering. Moreover, a fuzzy preordering which is antisymmetric, is called a fuzzy ordering relation.

A set equipped with a fuzzy order relation is called a fuzzy ordered set (foset).

Let $R$ be a fuzzy binary relation on a set $X$. The domain of $R$ is denoted by $\text{Dom}R$, whose membership function is defined by:

$$\text{Dom}R(x) = \bigvee_{y \neq x} \{ R(x, y) \mid y \in X \}.$$

Similarly, the range of $R$ is denoted by $\text{Ran}R$ and is defined by:

$$\text{Ran}R(y) = \bigvee_{x \neq y} \{ R(x, y) \mid x \in X \}.$$

The height of $R$ is denoted by $h(R)$ and is defined by:

$$h(R) = \bigvee_{\{(x, y)\mid x \neq y\}} \{ R(x, y) \}.$$

Let $X$ be a foset and $x \in X$. The fuzzy set $(\downarrow x)$ on $X$ is defined by:

$$(\downarrow x)(y) = R(y, x), \text{ for all } y \in X.$$

On the other hand $(\uparrow x)$ denotes the fuzzy set on $X$ which is given by:

$$(\uparrow x)(y) = R(x, y), \text{ for all } y \in X.$$

If $A$ is a subset of $X$, then we define

$$\uparrow A = \cup_{x \in A} (\uparrow x) \text{ and } \downarrow A = \cup_{x \in A} (\downarrow x).$$

Now, we recall the definition of lower and upper bounds, respectively, from [16, 17]. Let $A$ be a subset of a foset $X$. The upper bound $U(A)$ of $A$ is the fuzzy set on $X$ defined as follows:

$$U(A)(y) = \begin{cases} 0 & \text{if } R(x, y) = 0 \text{ for all } x \in A, \\ \land_{x \in A} R(x, y) & \text{otherwise}. \end{cases}$$

The lower bound $L(A)$ of $A$ is the fuzzy set on $X$ defined as follows:

$$L(A)(y) = \begin{cases} 0 & \text{if } R(y, x) = 0 \text{ for all } x \in A, \\ \land_{x \in A} R(y, x) & \text{otherwise}. \end{cases}$$

When $U(A)(y) \succ 0$ we write $y \in U(A)$. Similarly, for $L(A)$.

Let $E$ be a (crisp) subset of a non empty foset $X$. 
Lemma 3.1. and \( R \) (\( \{ 0 \} \) -lattice, then its dual space is defined by: 
\[
T(A) = (X, \tau, R_1)
\]
where 
\[
L(\delta) = \{ Y \subseteq X \mid Y \text{ is increasing and } \tau\text{-clopen} \}
\]
and \( r_1 \) is a fuzzy order adequately chosen.

**Lemma 3.1.** If \( (A, \lor, \land, R) \) is an \( F \)-D-lattice, then there exists two fuzzy orders \( R_1, R_2 \) such that:

1. \( T(A) = (X, \tau, R_1) \) is a fuzzy Priestley space,
2. \( (L(T(A)), \lor, \land, R_2) \) is an \( F \)-D-lattice.
Proof. (1) Let \( R_1 \) be such that:
\[
R_1(f, g) = \begin{cases} 
R \left( \land g^{-1}(1), \land f^{-1}(1) \right) & \text{if } f^{-1}(1) \subset g^{-1}(1) \\
0 & \text{otherwise},
\end{cases}
\]
where the symbol \( \land \) stands for an infimum with respect to the fuzzy relation \( R \).
We show that \( R_1 \) is a fuzzy order. We have \( R_1(f, f) = R \left( \land f^{-1}(1), \land f^{-1}(1) \right) = R(a, a) = 1 \) for all \( f \in X \) (\( R \)-reflexivity).
For all \( f, g \in X \) such that \( f \neq g \):
\[
R(f, g) \land R(g, f) = R \left( \land g^{-1}(1), \land f^{-1}(1) \right) \land R \left( \land f^{-1}(1), \land g^{-1}(1) \right) = R(a, b) \land R(b, a) = 0
\]
(because \( a \neq b \) otherwise, \( f = g \) (\( R \)-antisymmetry).
Now, for all \( f, g, h \in X \), we show that
\[
R_1(f, g) \land R_1(g, h) \leq R_1(f, h).
\]
The only case for investigating is
\[
f^{-1}(1) \subset g^{-1}(1) \text{ and } g^{-1}(1) \subset h^{-1}(1).
\]
By the transitivity of \( R \), for every \( a, b, c \) in \( A \), we have \( R(a, b) \land R(b, c) \leq R(a, c) \).
This yields
\[
R \left( \land g^{-1}(1), \land f^{-1}(1) \right) \land R \left( \land f^{-1}(1), \land g^{-1}(1) \right) \leq R \left( \land h^{-1}(1), \land f^{-1}(1) \right).
\]
The last inequality is true for every \( b \in A \). Then for all \( f, g, h \in X \), \( R_1(f, g) \land R_1(g, h) \leq R_1(f, h) \) holds, and \( R_1 \) is transitive. So, \( R_1 \) is a fuzzy order and by [11], \( T(A) = (X, \tau, R_1) \) is a Priestley space.

(2) Let \( m_0 = \land_x \land_y \{ r(x, y) \mid x \neq y \text{ and } r(x, y) > 0 \} \). We define \( R_2 \) by:
\[
R_2(H, D) = \begin{cases} 
1 & \text{if } H = D \\
R \left( \land_f f^{-1}(1), \land_g g^{-1}(1) \right) & \text{if } H \subset D \text{ and } H \neq \emptyset \\
m_0 & \text{if } H = \emptyset \\
0 & \text{otherwise},
\end{cases}
\]
where the symbol \( \land \) stands for an infimum with respect to the fuzzy relation \( R \).
First, we show that \( R_2 \) is a fuzzy order. We have \( R_2(A, A) = 1 \) (\( R \)-reflexivity) and \( R_2(A, B) \land R_2(B, A) = 0 \) whenever \( A \neq B \), i.e., \( R_2 \) is antisymmetric. In order to show the transitivity, we use the following truth table, where the proposition \( D \) is
\[
R_2(A, B) \land R_2(B, C) \leq R_2(A, C).
\]
Then $R_2$ is transitive.

Finally, the upper and lower bounds of $A$ and $B$ are denoted by $A \lor B$ and $A \land B$, respectively, and they are equal to $A \cup B$ and $A \cap B$, respectively. This shows that $(L(\delta), \lor, \land, R_2)$ is an $F$-$D$-lattice.

Note that in order to see the role of the topology in the proof of Lemma 3.1, it is sufficient to see that the dual of the Priestley space, $L(T(A))$ is defined by this topology, i.e., $L(T(A)) = \{Y \subset X : Y$ is increasing and $\tau$–clopem\}.

**Lemma 3.2.** If $\delta = (X, \tau, r)$ is a fuzzy finite Priestley space, then there exists two fuzzy orders $r_1$ and $r_2$ such that:

1. $(L(\delta), \lor, \land, r_1)$ is a $F$-$D$-lattice,
2. $(T(L(\delta)), \tau, r_2)$ is a fuzzy Priestley space.

**Proof.** (1) If $h(r) = 0$, then $X$ is an antichain and we can write $r_1$ as follows:

\[
r_1(A, B) = \begin{cases} 
1 & \text{if } A = B, \\
1 - \frac{\text{card } A}{\text{card } #} & \text{if } A \subset B, \\
0 & \text{otherwise}.
\end{cases}
\]

It is easy to show that $r_1$ is a fuzzy order and $A \lor B = A \cup B$ and $A \land B = A \cap B$ for every $A$ and $B$ from $L(\delta)$, where $(L(\delta), \lor, \land, r_1)$ is a fuzzy distributive lattice.

If $h(r) \neq 0$, then $X$ is not an antichain, we choose

\[m_0 = \land_x \land_y \{\mu_r(x, y) \mid x \neq y \text{ and } \mu_r(x, y) > 0\}.
\]

Then $m_0 \neq 0$ and we can take $r_1$ such that

\[
r_1(A, B) = \begin{cases} 
1 & \text{if } A = B, \\
\text{Max} \left(m_0, \lor_{a \in A, b \in B} r(a, b)\right) & \text{if } A \subset B, \\
0 & \text{otherwise}.
\end{cases}
\]

Similar to the previous lemma, $r_1$ is a fuzzy order and we can assume that $A \lor B = A \cup B$ and $A \land B = A \cap B$ for every $A$ and $B$ from $L(\delta)$, where $(L(\delta), \lor, \land, r_1)$ is
a fuzzy distributive lattice. For the second assertion, let
\[ r_2(f, g) = \begin{cases} 
1 & \text{if } f = g, \\
r \left( \bigwedge_{A \in f^{-1}(1)} A, \bigwedge_{B \in g^{-1}(1)} B \right) & \text{if } f^{-1}(1) \subset g^{-1}(1), \\
0 & \text{otherwise},
\end{cases} \]

where the first infimum \( \bigwedge \) is in the sense of the fuzzy relation \( r \) and the second infimum \( \bigwedge \) is in the sense of the fuzzy relation \( r_1 \). Note that \( r_2 \) is well defined:
\[ A_1 = \bigwedge_{A \in f^{-1}(1)} A, \]

where the symbol \( \bigwedge \) stands for an infimum with respect to the fuzzy relation \( r_1 \), it exists because \( L(\delta) \) is a lattice and \( a = A_1 \), where the symbol \( \bigwedge \) stands for an infimum with respect to the fuzzy relation \( r \), is unique, otherwise \( A_1 \) cannot be the minimal element of \( f^{-1}(1) \) \[11\]. Furthermore, \((T(L(\delta)), \tau, r_2)\) is a fuzzy Priestley space.

The following theorem shows that the category of finite fuzzy Priestley spaces is equivalent to the dual of the category of finite fuzzy distributive lattices.

In \[11\], Priestley remarked that the basis can be characterized by the fact that they are increasing according to inclusion of prime filters from \( A \) by taking the sets \( \{F_a : a \in A\} \) as basis, where \( F_A \) is the set of all lattice homomorphisms from \( A \) onto the chain \( \{0, 1\} \), non-identical mulls (taking 1 in \( a \)).

**Theorem 3.3.**

1. Let \( A \) be an \( F \)-D-lattice. The map \( F_A : A \rightarrow L(T(A)) \)
defined by \( F_A(a) = \{f \in X \mid f(a) = 1\} \) is a fuzzy lattice isomorphism.

2. If \( \delta = (X, \tau, r) \) is a finite Priestley space, then the map \( G_\delta : \delta \rightarrow T(L(\delta)) \)
defined by
\[ G_\delta(x)(Y) = \begin{cases} 
1 & \text{if } x \in Y, \\
0 & \text{if } x \notin Y,
\end{cases} \]

for all \( Y \in L(\delta) \) is an isomorphism of fuzzy Priestley space, i.e., a bijection and increasing map.

3. If \( f : A_2 \rightarrow A_2 \) is a fuzzy lattice homomorphism, then the map \( T(f) : T(A_2) \rightarrow T(A_2) \)
defined by \( T(f)(g) = g \circ f \) is a homomorphism of fuzzy Priestley space, i.e., a continuous and increasing map.

4. If \( h : \delta_1 \rightarrow \delta_2 \) is a homomorphism of fuzzy Priestley space, then the map \( L(h) : L(\delta_2) \rightarrow L(\delta_1) \)
defined by \( L(h)(y) = h^{-1}(y) \) for every \( y \in L(\delta_2) \) is a fuzzy lattice homomorphism.

5. If \( f \) and \( h \) are as in (3) and (4), then the following diagrams are commutative.

\[
\begin{array}{ccc}
A_1 & \xrightarrow{f_1} & A_2 \\
\downarrow_{F_{A_1}} & & \downarrow_{F_{A_2}} \\
L(T(A_1)) & \xrightarrow{T(f)} & L(T(A_2))
\end{array}
\]

and
Proof. (1) Let us show that the map $F_A(a) = \{ f \in X \mid f(a) = 1 \}$ is a fuzzy lattice isomorphism. We have $R(x, y) \leq R_2(F_A(x), F_A(y))$, where

$$R_2(F_A(x), F_A(y)) = \begin{cases} 1 & \text{if } F_A(x) = F_A(y) \\ R \left( \bigwedge_{f \in F_A(x)} f^{-1}(1), \bigwedge_{g \in F_A(y)} g^{-1}(1) \right) & \text{if } F_A(x) \subset F_A(y) \\ 0 & \text{otherwise}, \end{cases}$$

and the symbol $\land$ stands for an infimum with respect to the fuzzy relation $R$. Note that if $R(x, y) \succ 0$, then $F_A(x) \subset F_A(y)$ which implies that $R_2(F_A(x), F_A(y)) = R(x, y)$. This shows that $R(x, y) \leq R_2(F_A(x), F_A(y))$ and then the map $F_A$ is a fuzzy lattice isomorphism.

(2) According to [11], it suffices to show that $r(x, y) \leq r_2(G_\delta(x), G_\delta(y))$. If $x = y$, then

$$Z_0 = \bigcap_{A \in G_\delta^{-1}(x)(1)} A = \bigcap_{B \in G_\delta^{-1}(x)(1)} B = Z_1$$

and so $r(x, y) \leq r_2(G_\delta(x), G_\delta(y))$. If $x \neq y$, then there are two cases as follows:

- **Case 1**: if $r(x, y) = 0$, then we have $r(x, y) \leq r_2(G_\delta(x), G_\delta(y))$;
- **Case 2**: if $r(x, y) \succ 0$, then $y$ belongs to each $\tau$-clopen which contains $x$, so, $Z_0 \subset Z_1$, and then we have

$$r_2(G_\delta(x), G_\delta(y)) = r_2(\land Z_0, \land Z_1) = r(x, y),$$

where the symbol $\land$ stands for an infimum with respect to the fuzzy relation $r$. The remaining assertions are obtained by the same reasoning. □

Remark 3.4. We know that the minimum $t$-norm $T_M$ (Zadeh’s norm) [4] dominates any other $t$-norm. Then according to Lemma 2.4 (3) in [3], $T_M$ is stronger than any other $t$-norm. Consequently, the results can be extended to any other $t$-norm.

Example 3.5. Let $(A, \lor, \land, R)$ be a fuzzy distributive lattice, where $A = \{a, b, c, d, e, f\}$ and $R$ is a fuzzy relation defined by:
Then its dual is:

\[
T(A) = \text{The set of } 0\rightarrow 1 \text{ homomorphisms from } A \text{ onto } \{0, 1\} = \{f_1, f_2, f_3, f_4\},
\]
such that

\[
\begin{array}{c|cccccc}
A & f_1(x) & f_2(x) & f_3(x) & f_4(x) \\
\hline
a & 0 & 0 & 0 & 0 \\
b & 0 & 0 & 0 & 1 \\
c & 0 & 1 & 0 & 1 \\
d & 0 & 0 & 1 & 1 \\
e & 0 & 1 & 1 & 1 \\
f & 1 & 1 & 1 & 1 \\
\end{array}
\]

and its bidual is:

\[
L(T(A)) = \{\emptyset, \{f_4\}, \{f_2, f_4\}, \{f_3, f_4\}, \{f_2, f_3, f_4\}, X\},
\]

where \(R_2\) is given by:

\[
\begin{array}{c|cccccc}
R_2 & \emptyset & \{f_4\} & \{f_2, f_4\} & \{f_3, f_4\} & \{f_2, f_3, f_4\} & X \\
\hline
\emptyset & 1 & 0.1 & 0.3 & 0.3 & 0.5 & 0.7 \\
\{f_4\} & 0 & 1 & 0.2 & 0.2 & 0.4 & 0.7 \\
\{f_2, f_4\} & 0 & 0 & 1 & 0 & 0.2 & 0.6 \\
\{f_3, f_4\} & 0 & 0 & 0 & 1 & 0.2 & 0.3 \\
\{f_2, f_3, f_4\} & 0 & 0 & 0 & 0 & 1 & 0.3 \\
X & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{array}
\]

Finally, \(F_A : A \rightarrow L(T(A))\) is given by:

\[
\begin{array}{c|c}
A & F_A(a_i) \text{ for } i = 1 \text{ to } 6 \\
\hline
a & \emptyset \\
b & \{f_4\} \\
c & \{f_2, f_4\} \\
d & \{f_3, f_4\} \\
e & \{f_2, f_3, f_4\} \\
f & X \\
\end{array}
\]
Example 3.6. Let \((X, \tau, r)\) be a Priestley space, where \(X = \{x, y, z\}\) and \(r\) is given by:

\[
\begin{array}{cccc}
  r & x & y & z \\
  x & 1 & 0 & 0 \\
  y & 0 & 1 & 0 \\
  z & 0 & 0 & 1 \\
\end{array}
\]

Then \(L(X) = \{\emptyset, \{x\}, \{y\}, \{z\}, \{x, y\}, \{x, z\}, \{y, z\}, X\}\) and \(r_1\) is given by:

\[
r_1 (A, B) = \begin{cases} 
  1 & \text{if } A = B \\
  1 - \frac{\text{card}A}{\text{card}B} & \text{if } A \subset B \\
  0 & \text{otherwise.}
\end{cases}
\]

Then \(r_1\) will be given by:

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and the set of 0–1 homomorphisms from \(L(X)\) onto \(\{0, 1\}\), i.e., \(T(L(X))\) is equal to \(\{f_1, f_2, f_3\}\)

\[
L(X) | f_1 (X) | f_2 (X) | f_3 (X) \\
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And \(r_2\) will be given by:

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and the isomorphism \( G_X \) is defined by \( G_X : X \rightarrow T(L(X)) \), where

\[
\begin{array}{c|c|c|c|c|c}
X & G_X(X_i), X_i \in X \\
\hline
x & f_1 \\
\hline
y & f_2 \\
\hline
z & f_3 \\
\end{array}
\]

**Example 3.7.** Let \((X, \tau, r)\) be a Priestley space, where \(X = \{x, y, z, t\}\) and \(r\) is given by:

\[
\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c}
r & x & y & z & t \\
\hline
x & 1 & 0 & 0 & 0.3 \\
y & 0 & 1 & 0 & 0.4 \\
z & 0 & 0 & 1 & 0.7 \\
t & 0 & 0 & 0 & 1 \\
\end{array}
\]

and \(L(X) = \emptyset, \{t\}, \{x,t\}, \{y,t\}, \{z,t\}, \{x,y,t\}, \{x,z,t\}, \{y,z,t\}, X\) , where \(r_1\) is given by:

\[
\begin{array}{l|l|l|l|l|l|l|l|l|l|l|l|l}
r_1 & \emptyset & \{t\} & \{x,t\} & \{y,t\} & \{z,t\} & \{x,y,t\} & \{x,z,t\} & \{y,z,t\} & X \\
\hline
\emptyset & 1 & 0 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 \\
\{t\} & 0 & 1 & 0 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 \\
\{x,t\} & 0 & 0 & 1 & 0 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 \\
\{y,t\} & 0 & 0 & 0 & 1 & 0.4 & 0 & 0.7 & 0.7 & 0.7 \\
\{z,t\} & 0 & 0 & 0 & 0 & 1 & 0 & 0.7 & 0.7 & 0.7 \\
\{x,y,t\} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0.7 \\
\{x,z,t\} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0.7 \\
\{y,z,t\} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0.7 \\
X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{array}
\]

and \(T(L(X)) = \{f_1, f_2, f_3, f_4\}\) such that

\[
\begin{array}{l|l|l|l|l}
L(X) & f_1(X_i) & f_2(X_i) & f_3(X_i) & f_4(X_i) \\
\hline
\emptyset & 0 & 0 & 0 & 0 \\
\{t\} & 0 & 0 & 0 & 1 \\
\{x,t\} & 1 & 0 & 0 & 1 \\
\{y,t\} & 0 & 1 & 0 & 1 \\
\{z,t\} & 0 & 0 & 1 & 1 \\
\{x,y,t\} & 1 & 1 & 0 & 1 \\
\{x,z,t\} & 1 & 0 & 1 & 1 \\
\{y,z,t\} & 0 & 1 & 1 & 1 \\
X & 1 & 1 & 1 & 1 \\
\end{array}
\]
The isomorphism $G_X$ is defined as follows: $G_X : X \rightarrow T(L(X))$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>$f_2$</td>
</tr>
<tr>
<td>$z$</td>
<td>$f_3$</td>
</tr>
<tr>
<td>$t$</td>
<td>$f_4$</td>
</tr>
</tbody>
</table>

4. Conclusions and Open Problems

The Priestley duality comes from the classical Stone representation of distributive lattices.

Stone in [15], developed a representation theory for distributive lattices generalizing that for Boolean algebras. This he achieved by topologizing the set $X$ of prime ideals of a closed distributive lattice $A$ (with a first and last elements) by taking $\{I_a : a \in A\}$ as a base (where $I_a$ denotes the set of prime ideals of $A$ not containing $a$).

In 1970, H. A. Priestley developed a new duality for a closed distributive lattices by replacing $(I_a : a \in A)$ of prime ideals by $(F_a : a \in A)$ where $F_a$ is the set of all $0-1$ lattice homomorphisms from $A$ onto the chain $\{0, 1\}$ and taking $1$ in $a$.

The purpose of this paper is to establish a duality for closed finite fuzzy distributive lattices type Priestley extending the classical case. In this way, some results of [11, 12] are extended. The main theorem (Theorem 3.3) shows that the category of finite fuzzy Priestley spaces is equivalent to the dual of the category of finite fuzzy distributive lattices.

Now, we give two open problems:

(1) Is it possible to obtain such representation for an infinite fuzzy distributive lattice?

(2) Is it possible to obtain such representation if we change the definition of fuzzy set? In other words, what happens if we replace the unite interval $[0, 1]$ by any closed distributive lattice $L$?

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References


Abdelaziz Amroune*, Department of Mathematics, M’Sila University, P.O. Box 166, M’Sila 28000, Algeria
E-mail address: aamrounedz@yahoo.fr

Bijan Davvaz, Department of Mathematics, Yazd University, Yazd, Iran
E-mail address: davvaz@yazduni.ac.ir

*Corresponding author