ON ALGEBRAIC AND COALGEBRAIC CATEGORIES OF VARIETY-BASED TOPOLOGICAL SYSTEMS

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Abstract. Motivated by the recent study on categorical properties of lattice-valued topology, the paper considers a generalization of the notion of topological system introduced by S. Vickers, providing an algebraic and a coalgebraic category of the new structures. As a result, the nature of the category **TopSys** of S. Vickers gets clarified, and a metatheorem is stated, claiming that (lattice-valued) topology can be embedded into algebra.

1. Introduction

This paper studies a possible single framework in which to treat both variable-basis lattice-valued topological spaces [37] and the respective algebraic structures underlying their topologies. Originally suggested by both locale theory and many-valued set theory, the problem has a long history.

In 1959 D. Papert and S. Papert [29] presented an adjunction between the categories **Top** of topological spaces and $\mathbf{Frm}^{op}$, the dual of the category **Frm** of frames. The adjoint situation was described more succinctly by J. Isbell [20], who introduced the name *locale* for the objects of $\mathbf{Frm}^{op}$ and considered the category **Loc** of locales as a substitute for **Top**. In 1982 localic theory was given a coherent statement in the celebrated book of P. T. Johnstone "Stone Spaces" [22]. Using the logic of finite observations, S. Vickers [52] introduced the notion of *topological system* as a single framework for treating both spaces and locales (see also [53] for algebraic treatment of topology).

On the other hand, the pioneering papers of C. L. Chang [6], J. A. Goguen [14] and R. Lowen [26] started the theory of *fixed-basis* many-valued topological spaces. Using a point-free framework similar to localic theory, B. Hutton [19] proposed in 1980 the first variable-basis approach, which eventually resulted in a *variable-basis* category of *singleton* topological spaces. In 1983 S. E. Rodabaugh [35] introduced the first variable-basis category for topology in which the underlying sets of the spaces were non-singletons. Since then it is known as the category **C-Top** of variable-basis lattice-valued topological spaces [37]. Almost immediately, P. Eklund [12] began the study on categorical properties of variable-basis topology,

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thereby initiating categorical fuzzy topology, which has been developing rapidly ever since.

Both fixed- and variable-basis approaches induced many researchers to study their properties [8, 13, 16, 17, 21, 25, 41, 42, 43, 54]. In particular, [10, 15] considered functorial relationships between lattice-valued topology and topological systems. The use of *fuzzy* topological spaces and *crisp* topological systems appeared to be not flexible enough, that resulted in the concept of *lattice-valued topological system* over locales [9], significantly simplifying some results of [10]. For example, one easily produced an embedding of the category **Loc-Top** of lattice-valued topological spaces into the category **Loc-TopSys** of lattice-valued topological systems. This paper aims at providing a more general approach to the topic.

Motivated by our current slogan of doing fuzzy mathematics on arbitrary algebraic structures [44], we introduced in [49, 50, 51] the notions of *variety-based* topological space and topological system, generalizing the respective notions of [9, 37]. The cornerstone of the approach lies in replacing **Loc** with the dual category **LoA** of an arbitrary variety of algebras (in the obvious sense, as sets with operations, where no partial order is required) **A**. The new concepts appeared fruitful, i.e., that of system incorporated in itself not only the respective notion of S. Vickers, but also *state property systems* of D. Aerts [2] (serving as the basic mathematical structure in the Geneva-Brussels approach to foundations of physics) as well as *Chu spaces* (over sets) [30], whereas that of space provided a common framework for both the above-mentioned lattice-valued topological spaces of S. E. Rodabaugh and *closure spaces* [4] (made variable-basis on the way). Among other results, it was shown that the category **LoA-Top** of variety-based topological spaces is isomorphic to a full (regular mono)-coreflective subcategory of the category **LoA-TopSys** of variety-based topological systems. The relevance of the result is twofold: on one hand, it generalizes the above-mentioned embedding of **Loc-Top**, on the other, it provides an analogue of *spatialization procedure* for variety-based topological systems, introduced by S. Vickers [52] for the localic ones.

Developing the topic further, we introduced in [46] a variety-based modification of another procedure of S. Vickers [52] called *localification* of systems, the essence of which lies in representing the category of algebraic structures underlying the topologies (e.g., locales) as a particular subcategory of the category of topological systems. It appeared, however, that the machinery requires a modified category of systems, denoted **A-TopSys**. In the new setting, one can show that the product category $\mathbf{A} \times \mathbf{LoA}$ is isomorphic to a full reflective subcategory of the category **A-TopSys**, which (in general) is neither mono- nor epi-reflective. The simplest common point of both **LoA-TopSys** and **A-TopSys** is the fixed-basis approach generalizing the respective one proposed by S. Vickers.

It is important to notice that the word "topological" in the name chosen by S. Vickers for his structures suggested topological flavor in their behavior. Notwithstanding the expectations, J. T. Denniston and S. E. Rodabaugh [10] showed that the category **TopSys** of S. Vickers is not topological over its ground category, posing the question on its nature. In [51] we proved that the category **LoA-TopSys** is topological over its ground category if and only if the respective underlying functor is an isomorphism. The latter fact together with the result that the underlying functor of **LoA-TopSys** creates isomorphisms, gave rise to considering algebraic properties of the category in question. It is the aim of this paper to show that **LoA-TopSys** is essentially algebraic, whereas **A-TopSys** is coalgebraic over the respective ground category. As a consequence, we get that the motivating category **TopSys** is both algebraic and coalgebraic that suggests a possible change in the name of its objects, excluding the word "topological". Moreover, the above-mentioned embedding of **LoA-Top** into **LoA-TopSys** suggests a metatheorem claiming that (lattice-valued) topology can be embedded into algebra.

The necessary categorical background can be found in [1, 27, 28]. For algebraic notions we recommend [7, 28]. Although we tried to make the paper as much self-contained as possible, some details are still omitted and left to the reader.

2. **Variety-based Topological Spaces**

In this section, we recall the concept of *variety-based topological space*, which induced the notion of variety-based topological system. The concept provides a common framework for both variable-basis lattice-valued topological spaces [37] and closure spaces [4]. The approach stems from [49, 50] generalizing the respective one of S. E. Rodabaugh [37, 39]. For convenience of the reader, we begin with some algebraic preliminaries.

An algebra is to be thought of as a set with a family of operations defined on it, which satisfy certain identities, e.g., semigroup, monoid, group and also complete lattice, frame, quantale. In case of *finitary* algebras, i.e., algebras based on a set of finite operations, there exists the famous theorem of G. Birkhoff [5] representing them as *varieties*. This paper uses a modification of the concept, suitable to include infinitary cases as well and motivated by the notion of *equationally-definable class* [28].

Definition 2.1. Let $\Omega = (n_\lambda)_{\lambda \in \Lambda}$ be a (possibly proper) class of cardinal numbers. An Ω-*algebra* is a pair $(A, (\omega_\lambda^A)_{\lambda\in\Lambda})$ (denoted by A) consisting of a set A and a family of maps $A^{n_\lambda} \xrightarrow{\omega_\lambda^A} A$, called n_λ-*ary primitive operations* on A. An Ω-*homomorphism* $(A, (\omega_\lambda^A)_{\lambda\in\Lambda}) \xrightarrow{f} (B, (\omega_\lambda^B)_{\lambda\in\Lambda})$ is a map $A \xrightarrow{f} B$, making the diagram

$$\begin{array}{ccc} A^{n_\lambda} & \xrightarrow{f^{n_\lambda}} & B^{n_\lambda} \\ {\scriptstyle \omega_\lambda^A}\downarrow & & \downarrow{\scriptstyle \omega_\lambda^B} \\ A & \xrightarrow[f]{} & B \end{array}$$

commute for every $\lambda \in \Lambda$. The category of Ω-algebras and Ω-homomorphisms is denoted $\mathbf{Alg}(\Omega)$, with $|-|$ being the underlying functor to the ground category **Set** of sets and maps.

Let $\mathcal{M}$ (resp. $\mathcal{E}$) be the class of Ω-homomorphisms with injective (resp. surjective) underlying maps. A *variety of* Ω-*algebras* (also called a *variety*) is a full subcategory of $\mathbf{Alg}(\Omega)$ closed under the formation of products, $\mathcal{M}$-subobjects (subalgebras) and $\mathcal{E}$-quotients (homomorphic images). The objects (resp. morphisms) of a variety are called *algebras* (resp. *homomorphisms*).

As an example of varieties one can mention the constructs **Frm**, **SFrm** and **SQuant** of frames, semi-frames and semi-quantales (popular in lattice-valued topology) [31, 32, 33, 34, 39]. For convenience of the reader, as well as to feel free in using it throughout the paper, we recall the definition of the latter variety.

Definition 2.2. A *semi-quantale* (*s-quantale* for short) is a $\bigvee$-semilattice (partially ordered set having arbitrary $\bigvee$) equipped with a binary operation $\otimes$. An *s-quantale homomorphism* is a map preserving $\bigvee$ and $\otimes$. The variety of s-quantales and their homomorphisms is denoted **SQuant**.

It is important to underline that s-quantales are proposed in [39] as the basic mathematical structure for doing lattice-valued topology upon, since the obtained categories are topological over their ground categories. The crucial advantage of the new concept is that it incorporates the majority of lattice-like structures currently used in many-valued topology, e.g., the above-mentioned varieties **Frm** and **SFrm** are subcategories of **SQuant**. On the other hand, the structure in question is not so well-known as its counterpart *quantale*, which has two additional properties: associativity of $\otimes$ and its distributivity over $\bigvee$ from both sides [40].

From now on, we fix a variety $\mathbf{A}$ and use the following notations [10, 37, 39]. The dual of the category $\mathbf{A}$ (or, possibly, its subcategory) is denoted $\mathbf{LoA}$ (the "**Lo**" comes from "localic"). Its objects (resp. morphisms) are called *localic algebras* (resp. *homomorphisms*). Given a morphism φ of a variety $\mathbf{A}$, the respective morphism of $\mathbf{LoA}$ is denoted φ^{op} and vice versa. The reader should be aware that the notations for dual categories used in this paper follow the (already widely accepted) pattern of lattice-valued topology and not the category-theoretic one.

The cornerstone of our approach (just as in the classical case of [37]) are the so-called *image* and *preimage* operators (it was S. E. Rodabaugh [36, 38, 39] who fully realized their importance in topology). Given a map $X \xrightarrow{f} Y$, there exist the traditional image and preimage operators on the respective powersets $\mathcal{P}(X) \xrightarrow{f^{\rightarrow}} \mathcal{P}(Y)$ and $\mathcal{P}(Y) \xrightarrow{f^{\leftarrow}} \mathcal{P}(X)$ defined in the obvious way. Moreover, every algebra A provides the *fixed-basis* (*Zadeh*) *preimage operator* $A^Y \xrightarrow{f_A^{\leftarrow}} A^X$ defined by $f_A^{\leftarrow}(p) = p \circ f$ [55]. On the other hand, every homomorphism $A \xrightarrow{g} B$ can be lifted to a map $A^X \xrightarrow{g_{\rightarrow}^X} B^X$ defined by $g_{\rightarrow}^X(p) = g \circ p$ ([10, 36, 39] denote the latter map by $\langle g \rangle$). The next lemma contains an important property of the newly defined maps [50].

Lemma 2.3. *For every map $X \xrightarrow{f} Y$ and every homomorphism $A \xrightarrow{g} B$, both $A^Y \xrightarrow{f_A^{\leftarrow}} A^X$ and $A^X \xrightarrow{g_{\rightarrow}^X} B^X$ are homomorphisms.*

On the variable-basis side everything goes much the same. Suppose we are given a $\mathbf{Set} \times \mathbf{LoA}$-morphism (recall our notation for dual categories) $(X, A) \xrightarrow{(f,\varphi)} (Y, B)$. There exists the *variable-basis* (*Rodabaugh*) *preimage operator* $B^Y \xrightarrow{(f,\varphi)^{\leftarrow}} A^X$ defined by $(f, \varphi)^{\leftarrow}(p) = \varphi^{op} \circ p \circ f$ [36, 38, 39].

Lemma 2.4. *For every* $\mathbf{Set} \times \mathbf{LoA}$*-morphism* $(X, A) \xrightarrow{(f,\varphi)} (Y, B)$*, the diagram*

$$\begin{tikzcd} B^Y \arrow[r, "(\varphi^{op})^Y_{\rightarrow}"] \arrow[d, "f_B^{\leftarrow}"'] \arrow[dr, dotted, "(f,\varphi)^{\leftarrow}" description] & A^Y \arrow[d, "f_A^{\leftarrow}"] \\ B^X \arrow[r, "(\varphi^{op})^X_{\rightarrow}"'] & A^X \end{tikzcd}$$

commutes and, therefore, $B^Y \xrightarrow{(f,\varphi)^{\leftarrow}} A^X$ *is a homomorphism.*

It is important to notice that a more general approach to variety-based powerset theories has already been considered by the current author in [48]. Everything is on its place to introduce the category of variety-based topological spaces *à la* [37, 39].

Definition 2.5. Let $\mathbf{C}$ be a subcategory of $\mathbf{LoA}$. A $\mathbf{C}$-*topological space* ($\mathbf{C}$-*space* for short) is a triple $E = (\operatorname{pt} E, \Sigma\, E, \tau)$, where $(\operatorname{pt} E, \Sigma\, E)$ is a $\mathbf{Set} \times \mathbf{C}$-object and τ is a subalgebra of $(\Sigma\, E)^{\operatorname{pt} E}$ (called $\mathbf{C}$-*topology* on $(\operatorname{pt} E, \Sigma\, E)$). A $\mathbf{C}$-*continuous map* $E_1 \xrightarrow{f} E_2$ is a $\mathbf{Set} \times \mathbf{C}$-morphism $(\operatorname{pt} E_1, \Sigma\, E_1) \xrightarrow{f=(\operatorname{pt} f, (\Sigma\, f)^{op})} (\operatorname{pt} E_2, \Sigma\, E_2)$ such that $((\operatorname{pt} f, (\Sigma\, f)^{op})^{\leftarrow})^{\rightarrow}(\tau_2) \subseteq \tau_1$. $\mathbf{C}$-$\mathbf{Top}$ is the category of $\mathbf{C}$-topological spaces and $\mathbf{C}$-continuous maps, with the underlying functor $|-|$ to the ground category $\mathbf{Set} \times \mathbf{C}$ given by the formula $|E_1 \xrightarrow{f} E_2| = (\operatorname{pt} E_1, \Sigma\, E_1) \xrightarrow{(\operatorname{pt} f, \Sigma\, f)} (\operatorname{pt} E_2, \Sigma\, E_2)$.

The notations used in Definition 2.5 are different from those currently accepted in lattice-valued topology. The main reason for that is the fact (Theorem 3.7) that every space can be considered as a particular instance of topological system (Definition 3.1), the notations for which are already fixed (Remark 3.2).

In the following, we will consider a fixed-basis approach to variety-based topology and, therefore, we introduce its definition.

Definition 2.6. Given a $\mathbf{C}$-space E, $\Sigma\, E$ is called the *basis* of the space. Given a $\mathbf{C}$-object A, A-$\mathbf{Top}$ is the non-full subcategory of $\mathbf{C}$-$\mathbf{Top}$ with objects all spaces with the basis A and morphisms all $\mathbf{C}$-continuous maps f such that $\Sigma\, f = 1_A$.

The new notions can be illustrated by many examples, which clearly show their relations to the existing concepts in the literature.

Example 2.7. $\mathbf{Loc}$-$\mathbf{Top}$ is precisely the motivating example of lattice-valued topology from [9, 10]. The category 2-$\mathbf{Top}$ (2 being the two-element frame $\{\bot, \top\}$) provides the classical set-theoretic approach of, e.g., [23].

Example 2.8. $\mathbf{LoSQuant}$-$\mathbf{Top}$ provides the category for developing the topological theories of [39].

Example 2.9. Let $\mathbf{CQML}_{\vee}$ be the variety of *complete* $\vee$*-quasi-monoidal lattices* (cf. [37]), namely, s-quantales $(Q, \bigvee, \otimes)$ satisfying two additional conditions:

(1) $\otimes$ preserves finite $\bigvee$ in both arguments;
(2) $\top \otimes \top = \top$, where $\top$ is the upper bound of Q;

and $\top$-preserving s-quantale homomorphisms. The category $\mathbf{LoCQML}_{\vee}$**-Top** provides the classical approach to variable-basis topology of [37].

Example 2.10. Let $\mathbf{SLat}(\bigwedge, \perp)$ be the variety of $\{\bigwedge, \perp\}$-semilattices. The category $\mathbf{LoSLat}(\bigwedge, \perp)$**-Top** provides a variable-basis modification of the concept of *closure space* of [4]. In particular, **2-Top** (cf. the notations of Example 2.7) is isomorphic to the category **Cls** of closure spaces of [4]. Recall that a *closure space* $(X, \mathcal{F})$ consists of a set X and a family of subsets $\mathcal{F} \subseteq \mathcal{P}(X)$ satisfying the following conditions:

(1) $\varnothing \in \mathcal{F}$;
(2) if $(F_i)_{i \in I} \subseteq \mathcal{F}$, then $\bigcap_{i \in I} F_i \in \mathcal{F}$.

If $(X, \mathcal{F})$ and $(Y, \mathcal{G})$ are closure spaces, then a map $X \xrightarrow{f} Y$ is called *continuous* provided that $f^{\leftarrow}(G) \in \mathcal{F}$ for every $G \in \mathcal{G}$.

Although Definition 2.5 is provided in its general form, in this paper we restrict ourselves to the case $\mathbf{C} = \mathbf{LoA}$. For the sake of shortness, we call **LoA**-spaces by spaces and **LoA**-continuity by continuity.

3. Variety-based Topological Systems

In this section we introduce the main object of our study, i.e., the category of *variety-based topological systems*. The notion provides a common framework for topological systems introduced by S. Vickers [52], state property systems of D. Aerts [2] and Chu spaces (over sets) [30]. The approach comes from [45, 46, 51] generalizing those of [9, 52].

Definition 3.1. Let **C** be a subcategory of **LoA**. A **C**-*topological system* (**C**-*system* for short) is a tuple $D = (\mathrm{pt}\, D, \Sigma\, D, \Omega\, D, \models)$, where $(\mathrm{pt}\, D, \Sigma\, D, \Omega\, D)$ is a $\mathbf{Set} \times \mathbf{C} \times \mathbf{C}$-object and $\mathrm{pt}\, D \times \Omega\, D \xrightarrow{\models} \Sigma\, D$ is a map (called **C**-*satisfaction relation* on $(\mathrm{pt}\, D, \Sigma D, \Omega D)$) such that $\Omega\, D \xrightarrow{\models(x,-)} \Sigma\, D$ is a homomorphism for every $x \in \mathrm{pt}\, D$. A **C**-*continuous map* $D_1 \xrightarrow{f} D_2$ is a $\mathbf{Set} \times \mathbf{C} \times \mathbf{C}$-morphism $(\mathrm{pt}\, D_1, \Sigma\, D_1, \Omega\, D_1) \xrightarrow{f = (\mathrm{pt}\, f, (\Sigma\, f)^{op}, (\Omega\, f)^{op})} (\mathrm{pt}\, D_2, \Sigma\, D_2, \Omega\, D_2)$ such that for every $x \in \mathrm{pt}\, D_1$ and every $b \in \Omega\, D_2$, $\models_1(x, \Omega\, f(b)) = \Sigma\, f(\models_2(\mathrm{pt}\, f(x), b))$. **C-TopSys** is the category of **C**-topological systems and **C**-continuous maps, with the underlying functor to the ground category $\mathbf{Set} \times \mathbf{C} \times \mathbf{C}$ given by the formula $|D_1 \xrightarrow{f} D_2| = (\mathrm{pt}\, D_1, \Sigma\, D_1, \Omega\, D_1) \xrightarrow{(\mathrm{pt}\, f, (\Sigma\, f)^{op}, (\Omega\, f)^{op})} (\mathrm{pt}\, D_2, \Sigma\, D_2, \Omega\, D_2)$.

Remark 3.2. The notation $f = (\mathrm{pt}\, f, \ldots, (\Omega f)^{op})$ for continuous morphisms in Definition 3.1 is due to S. Vickers [52]. Since [10] uses it as well, we decided to do the same. Definition 3.1 adds one more component to f denoted by $(\Sigma f)^{op}$ and that is entirely our own invention, which does not go in line with [9], where the authors use completely different (even to [10]) notation. We would like to underline that $\mathrm{pt}\, f$, $(\Sigma f)^{op}$, $(\Omega f)^{op}$ are components of f and *not* new maps obtained from f.

In the following, we will rely heavily on a fixed-basis approach to variety-based topological systems and, therefore, we introduce its definition.

Definition 3.3. Given a $\mathbf{C}$-system D, $\Sigma\, D$ is called the basis of the system. Given a $\mathbf{C}$-object A, A-$\mathbf{TopSys}$ is the non-full subcategory of $\mathbf{C}$-$\mathbf{TopSys}$ with object all systems D with the basis A and morphisms all $\mathbf{C}$-continuous maps f such that $\Sigma\, f = 1_A$.

The new notion can be illustrated by the following examples, providing its relation to the existing concepts in the literature.

Example 3.4. $\mathbf{Loc}$-$\mathbf{TopSys}$ is precisely the category of lattice-valued topological systems introduced in [9]. The category $\mathbf{2}$-$\mathbf{TopSys}$ (cf. Example 2.7) is isomorphic to the category $\mathbf{TopSys}$ of S. Vickers [52].

Example 3.5. Given a set K, the subcategory K-$\mathbf{TopSys}$ of $\mathbf{LoSet}$-$\mathbf{TopSys}$ is isomorphic to the category $\mathsf{Chu}(\mathbf{Set}, K)$ comprising Chu spaces over K [30].

Example 3.6. Let $\mathbf{C}$-$\mathbf{SP}$ be the full subcategory of $\mathbf{C}$-$\mathbf{TopSys}$ consisting of all systems D such that for every $b_1, b_2 \in \Omega\, D$, $\models(-, b_1) = \models(-, b_2)$ implies $b_1 = b_2$. The full subcategory $\mathbf{2}$-$\mathbf{SP}$ of $\mathbf{Lo}(\mathbf{SLat}(\bigwedge, \bot))$-$\mathbf{SP}$ (cf. Examples 2.7 and 2.10) is isomorphic to the category $\mathbf{SP}$ of state property systems of D. Aerts [2] (the notion serves as the basic mathematical structure in the Geneva-Brussels approach to foundations of physics).

Although Definition 3.1 is provided in its general form, in this paper we restrict ourselves to the case $\mathbf{C} = \mathbf{LoA}$. For shortness sake we call $\mathbf{LoA}$-systems by systems and $\mathbf{LoA}$-continuity by continuity. The crucial property of the new category is contained in the fact that it provides a proper extension of the category $\mathbf{LoA}$-$\mathbf{Top}$ of variety-based topological spaces [51].

Theorem 3.7. *The category* $\mathbf{LoA}$-$\mathbf{Top}$ *is isomorphic to a full (regular mono)-coreflective subcategory of the category* $\mathbf{LoA}$-$\mathbf{TopSys}$.

Proof. There exists a full embedding $\mathbf{LoA}\text{-}\mathbf{Top} \xhookrightarrow{\mathrm{G}} \mathbf{LoA}\text{-}\mathbf{TopSys}$ with $\mathrm{G}\, E = (\mathrm{pt}\, E, \Sigma\, E, \tau, \models)$, $\models(x, p) = p(x)$ and $\mathrm{G}\, f = (\mathrm{pt}\, f, (\Sigma\, f)^{op}, ((\mathrm{pt}\, f, (\Sigma\, f)^{op})^{\leftarrow})^{op})$. There exists a functor $\mathbf{LoA}\text{-}\mathbf{TopSys} \xrightarrow{\mathrm{Spat}} \mathbf{LoA}\text{-}\mathbf{Top}$ with $\mathrm{Spat}\, D = (\mathrm{pt}\, D, \Sigma\, D, \tau)$, $\tau = \{\models(-, b) \mid b \in \Omega\, D\}$ and $\mathrm{Spat}\, f = (\mathrm{pt}\, f, (\Sigma\, f)^{op})$. Spat is a right-adjoint-left-inverse of G. $\square$

The above-mentioned functor G generalizes the embedding of $\mathbf{Loc}$-$\mathbf{Top}$ (lattice-valued topology) into the category of lattice-valued topological systems [9]. Moreover, Theorem 3.7 provides an analogue of the *spatialization procedure* for systems of S. Vickers [52, Theorem 5.3.4]. One should also notice that the equivalence between the categories of state property systems and closure spaces obtained in [3] is a direct consequence of Theorem 3.7 [47].

As was already mentioned in Introduction, to get a similar result for the underlying algebras of spaces, one has to modify the category of systems accordingly [46].

Definition 3.8. Given a subcategory $\mathbf{C}$ of $\mathbf{LoA}$, a $\mathbf{LoC}$-*topological system* is a tuple $D = (\mathrm{pt}\, D, \Sigma\, D, \Omega\, D, \models)$, where $(\mathrm{pt}\, D, \Sigma\, D, \Omega\, D)$ is a $\mathbf{Set} \times \mathbf{LoC} \times \mathbf{C}$-object and $\models$ is a $\mathbf{C}$-satisfaction relation on $(\mathrm{pt}\, D, \Sigma\, D, \Omega\, D)$. A $\mathbf{LoC}$-*continuous map* $D_1 \xrightarrow{f} D_2$ is a $\mathbf{Set} \times \mathbf{LoC} \times \mathbf{C}$-morphism $(\mathrm{pt}\, D_1, \Sigma D_1, \Omega D_1) \xrightarrow{f=(\mathrm{pt}\, f, \Sigma f, (\Omega f)^{op})} (\mathrm{pt}\, D_2, \Sigma D_2, \Omega D_2)$ such that for every $x \in \mathrm{pt}\, D_1$ and every $b \in \Omega D_2$, $\Sigma f(\models_1(x, \Omega f(b))) = \models_2(\mathrm{pt}\, f(x), b)$. $\mathbf{LoC}$-$\mathbf{TopSys}$ is the category of $\mathbf{LoC}$-topological systems and $\mathbf{LoC}$-continuous maps, with the underlying functor to the ground category $\mathbf{Set} \times \mathbf{LoC} \times \mathbf{C}$ denoted by $| - |$.

One should be aware of the following important relations between the above-mentioned two approaches to topological systems, based on the fact that $\mathbf{C}$ and $\mathbf{LoC}$ differ only on morphisms.

Remark 3.9. The categories $\mathbf{C}$-$\mathbf{TopSys}$ and $\mathbf{LoC}$-$\mathbf{TopSys}$ have eventually the same objects but the morphisms (as well as ground categories) are different. Given a $\mathbf{C}$-object A, A-$\mathbf{TopSys}$ is eventually a subcategory of both $\mathbf{C}$-$\mathbf{TopSys}$ and $\mathbf{LoC}$-$\mathbf{TopSys}$. If $\mathbf{E}$ is the subcategory of $\mathbf{C}$ with the same objects and with φ in $\mathbf{E}$ if and only if φ is an isomorphism, then the categories $\mathbf{E}$-$\mathbf{TopSys}$ and $\mathbf{LoE}$-$\mathbf{TopSys}$ are isomorphic.

Although Definition 3.8 is provided in its general form, in this paper we restrict ourselves to the case $\mathbf{C} = \mathbf{LoA}$. For the sake of shortness, as well as to distinguish between the categories $\mathbf{LoA}$-$\mathbf{TopSys}$ and $\mathbf{A}$-$\mathbf{TopSys}$, we call $\mathbf{A}$-systems by op-systems and $\mathbf{A}$-continuity by op-continuity.

Theorem 3.10. *The category* $\mathbf{A} \times \mathbf{LoA}$ *is isomorphic to a full reflective subcategory of the category* $\mathbf{A}$-$\mathbf{TopSys}$ *which (in general) is neither mono- nor epi-reflective.*

Proof. There exists a full embedding $\mathbf{A} \times \mathbf{LoA} \overset{\mathrm{F}}{\hookrightarrow} \mathbf{A}$-$\mathbf{TopSys}$ with $\mathrm{F}(A, B) = (\mathbf{A}(B, A), A, B, \models)$, $\models(p, b) = p(b)$ and $\mathrm{F}(\varphi, \psi) = ((|\psi^{op}|, \varphi^{op})^{\leftarrow}, \varphi, \psi)$. There exists a functor $\mathbf{A}$-$\mathbf{TopSys} \xrightarrow{\mathrm{Loc}} \mathbf{A} \times \mathbf{LoA}$ with $\mathrm{Loc}\, D = (\Sigma\, D, \Omega\, D)$, $\mathrm{Loc}\, f = (\Sigma\, f, (\Omega\, f)^{op})$. Loc is a left-adjoint-left-inverse of F. $\square$

Theorem 3.10 provides an analogue of the *localification procedure* for systems of S. Vickers [52, Theorem 5.4.3]. It will be worthwhile to underline once more that in the framework of variable-basis both spatialization and localification procedures require their own category of systems. The simplest common point of both categories is the fixed-basis approach generalizing that of S. Vickers [52].

4. Algebraic Category of Topological Systems

It is a well-known fact that the construct $\mathbf{Top}$ of topological spaces and continuous maps is topological [1, Example 21.8(1)]. There are several extensions of the result to the lattice-valued case [18, 37], differing in the approach to fuzziness and the algebraic structures underlying the topologies. Moreover, S. E. Rodabaugh [39] has come out with a general solution claiming that if the underlying lattices are s-quantales, then all the concrete categories for topology considered or constructed in [18, 37, 39] (and these include many well-known categories) are topological over their ground categories with respect to the underlying forgetful functors. Metamathematically restated, the conditions of s-quantales guarantee that one is doing topology when working in these categories. Motivated by these results, we proved in [49, Proposition 3.4] an analogous one for variety-based approach, partly covering the above-mentioned claim of [39] for s-quantales (the result of [11] claiming that the category **Cls** of closure spaces (Example 2.10) is topological over **Set** is also included).

Theorem 4.1. *The concrete category* **LoA**-**Top** *is topological over the ground category* **Set** $\times$ **LoA**.

The concept of topological system and the embedding of Theorem 3.7 raised the question on the nature of the category **LoA-TopSys**. In particular, since **TopSys** models the "topological" behavior coming from domain theory, it could be expected that **TopSys** would be topological over its ground category. Despite all hopes, J. T. Denniston *et al.* [10] showed that even sources comprising only one morphism need not have initial lifts. Moreover, the results of [51] claiming, firstly, that the category **LoA-TopSys** is topological if and only if the respective underlying functor $|-|$ is an isomorphism and, secondly, that $|-|$ creates isomorphisms, suggested an algebraic flavor in the behavior of the category. It is the purpose of this section to show that **LoA-TopSys** is essentially algebraic over its ground category **Set** $\times$ **LoA** $\times$ **LoA**. To verify the claim, we will use the so-called characterization theorem for essentially algebraic categories [1, Theorem 23.8].

Theorem 4.2. *A concrete category* $(\mathbf{C}, U)$ *is essentially algebraic if and only if the following conditions are satisfied:*

(1) *U creates isomorphisms,*
(2) *U is adjoint,*
(3) **C** *is (Epi, Mono-Source)-factorizable.*

We will check the required conditions in a row. The first one, probably the most obvious, can be easily shown as follows.

Lemma 4.3. *The functor* **LoA**-**TopSys** $\xrightarrow{|-|}$ **Set** $\times$ **LoA** $\times$ **LoA** *creates isomorphisms.*

Proof. Given a **Set** $\times$ **LoA** $\times$ **LoA**-isomorphism $(X, A, B) \xrightarrow{f} |D|$, the unique structure on (X, A, B), making f an isomorphism in **LoA-TopSys**, can be defined by $\overline{\models}(x, b) = \Sigma f(\models(\mathrm{pt}\, f(x), (\Omega f)^{-1}(b)))$. $\square$

The second condition is more sophisticated and requires a bit of computation.

Lemma 4.4. *The functor* **LoA**-**TopSys** $\xrightarrow{|-|}$ **Set** $\times$ **LoA** $\times$ **LoA** *is adjoint.*

Proof. For the sake of shortness, we denote the category **Set** $\times$ **LoA** $\times$ **LoA** by **X**. It will be enough to show that every **X**-object (X, A, B) has a $|-|$-universal arrow, i.e., an **X**-morphism $(X, A, B) \xrightarrow{\eta} |\overline{D}|$ such that for every **X**-morphism $(X, A, B) \xrightarrow{f} |D|$

there exists a unique continuous map $\overline{D} \xrightarrow{\overline{f}} D$ making the triangle

$$\begin{tikzcd} (X,A,B) \arrow[r, "\eta"] \arrow[dr, "f"'] & |\overline{D}| \arrow[d, dotted, "|\overline{f}|"] \\ & |D| \end{tikzcd}$$

commute.

Define the required system $\overline{D}$ by $\mathrm{pt}\,\overline{D} = X$, $\Sigma\,\overline{D} = A$, $\Omega\,\overline{D} = A^X \times B$, putting $\mathrm{pt}\,\overline{D} \times \Omega\,\overline{D} \xrightarrow{\overline{\models}} \Sigma\,\overline{D} = X \times (A^X \times B) \xrightarrow{\pi_{X \times A^X}} X \times A^X \xrightarrow{\mathrm{ev}} A$, where $\pi_{X\times A^X}$ is the projection map and $\mathrm{ev}(x,p) = p(x)$. To show that $\overline{\models}(x,-)$ is a homomorphism for every $x \in \mathrm{pt}\,\overline{D}$, notice that given $\lambda \in \Lambda$ and $(p_i, b_i) \in \Omega\,\overline{D}$ for $i \in n_\lambda$,

$$\overline{\models}(x, \omega_\lambda^{\Omega\,\overline{D}}(\langle (p_i,b_i)\rangle_{n_\lambda})) = \overline{\models}(x, (\omega_\lambda^{A^X}(\langle p_i\rangle_{n_\lambda}), \omega_\lambda^{B}(\langle b_i\rangle_{n_\lambda}))) = \mathrm{ev}(x, \omega_\lambda^{A^X}(\langle p_i\rangle_{n_\lambda})) = (\omega_\lambda^{A^X}(\langle p_i\rangle_{n_\lambda}))(x) = \omega_\lambda^{A}(\langle p_i(x)\rangle_{n_\lambda}) = \omega_\lambda^{\Sigma\,\overline{D}}(\langle \overline{\models}(x,(p_i,b_i))\rangle_{n_\lambda}).$$

Define the required **X**-morphism $(X,A,B) \xrightarrow{\eta} |\overline{D}|$ by $X \xrightarrow{\mathrm{pt}\,\eta} \mathrm{pt}\,\overline{D} = X \xrightarrow{1_X} X$, $\Sigma\,\overline{D} \xrightarrow{\Sigma\,\eta} A = A \xrightarrow{1_A} A$ and $\Omega\,\overline{D} \xrightarrow{\Omega\,\eta} B = A^X \times B \xrightarrow{\pi_B} B$, where π_B is the projection map. To show that $(X,A,B) \xrightarrow{\eta} |\overline{D}|$ is a $|-|$-universal arrow for (X,A,B), choose any **X**-morphism $(X,A,B) \xrightarrow{f} |D|$ and define $\overline{D} \xrightarrow{\overline{f}} D$ by $\mathrm{pt}\,\overline{D} \xrightarrow{\mathrm{pt}\,\overline{f}} \mathrm{pt}\,D = X \xrightarrow{\mathrm{pt}\,f} \mathrm{pt}\,D$, $\Sigma\,D \xrightarrow{\Sigma\,\overline{f}} \Sigma\,\overline{D} = \Sigma\,D \xrightarrow{\Sigma\,f} A$ and $\Omega\,D \xrightarrow{\Omega\,\overline{f}} \Omega\,\overline{D} = \Omega\,D \xrightarrow{\Omega\,\overline{f}} A^X \times B$, where $\Omega\,\overline{f}$ is the unique homomorphism making the diagram

$$\begin{tikzcd} & \Omega\,D \arrow[dl, "\phi"'] \arrow[d, dotted, "\Omega\,\overline{f}"] \arrow[dr, "\Omega\,f"] & \\ A^X & A^X \times B \arrow[l, "\pi_{A^X}"] \arrow[r, "\pi_B"'] & B \end{tikzcd}$$

commute, with ϕ in its turn given by the commutativity for every $x \in X$ of the diagram

$$\begin{tikzcd} \Omega\,D \arrow[r, "\models(\mathrm{pt}\,f(x),-)"] \arrow[d, dotted, "\phi"'] & \Sigma\,D \arrow[d, "\Sigma\,f"] \\ A^X \arrow[r, "\pi_x = \mathrm{ev}(x,-)"'] & A. \end{tikzcd}$$

To show continuity of $\overline{f}$ notice that given $x \in \mathrm{pt}\,\overline{D}$ and $b \in \Omega\,D$, $\Sigma\,\overline{f}(\models(\mathrm{pt}\,\overline{f}(x), b)) = \Sigma\,f(\models(\mathrm{pt}\,f(x), b)) = \mathrm{ev}(x, \phi(b)) = (\phi(b))(x) = \overline{\models}(x, (\phi(b), \Omega\,f(b))) = \overline{\models}(x, \Omega\,\overline{f}(b))$. On the other hand, the equality $|\overline{f}| \circ \eta = (\mathrm{pt}\,f, (\Sigma\,f)^{op}, (\Omega\,\overline{f})^{op}) \circ (1_X, 1_A, (\pi_B)^{op}) = (\mathrm{pt}\,f, (\Sigma\,f)^{op}, (\Omega\,f)^{op}) = f$ gives commutativity of the above-mentioned triangle.

For uniqueness of $\overline{f}$ notice that given any other $\overline{D} \xrightarrow{g} D$ such that $|g| \circ \eta = f$, it follows that $f = (\mathrm{pt}\,g, (\Sigma\,g)^{op}, (\pi_B \circ \Omega\,g)^{op})$ and, therefore, $\mathrm{pt}\,f = \mathrm{pt}\,g$, $\Sigma\,f = \Sigma\,g$ and $\Omega\,f = \pi_B \circ \Omega\,g$. Moreover, since g is continuous, $\Sigma\,f \circ \models(\mathrm{pt}\,f(x), -) =$

$\Sigma\, g(\models(\mathrm{pt}\, g(x), -)) = \overline{\models}(x, \Omega\, g(-)) = (\pi_{A^X} \circ \Omega\, g(-))(x) = \mathrm{ev}(x, -) \circ \pi_{A^X} \circ \Omega\, g$ and thus $\phi = \pi_{A^X} \circ \Omega\, g$. The desired equality $\Omega\, \overline{f} = \Omega\, g$ now follows. $\square$

The last condition of Theorem 4.2 uses the standard technique of universal algebra [7] applied to our slightly modified case.

Lemma 4.5. *The category* **LoA**-**TopSys** *is (Epi, Mono-Source)-factorizable.*

Proof. Let $\mathcal{S} = (D \xrightarrow{f_i} D_i)_{i \in I}$ be a source in **LoA-TopSys**. Define a system $\overline{D}$ by $\mathrm{pt}\, \overline{D} = \mathrm{pt}\, D/\sim$, where $\sim$ is the equivalence relation on $\mathrm{pt}\, D$ given by $x \sim y$ if and only if $\mathrm{pt}\, f_i(x) = \mathrm{pt}\, f_i(y)$ for every $i \in I$, $\Sigma\, \overline{D} = \langle \bigcup_{i \in I} (\Sigma\, f_i)^{\rightarrow}(\Sigma\, D_i) \rangle$ ($\langle S \rangle$ denotes the subalgebra generated by S), $\Omega\, \overline{D} = \langle \bigcup_{i \in I} (\Omega\, f_i)^{\rightarrow}(\Omega\, D_i) \rangle$ and $\overline{\models}([x]_\sim, b) = \models(x, b)$, where $[x]_\sim$ is the equivalence class of x, i.e., the set $\{y \mid y \in \mathrm{pt}\, D \text{ and } x \sim y\}$.

Define a continuous map $D \xrightarrow{e} \overline{D}$ as follows: $\mathrm{pt}\, D \xrightarrow{\mathrm{pt}\, e} \mathrm{pt}\, \overline{D}$, $\mathrm{pt}\, e(x) = [x]_\sim$, and $\Sigma\, \overline{D} \xhookrightarrow{\Sigma\, e} \Sigma\, D$, $\Omega\, \overline{D} \xhookrightarrow{\Omega\, e} \Omega\, D$ are the inclusion maps. Given $i \in I$, define a continuous map $\overline{D} \xrightarrow{m_i} D_i$ by $\mathrm{pt}\, \overline{D} \xrightarrow{\mathrm{pt}\, m_i} \mathrm{pt}\, D_i$ with $\mathrm{pt}\, m_i([x]_\sim) = \mathrm{pt}\, f_i(x)$, $\Sigma\, D_i \xrightarrow{\Sigma\, m_i} \Sigma\, \overline{D}$ with $\Sigma\, m_i(a) = \Sigma\, f_i(a)$ and $\Omega\, D_i \xrightarrow{\Omega\, m_i} \Omega\, \overline{D}$ with $\Omega\, m_i(b) = \Omega\, f_i(b)$, thus getting a source $\mathcal{M} = (\overline{D} \xrightarrow{m_i} D_i)_{i \in I}$ in **LoA-TopSys**. Easy computations show that $\mathcal{S} = \mathcal{M} \circ e$ is the required (Epi, Mono-Source)-factorization. $\square$

Lemmas 4.3, 4.4 and 4.5 together imply the main result of this section on the nature of the category of variety-based topological systems.

Theorem 4.6. *The concrete category* **LoA**-**TopSys** *is essentially algebraic over the ground category* **Set** $\times$ **LoA** $\times$ **LoA**.

Theorem 4.6 and [1, Example 23.6(4)] provide the result of [51] on highly non-topological nature of the category in question.

Corollary 4.7. *The concrete category* **LoA**-**TopSys** *is topological over the ground category* **Set** $\times$ **LoA** $\times$ **LoA** *if and only if the underlying functor is an isomorphism.*

Another result is not so obvious and requires some (straightforward and thus, omitted) computations.

Corollary 4.8. *Given a localic algebra A, the category A-***TopSys** *is essentially algebraic over its ground category* **Set** $\times$ **LoA**. *In particular, the category* **TopSys** *of S. Vickers is essentially algebraic over* **Set** $\times$ **Loc**.

We end the section with some remarks on a possible generalization of the obtained result. Recall from [1, Definition 23.19] the notion of *algebraic category.*

Definition 4.9. A concrete category $(\mathbf{C}, U)$ is called *algebraic* provided that it is essentially algebraic and U preserves extremal epimorphisms.

A natural question arises, whether one can show that the category **LoA-TopSys** is algebraic. Up to now, we have been able neither to prove nor disprove the claim. The only positive result in this direction is contained in the following lemma.

Lemma 4.10. *If* $D_1 \xrightarrow{e} D_2$ *is an extremal* **LoA-TopSys***-epimorphism, then the pair* $(\mathrm{pt}\, e, \Sigma\, e)$ *is an extremal epimorphism in* $\mathbf{Set} \times \mathbf{LoA}$ *and* $\Omega\, e$ *is injective. If* **A***-epimorphisms are surjective, then the category* **LoA-TopSys** *is algebraic.*

Proof. One part of the proof follows from our construction in Lemma 4.5, namely, if $D_1 \xrightarrow{e} D_2 = D_1 \xrightarrow{e'} \overline{D} \xrightarrow{m} D_2$ is an (Epi, Mono)-factorization of e, then m is an isomorphism and, therefore, $\mathrm{pt}\, D_1 \xrightarrow{\mathrm{pt}\, e} \mathrm{pt}\, D_2$ is surjective and both $\Sigma\, D_2 \xrightarrow{\Sigma\, e} \Sigma\, D_1$, $\Omega\, D_2 \xrightarrow{\Omega\, e} \Omega\, D_1$ are injective. It follows that $\mathrm{pt}\, e$ is an extremal epimorphism in **Set**. To show that $\Sigma\, e$ is an extremal monomorphism in **A** we proceed as follows.

Notice that injectivity of $\Sigma\, e$ implies the property of being a monomorphism. The only thing left to verify is the extremal condition. Suppose we are given a factorization $\Sigma\, D_2 \xrightarrow{\Sigma\, e} \Sigma\, D_1 = \Sigma\, D_2 \xrightarrow{\varphi} A \xrightarrow{\psi} \Sigma\, D_1$ with φ being an epimorphism. The factorization can be lifted to the category of systems as follows.

Define a system D by $\mathrm{pt}\, D = \mathrm{pt}\, D_2$, $\Sigma\, D = A$, $\Omega\, D = \Omega\, D_2$, putting $\mathrm{pt}\, D \times \Omega\, D \xrightarrow{\models} \Sigma\, D = \mathrm{pt}\, D_2 \times \Omega\, D_2 \xrightarrow{\models_2} \Sigma\, D_2 \xrightarrow{\varphi} A$. Moreover, define $D_1 \xrightarrow{g} D$ by $\mathrm{pt}\, D_1 \xrightarrow{\mathrm{pt}\, g} \mathrm{pt}\, D = \mathrm{pt}\, D_1 \xrightarrow{\mathrm{pt}\, e} \mathrm{pt}\, D_2$, $\Sigma\, D \xrightarrow{\Sigma\, g} \Sigma\, D_1 = A \xrightarrow{\psi} \Sigma\, D_1$ and $\Omega\, D \xrightarrow{\Omega\, g} \Omega\, D_1 = \Omega\, D_2 \xrightarrow{\Omega\, e} \Omega\, D_1$. To show that g is continuous notice that given $x \in \mathrm{pt}\, D_1$ and $b \in \Omega\, D$, $\Sigma\, g(\models(\mathrm{pt}\, g(x), b)) = \psi \circ \varphi(\models_2(\mathrm{pt}\, e(x), b)) = \Sigma\, e(\models_2(\mathrm{pt}\, e(x), b)) = \models_1(x, \Omega\, e(b)) = \models_1(x, \Omega\, g(b))$. Further, define $D \xrightarrow{f} D_2$ by $\mathrm{pt}\, D \xrightarrow{\mathrm{pt}\, f} \mathrm{pt}\, D_2 = \mathrm{pt}\, D_2 \xrightarrow{1_{\mathrm{pt}\, D_2}} \mathrm{pt}\, D_2$, $\Sigma\, D_2 \xrightarrow{\Sigma\, f} \Sigma\, D = \Sigma\, D_2 \xrightarrow{\varphi} A$ and $\Omega\, D_2 \xrightarrow{\Omega\, f} \Omega\, D = \Omega\, D_2 \xrightarrow{1_{\Omega\, D_2}} \Omega\, D_2$. For continuity of f notice that given $x \in \mathrm{pt}\, D$ and $b \in \Omega\, D_2$, it follows that $\Sigma\, f(\models_2(\mathrm{pt}\, f(x), b)) = \varphi(\models_2(x, b)) = \models(x, b) = \models(x, \Omega\, f(b))$. Since $D_1 \xrightarrow{e} D_2 = D_1 \xrightarrow{g} D \xrightarrow{f} D_2$ and f is a monomorphism in **LoA-TopSys**, f is an isomorphism and, therefore, φ must be as well.

The last statement of the lemma follows immediately from injectivity of $\Omega\, e$. $\square$

Notice that epimorphisms are surjective neither in the category **Frm** of frames nor in the category **Quant** of quantales [22, 24]. It follows that the last statement of Lemma 4.10 is not applicable in the most important cases of lattice-valued topology.

5. Coalgebraic Category of Topological Systems

In the previous section, we showed that the category **LoA-TopSys** is essentially algebraic. In Section 3 we introduced a modified version of the category denoted **A-TopSys** (Definition 3.8). It is the purpose of this section to show that the new category is coalgebraic. The reader should notice the difference from the case of the category **LoA-TopSys**, where we were able to show essential algebraicity only.

To obtain the result, we will use the duals of Theorem 4.2 and Definition 4.9. The following three lemmas provide the counterparts of the respective ones from the previous section.

Lemma 5.1. *The functor* $\mathbf{A}\text{-}\mathbf{TopSys} \xrightarrow{|-|} \mathbf{Set} \times \mathbf{A} \times \mathbf{LoA}$ *creates isomorphisms.*

Proof. Given a $\mathbf{Set} \times \mathbf{A} \times \mathbf{LoA}$-isomorphism $(X, A, B) \xrightarrow{f} |D|$, the unique structure on (X, A, B), making f an isomorphism in $\mathbf{A}$-$\mathbf{TopSys}$, can be defined by $\overline{\models}(x, b) = (\Sigma\, f)^{-1}(\models(\mathrm{pt}\, f(x), (\Omega\, f)^{-1}(b)))$. □

Lemma 5.2. *The functor* $\mathbf{A}$-$\mathbf{TopSys} \xrightarrow{|-|} \mathbf{Set} \times \mathbf{A} \times \mathbf{LoA}$ *is co-adjoint.*

Proof. For the sake of shortness, denote the category $\mathbf{Set} \times \mathbf{A} \times \mathbf{LoA}$ by $\mathbf{X}$. It will be enough to show that every $\mathbf{X}$-object (X, A, B) has a $|-|$-co-universal arrow, i.e., an $\mathbf{X}$-morphism $|\overline{D}| \xrightarrow{\varepsilon} (X, A, B)$ such that for every $\mathbf{X}$-morphism $|D| \xrightarrow{f} (X, A, B)$ there exists a unique op-continuous map $D \xrightarrow{\overline{f}} \overline{D}$ making the triangle

$$\begin{tikzcd} |D| \arrow[d, dotted, "|\overline{f}|"'] \arrow[dr, "f"] & \\ |\overline{D}| \arrow[r, "\varepsilon"'] & (X, A, B) \end{tikzcd}$$

commute.

Define the required op-system $\overline{D}$ by $\mathrm{pt}\,\overline{D} = X \times \mathbf{A}(B, A)$, $\Sigma\,\overline{D} = A$, $\Omega\,\overline{D} = B$, putting $\mathrm{pt}\,\overline{D} \times \Omega\,\overline{D} \xrightarrow{\models} \Sigma\,\overline{D} = (X \times \mathbf{A}(B, A)) \times B \xrightarrow{\pi_{\mathbf{A}(B,A)\times B}} \mathbf{A}(B, A) \times B \xrightarrow{\mathrm{ev}} A$, where $\pi_{\mathbf{A}(B,A)\times B}$ is the projection map and $\mathrm{ev}(p, b) = p(b)$ (cf. Lemma 4.4). To show that $\overline{\models}((x, p), -)$ is a homomorphism for every $(x, p) \in \mathrm{pt}\,\overline{D}$, notice that given $\lambda \in \Lambda$ and $b_i \in \Omega\,\overline{D}$ for $i \in n_\lambda$, $\overline{\models}((x, p), \omega_\lambda^{\Omega\,\overline{D}}(\langle b_i \rangle_{n_\lambda})) = p(\omega_\lambda^B(\langle b_i \rangle_{n_\lambda})) = \omega_\lambda^A(\langle p(b_i) \rangle_{n_\lambda}) = \omega_\lambda^{\Sigma\,\overline{D}}(\langle \overline{\models}((x, p), b_i) \rangle_{n_\lambda})$.

Define the required $\mathbf{X}$-morphism $|\overline{D}| \xrightarrow{\varepsilon} (X, A, B)$ by $\mathrm{pt}\,\overline{D} \xrightarrow{\mathrm{pt}\,\varepsilon} X = X \times \mathbf{A}(B, A) \xrightarrow{\pi_X} X$ with π_X the projection map, $B \xrightarrow{\Omega\,\varepsilon} \Omega\,\overline{D} = B \xrightarrow{1_B} B$ and $\Sigma\,\overline{D} \xrightarrow{\Sigma\,\varepsilon} A = A \xrightarrow{1_A} A$. To show that $|\overline{D}| \xrightarrow{\varepsilon} (X, A, B)$ is a $|-|$-co-universal arrow for (X, A, B), take any $\mathbf{X}$-morphism $|D| \xrightarrow{f} (X, A, B)$ and define $D \xrightarrow{\overline{f}} \overline{D}$ by $\mathrm{pt}\,D \xrightarrow{\mathrm{pt}\,\overline{f}} \mathrm{pt}\,\overline{D}$ with $\mathrm{pt}\,\overline{f}(y) = (\mathrm{pt}\,f(y), \Sigma\,f \circ \models(y, -) \circ \Omega\,f)$, $\Omega\,\overline{D} \xrightarrow{\Omega\,\overline{f}} \Omega\,D = B \xrightarrow{\Omega\,f} \Omega\,D$ and $\Sigma\,D \xrightarrow{\Sigma\,\overline{f}} \Sigma\,\overline{D} = \Sigma\,D \xrightarrow{\Sigma\,f} A$. Continuity of $\overline{f}$ follows from the fact that given $y \in \mathrm{pt}\,D$ and $b \in \Omega\,\overline{D}$, $\Sigma\,\overline{f}(\models(y, \Omega\,\overline{f}(b))) = \Sigma\,f \circ \models(y, -) \circ \Omega\,f(b) = \overline{\models}((\mathrm{pt}\,f(y), \Sigma\,f \circ \models(y, -) \circ \Omega\,f), b) = \overline{\models}(\mathrm{pt}\,\overline{f}(y), b)$.

For commutativity of the above-mentioned triangle, it will be enough to notice that given $y \in \mathrm{pt}\,D$, $\mathrm{pt}\,\varepsilon \circ \mathrm{pt}\,\overline{f}(y) = \mathrm{pt}\,\varepsilon(\mathrm{pt}\,f(y), \Sigma\,f \circ \models(y, -) \circ \Omega\,f) = \mathrm{pt}\,f(y)$.

For uniqueness of $\overline{f}$, use the fact that given another $D \xrightarrow{g} \overline{D}$ such that $\varepsilon \circ |g| = f$, it follows that $f = (\mathrm{pt}\,\varepsilon \circ \mathrm{pt}\,g, \Sigma\,g, (\Omega\,g)^{op})$ and, therefore, $\mathrm{pt}\,f = \mathrm{pt}\,\varepsilon \circ \mathrm{pt}\,g$, $\Sigma\,f = \Sigma\,g$ and $\Omega\,f = \Omega\,g$. Moreover, given $y \in \mathrm{pt}\,D$, $\mathrm{pt}\,g(y) = (\mathrm{pt}\,f(y), p)$, where $\Sigma\,f \circ \models(y, -) \circ \Omega\,f = \Sigma\,g(\models(y, \Omega\,g(-))) = \overline{\models}(\mathrm{pt}\,g(y), -) = \overline{\models}((\mathrm{pt}\,f(y), p), -) = p$. The desired equality $\mathrm{pt}\,\overline{f} = \mathrm{pt}\,g$ now follows. □

Lemma 5.3. *The category* $\mathbf{A}$-$\mathbf{TopSys}$ *is (Epi-Sink, Mono)-factorizable.*

Proof. Let $\mathcal{S} = (D_i \xrightarrow{f_i} D)_{i \in I}$ be a sink in $\mathbf{A}$-$\mathbf{TopSys}$. Define an op-system $\overline{D}$ by $\mathrm{pt}\,\overline{D} = \bigcup\limits_{i \in I} (\mathrm{pt}\,f_i)^{\rightarrow}(\mathrm{pt}\,D_i)$, $\Sigma\,\overline{D} = \langle \bigcup\limits_{i \in I} (\Sigma f_i)^{\rightarrow}(\Sigma D_i) \rangle$ (recall that $\langle S \rangle$ denotes the algebra generated by S), $\Omega\,\overline{D} = \Omega D/\sim$, where $\sim$ is the congruence relation on ΩD defined by $b_1 \sim b_2$ if an only if $\Omega f_i(b_1) = \Omega f_i(b_2)$ for every $i \in I$, and $\overline{\models}(x, [b]_\sim) = \models(x, b)$, where $[b]_\sim$ is the equivalence class of b, i.e., the set $\{c \mid c \in \Omega\, D \text{ and } b \sim c\}$.

Define an op-continuous map $\overline{D} \xrightarrow{m} D$ by $\mathrm{pt}\,\overline{D} \xhookrightarrow{\mathrm{pt}\,m} \mathrm{pt}\,D$, $\Sigma\,\overline{D} \xhookrightarrow{\Sigma\,m} \Sigma\,D$ being the inclusions and $\Omega\,D \xrightarrow{\Omega\,m} \Omega\,\overline{D}$ being the quotient map, i.e., $\Omega\,m(b) = [b]_\sim$. Given $i \in I$, define an op-continuous map $D_i \xrightarrow{e_i} \overline{D}$ by $\mathrm{pt}\,D_i \xrightarrow{\mathrm{pt}\,e_i} \mathrm{pt}\,\overline{D}$ with $\mathrm{pt}\,e_i(x) = \mathrm{pt}\,f_i(x)$, $\Sigma\,D_i \xrightarrow{\Sigma\,e_i} \Sigma\,\overline{D}$ with $\Sigma e_i(a) = \Sigma f_i(a)$ and $\Omega\,\overline{D} \xrightarrow{\Omega\,e_i} \Omega\,D_i$ with $\Omega e_i([b]_\sim) = \Omega f_i(b)$, thus getting a sink $\mathcal{E} = (D_i \xrightarrow{e_i} \overline{D})_{i\in I}$ in **A-TopSys**. Straightforward computations show that $\mathcal{S} = m \circ \mathcal{E}$ is the required (Epi-Sink, Mono)-factorization. $\square$

The above-mentioned Lemmas 5.1, 5.2 and 5.3 together imply the next result.

Theorem 5.4. *The concrete category* **A-TopSys** *is essentially coalgebraic over the ground category* $\mathbf{Set} \times \mathbf{A} \times \mathbf{LoA}$.

It appears that in the current setting the result can be strengthened as shows the following lemma.

Lemma 5.5. *The functor* $\mathbf{A}\text{-}\mathbf{TopSys} \xrightarrow{|-|} \mathbf{Set} \times \mathbf{A} \times \mathbf{LoA}$ *preserves extremal monomorphisms.*

Proof. Suppose $D_1 \xrightarrow{m} D_2$ is an extremal monomorphism in **A-TopSys**. Lemma 5.3 provides the (Epi, Mono)-factorization $D_1 \xrightarrow{m} D_2 = D_1 \xrightarrow{e} \overline{D} \xrightarrow{m'} D_2$ that together with the extremal property of m implies e being an isomorphism. Thus, $\mathrm{pt}\,m$, $\Sigma\,m$ are injective and $\Omega\,m$ is surjective. It follows that $\mathrm{pt}\,m$ is an extremal monomorphism in **Set**. To show the analogues for $\Sigma\,m$ and $\Omega\,m$, notice that injectivity of the first and surjectivity of the second map imply $\Sigma\,m$ being a monomorphism and $\Omega\,m$ being an epimorphism in the respective category. The only thing left to check is the extremal condition. The procedure is similar to that of Lemma 4.10 and, therefore, some details are left to the reader.

Suppose we are given a factorization $\Sigma D_1 \xrightarrow{\Sigma m} \Sigma D_2 = \Sigma D_1 \xrightarrow{\varphi} A \xrightarrow{\psi} \Sigma D_2$ with φ being an epimorphism. Define an op-system $D = (\mathrm{pt}\,D_1, A, \Omega D_1, \models)$ with $\mathrm{pt}\,D \times \Omega\,D \xrightarrow{\models} \Sigma\,D = \mathrm{pt}\,D_1 \times \Omega\,D_1 \xrightarrow{\models_1} \Sigma\,D_1 \xrightarrow{\varphi} A$ and get an (Epi, –)-factorization $D_1 \xrightarrow{m} D_2 = D_1 \xrightarrow{g=(1_{\mathrm{pt}\,D_1}, \varphi, 1_{\Omega D_1})} D \xrightarrow{f=(\mathrm{pt}\,m, \psi, (\Omega m)^{op})} D_2$. By the extremal property of m, g is an isomorphism and, therefore, φ must be as well.

Given a factorization $\Omega D_2 \xrightarrow{\Omega m} \Omega D_1 = \Omega D_2 \xrightarrow{\varphi} B \xrightarrow{\psi} \Omega D_1$ with ψ being a monomorphism, define an op-system $D = (\mathrm{pt}\,D_1, \Sigma D_1, B, \models)$ with $\mathrm{pt}\,D \times \Omega\,D \xrightarrow{\models} \Sigma\,D = \mathrm{pt}\,D_1 \times B \xrightarrow{1_{\mathrm{pt}\,D_1} \times \psi} \mathrm{pt}\,D_1 \times \Omega\,D_1 \xrightarrow{\models_1} \Sigma\,D_1$ and get an (Epi, –)-factorization $D_1 \xrightarrow{m} D_2 = D_1 \xrightarrow{g=(1_{\mathrm{pt}\,D_1}, 1_{\Sigma D_1}, \psi^{op})} D \xrightarrow{f=(\mathrm{pt}\,m, \Sigma m, \varphi^{op})} D_2$. By the extremal property of m, g is an isomorphism and, therefore, ψ must be as well. $\square$

Theorem 5.4 and Lemma 5.5 together imply the main result of this section on the nature of the category **A-TopSys**.

Theorem 5.6. *The concrete category* **A**-**TopSys** *is coalgebraic over the ground category* **Set** × **A** × **LoA**.

Easy calculations and Remark 3.9 imply the following result.

Corollary 5.7. *Given a localic algebra A, the category A-***TopSys** *is coalgebraic over its ground category* **Set**×**LoA**. *In particular, the category* **TopSys** *of S. Vickers is coalgebraic over* **Set** × **Loc**.

Corollaries 4.8 and 5.7 together imply that the category **TopSys** of S. Vickers [52] is both essentially algebraic and coalgebraic over its ground category.

6. Conclusion: Embedding Topology Into Algebra

In the paper, we presented a variety-based approach to both topological spaces and topological systems. It appears that the framework of varieties is quite fruitful, incorporating in itself the majority of the existing approaches to both fuzzy topology and fuzzy topological systems. Moreover, the embedding of Theorem 3.7 as well as Theorems 4.1, 4.6 together suggest an interesting and rather striking result.

Metatheorem 6.1. *(Lattice-valued) topology can be embedded into algebra.*

The embedding is not concrete since the categories **LoA-Top** and **LoA-TopSys** have different ground categories. More explicitly, the diagram

$$\begin{tikzcd} \mathbf{LoA\text{-}Top} \arrow[r, hook, "\mathrm{G}"] \arrow[d, "|-|"'] & \mathbf{LoA\text{-}TopSys} \arrow[d, "|-|"] \\ \mathbf{Set}\times\mathbf{LoA} & \mathbf{Set}\times\mathbf{LoA}\times\mathbf{LoA} \arrow[l, "\Pi"] \end{tikzcd} \tag{1}$$

commutes, where $\Pi((X,A,B) \xrightarrow{(f,\varphi,\psi)} (Y,C,D)) = (X,A) \xrightarrow{(f,\varphi)} (Y,C)$. An experienced reader will find out the embedding $\mathbf{Set}\times\mathbf{LoA} \overset{\mathrm{H}}{\hookrightarrow} \mathbf{Set}\times\mathbf{LoA}\times\mathbf{LoA}$, $\mathrm{H}((X,A) \xrightarrow{(f,\varphi)} (Y,B)) = (X,A,A^X) \xrightarrow{(f,\varphi,((f,\varphi)^{\leftarrow})^{op})} (Y,B,B^Y)$. Straightforward computations show that replacing Π with H in (6) provides a *non-commutative* diagram, since given a space E, $|\,\mathrm{G}\,E| = (\mathrm{pt}\,E, \Sigma\,E, \tau) \neq (\mathrm{pt}\,E, \Sigma\,E, (\Sigma\,E)^{(\mathrm{pt}\,E)}) = \mathrm{H}\,|E|$. The result is highly related to the problem of obtaining initial lifts of certain sources in the category **Loc-TopSys** raised in [9, Example 72]. A possible solution was based on the (unfortunately, wrong) assumption of commutativity of (6) with H instead of Π. Our discussion shows that the problem is still open.

The reader should also be aware of several other open problems suggested by the approach. Below we list the most important (by our opinion) ones.

Problem 6.2. By Theorem 4.6 the category **LoA-TopSys** is essentially algebraic over its ground category. Is it true that the category in question is algebraic?

Problem 6.3. Example 3.6 introduced the category **LoA-SP**, providing a generalization of the notion of state property system of D. Aerts [2]. What is the nature of the category in question (algebraic, coalgebraic)? What is its relation to the supercategory **LoA-TopSys** (reflective, coreflective subcategory)?

To continue, we recall the following result from [1, Proposition 23.12].

Problem 6.4. If a concrete category $(\mathbf{C}, U)$ is essentially algebraic over the category $\mathbf{X}$, then the following hold:

(1) If $\mathbf{X}$ is (strongly) complete, then $\mathbf{C}$ is (strongly) complete.
(2) If $\mathbf{X}$ has coproducts, then $\mathbf{C}$ is cocomplete.

Since every variety is complete and, moreover, has coequalizers, the following result holds.

Theorem 6.5. *The category* **LoA**-**TopSys** *is cocomplete. It is complete provided that* **A** *has coproducts. The category* **A**-**TopSys** *is both complete and cocomplete provided that* **A** *has coproducts.*

As an example, notice that the variety **CLat** of complete lattices does not have coproducts [1, Exercise 10S] and, therefore, the condition of Theorem 6.5 is essential in application of the theorem to different cases. On the other hand, the variety **Frm** of frames is cocomplete and, therefore, both **Loc-TopSys** and **Frm-TopSys** are complete and cocomplete.

In [52] S. Vickers constructed explicitly binary products and coproducts of topological systems. Moreover, in [3] D. Aerts *et al.* provided a construction of binary products of state property systems. The latter case, however, is simpler (from the categorical viewpoint) since the category **SP** of the structures in question is equivalent to the already mentioned (Example 2.10) category **Cls** of closure spaces [4], which is topological [11] over its ground category **Set** and, therefore, is both complete and cocomplete. In view of the above-mentioned remarks one can postulate the following problem.

Problem 6.6. What is the explicit limit and colimit structure of both **LoA-TopSys** and **A-TopSys**?

The problems raised in this section will be addressed to in our subsequent manuscripts.

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