ALGEBRAIC GENERATIONS OF SOME FUZZY POWERSET OPERATORS

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ABSTRACT. In this paper, let L be a complete residuated lattice, and let **Set** denote the category of sets and mappings, LF-**Pos** denote the category of LF-posets and LF-monotone mappings, and LF-**CSLat**($\sqcup$), LF-**CSLat**($\sqcap$) denote the category of LF-complete lattices and LF-join-preserving mappings and the category of LF-complete lattices and LF-meet-preserving mappings, respectively. It is proved that there are adjunctions between **Set** and LF-**CSLat**($\sqcup$), between LF-**Pos** and LF-**CSLat**($\sqcup$), and between LF-**Pos** and LF-**CSLat**($\sqcap$), that is, **Set**$\dashv$ LF-**CSLat**($\sqcup$), LF-**Pos**$\dashv$ LF-**CSLat**($\sqcup$), and LF-**Pos**$\dashv$ LF-**CSLat**($\sqcap$). And a usual mapping f generates the traditional Zadeh forward powerset operator $f_L^{\rightarrow}$ and the fuzzy forward powerset operators $\widetilde{f}^{\rightarrow}, \widetilde{f}_*^{\rightarrow}, \widetilde{f}^{*\rightarrow}$ defined by the author et al via these adjunctions. Moreover, it is also shown that all the fuzzy powerset operators mentioned above can be generated by the underlying algebraic theories.

1. Introduction

Given a usual mapping $f : X \longrightarrow Y$, the traditional powerset operators $f^{\rightarrow} : \mathcal{P}(X) \longrightarrow \mathcal{P}(Y)$ and $f^{\leftarrow} : \mathcal{P}(Y) \longrightarrow \mathcal{P}(X)$ given by

$$f^{\rightarrow}(A) = \{f(x) \mid x \in A\}, \quad f^{\leftarrow}(B) = \{x \mid f(x) \in B\}$$

play a critical role in ordinary mathematics. While it took more than a century of mathematics to "empirically" confirm that these powerset operators are the "correct" liftings of f to the powersets of X and Y, it is first verified mathematically and directly by E. G. Manes [9]. The differently formatted, but equivalent proof of S. E. Rodabaugh [11] creates $f^{\rightarrow}$ by proving the adjunction between **Set** (the category of sets and mappings) and **CSLat**($\bigvee$) (the category of complete join-semi-lattices and arbitrary-join-preserving maps), and then creates $f^{\leftarrow}$ from $f^{\rightarrow}$ by the Adjoint Function Theorem (AFT) for order-preserving mappings between posets. In fact, the powerset operator foundations of traditional mathematics may be viewed as entirely a consequence of the AFT. The fundamental importance of powerset operators for fuzzy sets is recognized from the beginning in L. A. Zadeh's pioneering paper [17] introducing fuzzy sets. That Zadeh puzzles over the definition of $f^{\rightarrow}$ (whether to use $\bigvee$ or $\bigwedge$) indicates it was not clear to Zadeh whether

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his powerset operators were the correct ones. But Manes [9] gives the first proof, using a monadic approach for a certain restricted class of lattices L, that the Zadeh operators were the right ones. Then in [10], S. E. Rodabaugh points out by the AFT that the traditional Zadeh forward powerset operator $f_L^{\rightarrow}$ is a left adjoint of Zadeh backward powerset operator $f_L^{\leftarrow}$ and the "Zadeh Extension Principle" is correct. After that he gives two different proofs for all complete lattices L vindicating Zadeh's definitions in [11], and extends these results to lattices taken from **CQMIL** (the category of complete quasi-monoidal integral lattices and the mappings of preserving $\otimes$, arbitrary $\bigvee$, and the top element $\top$) in [12]. Theorem 6.13 of [11] and Theorem 2.11 of [12] directly generate the traditional Zadeh forward powerset operator $f_L^{\rightarrow}$ from f via a universal construction. Such a universal construction is tantamount to having an adjunction between **Set** and some category. Thus [11] and [12] directly generate $f_L^{\rightarrow}$ from f via an adjunction in which the right-hand category is not named. However, his new paper [13] generates $f_L^{\rightarrow}$ from f by means of an algebraic theory built using L a unital quantale.

The first goal of this paper is to directly generate $f_L^{\rightarrow}$ from f via an adjunction between **Set** and a concrete category LF-**CSLat**$(\sqcup)$ (the category of L-fuzzy complete lattices and L-fuzzy join-preserving mappings) when L is a complete residuated lattice, then create the right 1-adjoint $f_L^{\leftarrow}$ by the AFT for L-fuzzy posets. It easily sees that the category LF-**CSLat**$(\sqcup)$ is exactly the category **CSLat**$(\bigvee)$ when L is the binary lattice **2**, and that the Zadeh powerset operators $f_L^{\rightarrow}, f_L^{\leftarrow}$ are respectively identical to the traditional powerset operators $f^{\rightarrow}, f^{\leftarrow}$, and that the 1-adjoint pair is the traditional adjoint pair, so this paper generalizes the corresponding work of [10-12] providing L is a complete residuated lattice. In addition, some fuzzy powerset operators $\widetilde{f}^{\rightarrow}, \widetilde{f}_*^{\rightarrow}, \widetilde{f}^{*\rightarrow}$ and $\widetilde{f}^{\leftarrow}, \widetilde{f}_*^{\leftarrow}, \widetilde{f}^{*\leftarrow}$ are suggested in [19, 20], they are not only generalizations of the traditional powerset operators but also of the Zadeh powerset operators. This paper also directly generates $\widetilde{f}^{\rightarrow}, \widetilde{f}_*^{\rightarrow}$ from f via adjunctions between LF-**Pos** (the category of L-fuzzy posets and L-fuzzy monotone mappings) and LF-**CSLat**$(\sqcup)$, and directly generates $\widetilde{f}^{*\rightarrow}$ from f via an adjunction between LF-**Pos** and LF-**CSLat**$(\sqcap)$ (the category of L-fuzzy complete lattices and L-fuzzy meet-preserving mappings). They create respectively $\widetilde{f}^{\leftarrow}, \widetilde{f}_*^{\leftarrow}, \widetilde{f}^{*\leftarrow}$ by the AFT for L-fuzzy posets.

On the other hand, similar to the idea in [13], we give the general fuzzy powerset theories in setting of fuzzy posets; through the aforesaid adjunctions we also build the underlying algebraic theories, and prove that all the above fuzzy powerset operators can be generated by these algebraic theories.

The content of the paper is as follows: Section 1 recalls some notions and results in [1-3, 18-20]. Section 2 directly generates $f_L^{\rightarrow}$ from f via an adjunction between **Set** and LF-**CSLat**$(\sqcup)$, generates $\widetilde{f}^{\rightarrow}, \widetilde{f}_*^{\rightarrow}$ from f via adjunctions between LF-**Pos** and LF-**CSLat**$(\sqcup)$, and generates $\widetilde{f}^{*\rightarrow}$ from f via an adjunction between LF-**Pos** and LF-**CSLat**$(\sqcap)$. And by the AFT, they respectively determine the unique right 1-adjoints $f_L^{\leftarrow}, \widetilde{f}^{\leftarrow}, \widetilde{f}_*^{\leftarrow}, \widetilde{f}^{*\leftarrow}$. Section 3 generates all the above fuzzy powerset operators by means of algebraic theories.

2. L-fuzzy Posets and L-fuzzy Complete Lattices

For a complete lattice L, let 0 denote the bottom element, 1 the top element, and for an $A \subseteq L$, let $\bigvee A$ denote the least upper bound of A and $\bigwedge A$ the greatest lower bound of A.

A *semi-quantale* $(L, \leq, \otimes)$ is a complete lattice $(L, \leq)$ equipped with a binary operation $\otimes : L \times L \longrightarrow L$, with no additional assumptions, called a *tensor product*. The category **SQuant** comprises all semi-quantales together with mappings preserving $\otimes$ and $\bigvee$. A *quantale* $(L, \leq, \otimes)$ is semi-quantale with $\otimes$ associative and distributive over arbitrary $\bigvee$ from both sides. A quantale is called *commutative* whenever its tensor is, and it is called *unital* if the tensor has a unit e, i.e. $p \otimes e = p = e \otimes p$ for all $p \in L$. A *strictly two-sided quantale*, abbreviated an *st-quantale*, is a unital quantale with $e = 1$.

A *complete residuated lattice* is an algebra $\langle L, \wedge, \vee, *, \rightarrow, 0, 1\rangle$ such that (R1) $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice with the least element 0 and the greatest element 1; (R2) $\langle L, *, 1\rangle$ is a commutative monoid; (R3) $*, \rightarrow$ form an adjoint pair, i.e. $x * y \leq z$ iff $x \leq y \rightarrow z$ holds for all $x, y, z \in L$. The operations $*$ and $\rightarrow$ are called *multiplication* and *residuum*, respectively.

From the above definitions, it easily follows that a complete residuated lattice is exactly a commutative st-quantale with $\otimes = *$, and a frame (or complete Heyting algebra) L can be viewed as a complete residuated lattice with $* = \wedge$ and that $\rightarrow$ is the implication in the frame L. Multiplication is monotone, residuum is monotone in the first and antitone in the second argument (w.r.t. lattice order $\leq$).

Let L denote a complete residuated lattice. For $a, b, c, d \in L$, $B \subseteq L$. The following results are often used in the proof.

$$a \rightarrow b = \bigvee\{c \in L \mid a * c \leq b\};$$

$$a \leq b \Leftrightarrow a \rightarrow b = 1, \quad a * b = 1 \Rightarrow a = 1, b = 1.$$

$$a = 1 \rightarrow a, \quad a * b \leq a \wedge b$$

$$b \leq c \Longrightarrow a \rightarrow b \leq a \rightarrow c, \quad c \rightarrow a \leq b \rightarrow a;$$

$$a * (a \rightarrow b) \leq b, \quad a \leq (a \rightarrow b) \rightarrow b, \quad (a \rightarrow b) * (b \rightarrow c) \leq a \rightarrow c;$$

$$a \rightarrow (b \rightarrow c) = (a * b) \rightarrow c, \quad (a \rightarrow b) * (c \rightarrow d) \leq a \rightarrow (c \rightarrow (b * d));$$

$$a \rightarrow (\bigwedge B) = \bigwedge\{a \rightarrow b \mid b \in B\}, \quad a \rightarrow (\bigvee B) \geq \bigvee\{a \rightarrow b \mid b \in B\};$$

$$(\bigvee B) \rightarrow a = \bigwedge\{b \rightarrow a \mid b \in B\}, \quad (\bigwedge B) \rightarrow a \geq \bigvee\{b \rightarrow a \mid b \in B\}.$$

The following definitions and results can be found in [1-3, 18-20] when L is a frame. Here we give the corresponding ones for L a complete residuated lattice. And we have the following denotations. If no other conditions are imposed, in the sequel L always denotes a complete residuated lattice, X denotes a non-empty set and L^X is the set of all L-fuzzy subsets of X, that is, the set of all mappings from X to L. Usually we write "LF-" instead of "L-fuzzy".

Definition 2.1. Let X be a non-empty set and $e : X \times X \longrightarrow L$ a mapping (called a degree function). Consider the following conditions:

(E1) $\forall x \in X$, $e(x, x) = 1$; (reflexivity)
(E2) $\forall x, y, z \in X$, $e(x, y) = e(y, x)$; (symmetry)
(E3) $\forall x, y, z \in X$, $e(x, y) * e(y, z) \leq e(x, z)$; (transitivity)
(E4) $\forall x, y \in X$, $e(x, y) = 1 = e(y, x) \Rightarrow x = y$. (antisymmetry)

(i) e is called an *L-fuzzy preorder* if it satisfies (E1) and (E3), and the pair (X, e) is called an *L-fuzzy preordered set*.

(ii) e is called an *L-fuzzy partial order* if it satisfies (E1), (E3) and (E4), and the pair (X, e) is called an *L-fuzzy partially ordered set* (or simply, *L-fuzzy poset*, or *fuzzy poset*).

(iii) e is called an *L-fuzzy equality* if it satisfies (E1), (E2), (E3) and (E4).

If (X, e) is an LF-poset, then the dual of (X, e) is the pair (X, e^{op}), where for all $x, y \in X$, $e^{op}(x, y) = e(y, x)$ and the symmetrization of (X, e) is the pair $((X, e^s)$, where for all $x, y \in X$, $e^s(x, y) = e(x, y) * e(y, x)$. It easily follows that (X, e^{op}) and (X, e^s) are LF-posets, and e^s is indeed an L-fuzzy equality on X.

Remark 2.2. An L-fuzzy preordered set can be viewed as an enriched category in [14] over a complete residuated lattice L.

Every complete residuated lattice L can be seen as an LF-poset by taking $e_L(a, b) = a \to b$. In what follows, the degree function in L will be always taken to be this map e_L.

Definition 2.3. Let $(X, e_X), (Y, e_Y)$ be LF-posets. Then a mapping $f : X \longrightarrow Y$ is called *L-fuzzy monotone* if $e_X(x, y) \leq e_Y(f(x), f(y))$ for all $x, y \in X$.

Remark 2.4. Let $(X, e_X), (Y, e_Y)$ be LF-posets. An L-fuzzy monotone mapping is exactly an Ω-morphism [14] between the two Ω-categories $(X, e_X), (Y, e_Y)$, where Ω denotes a unital commutative quantale.

Definition 2.5. For $a, b, \eta \in L$, put that $a \star b = (a \to b) * (b \to a)$ and postulate that

$$a \approx_\eta b \Longleftrightarrow \eta \leq a \star b.$$

Definition 2.6. Let (X, e_X) and (Y, e_Y) be LF-posets and $f : X \longrightarrow Y$, $g : Y \longrightarrow X$ LF-monotone mappings and $\eta \in L$. The pair (f, g) is called an *η-adjunction*, denoted by $f \dashv_\eta g$ if for all $x \in X$ and for all $y \in Y$, $e_Y(f(x), y) \approx_\eta e_X(x, g(y))$. In this case we call f a *left η-adjoint* of g and g a *right η-adjoint* of f.

Remark 2.7. It easily follows that $f \dashv_1 g \Longleftrightarrow e_Y(f(x), y) = e_X(x, g(y))$ for all $x \in X$ and $y \in Y$, so the 1-adjunction here is exactly an Ω-adjunction in [7, 14] and a fuzzy Galois connection in [15, 16], where Ω denotes a unital commutative quantale.

Definition 2.8. Let (X, e) be an LF-poset, $\phi \in L^X$. An $x_0 \in X$ is called a *join* (or a *supremum*) of ϕ (w.r.t. the L-fuzzy partial order e), and denoted by $\sqcup\phi$, if

(1) $\forall x \in X$, $\phi(x) \leq e(x, x_0)$;

(2) $\forall y \in X$, $\bigwedge_{x\in X}(\phi(x) \to e(x,y)) \le e(x_0,y)$.

And an $x_1 \in X$ is called a *meet* (or an *infimum*) of ϕ (w.r.t. the L-fuzzy partial order e), and denoted by $\sqcap\phi$, if

(1) $\forall x \in X$, $\phi(x) \le e(x_1,x)$;
(2) $\forall y \in X$, $\bigwedge_{x\in X}(\phi(x) \to e(y,x)) \le e(y,x_1)$.

Theorem 2.9. *Let (X,e) be an LF-poset, $x_0,x_1 \in X$, $\phi \in L^X$.*

(1) *x_0 is a join of $\phi \Longleftrightarrow \forall y \in X$, $e(x_0,y) = \bigwedge_{x\in X}(\phi(x) \to e(x,y))$.*
(2) *x_1 is a meet of $\phi \Longleftrightarrow \forall y \in X$, $e(y,x_1) = \bigwedge_{x\in X}(\phi(x) \to e(y,x))$.*

Definition 2.10. Let (X,e) be an LF-poset, $x \in X$. Define the mappings $\iota_x, \mu_x, s_x : X \longrightarrow L$ as follows: $\forall y \in X$,

$$\iota_x(y) = e(y,x), \quad \mu_x(y) = e(x,y), \quad s_x(y) = e(x,y) * e(y,x).$$

Remark 2.11. In [7] the ι_x, μ_x are denoted by $\mathbf{y}(x)$, $\mathbf{y}'(x)$, respectively. And in [15, 16] they are denoted by $\downarrow x, \uparrow x$, respectively.

Theorem 2.12. *Let (X,e) be an LF-poset. Then for all $x \in X$, $\sqcup\iota_x = \sqcup s_x = x$, $\sqcap\mu_x = \sqcap s_x = x$.*

Definition 2.13. An LF-poset (X,e) is called an *L-fuzzy complete lattice* if $\sqcup\phi$ and $\sqcap\phi$ exist for every L-fuzzy subset ϕ of X.

Theorem 2.14. *Let (X,e) be an LF-poset. Then*

(1) *(X,e) is an LF-complete lattice if and only if $\sqcup\phi$ exists for all $\phi \in L^X$.*
(2) *(X,e) is an LF-complete lattice if and only if $\sqcap\phi$ exists for all $\phi \in L^X$.*

Let X be a non-empty set. For all $\phi, \psi \in L^X$, define

$$\widetilde{e}(\phi,\psi) = \bigwedge_{x\in X} e_L(\phi(x),\psi(x)) = \bigwedge_{x\in X} (\phi(x) \to \psi(x)).$$

Theorem 2.15. *Let X be a non-empty set. Then $(L^X, \widetilde{e})$ is an LF-complete lattice, and for all $\Phi \in L^{L^X}$, $\sqcup\Phi$ and $\sqcap\Phi$ are given by $(\sqcup\Phi)(x) = \bigvee_{\phi\in L^X}(\Phi(\phi) * \phi(x))$ and $(\sqcap\Phi)(x) = \bigwedge_{\phi\in L^X}(\Phi(\phi) \to \phi(x))$ for every $x \in X$.*

Definition 2.16. Let (X,e) be an LF-poset. An L-fuzzy subset ϕ of X is called an *L-fuzzy lower set* (*L-fuzzy upper set, L-fuzzy sound set*) on X if $\phi(x) * e(y,x) \le \phi(y)$ ($\phi(x) * e(x,y) \le \phi(y)$, $\phi(x) * e(x,y) * e(y,x) \le \phi(y)$) for all $x,y \in X$. Let $\mathcal{L}_L(X), \mathcal{U}_L(X), \mathcal{S}_L(X)$ denote respectively the collection of all L-fuzzy lower sets on X and the collection of all L-fuzzy upper sets and the collection of all L-fuzzy sound sets on X.

Theorem 2.17. *Let (X,e) be an LF-poset. Then $(\mathcal{L}_L(X),\widetilde{e}), (\mathcal{U}_L(X),\widetilde{e}), (\mathcal{S}_L(X),\widetilde{e})$ are all LF-complete lattices. (Here the $\widetilde{e}$s are respectively the restrictions of $\widetilde{e}$ to $\mathcal{L}_L(X) \times \mathcal{L}_L(X)$ and $\mathcal{U}_L(X) \times \mathcal{U}_L(X)$ and $\mathcal{S}_L(X) \times \mathcal{S}_L(X)$). And for every L-fuzzy* *subset* Φ *of* $(\mathcal{L}_L(X),\widetilde{e})$ *or* $(\mathcal{U}_L(X),\widetilde{e})$ *or* $(\mathcal{S}_L(X),\widetilde{e})$, $\sqcup\Phi$ *and* $\sqcap\Phi$ *are respectively given by*

$$(\sqcup\Phi)(x)=\bigvee_{\phi\in\mathcal{L}_L(X)}(\Phi(\phi)*\phi(x)),\ \ (\sqcap\Phi)(x)=\bigwedge_{\phi\in\mathcal{L}_L(X)}(\Phi(\phi)\to\phi(x)),(\textit{ in }(\mathcal{L}_L(X),\widetilde{e}))$$

$$(\sqcup\Phi)(x)=\bigvee_{\phi\in\mathcal{U}_L(X)}(\Phi(\phi)*\phi(x)),\ \ (\sqcap\Phi)(x)=\bigwedge_{\phi\in\mathcal{U}_L(X)}(\Phi(\phi)\to\phi(x)),(\textit{ in }(\mathcal{U}_L(X),\widetilde{e}))$$

$$(\sqcup\Phi)(x)=\bigvee_{\phi\in\mathcal{S}_L(X)}(\Phi(\phi)*\phi(x)),\ \ (\sqcap\Phi)(x)=\bigwedge_{\phi\in\mathcal{S}_L(X)}(\Phi(\phi)\to\phi(x)),(\textit{ in }(\mathcal{S}_L(X),\widetilde{e}))$$

for all $x\in X$.

Definition 2.18. Let $f:X\longrightarrow Y$ be a mapping from a non-empty set X and an LF-poset (Y,e_Y). Then we define the fuzzy forward powerset operators $\widetilde{f}^{\to},\widetilde{f}^{\to}_{*},\widetilde{f}^{*\to}:L^X\longrightarrow L^Y$ as follows: $\forall\phi\in L^X$, $\forall y\in Y$,

$$\widetilde{f}^{\to}(\phi)(y)=\bigvee_{x\in X}(\phi(x)*e_Y(y,f(x))*e_Y(f(x),y)),$$

$$\widetilde{f}^{\to}_{*}(\phi)(y)=\bigvee_{x\in X}(\phi(x)*e_Y(y,f(x))),\quad \widetilde{f}^{*\to}(\phi)(y)=\bigvee_{x\in X}(\phi(x)*e_Y(f(x),y)).$$

And let $g:X\longrightarrow Y$ be a mapping from an LF-poset (X,e_X) to a non-empty set Y. Then we define the fuzzy backward powerset operators $\widetilde{g}^{\leftarrow},\widetilde{g}^{\leftarrow}_{*},\widetilde{g}^{*\leftarrow}:L^Y\longrightarrow L^X$ as follows: $\forall\psi\in L^Y$, $\forall x\in X$,

$$\widetilde{g}^{\leftarrow}(\psi)(x)=\bigvee_{x'\in X}(\psi(g(x'))*e_X(x,x')*e_X(x',x)),$$

$$\widetilde{g}^{\leftarrow}_{*}(\psi)(x)=\bigvee_{x'\in X}(\psi(g(x'))*e_X(x,x')),\quad \widetilde{g}^{*\leftarrow}(\psi)(x)=\bigvee_{x'\in X}(\psi(g(x'))*e_X(x',x)).$$

Proposition 2.19. *Let* (X,e_X), (Y,e_Y) *be LF-posets,* $f:X\longrightarrow Y$ *an LF-monotone mapping. Then*

$$\widetilde{f}^{\to},\widetilde{f}^{\to}_{*},\widetilde{f}^{*\to}:(L^X,\widetilde{e})\longrightarrow(L^Y,\widetilde{e}),\quad \widetilde{f}^{\leftarrow},\widetilde{f}^{\leftarrow}_{*},\widetilde{f}^{*\leftarrow}:(L^Y,\widetilde{e})\longrightarrow(L^X,\widetilde{e})$$

are all LF-monotone.

Theorem 2.20. *Let* $(X,e_X),(Y,e_Y)$ *be LF-posets,* $f:X\longrightarrow Y$ *a mapping,* $\phi\in L^X$.

(1) *If* $\sqcup\widetilde{f}^{\to}_{*}(\phi)$ *exists, then* $\sqcup\widetilde{f}^{\to}(\phi)$ *exists and* $\sqcup\widetilde{f}^{\to}(\phi)=\sqcup\widetilde{f}^{\to}_{*}(\phi)$; *conversely if* $\sqcup\widetilde{f}^{\to}(\phi)$ *exists, then* $\sqcup\widetilde{f}^{\to}_{*}(\phi)$ *exists and* $\sqcup\widetilde{f}^{\to}_{*}(\phi)=\sqcup\widetilde{f}^{\to}(\phi)$.

(2) *If* $\sqcap\widetilde{f}^{*\to}(\phi)$ *exists, then* $\sqcap\widetilde{f}^{\to}(\phi)$ *exists and* $\sqcap\widetilde{f}^{\to}(\phi)=\sqcap\widetilde{f}^{*\to}(\phi)$; *conversely if* $\sqcap\widetilde{f}^{\to}(\phi)$ *exists, then* $\sqcap\widetilde{f}^{*\to}(\phi)$ *exists and* $\sqcap\widetilde{f}^{*\to}(\phi)=\sqcap\widetilde{f}^{\to}(\phi)$.

Remark 2.21. Let $(X,e_X),(Y,e_Y)$ be LF-posets, $f:X\longrightarrow Y$ a mapping. By Theorem 2.20 it easily follows that for a $\phi\in L^X$,

(1) if $\sqcup\phi,\sqcup\widetilde{f}^{\to}(\phi),\sqcup\widetilde{f}^{\to}_{*}(\phi)$ exist, then $f(\sqcup\phi)=\sqcup\widetilde{f}^{\to}(\phi)\Longleftrightarrow f(\sqcup\phi)=\sqcup\widetilde{f}^{\to}_{*}(\phi)$;

(2) if $\sqcap\phi,\sqcap\widetilde{f}^{\to}(\phi),\sqcap\widetilde{f}^{*\to}(\phi)$ exist, then $f(\sqcap\phi)=\sqcap\widetilde{f}^{\to}(\phi)\Longleftrightarrow f(\sqcap\phi)=\sqcap\widetilde{f}^{*\to}(\phi)$.

Thus we have the following definition.

Definition 2.22. Let $(X, e_X), (Y, e_Y)$ be LF-posets. A mapping $f : X \longrightarrow Y$ is called *LF-join-preserving* if it satisfies $f(\sqcup\phi) = \sqcup\widetilde{f}_*^{\rightarrow}(\phi)$ for all $\phi \in L^X$, and called *LF-meet-preserving* if it satisfies $f(\sqcap\phi) = \sqcap\widetilde{f}^{*\rightarrow}(\phi)$ for all $\phi \in L^X$.

Proposition 2.23. *Let $(X, e_X), (Y, e_Y)$ be LF-posets and $f : X \longrightarrow Y$ a mapping. If f is LF-join-preserving, or LF-meet-preserving, then f is LF-monotone.*

3. Generate the Powerset Operators $f_L^{\rightarrow}, \widetilde{f}^{\rightarrow}, \widetilde{f}_*^{\rightarrow}, \widetilde{f}^{*\rightarrow}, f_L^{\leftarrow}, \widetilde{f}^{\leftarrow}, \widetilde{f}_*^{\leftarrow}, \widetilde{f}^{*\leftarrow}$ Directly Via Adjunctions and AFT

This section is to generate the powerset operators $f_L^{\rightarrow}, \widetilde{f}^{\rightarrow}, \widetilde{f}_*^{\rightarrow}, \widetilde{f}^{*\rightarrow}, f_L^{\leftarrow}, \widetilde{f}^{\leftarrow}, \widetilde{f}_*^{\leftarrow}, \widetilde{f}^{*\leftarrow}$ directly via adjunctions when L is a complete residuated lattice. The results and proofs of this section can be found in [21] when L is a frame. For a complete residuated lattice, the proofs are similar. However we also give most proofs for readability.

Let X, Y be non-empty sets and $f : X \longrightarrow Y$ a map. Then the traditional Zadeh forward powerset operator $f_L^{\rightarrow} : L^X \longrightarrow L^Y$ and the Zadeh backward powerset operator $f_L^{\leftarrow} : L^Y \longrightarrow L^X$ are defined respectively by

$$f_L^{\rightarrow}(\phi)(y) = \bigvee_{x \in f^{\leftarrow}(\{y\})} \phi(x), \quad \forall \phi \in L^X, \forall y \in Y,$$
$$f_L^{\leftarrow}(\psi)(x) = (\psi \circ f)(x), \quad \forall \psi \in L^Y, \forall x \in X.$$

By the definitions of the above fuzzy powerset operators we easily get the following fact:

$$f_L^{\rightarrow}(\chi_{\{x\}}) = \chi_{f(x)} \quad \widetilde{f}_*^{\rightarrow}(\iota_x) = \iota_{f(x)}, \quad \widetilde{f}^{\rightarrow}(s_x) = s_{f(x)}, \quad \widetilde{f}^{*\rightarrow}(\mu_x) = \mu_{f(x)}.$$

Proposition 3.1. *Let X be a non-empty set and (Y, e_Y) an LF-poset, $f : X \longrightarrow Y$ a mapping.*

(1) *If $\sqcup f_L^{\rightarrow}(\phi)$ and $\sqcup\widetilde{f}_*^{\rightarrow}(\phi)$ exist, then $\sqcup f_L^{\rightarrow}(\phi) = \sqcup\widetilde{f}_*^{\rightarrow}(\phi)$.*

(2) *If $\sqcap f_L^{\rightarrow}(\phi)$ and $\sqcap\widetilde{f}^{*\rightarrow}(\phi)$ exist, then $\sqcap f_L^{\rightarrow}(\phi) = \sqcap\widetilde{f}^{*\rightarrow}(\phi)$.*

Proof. (1) Let $y_0 = \sqcup\widetilde{f}_*^{\rightarrow}(\phi)$. We will prove that $y_0 = \sqcup f_L^{\rightarrow}(\phi)$. At first, for all $y \in Y$,

$$f_L^{\rightarrow}(\phi)(y) = \bigvee_{x \in f^{\leftarrow}(\{y\})} \phi(x) \le \bigvee_{x \in X} (\phi(x) * e_Y(y, f(x))) = \widetilde{f}_*^{\rightarrow}(\phi)(y) \le e_Y(y, y_0).$$

Secondly, for all $z \in Y$,

$$\begin{aligned}
& \textstyle\bigwedge_{y \in Y}(f_L^{\rightarrow}(\phi)(y) \to e_Y(y, z)) \\
\le\ & \textstyle\bigwedge_{y \in f(X)}(f_L^{\rightarrow}(\phi)(y) \to e_Y(y, z)) \\
=\ & \textstyle\bigwedge_{y \in f(X)}(\bigvee_{x' \in f^{\leftarrow}(\{y\})} \phi(x') \to e_Y(y, z)) \\
=\ & \textstyle\bigwedge_{f(x) \in f(X)}(\bigvee_{x' \in f^{\leftarrow}(\{f(x)\})} \phi(x') \to e_Y(f(x), z)) \\
\le\ & \textstyle\bigwedge_{x \in X}(\phi(x) \to e(f(x), z)) \\
=\ & \textstyle\bigwedge_{x \in X}(\phi(x) \to \bigwedge_{y \in Y}(e_Y(y, f(x)) \to e_Y(y, z))) \\
=\ & \textstyle\bigwedge_{x \in X}\bigwedge_{y \in Y}(\phi(x) \to (e_Y(y, f(x)) \to e_Y(y, z))) \\
=\ & \textstyle\bigwedge_{x \in X}\bigwedge_{y \in Y}((\phi(x) * e_Y(y, f(x))) \to e_Y(y, z)) \\
=\ & \textstyle\bigwedge_{x \in X}(\bigvee_{y \in Y}(\phi(x) * e_Y(y, f(x))) \to e_Y(y, z)) \\
=\ & \textstyle\bigwedge_{y \in Y}(\widetilde{f}_*^{\rightarrow}(\phi)(y) \to e_Y(y, z)) \\
\le\ & e_Y(y_0, z).
\end{aligned}$$

Thus we have $\sqcup f_L^{\rightarrow}(\phi) = y_0 = \sqcup \widetilde{f}_*^{\rightarrow}(\phi)$.

(2) can be proved dually. □

Corollary 3.2. *Let $(X, e_X), (Y, e_Y)$ be LF-posets, $f : X \longrightarrow Y$ a map. Then*

(1) *f is an LF-join-preserving map if and only if $f(\sqcup\phi) = \sqcup f_L^{\rightarrow}(\phi)$ for all $\phi \in L^X$;*

(2) *f is an LF-meet-preserving map if and only if $f(\sqcap\phi) = \sqcap f_L^{\rightarrow}(\phi)$ for all $\phi \in L^X$.*

The following Adjoint Functor Theorem (AFT) for L-fuzzy posets was proved in [21] when L is frame. It also holds for a complete residuated lattice. The result is also seen in [7, 15, 16]. In addition, from Proposition 3.1 and Corollary 3.2 we can get that the AFT here and Theorem 4.5 in [16] are just the same things.

Theorem 3.3. (**Adjoint Functor Theorem**) *Let $(X, e_X), (Y, e_Y)$ be LF-posets.*

(1) *Let $f : X \longrightarrow Y$ be LF-monotone. If f has a right 1-adjoint $g : Y \longrightarrow X$ (that is, $f \dashv_1 g$), then f preserves all joins which exist in X; conversely, if X is L-fuzzy complete and f preserves all joins, then f has a right 1-adjoint.*

(2) *Let $g : Y \longrightarrow X$ be LF-monotone. If g has a left 1-adjoint $f : X \longrightarrow Y$ (that is, $f \dashv_1 g$), then g preserves all meets which exist in Y; conversely, if Y is L-fuzzy complete and g preserves all meets, then g has a left 1-adjoint.*

Corollary 3.4. *Let $(X, e_X), (Y, e_Y)$ be LF-complete lattices and $f : X \longrightarrow Y$ an LF-monotone mapping.*

(1) *f is a left 1-adjoint if and only if f preserves joins of all L-fuzzy subsets.*

(2) *g is a right 1-adjoint if and only if g preserves meets of all L-fuzzy subsets.*

Let **Set** denote the category of sets and mappings, LF-**Pos** denote the category of LF-posets and LF-monotone mappings, and LF-**CSLat**$(\sqcup)$, LF-**CSLat**$(\sqcap)$ denote respectively the category of LF-complete lattices and LF-join-preserving mappings and the category of LF-complete lattices and LF-meet-preserving mappings.

Theorem 6.13 of [11] and Theorem 2.11 of [12] directly generate $f_L^{\rightarrow}$ from f via a universal construction for L a complete quasi-monoidal integral lattice. Such a universal construction is tantamount to having an adjunction between **Set** and some category. Thus [11] and [12] directly generate $f_L^{\rightarrow}$ from f via an adjunction in which the right-hand category is not named. In the following we will directly generate $f_L^{\rightarrow}$ from f via an adjunction between **Set** and LF-**CSLat**$(\sqcup)$ when L is a complete residuated lattice.

Lemma 3.5. (1) *Let $(X, e_X), (Y, e_Y)$ be LF-posets and $f : X \longrightarrow Y$ an LF-monotone mapping. Then*

$$\psi \in \mathcal{S}_L(Y) \Longrightarrow \widetilde{e}(\widetilde{f}^{\rightarrow} \circ \widetilde{f}^{\leftarrow}(\psi), \psi) = 1;$$

$$\psi \in \mathcal{L}_L(Y) \Longrightarrow \widetilde{e}(\widetilde{f}_*^{\rightarrow} \circ \widetilde{f}_*^{\leftarrow}(\psi), \psi) = 1; \quad \psi \in \mathcal{U}_L(Y) \Longrightarrow \widetilde{e}(\widetilde{f}^{*\rightarrow} \circ \widetilde{f}^{*\leftarrow}(\psi), \psi) = 1.$$

(2) *Let X, Y be non-empty sets and $f : X \longrightarrow Y$ a mapping. Then*

$$\psi \in L^Y \Longrightarrow \widetilde{e}_Y(f_L^{\rightarrow} \circ f_L^{\leftarrow}(\psi), \psi) = 1.$$

Proof. (1) We only prove the second argument, and the others can be shown similarly.

Suppose $\psi \in \mathcal{L}_L(Y)$. Then

$$
\begin{aligned}
& \widetilde{e}_Y(\widetilde{f_*}^{\rightarrow} \circ \widetilde{f_*}^{\leftarrow}(\psi), \psi) \\
= \ & \bigwedge_{y\in Y}(\widetilde{f_*}^{\rightarrow}(\widetilde{f_*}^{\leftarrow}(\psi))(y) \rightarrow \psi(y)) \\
= \ & \bigwedge_{y\in Y}(\bigvee_{x\in X}(\widetilde{f_*}^{\leftarrow}(\psi)(x) * e_Y(y, f(x))) \rightarrow \psi(y)) \\
= \ & \bigwedge_{y\in Y}\bigwedge_{x\in X}((\bigvee_{x'\in X}(\psi(f(x')) * e_X(x, x')) * e_Y(y, f(x))) \rightarrow \psi(y)) \\
= \ & \bigwedge_{y\in Y}\bigwedge_{x\in X}(\bigvee_{x'\in X}(\psi(f(x')) * e_X(x, x') * e_Y(y, f(x))) \rightarrow \psi(y)) \\
= \ & \bigwedge_{y\in Y}\bigwedge_{x\in X}\bigwedge_{x'\in X}((\psi(f(x')) * e_X(x, x') * e_Y(y, f(x))) \rightarrow \psi(y)) \\
= \ & \bigwedge_{y\in Y}\bigwedge_{x\in X}\bigwedge_{x'\in X}(\psi(f(x')) \rightarrow ((e_X(x, x') * e_Y(y, f(x))) \rightarrow \psi(y))) \\
= \ & \bigwedge_{x'\in X}(\psi(f(x') \rightarrow \bigwedge_{y\in Y}\bigwedge_{x\in X}((e_X(x, x') * e_Y(y, f(x))) \rightarrow \psi(y))) \\
\geq \ & \bigwedge_{x'\in X}(\psi(f(x')) \rightarrow \bigwedge_{y\in Y}\bigwedge_{x\in X}(e_Y(f(x), f(x')) * e_Y(y, f(x))) \rightarrow \psi(y))) \\
= \ & \bigwedge_{x'\in X}(\psi(f(x')) \rightarrow \bigwedge_{y\in Y}(\bigvee_{x\in X}(e_Y(f(x), f(x')) * e_Y(y, f(x))) \rightarrow \psi(y))) \\
\geq \ & \bigwedge_{x'\in X}(\psi(f(x')) \rightarrow \bigwedge_{y\in Y}(e_Y(y, f(x')) \rightarrow \psi(y))) \\
= \ & \bigwedge_{y\in Y}\bigwedge_{x'\in X}(\psi(f(x')) \rightarrow (e_Y(y, f(x')) \rightarrow \psi(y))) \\
= \ & \bigwedge_{y\in Y}\bigwedge_{x'\in X}((\psi(f(x')) * e_Y(y, f(x'))) \rightarrow \psi(y)) \\
= \ & \bigwedge_{y\in Y}(\bigvee_{x'\in X}(\psi(f(x')) * e_Y(y, f(x'))) \rightarrow \psi(y)) \\
\geq \ & \bigwedge_{y\in Y}(\psi(y) \rightarrow \psi(y)) \qquad (\psi \text{ is } LF\text{-lower}) \\
= \ & 1.
\end{aligned}
$$

(2)

$$
\begin{aligned}
& \widetilde{e}_Y(f_L^{\rightarrow} \circ f_L^{\leftarrow}(\psi), \psi) \\
= \ & \bigwedge_{y\in Y}(f_L^{\rightarrow}(f_L^{\leftarrow}(\psi))(y) \rightarrow \psi(y)) \\
= \ & \bigwedge_{y\in Y}(\bigvee_{x\in f^{\leftarrow}(\{y\})} f_L^{\leftarrow}(\psi)(x) \rightarrow \psi(y)) \\
= \ & \bigwedge_{y\in Y}(\bigvee_{x\in f^{\leftarrow}(\{y\})} \psi(f(x)) \rightarrow \psi(y)) \\
= \ & \bigwedge_{y\in Y}(\psi(y) \rightarrow \psi(y)) = 1.
\end{aligned}
$$

□

Definition 3.6. (**Adjunction Between Categories**) [12] Let **C** and **D** be two categories, and $F : \mathbf{C} \longrightarrow \mathbf{D}$, $G : \mathbf{D} \longrightarrow \mathbf{C}$ functors. We say F is *left-adjoint* to G iff the following two criteria are satisfied in the order stated:

(1) **Lifting/Continuity criterion:**

$$\forall A \in |\mathbf{C}|, \exists \eta : A \longrightarrow GF(A), \forall B \in \mathbf{D}, \forall f : A \longrightarrow G(B),$$
$$\exists! \overline{f} : F(A) \longrightarrow B, f = G(\overline{f}) \circ \eta$$

(2) **Naturality criterion:**

$$\forall f : A_1 \longrightarrow A_2 \in \mathbf{C}, F(f) = \overline{\eta_{A_2} \circ f}$$

We also say that G is *right-adjoint* to F or that (F, G) is an *adjunction*, or we may write $F \dashv G$ and $\mathbf{C} \dashv \mathbf{D}$. The map η is the *unit* of the adjunction; and the **D** morphism ε dual to η in the duals of the above statements is the *counit* of the adjunction.

Theorem 3.7. (**Set** $\dashv$ LF-**CSLat**($\sqcup$) **and generation of Zadeh powerset operators**) *Let* $\mathbf{P}_L$*:* **Set**$\longrightarrow$ LF-**CSLat**($\sqcup$) *and* $\mathbf{V}$*:* LF-**CSLat**($\sqcup$) $\longrightarrow$ **Set** *be defined by*

$$\mathbf{P}_L(X) = (L^X, \widetilde{e}), \quad \mathbf{V}(X, e) = X, \quad \mathbf{V}(f) = f.$$

Then the followings hold:

(1) $\forall X \in |\mathbf{Set}|$, $\exists \chi_X : X \longrightarrow \mathbf{P}_L(X)$ *defined by* $\chi_X(x) = \chi_{\{x\}}$, $\forall (Y, e_Y) \in |LF$-**CSLat**$(\sqcup)|$, $\forall f : X \longrightarrow Y$ *in* **Set**, $\exists ! \overline{f} : \mathbf{P}_L(X) \longrightarrow (Y, e_Y)$ *in* LF-**CSLat**($\sqcup$), $f = \mathbf{V}(\overline{f}) \circ \chi_X$.

(2) *If* $f : X \longrightarrow Y$ *in* **Set** *is given and* $\mathbf{P}_L(f)$ *is stipulated to be* $\overline{\chi_Y \circ f}$*, then* $\mathbf{P}_L$ *is a functor and* $\mathbf{P}_L \dashv \mathbf{V}$.

(3) $\overline{\chi_Y \circ f} : \mathbf{P}_L(X) \longrightarrow \mathbf{P}_L(Y)$ *is the traditional Zadeh forward powerset operator* $f_L^{\rightarrow}$*, i.e. for all* $\phi \in L^X$ *and for all* $y \in Y$, $\overline{\chi_Y \circ f}(\phi)(y) = \bigvee\{\phi(x) \mid x \in f^{\leftarrow}(\{y\})\}$.

(4) *Since* $\overline{\chi_Y \circ f}$ *is an* LF-**CSLat**($\sqcup$) *morphism, then* $f_L^{\rightarrow}$ *preserves arbitrary joins of L-fuzzy subsets and so has a right* 1*-adjoint* g (*by the AFT*) *which is the traditional Zadeh backward powerset operator* $f_L^{\leftarrow}$*, i.e. for all* $\psi \in L^Y$ *and for all* $x \in X$, $g(\psi)(x) = \psi \circ f(x)$.

Proof. (1) Let $\overline{f} : L^X \longrightarrow Y$ be defined by $\overline{f}(\phi) = \sqcup f_L^{\rightarrow}(\phi)$. Then $\overline{f}$ is an LF-**CSLat**($\sqcup$) morphism, i.e. $\overline{f}(\sqcup \Phi) = \sqcup \widetilde{\overline{f}}_*^{\rightarrow}(\Phi)$ for all $\Phi \in L^{L^X}$. In fact, suppose Φ is an L-fuzzy subset of L^X. Then $\sqcup \Phi \in L^X$ and for all $x \in X$, $(\sqcup \Phi)(x) = \bigvee_{\phi \in L^X} (\Phi(\phi) * \phi(x))$ by Theorem 2.15. So for all $y \in Y$,

$$\begin{aligned} f_L^{\rightarrow}(\sqcup \Phi)(y) &= \textstyle\bigvee_{x \in f^{\leftarrow}(\{y\})} (\sqcup \Phi)(x) \\ &= \textstyle\bigvee_{x \in f^{\leftarrow}(\{y\})} (\bigvee_{\phi \in L^X} (\Phi(\phi) * \phi(x))) \\ &= \textstyle\bigvee_{\phi \in L^X} (\Phi(\phi) * (\bigvee_{x \in f^{\leftarrow}(\{y\})} \phi(x))) \\ &= \textstyle\bigvee_{\phi \in L^X} (\Phi(\phi) * f_L^{\rightarrow}(\phi)(y)). \end{aligned}$$

Let $y_0 = \sqcup \widetilde{\overline{f}}_*^{\rightarrow}(\Phi)$. We will prove $y_0 = \sqcup f_L^{\rightarrow}(\sqcup \Phi)$. In fact, by Theorem 2.9, for all $z \in Y$,

$$\begin{aligned} e_Y(y_0, z) &= \textstyle\bigwedge_{y \in Y} (\widetilde{\overline{f}}_*^{\rightarrow}(\Phi)(y) \rightarrow e_Y(y, z)) \\ &= \textstyle\bigwedge_{y \in Y} (\bigvee_{\phi \in L^X} (\Phi(\phi) * e_Y(y, \overline{f}(\phi))) \rightarrow e_Y(y, z)) \\ &= \textstyle\bigwedge_{y \in Y} (\bigvee_{\phi \in L^X} (\Phi(\phi) * e_Y(y, \sqcup f_L^{\rightarrow}(\phi))) \rightarrow e_Y(y, z)) \\ &= \textstyle\bigwedge_{y \in Y} \bigwedge_{\phi \in L^X} ((\Phi(\phi) * e_Y(y, \sqcup f_L^{\rightarrow}(\phi))) \rightarrow e_Y(y, z)) \\ &= \textstyle\bigwedge_{y \in Y} \bigwedge_{\phi \in L^X} (\Phi(\phi) \rightarrow (e_Y(y, \sqcup f_L^{\rightarrow}(\phi)) \rightarrow e_Y(y, z))) \\ &= \textstyle\bigwedge_{\phi \in L^X} (\Phi(\phi) \rightarrow \bigwedge_{y \in Y} (e_Y(y, \sqcup f_L^{\rightarrow}(\phi)) \rightarrow e_Y(y, z))) \\ &= \textstyle\bigwedge_{\phi \in L^X} (\Phi(\phi) \rightarrow e_Y(\sqcup f_L^{\rightarrow}(\phi), z)) \\ &= \textstyle\bigwedge_{\phi \in L^X} (\Phi(\phi) \rightarrow \bigwedge_{y \in Y} (f_L^{\rightarrow}(\phi)(y) \rightarrow e_Y(y, z))) \\ &= \textstyle\bigwedge_{\phi \in L^X} \bigwedge_{y \in Y} (\Phi(\phi) \rightarrow (f_L^{\rightarrow}(\phi)(y) \rightarrow e_Y(y, z))) \\ &= \textstyle\bigwedge_{y \in Y} \bigwedge_{\phi \in L^X} ((\Phi(\phi) * f_L^{\rightarrow}(\phi)(y)) \rightarrow e_Y(y, z)) \\ &= \textstyle\bigwedge_{y \in Y} (\bigvee_{\phi \in L^X} (\Phi(\phi) * f_L^{\rightarrow}(\phi)(y)) \rightarrow e_Y(y, z)) \\ &= \textstyle\bigwedge_{y \in Y} (f_L^{\rightarrow}(\sqcup \Phi)(y) \rightarrow e_Y(y, z)). \end{aligned}$$

Thus $\overline{f}(\sqcup\Phi) = \sqcup f_L^{\rightarrow}(\sqcup\Phi) = y_0 = \sqcup\overrightarrow{\widetilde{\overline{f}}_*}(\Phi)$, that is, $\overline{f}$ is a join-preserving mapping.

Secondly, for all $x \in X$,

$$\mathbf{V}(\overline{f}) \circ \chi_X(x) = \overline{f}(\chi_{\{x\}}) = \sqcup f_L^{\rightarrow}(\chi_{\{x\}}) = \sqcup\chi_{\{f(x)\}} = f(x).$$

That is, $\mathbf{V}(\overline{f}) \circ \chi_X = f$.

Moreover, $\overline{f}$ is the unique LF-$\mathbf{CSLat}(\sqcup)$ morphism satisfying $\mathbf{V}(\overline{f}) \circ \chi_X = f$. In fact, if g is also an LF-$\mathbf{CSLat}(\sqcup)$ morphism satisfying $\mathbf{V}(g) \circ \chi_X = f$, then for every $\phi \in L^X$, we define $\widehat{\phi} \in L^{L^X}$ as follows: for all $\psi \in L^X$,

$$\widehat{\phi}(\psi) = \begin{cases} \phi(x), & \exists x \in X, \psi = \chi_{\{x\}}, \\ 0, & \text{otherwise.} \end{cases}$$

By Theorem 2.15, for all $x \in X$,

$$\begin{aligned} (\sqcup\widehat{\phi})(x) &= \bigvee_{\psi\in L^X}(\widehat{\phi}(\psi) * \psi(x)) \\ &= \bigvee_{\psi=\chi_{\{x'\}}, x'\in X}(\phi(x') * \psi(x)) \\ &= \bigvee_{x'\in X}(\phi(x') * \chi_{\{x'\}}(x)) \\ &= \phi(x), \end{aligned}$$

that is, $\sqcup\widehat{\phi} = \phi$, we have $g(\phi) = g(\sqcup\widehat{\phi}) = \sqcup\widetilde{g}_*^{\rightarrow}(\widehat{\phi})$. However, for all $y \in Y$,

$$\begin{aligned} \widetilde{g}_*^{\rightarrow}(\widehat{\phi})(y) &= \bigvee_{\psi\in L^X}(\widehat{\phi}(\psi) * e_Y(y, g(\psi))) \\ &= \bigvee_{\psi=\chi_{\{x\}}, x\in X}(\phi(x) * e_Y(y, g(\psi))) \\ &= \bigvee_{x\in X}(\phi(x) * e_Y(y, g(\chi_{\{x\}}))) \\ &= \bigvee_{x\in X}(\phi(x) * e_Y(y, f(x))) \qquad (\text{since } V(g) \circ \chi_X = f) \\ &= \widetilde{f}_*^{\rightarrow}(\phi)(y), \end{aligned}$$

that is, $\widetilde{g}_*^{\rightarrow}(\widehat{\phi}) = \widetilde{f}_*^{\rightarrow}(\phi)$, so $\sqcup\widetilde{g}_*^{\rightarrow}(\widehat{\phi}) = \sqcup\widetilde{f}_*^{\rightarrow}(\phi)$. By Proposition 3.1 we know that $g(\phi) = \sqcup\widetilde{g}_*^{\rightarrow}(\widehat{\phi}) = \sqcup\widetilde{f}_*^{\rightarrow}(\phi) = \sqcup f_L^{\rightarrow}(\phi) = \overline{f}(\phi)$. Thus $g = \overline{f}$.

(2) It is easily proved that $\mathbf{P}_L$ is a functor by (3), and by (1) we know that $\mathbf{P}_L \dashv \mathbf{V}$.

(3) By the definition of $\overline{()}$ we know that $\overline{\chi_Y \circ f}(\phi) = \sqcup(\chi_Y \circ f)_L^{\rightarrow}(\phi)$ for all $\phi \in L^X$. However, by Theorem 2.15 we have for all $y \in Y$,

$$\begin{aligned} (\sqcup(\chi_Y \circ f)_L^{\rightarrow}(\phi))(y) &= \bigvee_{\psi\in L^Y}((\chi_Y \circ f)_L^{\rightarrow}(\phi)(\psi) * \psi(y)) \\ &= \bigvee_{\psi\in L^Y}(\bigvee_{x\in(\chi_Y\circ f)^{\leftarrow}(\{\psi\})} \phi(x) * \psi(y)) \\ &= \bigvee_{\psi\in L^Y}(\bigvee_{\psi=\chi_{f(x)}, x\in X} \phi(x) * \psi(y)) \\ &= \bigvee_{\psi\in\{\chi_{f(x')}\mid x'\in X\}}(\bigvee_{\psi=\chi_{f(x)}, x\in X} \phi(x) * \psi(y)) \\ &= \bigvee_{x'\in X}(\bigvee_{\chi_{f(x')}=\chi_{f(x)}, x\in X} \phi(x) * \chi_{\{f(x')\}}(y)) \\ &= \bigvee_{x'\in f^{\leftarrow}(\{y\})} \bigvee_{x\in f^{\leftarrow}(\{f(x')\})} \phi(x) \\ &= \bigvee_{x\in f^{\leftarrow}(\{y\})} \phi(x) = f_L^{\rightarrow}(\phi)(y). \end{aligned}$$

Hence for all $\phi \in L^X$, $\overline{\chi_Y \circ f}(\phi) = \sqcup(\chi_Y \circ f)_L^{\rightarrow}(\phi) = f_L^{\rightarrow}(\phi)$, that is, $\overline{\chi_Y \circ f} = f_L^{\rightarrow}$ is the traditional Zadeh forward powerset operator.

(4) By (1) we know that $\overline{\chi_Y \circ f}$ is an LF-**CSLat**$(\sqcup)$ morphism, consequently it preserves arbitrary joins of L-fuzzy subsets, which implies that the traditional Zadeh forward powerset operator $f_L^{\rightarrow}$ preserves arbitrary joins of L-fuzzy subsets. By the AFT, $f_L^{\rightarrow}$ has a right 1-adjoint $g : L^Y \longrightarrow L^X$, which is given by $g(\psi) = \sqcup\Phi_\psi$, where $\Phi_\psi \in L^{L^X}$ is defined by $\Phi_\psi(\phi) = \widetilde{e}(f_L^{\rightarrow}(\phi), \psi)$ for all $\phi \in L^X$. Now we prove that g is exactly the traditional Zadeh backward powerset operator $f_L^{\leftarrow}$. At first, for all $x \in X$,

$$\begin{aligned} g(\psi)(x) &= (\sqcup\Phi_\psi)(x) \\ &= \bigvee_{\phi\in L^X} (\Phi_\psi(\phi) * \phi(x)) \\ &= \bigvee_{\phi\in L^X} (\widetilde{e}(f_L^{\rightarrow}(\phi), \psi) * \phi(x)) \\ &\leq \bigvee_{\phi\in L^X} (\widetilde{e}(f_L^{\rightarrow}(\phi), \psi) * f_L^{\rightarrow}(\phi)(f(x))) \\ &= \bigvee_{\phi\in L^X} (\bigwedge_{y\in Y} (f_L^{\rightarrow}(\phi)(y) \rightarrow \psi(y)) * f_L^{\rightarrow}(\phi)(f(x))) \\ &\leq \bigvee_{\phi\in L^X} ((f_L^{\rightarrow}(\phi)(f(x)) \rightarrow \psi(f(x))) * f_L^{\rightarrow}(\phi)(f(x))) \\ &\leq \psi(f(x)). \end{aligned}$$

Conversely,

$$\begin{aligned} g(\psi)(x) &= (\sqcup\Phi_\psi)(x) \\ &= \bigvee_{\phi\in L^X} (\Phi_\psi(\phi) * \phi(x)) \\ &= \bigvee_{\phi\in L^X} (\widetilde{e}(f_L^{\rightarrow}(\phi), \psi) * \phi(x)) \\ &\geq \widetilde{e}(f_L^{\rightarrow}(f_L^{\leftarrow}(\psi), \psi) * f_L^{\leftarrow}(\psi)(x)) \qquad \text{(by taking } \phi = f_L^{\leftarrow}(\psi)) \\ &= f_L^{\leftarrow}(\psi)(x). \qquad \text{(by Lemma 2.5)} \end{aligned}$$

Thus $g(\psi) = f_L^{\leftarrow}(\psi)$ for all $\psi \in L^Y$, that is, $g = f_L^{\leftarrow}$. □

Remark 3.8. Theorem 3.7 directly generates $f_L^{\rightarrow}$ from f via an adjunction between the categories **Set** and LF-**CSLat**$(\sqcup)$, and creates $f_L^{\leftarrow}$ by the AFT for L-fuzzy posets. It easily sees that the category LF-**CSLat**$(\sqcup)$ is exactly the category **CSLat**$(\bigvee)$ when L is the binary lattice **2**, and that the Zadeh powerset operators $f_L^{\rightarrow}, f_L^{\leftarrow}$ are respectively identical to the traditional powerset operators $f^{\rightarrow}, f^{\leftarrow}$, and that the 1-adjoint pair is the traditional adjoint pair, so this theorem generalizes the corresponding work of [10-12] providing L is a complete residuated lattice.

Lemma 3.9. (**Yoneda Lemma**) [6, 20] *Let (X, e) be an LF-poset. Then for every $x \in X$, $\phi \in \mathcal{L}_L(X)$ implies $\widetilde{e}(\iota_x, \phi) = \phi(x)$, and $\phi \in \mathcal{U}_L(X)$ implies $\widetilde{e}(\mu_x, \phi) = \phi(x)$, and $\phi \in \mathcal{S}_L(X)$ implies $\widetilde{e}(s_x, \phi) = \phi(x)$.*

Corollary 3.10. *Let (X, e) be an LF-poset. Then*

$$e(x, y) = \widetilde{e}(\iota_x, \iota_y), \quad e(x, y) = \widetilde{e}(\mu_y, \mu_x), \quad e(x, y) = \widetilde{e}(s_x, s_y)$$

for all $x, y \in X$.

Theorem 3.11. (LF-**Pos**$\dashv$ LF-**CSLat**$(\sqcup)$ **and generation of fuzzy powerset operators** $\widetilde{f}_*^{\rightarrow}, \widetilde{f}_*^{\leftarrow}$) *Let* $\mathbf{L}_L$: LF-**Pos**$\longrightarrow$ LF-**CSLat**$(\sqcup)$ *and* $\mathbf{F}$: LF-**CSLat**$(\sqcup) \longrightarrow$ LF-**Pos** *be defined by*

$$\mathbf{L}_L(X, e) = (\mathcal{L}_L(X), \widetilde{e}), \quad \mathbf{F}(X, e) = (X, e), \quad \mathbf{F}(f) = f.$$

Then the followings hold:

(1) $\forall(X, e_X) \in |LF\text{-}\mathbf{Pos}|$, $\exists \iota_X : (X, e_X) \longrightarrow (\mathcal{L}_L(X), \widetilde{e}_X)$ *defined by* $\iota_X(x) = \iota_x$, $\forall(Y, e_Y) \in |LF\text{-}\mathbf{CSLat}(\sqcup)|$, $\forall f : (X, e_X) \longrightarrow (Y, e_Y)$ *in* LF-$\mathbf{Pos}$, $\exists! \overline{f} : (\mathcal{L}_L(X), \widetilde{e}_X) \longrightarrow (Y, e_Y)$ *in* LF-$\mathbf{CSLat}(\sqcup)$, $f = \mathbf{F}(\overline{f}) \circ \iota_X$.

(2) *If* $f : (X, e_X) \longrightarrow (Y, e_Y)$ *in* LF-$\mathbf{Pos}$ *is given and* $\mathbf{L}_L(f)$ *is stipulated to be* $\overline{\iota_Y \circ f}$, *then* $\mathbf{L}_L$ *is a functor and* $\mathbf{L}_L \dashv \mathbf{F}$.

(3) $\overline{\iota_Y \circ f} : (\mathcal{L}_L(X), \widetilde{e}_X) \longrightarrow (\mathcal{L}_L(Y), \widetilde{e}_Y)$ *is* $\widetilde{f}_*^{\rightarrow}$, *i.e. for all* $\phi \in \mathcal{L}_L(X)$ *and for all* $y \in Y$, $\overline{\iota_Y \circ f}(\phi)(y) = \bigvee_{x \in X}(\phi(x) * e_Y(y, f(x)))$.

(4) *Since* $\overline{\iota_Y \circ f}$ *is an* LF-$\mathbf{CSLat}(\sqcup)$ *morphism, then* $\widetilde{f}_*^{\rightarrow}$ *preserves arbitrary joins of L-fuzzy subsets and so has a right* 1*-adjoint* g (*by the AFT*) *which is* $\widetilde{f}_*^{\leftarrow}$, *i.e. for all* $\psi \in \mathcal{L}_L(Y)$ *and for all* $x \in X$, $g(\psi)(x) = \bigvee_{x' \in X}(\psi(f(x')) * e_X(x, x'))$.

Proof. (1) Let $\overline{f} : \mathcal{L}_L(X) \longrightarrow Y$ be defined by $\overline{f}(\phi) = \sqcup \widetilde{f}_*^{\rightarrow}(\phi)$. Then $\overline{f}$ is an LF-$\mathbf{CSLat}(\sqcup)$ morphism, i.e. for all $\Phi \in L^{\mathcal{L}_L(X)}$, $\overline{f}(\sqcup\Phi) = \sqcup \widetilde{\overline{f}}_*^{\rightarrow}(\Phi)$. In fact, suppose Φ is an L-fuzzy subset of $\mathcal{L}_L(X)$. Then $\sqcup\Phi \in \mathcal{L}_L(X)$ and for all $x \in X$, $(\sqcup\Phi)(x) = \bigvee_{\phi \in \mathcal{L}_L(X)}(\Phi(\phi) * \phi(x))$ by Theorem 2.17. So for all $y \in Y$,

$$\begin{aligned}
\widetilde{f}_*^{\rightarrow}(\sqcup\Phi)(y) &= \textstyle\bigvee_{x \in X}((\sqcup\Phi)(x) * e_Y(y, f(x))) \\
&= \textstyle\bigvee_{x \in X}((\bigvee_{\phi \in \mathcal{L}_L(X)}(\Phi(\phi) * \phi(x))) * e_Y(y, f(x))) \\
&= \textstyle\bigvee_{x \in X} \bigvee_{\phi \in \mathcal{L}_L(X)}(\Phi(\phi) * \phi(x) * e_Y(y, f(x))) \\
&= \textstyle\bigvee_{\phi \in \mathcal{L}_L(X)}(\Phi(\phi) * \bigvee_{x \in X}(\phi(x) * e_Y(y, f(x)))) \\
&= \textstyle\bigvee_{\phi \in \mathcal{L}_L(X)}(\Phi(\phi) * \widetilde{f}_*^{\rightarrow}(\phi)(y)).
\end{aligned}$$

Let $y_0 = \sqcup \widetilde{\overline{f}}_*^{\rightarrow}(\Phi)$. We will prove $y_0 = \sqcup \widetilde{f}_*^{\rightarrow}(\sqcup\Phi)$. In fact, by Theorem 2.9, for all $z \in Y$,

$$\begin{aligned}
e_Y(y_0, z) &= \textstyle\bigwedge_{y \in Y}(\widetilde{\overline{f}}_*^{\rightarrow}(\Phi)(y) \rightarrow e_Y(y, z)) \\
&= \textstyle\bigwedge_{y \in Y}(\bigvee_{\phi \in \mathcal{L}_L(X)}(\Phi(\phi) * e_Y(y, \overline{f}(\phi))) \rightarrow e_Y(y, z)) \\
&= \textstyle\bigwedge_{y \in Y}(\bigvee_{\phi \in \mathcal{L}_L(X)}(\Phi(\phi) * e_Y(y, \sqcup \widetilde{f}_*^{\rightarrow}(\phi))) \rightarrow e_Y(y, z)) \\
&= \textstyle\bigwedge_{y \in Y} \bigwedge_{\phi \in \mathcal{L}_L(X)}((\Phi(\phi) * e_Y(y, \sqcup \widetilde{f}_*^{\rightarrow}(\phi))) \rightarrow e_Y(y, z)) \\
&= \textstyle\bigwedge_{y \in Y} \bigwedge_{\phi \in \mathcal{L}_L(X)}(\Phi(\phi) \rightarrow (e_Y(y, \sqcup \widetilde{f}_*^{\rightarrow}(\phi)) \rightarrow e_Y(y, z))) \\
&= \textstyle\bigwedge_{\phi \in \mathcal{L}_L(X)}(\Phi(\phi) \rightarrow \bigwedge_{y \in Y}(e_Y(y, \sqcup \widetilde{f}_*^{\rightarrow}(\phi)) \rightarrow e_Y(y, z))) \\
&= \textstyle\bigwedge_{\phi \in \mathcal{L}_L(X)}(\Phi(\phi) \rightarrow e_Y(\sqcup \widetilde{f}_*^{\rightarrow}(\phi), z)) \\
&= \textstyle\bigwedge_{\phi \in \mathcal{L}_L(X)}(\Phi(\phi) \rightarrow \bigwedge_{y \in Y}(\widetilde{f}_*^{\rightarrow}(\phi)(y) \rightarrow e_Y(y, z))) \\
&= \textstyle\bigwedge_{\phi \in \mathcal{L}_L(X)} \bigwedge_{y \in Y}(\Phi(\phi) \rightarrow (\widetilde{f}_*^{\rightarrow}(\phi)(y) \rightarrow e_Y(y, z))) \\
&= \textstyle\bigwedge_{y \in Y} \bigwedge_{\phi \in \mathcal{L}_L(X)}((\Phi(\phi) * \widetilde{f}_*^{\rightarrow}(\phi)(y)) \rightarrow e_Y(y, z)) \\
&= \textstyle\bigwedge_{y \in Y}(\bigvee_{\phi \in \mathcal{L}_L(X)}(\Phi(\phi) * \widetilde{f}_*^{\rightarrow}(\phi)(y)) \rightarrow e_Y(y, z)) \\
&= \textstyle\bigwedge_{y \in Y}(\widetilde{f}_*^{\rightarrow}(\sqcup\Phi)(y) \rightarrow e_Y(y, z)).
\end{aligned}$$

Thus $\overline{f}(\sqcup\Phi) = \sqcup \widetilde{f}_*^{\rightarrow}(\sqcup\Phi) = y_0 = \sqcup \widetilde{\overline{f}}_*^{\rightarrow}(\Phi)$, that is, $\overline{f}$ is a join-preserving mapping.

Secondly, for all $x \in X$,

$$\mathbf{F}(\overline{f}) \circ \iota_X(x) = \overline{f}(\iota_x) = \sqcup \widetilde{f}_*^{\rightarrow}(\iota_x) = \sqcup \iota_{f(x)} = f(x).$$

That is, $\mathbf{F}(\overline{f}) \circ \iota_X = f$.

Moreover, $\overline{f}$ is the unique LF-**CSLat**($\sqcup$) morphism satisfying $\mathbf{F}(\overline{f}) \circ \iota_X = f$. In fact, if g is also an LF-**CSLat**($\sqcup$) morphism satisfying $\mathbf{F}(g) \circ \iota_X = f$, then for every $\phi \in \mathcal{L}_L(X)$, we define $\widehat{\phi} \in L^{\mathcal{L}_L(X)}$ as follows: for all $\psi \in \mathcal{L}_L(X)$,

$$\widehat{\phi}(\psi) = \bigvee_{x \in X} (\phi(x) * \widetilde{e}(\psi, \iota_x))$$

Since for all $x \in X$,

$$\begin{aligned}
(\sqcup\widehat{\phi})(x) &= \textstyle\bigvee_{\psi \in \mathcal{L}_L(X)} (\widehat{\phi}(\psi) * \psi(x)) \\
&= \textstyle\bigvee_{\psi \in \mathcal{L}_L(X)} (\bigvee_{x' \in X} (\phi(x') * \widetilde{e}(\psi, \iota_{x'})) * \psi(x)) \\
&= \textstyle\bigvee_{\psi \in \mathcal{L}_L(X)} \bigvee_{x' \in X} (\phi(x') * \widetilde{e}(\psi, \iota_{x'}) * \psi(x)) \\
&= \textstyle\bigvee_{x' \in X} (\phi(x') * \bigvee_{\psi \in \mathcal{L}_L(X)} (\widetilde{e}(\psi, \iota_{x'}) * \psi(x))) \\
&= \textstyle\bigvee_{x' \in X} (\phi(x') * \bigvee_{\psi \in \mathcal{L}_L(X)} (\widetilde{e}(\psi, \iota_{x'}) * \widetilde{e}(\iota_x, \psi))) \qquad \text{(by Yoneda Lemma)} \\
&= \textstyle\bigvee_{x' \in X} (\phi(x') * \widetilde{e}(\iota_x, \iota_{x'})) \\
&= \textstyle\bigvee_{x' \in X} (\phi(x') * e(x, x')) \qquad \text{(by Corollary 2.10)} \\
&= \phi(x) \qquad \text{(since } \phi \text{ is an } LF\text{-lower set)},
\end{aligned}$$

that is, $\sqcup\widehat{\phi} = \phi$, we have $g(\phi) = g(\sqcup\widehat{\phi}) = \sqcup\widetilde{g}_*^{\rightarrow}(\widehat{\phi})$. However, for all $y \in Y$,

$$\begin{aligned}
\widetilde{g}_*^{\rightarrow}(\widehat{\phi})(y) &= \textstyle\bigvee_{\psi \in \mathcal{L}_L(X)} (\widehat{\phi}(\psi) * e_Y(y, g(\psi))) \\
&= \textstyle\bigvee_{\psi \in \mathcal{L}_L(X)} (\bigvee_{x \in X} (\phi(x) * \widetilde{e}(\psi, \iota_x)) * e_Y(y, g(\psi))) \\
&= \textstyle\bigvee_{x \in X} (\phi(x) * \bigvee_{\psi \in \mathcal{L}_L(X)} (\widetilde{e}(\psi, \iota_x) * e_Y(y, g(\psi)))) \\
&= \textstyle\bigvee_{x \in X} (\phi(x) * e_Y(y, g(\iota_x))) \qquad \text{(since } g \text{ is } LF\text{-monotone)} \\
&= \textstyle\bigvee_{x \in X} (\phi(x) * e_Y(y, f(x))) \qquad \text{(since } F(g) \circ \iota_X = f) \\
&= \widetilde{f}_*^{\rightarrow}(\phi)(y),
\end{aligned}$$

that is, $\widetilde{g}_*^{\rightarrow}(\widehat{\phi}) = \widetilde{f}_*^{\rightarrow}(\phi)$, so $g(\phi) = \sqcup\widetilde{g}_*^{\rightarrow}(\widehat{\phi}) = \sqcup\widetilde{f}_*^{\rightarrow}(\phi) = \overline{f}(\phi)$. Thus $g = \overline{f}$.

(2) It is easily proved that $\mathbf{L}_L$ is a functor by (3), and by (1) we know that $\mathbf{L}_L \dashv \mathbf{F}$.

(3) By the definition of $\overline{()}$ we know that $\overline{\iota_Y \circ f}(\phi) = \sqcup\widetilde{(\iota_Y \circ f)}_*^{\rightarrow}(\phi)$ for all $\phi \in \mathcal{L}_L(X)$. However by Theorem 2.17, for all $y \in Y$,

$$\begin{aligned}
&(\sqcup\widetilde{(\iota_Y \circ f)}_*^{\rightarrow}(\phi))(y) \\
= \ & \textstyle\bigvee_{\psi \in \mathcal{L}_L(Y)} (\widetilde{(\iota_Y \circ f)}_*^{\rightarrow}(\phi)(\psi) * \psi(y)) \\
= \ & \textstyle\bigvee_{\psi \in \mathcal{L}_L(Y)} (\bigvee_{x \in X} (\phi(x) * \widetilde{e}(\psi, (\iota_Y \circ f)(x))) * \psi(y)) \\
= \ & \textstyle\bigvee_{\psi \in \mathcal{L}_L(Y)} (\bigvee_{x \in X} (\phi(x) * \widetilde{e}(\psi, \iota_{f(x)})) * \psi(y)) \\
= \ & \textstyle\bigvee_{x \in X} (\phi(x) * \bigvee_{\psi \in \mathcal{L}_L(Y)} (\widetilde{e}(\psi, \iota_{f(x)}) * \psi(y))) \\
= \ & \textstyle\bigvee_{x \in X} (\phi(x) * \bigvee_{\psi \in \mathcal{L}_L(Y)} (\widetilde{e}(\psi, \iota_{f(x)})) * \widetilde{e}(\iota_y, \psi)) \qquad \text{(by Yoneda Lemma)} \\
= \ & \textstyle\bigvee_{x \in X} (\phi(x) * \widetilde{e}(\iota_y, \iota_{f(x)})) \\
= \ & \textstyle\bigvee_{x \in X} (\phi(x) * e_Y(y, f(x))) \qquad \text{(by Corollary 2.10)} \\
= \ & \widetilde{f}_*^{\rightarrow}(\phi)(y).
\end{aligned}$$

Hence for all $\phi \in L^X$, $\overline{\iota_Y \circ f}(\phi) = \sqcup\widetilde{(\iota_Y \circ f)}_*^{\rightarrow}(\phi) = \widetilde{f}_*^{\rightarrow}(\phi)$, that is, $\overline{\iota_Y \circ f} = \widetilde{f}_*^{\rightarrow}$.

(4) By (1) we know that $\overline{\iota_Y \circ f}$ is an LF-**CSLat**($\sqcup$) morphism, consequently it preserves arbitrary joins of L-fuzzy subsets, which implies that the fuzzy forward powerset operator $\widetilde{f_*^{\rightarrow}}$ preserves arbitrary joins of L-fuzzy subsets. By AFT, $\widetilde{f_*^{\rightarrow}}$ has a right 1-adjoint $g : \mathcal{L}_L(Y) \longrightarrow \mathcal{L}_L(X)$, which is given by $g(\psi) = \sqcup\Phi_\psi$, where $\Phi_\psi \in L^{\mathcal{L}_L(X)}$ is defined by $\Phi_\psi(\phi) = \widetilde{e}(\widetilde{f_*^{\rightarrow}}(\phi), \psi)$ for all $\phi \in \mathcal{L}_L(X)$. Now we prove that g is exactly the fuzzy backward powerset operator $\widetilde{f_*^{\leftarrow}}$. At first, for all $x \in X$,

$$\begin{aligned} g(\psi)(x) &= \sqcup\Phi_\psi(x) \\ &= \textstyle\bigvee_{\phi\in\mathcal{L}_L(X)}(\Phi_\psi(\phi) * \phi(x)) \\ &= \textstyle\bigvee_{\phi\in\mathcal{L}_L(X)}(\widetilde{e}(\widetilde{f_*^{\rightarrow}}(\phi), \psi) * \phi(x)) \\ &= \textstyle\bigvee_{\phi\in\mathcal{L}_L(X)}(\widetilde{e}(\widetilde{f_*^{\rightarrow}}(\phi), \psi) * \widetilde{e}(\iota_x, \phi)) \qquad \text{(by Yoneda Lemma)} \\ &= \widetilde{e}(\widetilde{f_*^{\rightarrow}}(\iota_x), \psi) \qquad (\text{since } \widetilde{f_*^{\rightarrow}} \text{ is } LF\text{-monotone}) \\ &= \widetilde{e}(\iota_{f(x)}, \psi) \\ &= \psi(f(x)) \qquad \text{(by Yoneda Lemma)} \\ &= \textstyle\bigvee_{x'\in X}(\psi(f(x')) * e(x, x')) \qquad (\text{since } f \text{ is } LF\text{-monotone and } \psi \text{ is } LF\text{-lower}) \\ &= \widetilde{f_*^{\leftarrow}}(\psi)(x). \end{aligned}$$

Thus $g(\psi) = \widetilde{f_*^{\leftarrow}}(\psi)$ for all $\psi \in \mathcal{L}_L(Y)$, that is, $g = \widetilde{f_*^{\leftarrow}}$. □

Remark 3.12. Theorem 3.11 indicates that f can directly generate $\widetilde{f_*^{\rightarrow}}$ via an adjunction between the categories LF-**Pos** and LF-**CSLat**($\sqcup$) and create $\widetilde{f_*^{\leftarrow}}$ by the AFT for L-fuzzy posets. Similarly we can prove that f can directly generate $\widetilde{f^{\rightarrow}}$ via an adjunction between the categories LF-**Pos** and LF-**CSLat**($\sqcup$) and create $\widetilde{f^{\leftarrow}}$ by the AFT for L-fuzzy posets. See the following theorem.

Theorem 3.13. (LF-**Pos**$\dashv$ LF-**CSLat**($\sqcup$) **and generation of fuzzy powerset operators** $\widetilde{f^{\rightarrow}}, \widetilde{f^{\leftarrow}}$) *Let* $\mathbf{S}_L$*:* LF-**Pos**$\longrightarrow$ LF-**CSLat**($\sqcup$) *and* $\mathbf{F}$*:* LF-**CSLat**($\sqcup$) $\longrightarrow$ LF-**Pos** *be defined by*

$$\mathbf{S}_L(X, e) = (\mathcal{S}_L(X), \widetilde{e}), \quad \mathbf{F}(X, e) = (X, e), \quad \mathbf{F}(f) = f.$$

Then the followings hold:

(1) $\forall (X, e_X) \in |LF\text{-}\mathbf{Pos}|$, $\exists s_X : (X, e_X) \longrightarrow (\mathcal{S}_L(X), \widetilde{e}_X)$ *defined by* $s_X(x) = s_x$, $\forall (Y, e_Y) \in |LF\text{-}\mathbf{CSLat}(\sqcup)|$, $\forall f : (X, e_X) \longrightarrow (Y, e_Y)$ *in* LF-**Pos**, $\exists! \overline{f} : (\mathcal{S}_L(X), \widetilde{e}_X) \longrightarrow (Y, e_Y)$ *in* LF-**CSLat**($\sqcup$), $f = \mathbf{F}(\overline{f}) \circ s_X$.

(2) *If* $f : (X, e_X) \longrightarrow (Y, e_Y)$ *in* LF-**Pos** *is given and* $\mathbf{S}_L(f)$ *is stipulated to be* $\overline{s_Y \circ f}$, *then* $\mathbf{S}_L$ *is a functor and* $\mathbf{S}_L \dashv \mathbf{F}$.

(3) $\overline{s_Y \circ f} : (\mathcal{S}_L(X), \widetilde{e}_X) \longrightarrow (\mathcal{S}_L(Y), \widetilde{e}_Y)$ *is* $\widetilde{f^{\rightarrow}}$, *i.e. for all* $\phi \in \mathcal{S}_L(X)$ *and for all* $y \in Y$, $\overline{s_Y \circ f}(\phi)(y) = \bigvee_{x\in X}(\phi(x) * e_Y(y, f(x)) * e_Y(f(x), y))$.

(4) *Since* $\overline{s_Y \circ f}$ *is an* LF-**CSLat**($\sqcup$) *morphism, then* $\widetilde{f^{\rightarrow}}$ *preserves arbitrary joins of* L*-fuzzy subsets and so has a right* 1*-adjoint* g (*by the AFT*) *which is* $\widetilde{f^{\leftarrow}}$, *i.e. for all* $\psi \in \mathcal{S}_L(Y)$ *and for all* $x \in X$, $g(\psi)(x) = \bigvee_{x'\in X}(\psi(f(x')) * e_X(x, x') * e_X(x'x))$.

Lemma 3.14. *Let (X, e) be an LF-complete lattice and $\phi \in L^X$ and let $\sqcup\phi, \sqcup^{op}\phi$ denote respectively the join of ϕ in (X, e) and (X, e^{op}), and $\sqcap\phi, \sqcap^{op}\phi$ denote respectively the meet of ϕ in (X, e) and (X, e^{op}). Then $\sqcap^{op}\phi = \sqcup\phi$ and $\sqcup^{op}\phi = \sqcap\phi$, which implies the dual (X, e^{op}) of (X, e) is also an LF-complete lattice.*

Proof. We only prove the equality $\sqcap^{op}\phi = \sqcup\phi$, and the equality $\sqcup^{op}\phi = \sqcap\phi$ can be shown dually.

Let $x_0 = \sqcup\phi$. For all $y \in X$, since $e^{op}(y, x_0) = e(x_0, y) = \bigwedge_{x\in X}(\phi(x) \to e(x, y)) = \bigwedge_{x\in X}(\phi(x) \to e^{op}(y, x))$, we have $x_0 = \sqcap^{op}\phi$ by Theorem 2.9. Thus $\sqcap^{op}\phi = \sqcup\phi$. □

Corollary 3.15. *Let X be a non-empty set and (Y, e) an LF-complete lattice and $f : X \longrightarrow Y$ a mapping. For $\phi \in L^X$, let $\widetilde{f}^{\rightarrow}_*(\phi)^{op}, \widetilde{f}^{*\rightarrow}(\phi)^{op}$ be respectively defined by*

$$\widetilde{f}^{\rightarrow}_*(\phi)^{op}(y) = \bigvee_{x\in X}(\phi(x) * e^{op}(y, f(x))), \quad \widetilde{f}^{*\rightarrow}(\phi)^{op}(y) = \bigvee_{x\in X}(\phi(x) * e^{op}(f(x), y)), \quad \forall y \in Y.$$

Then $\sqcap^{op}\widetilde{f}^{*\rightarrow}(\phi)^{op} = \sqcup\widetilde{f}^{\rightarrow}_*(\phi)$ and $\sqcup^{op}\widetilde{f}^{\rightarrow}_*(\phi)^{op} = \sqcap\widetilde{f}^{*\rightarrow}(\phi)$.

Proof. It easily follows that $\widetilde{f}^{*\rightarrow}(\phi)^{op} = \widetilde{f}^{\rightarrow}_*(\phi)$ and $\widetilde{f}^{\rightarrow}_*(\phi)^{op} = \widetilde{f}^{*\rightarrow}(\phi)$ by their definitions. So by Lemma 3.14 we have $\sqcap^{op}\widetilde{f}^{*\rightarrow}(\phi)^{op} = \sqcup\widetilde{f}^{*\rightarrow}(\phi)^{op} = \sqcup\widetilde{f}^{\rightarrow}_*(\phi)$ and $\sqcup^{op}\widetilde{f}^{\rightarrow}_*(\phi)^{op} = \sqcap\widetilde{f}^{\rightarrow}_*(\phi)^{op} = \sqcap\widetilde{f}^{*\rightarrow}(\phi)$. □

By Lemma 3.14 and Corollary 3.15 we can easily obtain the following theorem.

Theorem 3.16. *The assignment $(X, e) \mapsto (X, e^{op})$, $f : (X, e_X) \longrightarrow (Y, e_Y) \mapsto f : (X, e^{op}_X) \longrightarrow (Y, e^{op}_Y)$ give an isomorphism between the categories LF-**CSLat**$(\sqcup)$ and LF-**CSLat**$(\sqcap)$.*

Theorem 3.17. (*LF*-**Pos**$\dashv$ *LF*-**CSLat**$(\sqcap)$ **and generation of fuzzy powerset operators** $\widetilde{f}^{*\rightarrow}, \widetilde{f}^{*\leftarrow}$)) *Let $\mathbf{U}_L$: LF-**Pos**$\longrightarrow$ LF-**CSLat**$(\sqcap)$ and $\mathbf{F}$: LF-**CSLat**$(\sqcap) \longrightarrow$ LF-**Pos** be defined by*

$$\mathbf{U}_L(X, e) = (\mathcal{U}_L(X), \widetilde{e}^{op}), \quad \mathbf{F}(X, e) = (X, e), \quad \mathbf{F}(f) = f.$$

Then the followings hold:

(1) *$\forall(X, e_X) \in |LF\text{-}\mathbf{Pos}|$, $\exists\mu_X : (X, e_X) \longrightarrow (\mathcal{U}_L(X), \widetilde{e}^{op}_X)$ defined by $\mu_X(x) = \mu_x$, $\forall(Y, e_Y) \in LF\text{-}\mathbf{CSLat}(\sqcap)$, $\forall f : (X, e_X) \longrightarrow (Y, e_Y)$ in LF-**Pos**, $\exists!\overline{f} : (\mathcal{U}_L(X), \widetilde{e}^{op}_X) \longrightarrow (Y, e_Y)$ in LF-**CSLat**$(\sqcap)$, $f = \mathbf{F}(\overline{f}) \circ \mu_X$.*

(2) *If $f : (X, e_X) \longrightarrow (Y, e_Y)$ in LF-**Pos** is given and $\mathbf{U}_L(f)$ is stipulated to be $\overline{\mu_Y \circ f}$, then $\mathbf{U}_L$ is a functor and $\mathbf{U}_L \dashv \mathbf{F}$.*

(3) *$\overline{\mu_Y \circ f} : (\mathcal{U}_L(X), \widetilde{e}^{op}_X) \longrightarrow (\mathcal{U}_L(Y), \widetilde{e}^{op}_Y)$ is $\widetilde{f}^{*\rightarrow}$, i.e. for all $\phi \in \mathcal{U}_L(X)$ and for all $y \in Y$, $\overline{\mu_Y \circ f}(\phi)(y) = \bigvee_{x\in X}(\phi(x) * e_Y(f(x), y))$.*

(4) *Since $\overline{\mu_Y \circ f}$ is an LF-**CSLat**$(\sqcap)$ morphism, $\widetilde{f}^{*\rightarrow}$ preserves arbitrary meets of L-fuzzy subsets and so has a left 1-adjoint g (by the AFT) which is $\widetilde{f}^{*\leftarrow}$, i.e. for all $\psi \in \mathcal{U}_L(Y)$ and for all $x \in X$, $g(\psi)(x) = \bigvee_{x'\in X}(\psi(f(x')) * e_X(x', x))$.*

Proof. (1) Let $\overline{f} : \mathcal{U}_L(X) \longrightarrow Y$ be defined by $\overline{f}(\phi) = \sqcap \widetilde{f}^{*\rightarrow}(\phi)$. Then $\overline{f}$ is an LF-$\mathbf{CSLat}(\sqcap)$ morphism, i.e. for all $\Phi \in L^{\mathcal{U}_L(X)}$, $\overline{f}(\sqcap^{op}\Phi) = \sqcap \widetilde{\overline{f}}^{*\rightarrow}(\Phi)$. In fact, suppose Φ is an L-fuzzy subset of $\mathcal{U}_L(X)$. Then by Theorem 2.17 and Lemma 3.14, $\sqcap^{op}\Phi \in \mathcal{U}_L(X)$ and for all $x \in X$, $(\sqcap^{op}\Phi)(x) = (\sqcup\Phi)(x) = \bigvee_{\phi\in\mathcal{U}_L(X)}(\Phi(\phi) * \phi(x))$. So for all $y \in Y$,

$$\begin{aligned}
\widetilde{f}^{*\rightarrow}(\sqcap^{op}\Phi)(y) &= \widetilde{f}^{*\rightarrow}(\sqcup\Phi)(y) \\
&= \textstyle\bigvee_{x\in X}((\sqcup\Phi)(x) * e_Y(f(x), y)) \\
&= \textstyle\bigvee_{x\in X}((\bigvee_{\phi\in\mathcal{U}_L(X)}(\Phi(\phi) * \phi(x))) * e_Y(f(x), y)) \\
&= \textstyle\bigvee_{x\in X}\bigvee_{\phi\in\mathcal{U}_L(X)}(\Phi(\phi) * \phi(x) * e_Y(f(x), y)) \\
&= \textstyle\bigvee_{\phi\in\mathcal{U}_L(X)}(\Phi(\phi) * \bigvee_{x\in X}(\phi(x) * e_Y(f(x), y))) \\
&= \textstyle\bigvee_{\phi\in\mathcal{U}_L(X)}(\Phi(\phi) * \widetilde{f}^{*\rightarrow}(\phi)(y)).
\end{aligned}$$

Let $y_1 = \sqcap \widetilde{\overline{f}}^{*\rightarrow}(\Phi)$. We will prove $y_1 = \sqcap \widetilde{f}^{*\rightarrow}(\sqcup\Phi)$. In fact, by Theorem 2.9, for all $z \in Y$,

$$\begin{aligned}
e_Y(z, y_1) &= \textstyle\bigwedge_{y\in Y}(\widetilde{\overline{f}}^{*\rightarrow}(\Phi)(y) \rightarrow e_Y(z, y)) \\
&= \textstyle\bigwedge_{y\in Y}(\bigvee_{\phi\in\mathcal{U}_L(X)}(\Phi(\phi) * e_Y(\overline{f}(\phi), y)) \rightarrow e_Y(z, y)) \\
&= \textstyle\bigwedge_{y\in Y}(\bigvee_{\phi\in\mathcal{U}_L(X)}(\Phi(\phi) * e_Y(\sqcap\widetilde{f}^{*\rightarrow}(\phi), y)) \rightarrow e_Y(z, y)) \\
&= \textstyle\bigwedge_{y\in Y}\bigwedge_{\phi\in\mathcal{U}_L(X)}((\Phi(\phi) * e_Y(\sqcap\widetilde{f}^{*\rightarrow}(\phi), y)) \rightarrow e_Y(z, y)) \\
&= \textstyle\bigwedge_{y\in Y}\bigwedge_{\phi\in\mathcal{U}_L(X)}(\Phi(\phi) \rightarrow (e_Y(\sqcap\widetilde{f}^{*\rightarrow}(\phi), y) \rightarrow e_Y(z, y))) \\
&= \textstyle\bigwedge_{\phi\in\mathcal{U}_L(X)}(\Phi(\phi) \rightarrow \bigwedge_{y\in Y}(e_Y(\sqcap\widetilde{f}^{*\rightarrow}(\phi), y) \rightarrow e_Y(z, y))) \\
&= \textstyle\bigwedge_{\phi\in\mathcal{U}_L(X)}(\Phi(\phi) \rightarrow e_Y(z, \sqcap\widetilde{f}^{*\rightarrow}(\phi))) \\
&= \textstyle\bigwedge_{\phi\in\mathcal{U}_L(X)}(\Phi(\phi) \rightarrow \bigwedge_{y\in Y}(\widetilde{f}^{*\rightarrow}(\phi)(y) \rightarrow e_Y(z, y))) \\
&= \textstyle\bigwedge_{\phi\in\mathcal{U}_L(X)}\bigwedge_{y\in Y}(\Phi(\phi) \rightarrow (\widetilde{f}^{*\rightarrow}(\phi)(y) \rightarrow e_Y(z, y))) \\
&= \textstyle\bigwedge_{y\in Y}\bigwedge_{\phi\in\mathcal{U}_L(X)}((\Phi(\phi) * \widetilde{f}^{*\rightarrow}(\phi)(y)) \rightarrow e_Y(z, y)) \\
&= \textstyle\bigwedge_{y\in Y}(\bigvee_{\phi\in\mathcal{U}_L(X)}(\Phi(\phi) * \widetilde{f}^{*\rightarrow}(\phi)(y)) \rightarrow e_Y(z, y)) \\
&= \textstyle\bigwedge_{y\in Y}(\widetilde{f}^{*\rightarrow}(\sqcap^{op}\Phi)(y) \rightarrow e_Y(z, y)).
\end{aligned}$$

Thus $\overline{f}(\sqcap^{op}\Phi) = \sqcap\widetilde{f}^{*\rightarrow}(\sqcap^{op}\Phi) = y_1 = \sqcap\widetilde{\overline{f}}^{*\rightarrow}(\Phi)$, that is, $\overline{f}$ is a meet-preserving mapping.

Secondly, for all $x \in X$,

$$\mathbf{F}(\overline{f}) \circ \mu_X(x) = \overline{f}(\mu_x) = \sqcap\widetilde{f}^{*\rightarrow}(\mu_x) = \sqcap\mu_{f(x)} = f(x).$$

That is, $\mathbf{F}(\overline{f}) \circ \mu_X = f$.

Moreover, $\overline{f}$ is the unique LF-$\mathbf{CSLat}(\sqcap)$ morphism satisfying $\mathbf{F}(\overline{f}) \circ \mu_X = f$. In fact, if g is also an LF-$\mathbf{CSLat}(\sqcap)$ morphism satisfying $\mathbf{F}(g) \circ \mu_X = f$, then for every $\phi \in \mathcal{U}_L(X)$, we define $\widehat{\phi} \in L^{\mathcal{U}_L(X)}$ as follows: for all $\psi \in \mathcal{U}_L(X)$,

$$\widehat{\phi}(\psi) = \bigvee_{x\in X}(\phi(x) * \widetilde{e}(\psi, \mu_x))$$

Since for all $x \in X$, by Theorem 2.17 and Lemma 3.14,

$$\begin{aligned}
(\sqcap^{op}\widehat{\phi})(x) = (\sqcup\widehat{\phi})(x) &= \textstyle\bigvee_{\psi\in\mathcal{U}_L(X)}(\widehat{\phi}(\psi) * \psi(x)) \\
&= \textstyle\bigvee_{\psi\in\mathcal{U}_L(X)}(\bigvee_{x'\in X}(\phi(x') * \widetilde{e}(\psi, \mu_{x'})) * \psi(x)) \\
&= \textstyle\bigvee_{x'\in X}(\phi(x') * \bigvee_{\psi\in\mathcal{U}_L(X)}(\widetilde{e}(\psi, \mu_{x'}) * \psi(x))) \\
&= \textstyle\bigvee_{x'\in X}(\phi(x') * \bigvee_{\psi\in\mathcal{U}_L(X)}(\widetilde{e}(\psi, \mu_{x'}) * \widetilde{e}(\mu_x, \psi))) \quad \text{(by Yoneda Lemma)} \\
&= \textstyle\bigvee_{x'\in X}(\phi(x') * \widetilde{e}(\mu_x, \mu_{x'})) \\
&= \textstyle\bigvee_{x'\in X}(\phi(x') * e(x', x)) \quad \text{(by Corollary 3.10)} \\
&= \phi(x) \quad \text{(since } \phi \text{ is an } LF\text{-upper set),}
\end{aligned}$$

that is, $\sqcap^{op}\widehat{\phi} = \phi$, we have $g(\phi) = g(\sqcap^{op}\widehat{\phi}) = \sqcap\widetilde{g}^{*\rightarrow}(\widehat{\phi})$. However, for all $y \in Y$,

$$\begin{aligned} \widetilde{g}^{*\rightarrow}(\widehat{\phi})(y) &= \bigvee_{\psi\in\mathcal{U}_L(X)}(\widehat{\phi}(\psi) * e_Y(g(\psi), y)) \\ &= \bigvee_{\psi\in\mathcal{U}_L(X)}(\bigvee_{x\in X}(\phi(x) * \widetilde{e}(\psi, \mu_x)) * e_Y(g(\psi), y)) \\ &= \bigvee_{x\in X}(\phi(x) * \bigvee_{\psi\in\mathcal{U}_L(X)}(\widetilde{e}^{op}(\mu_x, \psi) * e_Y(g(\psi), y))) \\ &= \bigvee_{x\in X}(\phi(x) * e_Y(g(\mu_x), y)) \quad \text{(since } g \text{ is } LF\text{-monotone)} \\ &= \bigvee_{x\in X}(\phi(x) * e_Y(f(x), y)) \quad \text{(since } \mathbf{F}(g)\circ\mu_X = f) \\ &= \widetilde{f}^{*\rightarrow}(\phi)(y), \end{aligned}$$

that is, $\widetilde{g}^{*\rightarrow}(\widehat{\phi}) = \widetilde{f}^{*\rightarrow}(\phi)$, so $g(\phi) = \sqcap\widetilde{g}^{*\rightarrow}(\widehat{\phi}) = \sqcap\widetilde{f}^{*\rightarrow}(\phi) = \overline{f}(\phi)$. Thus $g = \overline{f}$.

(2) It is easily proved that $\mathbf{U}_L$ is a functor by (3), and by (1) we know that $\mathbf{U}_L \dashv \mathbf{F}$.

(3) By the definition of $\overline{()}$ we know that $\overline{\mu_Y \circ f}(\phi) = \sqcap^{op}(\widetilde{\mu_Y \circ f})^{*\rightarrow}(\phi)^{op}$ for all $\phi \in \mathcal{U}_L(X)$. However, for all $y \in Y$, by Corollary 3.15

$$\begin{aligned} & \sqcap^{op}(\widetilde{\mu_Y \circ f})^{*\rightarrow}(\phi)^{op}(y) \\ = \ & \sqcup(\widetilde{\mu_Y \circ f})_*^{\rightarrow}(\phi)(y) \\ = \ & \bigvee_{\psi\in\mathcal{U}_L(Y)}((\widetilde{\mu_Y \circ f})_*^{\rightarrow}(\phi)(\psi) * \psi(y)) \\ = \ & \bigvee_{\psi\in\mathcal{U}_L(Y)}(\bigvee_{x\in X}(\phi(x) * \widetilde{e}(\psi, (\mu_Y \circ f)(x))) * \psi(y)) \\ = \ & \bigvee_{\psi\in\mathcal{U}_L(Y)}(\bigvee_{x\in X}(\phi(x) * \widetilde{e}(\psi, \mu_{f(x)})) * \psi(y)) \\ = \ & \bigvee_{x\in X}(\phi(x) * \bigvee_{\psi\in\mathcal{U}_L(Y)}(\widetilde{e}(\psi, \mu_{f(x)}) * \psi(y))) \\ = \ & \bigvee_{x\in X}(\phi(x) * \bigvee_{\psi\in\mathcal{U}_L(Y)}(\widetilde{e}(\psi, \mu_{f(x)})) * \widetilde{e}(\mu_y, \psi)) \quad \text{(by Yoneda Lemma)} \\ = \ & \bigvee_{x\in X}(\phi(x) * \widetilde{e}(\mu_y, \mu_{f(x)})) \\ = \ & \bigvee_{x\in X}(\phi(x) * e_Y(f(x), y)) \quad \text{(by Corollary 2.10)} \\ = \ & \widetilde{f}^{*\rightarrow}(\phi)(y). \end{aligned}$$

Hence for all $\phi \in L^X$, $\overline{\mu_Y \circ f}(\phi) = \sqcap^{op}(\widetilde{\mu_Y \circ f})^{*\rightarrow}(\phi)^{op} = \widetilde{f}^{*\rightarrow}(\phi)$, that is, $\overline{\mu_Y \circ f} = \widetilde{f}^{*\rightarrow}$.

(4) By (1) and (3) we know that $\widetilde{f}^{*\rightarrow} = \overline{\mu_Y \circ f} : (\mathcal{U}_L(X), \widetilde{e}^{op}) \longrightarrow (\mathcal{U}_L(Y), \widetilde{e}^{op})$ is an LF-**CSLat**($\sqcap$) morphism, consequently it preserves arbitrary meets of L-fuzzy subsets, which implies that $\widetilde{f}^{*\rightarrow} : (\mathcal{U}_L(X), \widetilde{e}) \longrightarrow (\mathcal{U}_L(Y), \widetilde{e})$ preserves arbitrary joins of L-fuzzy subsets. By the AFT, $\widetilde{f}^{*\rightarrow}$ has a right 1-adjoint $g : (\mathcal{U}_L(Y), \widetilde{e}) \longrightarrow (\mathcal{U}_L(X), \widetilde{e})$ (in fact, $\widetilde{f}^{*\rightarrow}$ has a left 1-adjoint $g : (\mathcal{U}_L(Y), \widetilde{e}^{op}) \longrightarrow (\mathcal{U}_L(X), \widetilde{e}^{op})$ from Corollary 3.20), which is given by $g(\psi) = \sqcup\Phi_\psi$, where $\Phi_\psi \in L^{\mathcal{U}_L(X)}$ is defined by $\Phi_\psi(\phi) = \widetilde{e}(\widetilde{f}^{*\rightarrow}(\phi), \psi)$ for all $\phi \in \mathcal{U}_L(X)$. Now we prove that g is exactly the fuzzy backward powerset operator $\widetilde{f}^{*\leftarrow}$. At first, for all $x \in X$,

$$\begin{aligned} g(\psi)(x) &= \sqcup\Phi_\psi(x) \\ &= \bigvee_{\phi\in\mathcal{U}_L(X)}(\Phi_\psi(\phi) * \phi(x)) \\ &= \bigvee_{\phi\in\mathcal{U}_L(X)}(\widetilde{e}(\widetilde{f}^{*\rightarrow}(\phi), \psi) * \phi(x)) \\ &= \bigvee_{\phi\in\mathcal{U}_L(X)}(\widetilde{e}(\widetilde{f}^{*\rightarrow}(\phi), \psi) * \widetilde{e}(\mu_x, \phi)) \quad \text{(by Yoneda Lemma)} \\ &= \widetilde{e}(\widetilde{f}^{*\rightarrow}(\mu_x), \psi) \quad \text{(since } \widetilde{f}^{*\rightarrow} \text{ is } LF\text{-monotone)} \\ &= \widetilde{e}(\mu_{f(x)}, \psi) \\ &= \psi(f(x)) \quad \text{(by Yoneda Lemma)} \\ &= \bigvee_{x'\in X}(\psi(f(x')) * e(x', x)) \quad \text{(since } f \text{ is } LF\text{-monotone and } \psi \text{ is } LF\text{-upper)} \\ &= \widetilde{f}^{*\leftarrow}(\psi)(x). \end{aligned}$$

Thus $g(\psi) = \widetilde{f}^{*\leftarrow}(\psi)$ for all $\psi \in \mathcal{U}_L(Y)$, that is, $g = \widetilde{f}^{*\leftarrow}$. □

Remark 3.18. (1) Theorem 3.17 indicates that f can directly generate $\widetilde{f}^{*\rightarrow}$ via an adjunction between the categories LF-**Pos** and LF-**CSLat**($\sqcap$) and create $\widetilde{f}^{*\leftarrow}$ by the AFT for L-fuzzy posets.

(2) In [20] it is proved that if $f : X \longrightarrow Y$ is an LF-monotone mapping then $\widetilde{f}^{\leftarrow}(\psi) = f_L^{\leftarrow}(\psi)$ for all $\psi \in \mathcal{S}_L(Y)$ and $\widetilde{f}_*^{\leftarrow}(\psi) = f_L^{\leftarrow}(\psi)$ for all $\psi \in \mathcal{L}_L(Y)$ and $\widetilde{f}^{*\leftarrow}(\psi) = f_L^{\leftarrow}(\psi)$ for all $\psi \in \mathcal{U}_L(Y)$. Thus Theorems 3.11, 3.13 and 3.17 indeed create $f_L^{\leftarrow}$ from $\widetilde{f}^{\rightarrow}, \widetilde{f}_*^{\rightarrow}, \widetilde{f}^{*\rightarrow}$ by the AFT for L-fuzzy posets.

Lemma 3.19. *Let (X, e_X) and (Y, e_Y) be LF-posets and let $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ be L-fuzzy monotone mappings.*

(1) *If f has a right 1-adjoint, then the right 1-adjoint is unique.*
(2) *If g has a left 1-adjoint, then the left 1-adjoint is unique.*

Proof. (1) Suppose g_1 and g_2 are all right 1-adjoints of f. Then for all $x \in X$ and for all $y \in Y$, $e_X(x, g_1(y)) = e_Y(f(x), y) = e_X(x, g_2(y))$. Taking $x = g_1(y)$ and $x = g_2(y)$, we have $e_X(g_2(y), g_1(y)) = 1 = e_X(g_1(y), g_2(y))$. Thus $g_1(y) = g_2(y)$ for all $y \in Y$, that is, $g_1 = g_2$.
(2) Similar to (1). □

Corollary 3.20. *Let (X, e_X) and (Y, e_Y) be LF-posets, $f : (X, e_X) \longrightarrow (Y, e_Y)$ an LF-join-preserving mapping and $g : (Y, e_Y) \longrightarrow (X, e_X)$ an LF-meet-preserving mapping. Then*

(1) *the right 1-adjoint of $f : (X, e_X) \longrightarrow (Y, e_Y)$ is exactly the left 1-adjoint of $f : (X, e_X^{op}) \longrightarrow (Y, e_Y^{op})$;*

(2) *the left 1-adjoint of $g : (Y, e_Y) \longrightarrow (X, e_X)$ is exactly the right 1-adjoint of $g : (Y, e_Y^{op}) \longrightarrow (X, e_X^{op})$.*

Proof. (1) Let h is the unique right 1-adjoint of $f : (X, e_X) \longrightarrow (Y, e_Y)$. Then $e_X(f(x), y) = e_Y(x, h(y))$ for all $x \in X$ and for all $y \in Y$. Whence $e_Y^{op}(h(y), x) = e_Y(x, h(y)) = e_X(f(x), y) = e_X^{op}(y, f(x))$ for all $x \in X$ and for all $y \in Y$, which implies that h is the unique left 1-adjoint of $f : (X, e_X^{op}) \longrightarrow (Y, e_Y^{op})$ by Lemma 3.19.
(2) can be proved similarly to (1). □

Lemma 3.21. *Let $(X, e_X), (Y, e_Y)$ and (Z, e_Z) be LF-posets and let $f_1 : X \longrightarrow Y$ and $f_2 : Y \longrightarrow Z$ be LF-monotone mappings. If g_1 is the right 1-adjoint of f_1 and g_2 is the right 1-adjoint of f_2, then $g_1 \circ g_2$ is the right 1-adjoint of $f_2 \circ f_1$.*

Proof. Suppose g_1 is the right 1-adjoint of f_1 and g_2 is the right 1-adjoint of f_2. Then for all $x \in X$ and for all $y \in Y$ and for all $z \in Z$, $e_Y(f_1(x), y) = e_X(x, g_1(y))$ and $e_Y(f_2(y), z) = e_X(y, g_2(z))$. So for all $x \in X$ and for all $z \in Z$, $e_Z(f_2 \circ f_1(x), z) = e_Z(f_2(f_1(x)), z) = e_Y(f_1(x), g_2(z)) = e_X(x, g_1(g_2(z))) = e_X(x, g_1 \circ g_2(z))$. Thus $g_1 \circ g_2$ is the right 1-adjoint of $f_2 \circ f_1$. □

Theorem 3.22. *The category LF-$\mathbf{CSLat}(\sqcup)$ is isomorphic to its opposite LF-$\mathbf{CSLat}(\sqcup)^{op}$ and the category LF-$\mathbf{CSLat}(\sqcap)$ is isomorphic to its opposite LF-$\mathbf{CSLat}(\sqcap)^{op}$.*

Proof. Define $F : LF\text{-}\mathbf{CSLat}(\sqcup) \longrightarrow LF\text{-}\mathbf{CSLat}(\sqcup)^{op}$ as follows: F sends an LF-complete lattice (X, e) to its dual (X, e^{op}) and an LF-join-preserving mapping $f : (X, e_X) \longrightarrow (Y, e_Y)$ to $g^{op} : (X, e_X^{op}) \longrightarrow (Y, e_Y^{op})$, where $g : (Y, e_Y) \longrightarrow (X, e_X)$ is the right 1-adjoint of f. And define $G : LF\text{-}\mathbf{CSLat}(\sqcup)^{op} \longrightarrow LF\text{-}\mathbf{CSLat}(\sqcup)$ as follows: G sends an LF-complete lattice (X, e) to its dual (X, e^{op}) and a mapping $f^{op} : (Y, e_Y) \longrightarrow (X, e_X)$ (where $f : (X, e_X) \longrightarrow (Y, e_Y)$ is in LF-$\mathbf{CSLat}(\sqcup)$) to the right 1-adjoint $g : (Y, e_Y^{op}) \longrightarrow (X, e_X^{op})$ of f. By Lemma 3.19 and Lemma 3.21 it easily shows that F and G are all covariant functors. Moreover $G \circ F = Id_{LF\text{-}\mathbf{CSLat}(\sqcup)}$ and $F \circ G = Id_{LF\text{-}\mathbf{CSLat}(\sqcup)^{op}}$ by Lemma 3.14 and Corollary 3.15 and Corollary 3.20. Thus the category LF-$\mathbf{CSLat}(\sqcup)$ is isomorphic to its opposite LF-$\mathbf{CSLat}(\sqcup)^{op}$.

One can prove that the category LF-$\mathbf{CSLat}(\sqcap)$ is isomorphic to its opposite LF-$\mathbf{CSLat}(\sqcap)^{op}$ dually. □

4. Generate the Powerset Operators $f_L^{\rightarrow}, \widetilde{f}^{\rightarrow}, \widetilde{f}_*^{\rightarrow}, \widetilde{f}^{*\rightarrow}, f_L^{\leftarrow}, \widetilde{f}^{\leftarrow}, \widetilde{f}_*^{\leftarrow}, \widetilde{f}^{*\leftarrow}$ by Means of Algebraic Theories

Definition 4.1. [9, 13] $\mathbf{T} = (T, \eta, \diamond)$ is an *algebraic theory* (in clone form) in ground category $\mathbf{K}$ providing we have the following data subject to the following axioms:

(D1) $T : |\mathbf{K}| \longrightarrow |\mathbf{K}|$ is an object function on $\mathbf{K}$.

(D2) η assigns a $\mathbf{K}$ morphism $\eta_A : A \longrightarrow T(A)$ to each $A \in |\mathbf{K}|$.

(D3) $\diamond$ assigns a $\mathbf{K}$ morphism $g \diamond f : A \longrightarrow T(C)$ to each pair of $\mathbf{K}$ morphisms $f : A \longrightarrow T(B), g : B \longrightarrow T(C)$ and is called the *clone composition.*

(A1) $\diamond$ is associative: $\forall f : A \longrightarrow T(B), g : B \longrightarrow T(C), h : C \longrightarrow T(D)$,

$$h \diamond (g \diamond f) = (h \diamond g) \diamond f.$$

(A2) η furnishes (left) identities: $\forall f : A \longrightarrow T(B)$,

$$\eta_B \diamond f = f.$$

(A3) $\diamond$ is compatible with the composition $\circ$ of $\mathbf{K}$ morphisms: $\forall f : A \longrightarrow B, g : B \longrightarrow T(C)$, and setting $f^{\triangle} : A \longrightarrow T(B)$ by $f^{\triangle} = \eta_B \circ f$, it is the case that

$$g \diamond f^{\triangle} = g \circ f.$$

Remark 4.2. (1) Axiom (A2) only specifies that η gives left-hand identities. But in fact:

(A2′) η furnishes identities on both sides for $\diamond$: $\forall f : A \longrightarrow T(B)$,

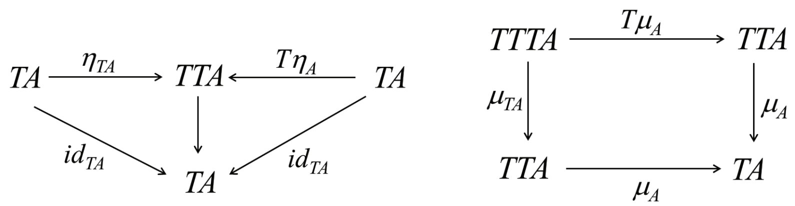
$$\eta_B \diamond f = f, f \diamond \eta_A = f.$$

(2) Each $\mathbf{K}$ morphism $f : A \longrightarrow B$ induces a $\mathbf{K}$ morphism $T(f) : T(A) \longrightarrow T(B)$, lifting f, by

$$T(f) = f^{\triangle} \diamond id_{T(A)}.$$

In fact, $T : \mathbf{K} \longrightarrow \mathbf{K}$ is functor and η is a natural transformation from $Id_{\mathbf{K}}$ to T.

Definition 4.3. [8] A *triple* or *monad* $\mathbf{T} = (T, \eta, \mu)$ on a category $\mathbf{K}$ consists of a functor $T : \mathbf{K} \longrightarrow \mathbf{K}$, together with two natural transformation $\eta : Id_{\mathbf{K}} \longrightarrow T$ and $\mu : TT \longrightarrow T$ for which the following diagrams commute for every object A of $\mathbf{K}$:



In [9], a triple or monad $\mathbf{T} = (T, \eta, \mu)$ is called an *algebraic theory* (in monoid form).

Theorem 4.4. [9] *In any category* $\mathbf{K}$*, the passage from algebraic theories* $\mathbf{T} = (T, \eta, \diamond)$ *to monads* $\mathbf{T} = (T, \eta, \mu)$ *defined as follows is bijective:*

- *for an algebraic theory* $\mathbf{T} = (T, \eta, \diamond)$*, let* $\mu_\diamond$ *be defined by*

$$\mu_{\diamond A} = id_{T(A)} \diamond id_{TT(A)} : TT(A) \longrightarrow T(A), \quad \forall A \in |\mathbf{K}|,$$

then $\mathbf{T} = (T, \eta, \mu_\diamond)$ *is a monad.*

In fact, the inverse passage can be achieved as follows:

- *for a monad* $\mathbf{T} = (T, \eta, \mu)$*, let* $\diamond_\mu$ *be defined by*

$$g \diamond_\mu f = \mu_C \circ T(g) \circ f : A \longrightarrow T(C), \quad \forall f : A \longrightarrow T(B), \forall g : B \longrightarrow T(C),$$

then $\mathbf{T} = (T, \eta, \diamond_\mu)$ *is an algebraic theory.*

Theorem 4.5. [13] *Let* $(L, \leq, \otimes) \in \mathbf{SQuant}$ *and* $\mathbf{T} = (T, \eta, \diamond)$ *be as follows:*

$D1$. $T : |\mathbf{Set}| \longrightarrow |\mathbf{Set}|$ *by* $T(X) = L^X$*;*
$D2$. $\forall X \in |\mathbf{Set}|$*, the component* $\eta_X : X \longrightarrow L^X$ *of* η *is defined by* $\eta_X(x) = \chi_{\{x\}}$*;*
$D3$. $\forall f : X \longrightarrow L^Y$*,* $\forall g : Y \longrightarrow L^Z$ *in* $\mathbf{Set}$*, define* $g \diamond f : X \longrightarrow L^Z$ *by*

$$[(g \diamond f)(x)](z) = \bigvee_{y \in Y} [(f(x))(y) \otimes (g(y))(z)].$$

Then $\mathbf{T}$ *is an algebraic theory in* $\mathbf{Set}$ *if and only if* $(L, \leq, \otimes)$ *is an st-quantale.*

Theorem 4.6. [8] *Every adjunction* $\langle F, G, \eta, \varepsilon \rangle$ *gives rise to a monad* $\mathbf{T} = (T, \eta, \mu)$ *by* $T = GF$*,* $\eta = \eta$*,* $\mu = G\varepsilon F$*. Therefore, every adjunction gives rise to an algebraic theory by Theorem* 4.4.

In the following L always denotes a complete residuated lattice, which is a commutative st-quantale with $\otimes = *$.

Corollary 4.7. *The algebraic theory* $\mathbf{T}^z = \langle T^z, \eta^z, \diamond^z \rangle$ *determined by the adjunction* $\langle \mathbf{P}_L, \mathbf{V}, \eta^z, \varepsilon \rangle$ *between* $\mathbf{Set}$ *and* LF-$\mathbf{CSLat}(\sqcup)$ *in Theorem* 3.7*, where* $\eta^z_X = \chi_X : X \longrightarrow L^X$ *for every* $X \in |\mathbf{Set}|$ *and* $\varepsilon^z : (L^X, \widetilde{e}) \rightarrow (X, e)$*,* $\phi \mapsto \sqcup \phi$ *for every* $(X, e) \in |LF$-$\mathbf{CSLat}(\sqcup)|$*, is identical to the algebraic theory in Theorem* 4.5.

Proof. Let $\mathbf{T}^z = \langle T^z, \eta^z, \mu^z \rangle$ denote the monad generated by the the adjunction $\langle \mathbf{P}_L, \mathbf{V}, \eta^z, \varepsilon^z \rangle$ between **Set** and LF-**CSLat**($\sqcup$) in Theorem 3.7. From the proof of this theorem we know that

- $\forall X \in |\mathbf{Set}|$ and $\forall f : X \longrightarrow Y$ in **Set**,

$$T^z(X) = \mathbf{VP}_L(X) = \mathbf{V}(L^X, \widetilde{e}) = L^X, \quad T^z(f) = \mathbf{VP}_L(f) = \mathbf{V}(f_L^{\rightarrow}) = f_L^{\rightarrow};$$

- $\forall X \in |\mathbf{Set}|$, the component η^z_X of the natural transformation η^z is $\eta^z_X = \chi_X : X \longrightarrow L^X$. It is exactly the η_X defined in Theorem 4.5;
- $\forall X \in |\mathbf{Set}|$, the component $\mu^z_X = (\mathbf{V}\varepsilon^z\mathbf{P}_L)_X : L^{L^X} \longrightarrow L^X$ of the natural transformation μ^z is defined by $\mu^z_X(\Phi) = \sqcup\Phi$ for all $\Phi \in L^{L^X}$.

Furthermore, by Theorem 4.4 the monad $\langle T^z, \eta^z, \mu^z \rangle$ in **Set** gives rise to the algebraic theory $\mathbf{T}^z = (T^z, \eta^z, \diamond^z)$ in **Set** by defining $\diamond^z$ as follows:

$$g \diamond^z f = \mu^z \circ T^z(g) \circ f, \quad \forall f : X \longrightarrow L^Y, \forall g : Y \longrightarrow L^Z.$$

Meanwhile, for $x \in X$ and $z \in Z$,

$$\begin{array}{ll}
 & [(g \diamond^z f)(x)](z) \\
= & [(\mu^z \circ T^z(g) \circ f)(x)](z) \\
= & [(\mu^z \circ T^z(g))(f(x))](z) \\
= & [\mu^z(g_L^{\rightarrow}(f(x)))](z) \\
= & [\sqcup g_L^{\rightarrow}(f(x))](z) \\
= & \bigvee_{\psi \in L^Z} (g_L^{\rightarrow}(f(x))(\psi) * \psi(z)) \qquad \text{(by Theorem 2.15)} \\
= & \bigvee_{\psi \in L^Z} ([\bigvee_{y \in g^{\leftarrow}(\{\psi\})} f(x)(y)] * \psi(z)) \\
= & \bigvee_{\psi \in g(Y)} ([\bigvee_{y \in g^{\leftarrow}(\{\psi\})} f(x)(y)] * \psi(z)) \\
= & \bigvee_{y' \in Y} ([\bigvee_{y \in g^{\leftarrow}(\{g(y')\})} f(x)(y)] * g(y')(z)) \\
= & \bigvee_{y' \in Y} \bigvee_{y \in g^{\leftarrow}(\{g(y')\})} (f(x)(y) * g(y)(z)) \\
= & \bigvee_{y \in Y} (f(x)(y) * g(y)(z)) \\
= & [(g \diamond f)(x)](z) \ \ \text{(in Theorem 4.5).}
\end{array}$$

□

Corollary 4.8. *The algebraic theory $\mathbf{T}^l = \langle T^l, \eta^l, \diamond^l \rangle$ determined by the adjunction $\langle \mathbf{L}_L, \mathbf{F}, \eta^l, \varepsilon^l \rangle$ between LF-**Pos** and LF-**CSLat**($\sqcup$) in Theorem 3.11, where $\eta^l_X = \iota_X : (X, e) \longrightarrow (\mathcal{L}_L(X), \widetilde{e})$ for every $(X, e) \in |LF\text{-}\mathbf{Pos}|$ and $\varepsilon^l : (\mathcal{L}_L(X), \widetilde{e}) \rightarrow (X, e)$, $\phi \mapsto \sqcup\phi$ for every $(X, e) \in |LF\text{-}\mathbf{CSLat}(\sqcup)|$, is as follows:*

L1. $T^l : |LF\text{-}\mathbf{Pos}| \longrightarrow |LF\text{-}\mathbf{Pos}|$ *by* $T((X, e)) = (\mathcal{L}_L(X), \widetilde{e})$*;*

L2. $\forall (X, e) \in |LF\text{-}\mathbf{Pos}|$*, the component* η^l_X *of the natural transformation* η^l *is* $\eta^l_X = \iota_X : (X, e) \longrightarrow (\mathcal{L}_L(X), \widetilde{e})$*;*

L3. $\forall f : (X, e_X) \longrightarrow (\mathcal{L}_L(Y), \widetilde{e}_Y)$*,* $\forall g : (Y, e_Y) \longrightarrow (\mathcal{L}_L(Z), \widetilde{e}_Z)$ *in* LF-**Pos***, define* $g \diamond^l f : (X, e_X) \longrightarrow (\mathcal{L}_L(Z), \widetilde{e}_Z)$ *by*

$$[(g \diamond^l f)(x)](z) = \bigvee_{y \in Y} [(f(x))(y) * (g(y))(z)].$$

Proof. First, the monad $\langle T^l, \eta^l, \mu^l \rangle$ generated by the adjunction $\langle \mathbf{L}_L, \mathbf{F}, \eta^l, \varepsilon^l \rangle$ between LF-**Pos** and LF-**CSLat**($\sqcup$) in Theorem 3.11 is as follows:

- $\forall (X,e)\in |LF\text{-}\mathbf{Pos}|$ and $\forall f:(X,e_X)\longrightarrow (Y,e_Y)$ in LF-**Pos**,

$$T^l(X)=\mathbf{FL}_L((X,e))=\mathbf{F}((\mathcal{L}_L(X),\widetilde{e}))=(\mathcal{L}_L(X),\widetilde{e}),\ \ T^l(f)=\mathbf{FL}_L(f)=\mathbf{F}(\widetilde{f_*^{\to}})=\widetilde{f_*^{\to}};$$

- $\forall (X,e)\in |LF\text{-}\mathbf{Pos}|$, the component η^l_X of the natural transformation η^l is $\eta^l_X=\iota_X:(X,e)\longrightarrow(\mathcal{L}_L(X),\widetilde{e})$;

- $\forall (X,e)\in |LF\text{-}\mathbf{Pos}|$, the component $\mu^l_X=(\mathbf{F}\varepsilon^l\mathbf{L}_L)_X:\mathcal{L}_L(\mathcal{L}_L(X))\longrightarrow\mathcal{L}_L(X)$ of the natural transformation μ^l is defined by $\mu^l_X(\Phi)=\sqcup\Phi$ for all $\Phi\in\mathcal{L}_L(\mathcal{L}_L(X))$.

Second, by Theorem 4.4 the monad $\langle T^l,\eta^l,\mu^l\rangle$ in LF-**Pos** gives rise to the algebraic theory $\mathbf{T}^l=(T^l,\eta^l,\diamond^l)$ in LF-**Pos** by defining $\diamond^l$ as follows:

$$g\diamond^l f=\mu^l\circ T^l(g)\circ f,\ \ \forall f:(X,e_X)\longrightarrow(\mathcal{L}_L(Y),\widetilde{e}_Y),\forall g:(Y,e_Y)\longrightarrow(\mathcal{L}_L(Z),\widetilde{e}_Z).$$

Meanwhile, for $x\in X$ and $z\in Z$,

$$\begin{aligned}
& [(g\diamond^l f)(x)](z)\\
= & [(\mu^l\circ T^l(g)\circ f)(x)](z)\\
= & [(\mu^l\circ T^l(g))(f(x))](z)\\
= & [\mu^l(\widetilde{g}_*^{\to}(f(x)))](z)\\
= & [\sqcup\widetilde{g}_*^{\to}(f(x))](z)\\
= & \textstyle\bigvee_{\psi\in\mathcal{L}_L(Z)}(\widetilde{g}_*^{\to}(f(x))(\psi)*\psi(z)) \qquad \text{(by Theorem 2.17)}\\
= & \textstyle\bigvee_{\psi\in\mathcal{L}_L(Z)}([\bigvee_{y\in Y}(f(x)(y)*\widetilde{e}(\psi,g(y)))]*\psi(z))\\
= & \textstyle\bigvee_{y\in Y}(f(x)(y)*\bigvee_{\psi\in\mathcal{L}_L(Z)}(\widetilde{e}(\psi,g(y))*\widetilde{e}(\iota_z,\psi))) \qquad \text{(by Yoneda Lemma)}\\
= & \textstyle\bigvee_{y\in Y}(f(x)(y)*\widetilde{e}(\iota_z,g(y)))\\
= & \textstyle\bigvee_{y\in Y}(f(x)(y)*g(y)(z)). \qquad \text{(by Yoneda Lemma)}
\end{aligned}$$

□

Similar to the proof of Corollary 4.8 we can get the following corollary.

Corollary 4.9. *The algebraic theory* $\mathbf{T}^s=\langle T^s,\eta^s,\diamond^s\rangle$ *determined by the adjunction* $\langle\mathbf{S}_L,\mathbf{F},\eta^s,\varepsilon^s\rangle$ *between* LF-**Pos** *and* LF-**CSLat**$(\sqcup)$ *in Theorem* 3.13*, where* $\eta^s_X=s_X:(X,e)\longrightarrow(\mathcal{S}_L(X),\widetilde{e})$ *for every* $(X,e)\in|LF\text{-}\mathbf{Pos}|$ *and* $\varepsilon^s:(\mathcal{S}_L(X),\widetilde{e})\to(X,e)$, $\phi\mapsto\sqcup\phi$ *for every* $(X,e)\in|LF\text{-}\mathbf{CSLat}(\sqcup)|$*, is as follows:*

$S1$. $T^s:|LF\text{-}\mathbf{Pos}|\longrightarrow|LF\text{-}\mathbf{Pos}|$ *by* $T((X,e))=(\mathcal{S}_L(X),\widetilde{e})$*;*

$S2$. $\forall(X,e)\in|LF\text{-}\mathbf{Pos}|$*, the component* η^s_X *of the natural transformation* η^s *is* $\eta^s_X=s_X:(X,e)\longrightarrow(\mathcal{S}_L(X),\widetilde{e})$*;*

$S3$. $\forall f:(X,e_X)\longrightarrow(\mathcal{S}_L(Y),\widetilde{e}_Y)$, $\forall g:(Y,e_Y)\longrightarrow(\mathcal{S}_L(Z),\widetilde{e}_Z)$ *in* LF-**Pos**, *define* $g\diamond^l f:(X,e_X)\longrightarrow(\mathcal{S}_L(Z),\widetilde{e}_Z)$ *by*

$$[(g\diamond^s f)(x)](z)=\bigvee_{y\in Y}[(f(x))(y)*(g(y))(z)].$$

Corollary 4.10. *The algebraic theory* $\mathbf{T}^u=\langle T^u,\eta^u,\diamond^u\rangle$ *determined by the adjunction* $\langle\mathbf{U}_L,\mathbf{F},\eta^u,\varepsilon^u\rangle$ *between* LF-**Pos** *and* LF-**CSLat**$(\sqcap)$ *in Theorem* 3.17*, where* $\eta^u_X=\mu_X:(X,e)\longrightarrow(\mathcal{U}_L(X),\widetilde{e}^{op})$ *for every* $(X,e)\in|LF\text{-}\mathbf{Pos}|$ *and* $\varepsilon^u:(\mathcal{U}_L(X),\widetilde{e}^{op})\to(X,e)$, $\phi\mapsto\sqcap^{op}\phi$ *for every* $(X,e)\in|LF\text{-}\mathbf{CSLat}(\sqcap)|$*, is as follows:*

$U1$. $T^u:|LF\text{-}\mathbf{Pos}|\longrightarrow|LF\text{-}\mathbf{Pos}|$ *by* $T((X,e))=(\mathcal{U}_L(X),\widetilde{e}^{op})$*;*

U2. $\forall (X,e) \in |LF\text{-}\mathbf{Pos}|$, *the component* η^u_X *of the natural transformation* η^u *is* $\eta^u_X = \mu_X : (X,e) \longrightarrow (\mathcal{U}_L(X), \widetilde{e}^{op})$;

U3. $\forall f : (X, e_X) \longrightarrow (\mathcal{U}_L(Y), \widetilde{e}^{op}_Y)$, $\forall g : (Y, e_Y) \longrightarrow (\mathcal{U}_L(Z), \widetilde{e}^{op}_Z)$ *in* $LF\text{-}\mathbf{Pos}$, *define* $g \diamond^u f : (X, e_X) \longrightarrow (\mathcal{U}_L(Z), \widetilde{e}^{op}_Z)$ *by*

$$[(g \diamond^u f)(x)](z) = \bigvee_{y \in Y} [(f(x))(y) * (g(y))(z)].$$

Proof. First, the monad $\langle T^u, \eta^u, \mu^u \rangle$ generated by the adjunction $\langle \mathbf{U}_L, \mathbf{F}, \eta^u, \varepsilon^u \rangle$ between $LF\text{-}\mathbf{Pos}$ and $LF\text{-}\mathbf{CSLat}(\sqcup)$ in Theorem 3.17 is as follows:

- $\forall (X,e) \in |LF\text{-}\mathbf{Pos}|$ and $\forall f : (X, e_X) \longrightarrow (Y, e_Y)$ in $LF\text{-}\mathbf{Pos}$,

$T^l(X) = \mathbf{F}\mathbf{L}_L((X,e)) = \mathbf{F}((\mathcal{U}_L(X), \widetilde{e}^{op})) = (\mathcal{U}_L(X), \widetilde{e}^{op})$, $T^l(f) = \mathbf{F}\mathbf{U}_L(f) = \mathbf{F}(\widetilde{f}^{*\rightarrow}) = \widetilde{f}^{*\rightarrow}$;

- $\forall (X,e) \in |LF\text{-}\mathbf{Pos}|$, the component η^u_X of the natural transformation η^l is $\eta^u_X = \mu_X : (X,e) \longrightarrow (\mathcal{U}_L(X), \widetilde{e}^{op})$;

- $\forall (X,e) \in |LF\text{-}\mathbf{Pos}|$, the component $\mu^l_X = (\mathbf{F}\varepsilon^u \mathbf{U}_L)_X : \mathcal{U}_L(\mathcal{U}_L(X)) \longrightarrow \mathcal{U}_L(X)$ of the natural transformation μ^u is defined by $\mu^u_X(\Phi) = \sqcap^{op}\Phi = \sqcup\Phi$ for all $\Phi \in \mathcal{U}_L(\mathcal{U}_L(X))$.

Second, by Theorem 4.4 the monad $\langle T^u, \eta^u, \mu^u \rangle$ in $LF\text{-}\mathbf{Pos}$ gives rise to the algebraic theory $\mathbf{T}^u = (T^u, \eta^u, \diamond^u)$ in $LF\text{-}\mathbf{Pos}$ by defining $\diamond^l$ as follows:

$g \diamond^u f = \mu^u \circ T^u(g) \circ f$, $\forall f : (X, e_X) \longrightarrow (\mathcal{U}_L(Y), \widetilde{e}^{op}_Y), \forall g : (Y, e_Y) \longrightarrow (\mathcal{L}_L(Z), \widetilde{e}^{op}_Z)$.

Meanwhile, for $x \in X$ and $z \in Z$,

$$\begin{array}{ll} & [(g \diamond^u f)(x)](z) \\ = & [(\mu^u \circ T^u(g) \circ f)(x)](z) \\ = & [(\mu^u \circ T^u(g))(f(x))](z) \\ = & [\mu^u(\widetilde{g}^{*\rightarrow}(f(x)))](z) \\ = & [\sqcup \widetilde{g}^{*\rightarrow}(f(x))](z) \\ = & \bigvee_{\psi \in \mathcal{U}_L(Z)} (\widetilde{g}^{*\rightarrow}(f(x))(\psi) * \psi(z)) \qquad \text{(by Theorem 2.17)} \\ = & \bigvee_{\psi \in \mathcal{U}_L(Z)} ([\bigvee_{y \in Y} (f(x)(y) * \widetilde{e}(g(y), \psi))] * \psi(z)) \\ = & \bigvee_{y \in Y} (f(x)(y) * \bigvee_{\psi \in \mathcal{U}_L(Z)} (\widetilde{e}(\psi, g(y)) * \widetilde{e}(\mu_z, \psi))) \qquad \text{(by Yoneda Lemma)} \\ = & \bigvee_{y \in Y} (f(x)(y) * \widetilde{e}(\mu_z, g(y))) \\ = & \bigvee_{y \in Y} (f(x)(y) * g(y)(z)). \qquad \text{(by Yoneda Lemma)} \end{array}$$

□

In the following, similar to [13], we give the axioms for fuzzy powerset theories in our settings. Then we will prove the fuzzy powerset operators defined above all can be generated by algebraic theories.

Definition 4.11. Let a category $\mathbf{K}$ be given, called a ground category. We consider the following conditions:

(P1) **Powerset generator:** $P : |\mathbf{K}| \longrightarrow |LF\text{-}\mathbf{CSLat}(\sqcup)|$ is an object-mapping.

(P2) **Forward/image powerset operator:** assuming (P1), there is an operator $\rightarrow$ such that $\forall f : A \longrightarrow B$ in $\mathbf{K}$, $\exists f^{\rightarrow}_{\mathbf{P}} : P(A) \longrightarrow P(B)$ in $LF\text{-}\mathbf{Pos}$.

(P3) **Backward/image powerset operator:** assuming (P1), there is an operator $\leftarrow$ such that $\forall f : A \longrightarrow B$ in $\mathbf{K}$, $\exists f^{\leftarrow}_{\mathbf{P}} : P(B) \longrightarrow P(A)$ in LF-**Pos**.

(Ad) **Adjunction:** assuming (P1-P3), $\forall f : A \longrightarrow B$ in $\mathbf{K}$, $f^{\rightarrow}_{\mathbf{P}} \dashv_1 f^{\leftarrow}_{\mathbf{P}}$.

(C) **Concreteness:** assuming (P1, P2), $\exists$ a concrete functor $V : \mathbf{K} \longrightarrow \mathbf{Set}$ and an insertion map η which determines for each $A \in |\mathbf{K}|$ a **Set** morphism $\eta_A : V(A) \longrightarrow P(A)$.

(N) **Naturality:** assuming (P1, P2, C) and $f : A \longrightarrow B$ in $\mathbf{K}$, then in **Set** $f^{\rightarrow}_{\mathbf{P}} \circ \eta_A = \eta_B \circ V(f)$.

(T) **Topological criterion**: assuming (P1, P3), this criterion comprises the following conditions:

(T1) $\forall f : A \longrightarrow B$ in $\mathbf{K}$, $f^{\leftarrow}_{\mathbf{P}} : P(B) \longrightarrow P(A)$ is in LF-$\mathbf{CSLat}(\sqcup)$;
(T2) $\forall f : A \longrightarrow B, g : B \longrightarrow C$ in $\mathbf{K}$, $(g \circ f)^{\leftarrow}_{\mathbf{P}} = f^{\leftarrow}_{\mathbf{P}} \circ g^{\leftarrow}_{\mathbf{P}}$;
(T3) $\forall A$ in $\mathbf{K}$, $(id_A)^{\leftarrow}_{\mathbf{P}} = id_{P(A)}$.

Definition 4.12. Let a category $\mathbf{K}$ be given, called a ground category.

(1) $\mathbf{P} = (P, \rightarrow)$ is a *forward LF-powerset theory in* $\mathbf{K}$ if (P1, P2) are satisfied.

(2) $\mathbf{P} = (P, \leftarrow)$ is a *backward LF-powerset theory in* $\mathbf{K}$ if (P1, P3) are satisfied.

(3) $\mathbf{P} = (P, \rightarrow, \leftarrow)$ is a *balanced LF-powerset theory in* $\mathbf{K}$ if (P1-P3) are satisfied.

(4) $\mathbf{P} = (P, \rightarrow, \leftarrow)$ is an *adjunctive LF-powerset theory in* $\mathbf{K}$ if (P1-P3, Ad) are satisfied.

(5) $\mathbf{P} = (P, \rightarrow, V, \eta)$ is a *concrete LF-powerset theory in* $\mathbf{K}$ if (P1, P2, C) are satisfied; and $\mathbf{P}$ is *Natural* if additionally (N) is satisfied.

(6) $\mathbf{P} = (P, \leftarrow)$ is a *topological LF-powerset theory in* $\mathbf{K}$ if (P1, P3, T) are satisfied.

From the above definitions and by Theorem 3.3 (AFT) we are not difficult to obtain the following proposition.

Proposition 4.13. *Let a category* $\mathbf{K}$ *be a ground category, and* P *satisfies* $(P1)$. *The following hold:*

(1) *If* $(P2)$ *is satisfied, then* $(\forall f : A \longrightarrow B$ *in* $\mathbf{K}$, $f^{\rightarrow}_{\mathbf{P}} : P(A) \longrightarrow P(B)$ *in LF-*$\mathbf{CSLat}(\sqcup))$ *if and only if* $(\forall f : A \longrightarrow B$ *in* $\mathbf{K}$, $f^{\leftarrow}_{\mathbf{P}} : P(B) \longrightarrow P(A)$ *is uniquely determined such that* $\mathbf{P} = (P, \rightarrow, \leftarrow)$ *is an adjunctive LF-powerset theory*$)$.

(2) *If* $(P3)$ *is satisfied, then* $(\forall f : A \longrightarrow B$ *in* $\mathbf{K}$, $f^{\leftarrow}_{\mathbf{P}} : P(B) \longrightarrow P(A)$ *in LF-*$\mathbf{CSLat}(\sqcap))$ *if and only if* $(\forall f : A \longrightarrow B$ *in* $\mathbf{K}$, $f^{\rightarrow}_{\mathbf{P}} : P(A) \longrightarrow P(B)$ *is uniquely determined such that* $\mathbf{P} = (P, \rightarrow, \leftarrow)$ *is an adjunctive LF-powerset theory*$)$.

(3) *There is an operator* $\rightarrow$ *such that* $\mathbf{P} = (P, \leftarrow)$ *is a topological L-powerset theory if and only if the object mapping* P *extends to a contravariant functor* $P_{\leftarrow} : \mathbf{K} \longrightarrow LF$-$\mathbf{CSLat}(\sqcup)$.

Example 4.14. (**Zadeh Fuzzy Powerset Theories in the Category Set**) Let $\mathbf{K} = \mathbf{Set}$. Put $P^z : |\mathbf{K}| \longrightarrow |LF\text{-}\mathbf{CSLat}(\sqcup)|$ by $P^z(X) = (L^X, \widetilde{e})$,

- for $f : X \longrightarrow Y$ put $f^{\rightarrow}_{\mathbf{P}} = f^{\rightarrow}_L : (L^X, \widetilde{e}_X) \longrightarrow (L^Y, \widetilde{e}_Y)$, $f^{\leftarrow}_{\mathbf{P}} = f^{\leftarrow}_L : (L^Y, \widetilde{e}_Y) \longrightarrow (L^X, \widetilde{e}_X)$;

- put $V : \mathbf{K} \longrightarrow \mathbf{Set}$ by $V = Id_{\mathbf{Set}}$;
- for $X \in |\mathbf{Set}|$ define $\eta^z_X : V(X) \longrightarrow P(X)$ by $\eta^z_X = \chi_X$.

Then by the definitions and Theorem 3.7 we easily know that $\mathbf{P}^z = (P^z, ()^{\rightarrow}_L, ()^{\leftarrow}_L, V, \eta^z)$ is an adjunctive, concrete, natural topological LF-powerset theory in **Set**.

Example 4.15. (**Some Fuzzy Powerset Theories in** LF-**Pos**) Let $\mathbf{K} = LF$-**Pos**.

(1) Put $P^l : |\mathbf{K}| \longrightarrow |LF\text{-}\mathbf{CSLat}(\sqcup)|$ by $P^l(X) = (\mathcal{L}_L(X), \widetilde{e})$,

- for $f : (X, e_X) \longrightarrow (Y, e_Y)$ in $\mathbf{K}$, put $f^{\rightarrow}_{\mathbf{P}^l} = \widetilde{f}^{\rightarrow}_* : (\mathcal{L}_L(X), \widetilde{e}_X) \longrightarrow (\mathcal{L}_L(Y), \widetilde{e}_Y)$, $f^{\leftarrow}_{\mathbf{P}^l} = \widetilde{f}^{\leftarrow}_* : (\mathcal{L}_L(Y), \widetilde{e}_Y) \longrightarrow (\mathcal{L}_L(X), \widetilde{e}_X)$;
- put $V : \mathbf{K} \longrightarrow \mathbf{Set}$ by $V((X, e)) = X, V(f) = f$;
- for $X \in |\mathbf{Set}|$ define $\eta_X : V(X) \longrightarrow P^l(X)$ by $\eta^l_X = \iota_X$.

Then by the definitions and the proof of Theorem 3.11 we know that $\mathbf{P}^l = (P^l, \widetilde{()}^{\rightarrow}_*, \widetilde{()}^{\leftarrow}_*, V, \eta^l)$ is an adjunctive, concrete, natural topological LF-powerset theory in LF-**Pos**.

(2) Put $P^s : |\mathbf{K}| \longrightarrow |LF\text{-}\mathbf{CSLat}(\sqcup)|$ by $P^s(X) = (\mathcal{S}_L(X), \widetilde{e})$,

- for $f : (X, e_X) \longrightarrow (Y, e_Y)$ in $\mathbf{K}$, put $f^{\rightarrow}_{\mathbf{P}^s} = \widetilde{f}^{\rightarrow}_* : (\mathcal{S}_L(X), \widetilde{e}_X) \longrightarrow (\mathcal{S}_L(Y), \widetilde{e}_Y)$, $f^{\leftarrow}_{\mathbf{P}^s} = \widetilde{f}^{\leftarrow}_* : (\mathcal{S}_L(Y), \widetilde{e}_Y) \longrightarrow (\mathcal{S}_L(X), \widetilde{e}_X)$;
- put $V : \mathbf{K} \longrightarrow \mathbf{Set}$ by $V((X, e)) = X, V(f) = f$;
- for $X \in |\mathbf{Set}|$ define $\eta_X : V(X) \longrightarrow P^s(X)$ by $\eta^s_X = s_X$.

Then by the definitions and Theorem 3.13 we know that $\mathbf{P}^s = (P^s, \widetilde{()}^{\rightarrow}, \widetilde{()}^{\leftarrow}, V, \eta^s)$ is an adjunctive, concrete, natural topological LF-powerset theory in LF-**Pos**.

(3) Put $P^u : |\mathbf{K}| \longrightarrow |LF\text{-}\mathbf{CSLat}(\sqcup)|$ by $P^u(X) = (\mathcal{U}_L(X), \widetilde{e})$,

- for $f : (X, e_X) \longrightarrow (Y, e_Y)$ in $\mathbf{K}$, put $f^{\rightarrow}_{\mathbf{P}^u} = \widetilde{f}^{*\rightarrow} : (\mathcal{U}_L(X), \widetilde{e}_X) \longrightarrow (\mathcal{U}_L(Y), \widetilde{e}_Y)$, $f^{\leftarrow}_{\mathbf{P}^u} = \widetilde{f}^{*\leftarrow} : (\mathcal{U}_L(Y), \widetilde{e}_Y) \longrightarrow (\mathcal{U}_L(X), \widetilde{e}_X)$;
- put $V : \mathbf{K} \longrightarrow \mathbf{Set}$ by $V((X, e)) = X, V(f) = f$;
- for $X \in |\mathbf{Set}|$ define $\eta_X : V(X) \longrightarrow P^u(X)$ by $\eta^u_X = \mu_X$.

Then by the definitions and the proof of Theorem 3.17 we know that $\mathbf{P}^u = (P^u, \widetilde{()}^{*\rightarrow}, \widetilde{()}^{*\leftarrow}, V, \eta^u)$ is an adjunctive, concrete, natural topological LF-powerset theory in LF-**Pos**.

Definition 4.16. [13] An algebraic theory $\mathbf{T} = (T, \hat{\eta}, \diamond)$ in a category $\mathbf{K}$ generates a concrete LF-powerset theory $\mathbf{P} = (P, \rightarrow, V, \eta)$ if the following are satisfied:

(G1) **Compatibility of objective functions:** $\forall A \in |\mathbf{K}|$, $V(T(A)) = P(A)$.

(G2) **Compatibility of insertion morphisms:** $\forall A \in |\mathbf{K}|$, $V(\hat{\eta}_A) = \eta_A$.

(G3) **Generation of forward/image powerset operator:** $\forall A \in |\mathbf{K}|$, the operator $f^{\rightarrow}_{\mathbf{T}} : V(T(A)) \longrightarrow V(T(B))$ defined by setting $f^{\rightarrow}_{\mathbf{T}} = V(T(f))$ is precisely the image operator $f^{\rightarrow}_{\mathbf{P}} : P(A) \longrightarrow P(B)$ of $\mathbf{P}$, where $T(f) : T(A) \longrightarrow T(B)$ is arrow induced by $\mathbf{T}$.

If $\mathbf{P} = (P, \rightarrow, \leftarrow, V, \eta)$ is a balanced, concrete LF-powerset theory, then $\mathbf{P}$ is generated from an algebraic theory $\mathbf{T}$ if $\mathbf{T}$ generates $(P, \rightarrow, V, \eta)$ and the following additional condition holds:

(G4) **Generation of forward/image powerset operator:** $f^{\leftarrow}_{\mathbf{P}}$ is always uniquely determined from $f^{\rightarrow}_{\mathbf{P}}$ so that (P3) and (Ad) are satisfied.

Theorem 4.17. *The algebraic theory* $\mathbf{T}^z$ *in Corollary* 4.7 *generates the natural topological* LF*-powerset theory* $\mathbf{P}^z$ *in* **Set** *of Example* 4.14*.*

Proof. From the definitions we easily get (G1) and (G2), (G4) is obtained by Theorem 3.7. For (G3) let $f : X \longrightarrow Y$, by the definition of $f^{\rightarrow}_{\mathbf{T}^z}$, we have for any $\phi \in L^X$ and $y \in Y$,

$$\begin{aligned} f_{\mathbf{T}^z}(\phi)(y) &= (T^z(f))(\phi)(y) \\ &= [(f^{\triangle} \diamond^z id_{T(X)})(\phi)](y) \\ &= \textstyle\bigvee_{x\in X}(id_{T(X)}(\phi)(x) * (\eta^z \circ f)(x)(y)) \\ &= \textstyle\bigvee_{x\in X}(\phi(x) * \chi_{\{f(x)\}}(y)) \\ &= f^{\rightarrow}_L(\phi)(y) = f^{\rightarrow}_{\mathbf{P}^z}(\phi)(y) \end{aligned}$$

□

Theorem 4.18. *The algebraic theories* $\mathbf{T}^l, \mathbf{T}^s, \mathbf{T}^u$ *in Corollaries* 4.8*,* 4.9*,* 4.10 *generate the natural topological* LF*-powerset theories* $\mathbf{P}^l, \mathbf{P}^s, \mathbf{P}^u$ *in* LF-**Pos** *of Example* 4.15 $(1), (2), (3)$*, respectively.*

Proof. We only prove the third case, and other cases are similar.

(G1), (G2) and (G4) can be obtained by the definitions and Theorem 3.17. For (G3) let $f : (X, e_X) \longrightarrow (Y, e_Y)$ in LF-**Pos**, by the definition of $f^{\rightarrow}_{\mathbf{T}^u}$, we have for any $\phi \in \mathcal{U}_L(X)$ and $y \in Y$,

$$\begin{aligned} f_{\mathbf{T}^u}(\phi)(y) &= (T^u(f))(\phi)(y) \\ &= [(f^{\triangle} \diamond^u id_{T(X)})(\phi)](y) \\ &= \textstyle\bigvee_{x\in X}(id_{T(X)}(\phi)(x) * (\eta^u \circ f)(x)(y)) \\ &= \textstyle\bigvee_{x\in X}(\phi(x) * \mu_{\{f(x)\}}(y)) \\ &= \textstyle\widetilde{\bigvee}_{x\in X}(\phi(x) * e_Y(f(x), y)) \\ &= \widetilde{f}^{*\rightarrow}(\phi)(y) = f^{\rightarrow}_{\mathbf{P}^u}(\phi)(y) \end{aligned}$$

□

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