THE CONNECTION BETWEEN SOME EQUIVALENCE RELATIONS ON FUZZY SUBGROUPS

A. IRANMANESH AND H. NARAGHI

Abstract. This paper, deals with some equivalence relations in fuzzy subgroups. Further the probability of commuting two fuzzy subgroups of some finite abelian groups is defined.

1. **Introduction**

The classification of fuzzy subgroups of a finite group of any given order is considered as a fundamental general problem [3,7,8]. Recently, many authors have defined some equivalence relations on $F(G)$ where $F(G)$ is the set of all fuzzy subgroups of the group G, in order to study the equivalent fuzzy subgroups [9,10,12,15].

Let $F_1(G)$ be the set of all fuzzy subgroups μ of the group G such that $\mu(e) = 1$. In the present paper, we will define a new equivalence relation on $F_1(G)$ and obtain the connection between this equivalence relation and other equivalence relations defined in the above mentioned papers. This helps us to count the number of nonequivalent fuzzy subgroups of a finite group. For instance, Tărnăuceanu [14] and Muralli and Makamba [11] have obtained some results on counting the number of fuzzy subgroups of finite abelian groups, specially for $Z_{p^n q^m}$, as one of the most interesting groups with cyclic subgroups.

Actually, the probability of commuting two elements of a finite group is an important subject in group theory and many papers have been published in this area. For example, see [4] and [13]. But as an inquiry someone may wonder if it is possible to extend this concept to fuzzy groups and obtain similar results as in group theory. Surely the possibility of counting the number of fuzzy subgroups must be useful in computing this probability.

In the last section we will focus on defining the probability of commuting fuzzy subgroups of $Z_{p^n q^m}$. Also we will obtain some interesting results of this group using the previous findings and will give more information about the structure of its fuzzy subgroups. These results can be generalized for every finite abelian group and may lead to new approaches for connecting the fuzzy subgroup and finite group theories.

2. **Preliminaries**

We use [0,1]=I, the real unit interval, as a chain with the usual ordering, which $\wedge$ stands for infimum (or intersection) and $\vee$ stands for supremum (or union)

Received: September 2009; Accepted: December 2010

Key words and phrases: Dihedral group, Equivalence relation, Fuzzy subgroups, Finite cyclic group, $p - group$, Probability.

as the degree of membership. A fuzzy subset of a set X is defined as a mapping $\mu : X \to [0,1]$. The union and intersection of two fuzzy subsets are defined using point wise sup and inf . We denote the set of all fuzzy subsets of X by I^X. Further, we denote fuzzy subsets by the Greek letters μ, ν, η, etc. Let $\mu, \nu \in I^X$. If $\mu(x) \leq \nu(x)$, for all $x \in X$, then we say that μ is contained in ν (or ν contains μ) and we write $\mu \subseteq \nu$. Let $\mu \in I^X$ for $a \in I$, then the a-cut (or a-level) subset of μ denoted by μ_a can be defined as $\mu_a = \{x \mid x \in X, \mu(x) \geq a\}$.
It is easy to verify the following properties for any $\mu, \nu \in I^X$:

(1) $\mu \subseteq \nu, a \in I \Rightarrow \mu_a \subseteq \nu_a$.
(2) $a \leq b, a, b \in I \Rightarrow \mu_b \subseteq \mu_a$.
(3) $\mu = \nu \Leftrightarrow \mu_a = \nu_a$ for all $a \in I$.

Let G be an arbitrary group with a multiplicative binary operation and identity element. The binary operation o on I^G is defined for all $\mu, \nu \in I^G$ and all $x \in G$ as
$(\mu o \nu)(x) = \vee\{\mu(y) \wedge \nu(z) \mid y, z \in G, yz = x\}$.
We call $\mu o \nu$ the product of μ and ν. The fuzzy subset μ of the group G is called a fuzzy subgroup of G if:

i) $\mu(xy) \geq \mu(x) \wedge \mu(y)$, for all $x, y \in G$;
ii) $\mu(x^{-1}) \geq \mu(x)$, for all $x \in G$.

Also the set of all fuzzy subgroups of a group G is denoted by $F(G)$.

Proposition 2.1. [9] *Let $\mu \in I^G$, then μ is a fuzzy subgroup of G, if and only if μ_a is a subgroup of G for all $a \in \mu(G) \bigcup \{b \in I \mid b \leq \mu(e)\}$.*

Theorem 2.2. [9] *Let $\mu \in I^G$, then $\mu o \nu$ is a fuzzy subgroup, if and only if $\mu o \nu = \nu o \mu$.*

Definition 2.3. Let G be a group and $\mu \in F(G)$. The set $\{x \in G | \mu(x > 0)\}$ is called the support of μ and is denoted by supp μ.

Let G be a group and $\mu \in F(G)$. We would use Imμ for representing the image set of μ and also F_μ for the family $\{\mu_t | t \in \text{Im}\mu\}$.

Definition 2.4. [11] Let G be a group. A chain of maximal subgroups of G

$$\varphi : \ (e) \subset G_1 \subset G_2 \subset \ldots \subset G_n = G$$

is called a flag and is denoted by φ. Furthermore, the length of the flag φ for the group G is set to be equal to n.

It is trivial that if G is an abelian group, then every flag of G is a composite series of G. Therefore, if G is an abelian group of order $p_1^{n_1} p_2^{n_2} \ldots p_r^{n_r}$, then the length of every flag is equal to $n_1 + n_2 + \ldots + n_r$.
Let G be a group and $\varphi : (e) = G_0 \subset G_1 \subset \ldots \subset G_n = G$ be a flag of G. Then for every $\lambda_i \in [0,1], i \in \{1, \ldots, n\}$, $1 \geq \lambda_1 \geq \ldots \geq \lambda_n$, the function μ from G to $[0,1]$

is a fuzzy subgroup of G, where:

$$\mu(x) = \begin{cases} 1 & x \in G_0, \\ \lambda_1 & x \in G_1 - G_o, \\ \lambda_2 & x \in G_2 - G_1, \\ \vdots & \vdots \\ \lambda_n & x \in G_n - G_{n-1}. \end{cases}$$

We denote the above μ by $\mu = (1\lambda_1\lambda_2 \cdots \lambda_n)\varphi$.
Suppose G is a finite group and $\mu \in F(G)$ such that $\mathrm{Im}\mu = \{1, \lambda_1, \cdots, \lambda_{\mathrm{r}}\}$, where, $1 > \lambda_1 > \cdots > \lambda_r$. Then, there exists a flag of G with the length n such that $n_0 + n_1 + \cdots + n_r = n$, $1 \leq i \leq n$.

$$\begin{aligned} \varphi:\ & G_0 = (e) \subset G_1 \subset \cdots \subset G_{n_0} \\ & \subset G_{n_0+1} \subset \cdots \subset G_{n_1} \\ & \subset G_{n_1+1} \subset \cdots \subset G_{n_2} \\ & \vdots \\ & \subset G_{n_{(r-1)}+1} \subset \cdots \subset G_{n_r} \subset G \end{aligned}$$

such that:

$$\mu = (\underbrace{1 \cdots 1}_{n'_0} \underbrace{\lambda_1 \cdots \lambda_1}_{n'_1} \cdots \underbrace{\lambda_r \cdots \lambda_r}_{n'_r})\varphi$$

where, $n'_0 = n_0, n'_1 = n_1 - n_0$, and for $i \in \{2, \cdots, s\}$, we have $n'_i = n_i - \sum_{k=1}^{i-1} n_k$, $i \in \{0, \cdots, s\}$, and $\mu_{\alpha_i} = G_{n_i}$

3. **Some Equivalent Relations on the Fuzzy Subgroups of a Group**

Let G be a group and $\mu, \nu \in F(G)$. In [3], [10] and [7,8] three equivalence relations are defined respectively as follows:

i) We say that μ is equivalent to ν, written as $\mu \approx \nu$, if $F_\mu = F_\nu$.

ii) We say that μ is equivalent to ν, written as $\mu \sim \nu$, if we have

$$\mu(x) > \mu(y) \Leftrightarrow \nu(x) > \nu(y),\ for\ all\ x, y \in G,$$

and

$$\mu(x) = 0 \Leftrightarrow \nu(x) = 0,\ for\ all\ x \in G.$$

Note that the condition "$\mu(x) = 0$ holds if and only if $\nu(x) = 0$" simply says that the supports of μ and ν are equal.

iii) We say that μ is equivalent to ν, written as $\mu \simeq_t \nu$, if there exists an isomorphism f from suppμ to suppν, such that for all $x, y \in$ suppμ we have

$$\mu(x) > \mu(y) \Leftrightarrow \nu(f(x)) > \nu(f(y)).$$

Theorem 3.1. [6] *Let G be a group and $\mu, \nu \in F(G)$. Then $\mu \sim \nu$ if and only if we have $F_\mu = F_\nu$, and* $\mathrm{Supp}\mu = \mathrm{Supp}\nu$.

Definition 3.2. Let G be a group and $\mu, \nu \in F(G)$. Then μ is equivalent to ν, written as $\mu \simeq_k \nu$, if there exist a one to one and onto function f from F_μ to F_ν such that for all $\mu_t \in F_\mu$, $\mu_t \cong f(\mu_t)$.

Moreover for an arbitrary group G and $\mu \in F(G)$, we can define the set

$$F_\mu^* = \{\mu_t | t \in \mathrm{Im}\mu - \{0\}\}.$$

Lemma 3.3. *Let G be a group, $\mu, \nu \in F(G)$ and $\mu \simeq_t \nu$. Then there exist a one to one and onto corresponding $f^* : F_\mu^* \to F_\nu^*$ such that for all $\mu_\alpha \in F_\mu^*$, we have $\mu_\alpha \cong f^*(\mu_\alpha)$.*

Proof. Since $\mu \simeq_t \nu$, there exists an isomorphism $f : \mathrm{supp}\mu \to \mathrm{supp}\nu$ such that

$$\forall x, y \in \mathrm{supp}\mu,\ \mu(\mathrm{x}) > \mu(\mathrm{y}) \Leftrightarrow \nu(\mathrm{f(x)}) > \nu(\mathrm{f(y)})$$

We define a one to one and onto function $f^* : F_\mu^* \to F_\nu^*$ such that $f^*(\mu_\alpha) = f(\mu_\alpha)$. Also the map $f_{|\mu_\alpha}$ from μ_α to $f(\mu_\alpha)$ is a group isomorphism. Therefore, $\mu_\alpha \cong f_{|\mu_\alpha}(\mu_\alpha) = f^*(\mu_\alpha)$. □

Theorem 3.4. *Let G be a finite group and $\mu \simeq_t \nu$, where $\mu, \nu \in F(G)$, then $\mu \simeq_k \nu$.*

Proof. If $0 \notin \mathrm{Im}\mu$, since $\mu \simeq_t \nu$, by the above lemma, there is a one to one function $f^* : F_\mu^* \to F_\nu^*$ such that for all $\alpha \in \mathrm{Im}\mu$, $\mu_\alpha = f^*(\mu_\alpha)$. If $0 \in \mathrm{Im}\nu$, and we set $\alpha = min\{\mu(x)|\mu(x) > 0\}$, then $G = \mu_\alpha$, but if $\mu_\alpha \in F_\mu^*$, since G is finite, then $f^*(\mu_\alpha) = G$ and so $f(\mu_\alpha) = G$. We know that there exists an element a in G such that $\mu(a) = \alpha$, so $f(\mu_\alpha) = \nu_{\beta_\alpha}$ where $\beta_\alpha = \nu(f(a))$, hence $\beta_\alpha > 0$, and since $\nu_{\beta_\alpha} \subset \nu_0$ we obtain $G \subset G$, which is a contradiction, therefore $0 \notin \mathrm{Im}\nu$. We have $F_\mu^* = F_\mu$ and $F_\nu^* = F_\nu$, so $f^* : F_\mu \to F_\nu$ is a one to one and onto function such that for all $\mu_\alpha \in F_\mu$, we have $\mu_\alpha \simeq f^*(\mu_\alpha)$, therefore $\mu \simeq_k \nu$.
If $0 \in \mathrm{Im}\mu$, then $0 \in \mathrm{Im}\nu$, and we have the function $f : F_\mu \to F_\nu$ such that

$$f(\mu_\alpha) = \begin{cases} f^*(\mu_\alpha) & \alpha \in \mathrm{Im}\mu - \{0\} \\ \nu_0 & \alpha = 0 \end{cases}$$

Then f is a one to one and onto function and for all $\alpha \in \mathrm{Im}\mu$, we have $\mu_\alpha \cong f(\mu_\alpha)$, hence $\mu \simeq_k \nu$. □

Corollary 3.5. *Let G be a finite group and $\mu \simeq_t \nu$, for $\mu, \nu \in F(G)$, then we will obtain* $|\mathrm{Im}\mu| = |\mathrm{Im}\nu|$.

Proof. Since $\mu \simeq_t \nu$ and by Theorem 3.4, $\mu \simeq_k \nu$, we can say that there exists a one to one and onto function from F_μ to F_ν, i.e., $|F_\mu| = |F_\nu|$, but since $|F_\mu| = |\mathrm{Im}\mu|$ and $|F_\nu| = |\mathrm{Im}\nu|$, we have $|\mathrm{Im}\mu| = |\mathrm{Im}\nu|$. □

Suppose G is a group and $\mu \in F(G)$. Let us call $\mu(e)$ the tip of μ.

Definition 3.6. [12] Let $\mu \in F(G)$ and $\nu \in F(G')$ with the tips α and β, respectively. Then μ and ν are said to be isomorphic fuzzy groups, if there exists a one to one correspondence f form Imμ onto Imν such that:

(1) If $\tau_1 < \tau_2 \in \mathrm{Im}\mu$, then $f(\tau_1) < f(\tau_2)$,
(2) $\mu^{-1}[\gamma, \alpha] \approx \nu^{-1}[f(\gamma), \beta], for\ all\ \gamma \in \mathrm{Im}\mu$,
(3) $\mu^{-1}(0, \alpha] \approx \nu^{-1}(0, \beta]$.

Note. The Definition 3.2 is the same as Definition 3.6 ignoring the third condition in the Definition 3.6. Let $F_1(G)$ be the set of all fuzzy subgroups μ of group G such that $\mu(e) = 1$.

Now if $\sim_R$ be an equivalence relation on $F_1(G)$, then we can specify the set $\{\nu \in F_1(G) | \nu \sim_{R} \mu\}$ by $\frac{\mu}{\sim_R}$ and also the set $\{\frac{\mu}{\sim_R} | \mu \in F_1(G)\}$ by $\frac{F_1(G)}{\sim_R}$.
In addition the number of equivalence classes $\approx$ on $F_1(G)$ and the number of equivalence classes $\sim$ on $F_1(G)$ are denoted by $r(G)$ and $s(G)$ respectively. Also we denote the number of the equivalence classes $\cong$ on $F_1(G)$ by $I(G)$.

Theorem 3.7. *For any finite group G, we have $r(G) = \frac{s(G)+1}{2}$.*

Proof. Let $A_s(G) = \{\frac{\mu}{\sim} | \mathrm{supp}\mu = G\}$, and suppose $\frac{\mu}{\approx} \in \frac{F(G)}{\approx}$. Since G is finite, we have $\mathrm{Im}\mu = \{1, \alpha_1, \cdots, \alpha_{\mathrm{r}}\}$, where $1 > \alpha_1 > \cdots > \alpha_r \geq 0$. If $\alpha_r \neq 0$, then we define the fuzzy subgroup ν_μ as $\nu_\mu : G \to [0, 1]$, such that $\nu_\mu(x) = \mu(x)$. If $\alpha_r = 0$, then we define the fuzzy subgroup ν_μ as

$$\nu_\mu : G \to [0, 1]$$

$$\nu_\mu(x) = \begin{cases} 1 & x \in \mu_1 \\ \alpha_1 & x \in \mu_{\alpha_1} - \mu_1 \\ \vdots & \vdots \\ \alpha_{r-1} & x \in \mu_{\alpha_{r-1}} - \mu_{\alpha_{r-2}} \\ \frac{\alpha_{r-1}}{2} & x \in \mu_{\alpha_r} - \mu_{\alpha_{r-1}} \end{cases}$$

Hence the function $f : \frac{F_1(G)}{\approx} \to A_s(G)$ which is defined by $f(\frac{\mu}{\approx}) = \frac{\nu_\mu}{\sim}$ is a bijection function, thus $|\frac{F_1(G)}{\approx}| = |A_s(G)|$. □

Theorem 3.8. *Let G be a finite cyclic group and μ and $\nu \in F_1(G)$, Then*

(1) $\mu \sim \nu \Leftrightarrow \mu \simeq_t \nu$
(2) $\mu \simeq_k \nu \Leftrightarrow \mu \approx \nu$
(3) $\mu \simeq_t \nu \Leftrightarrow \mu \cong \nu$.

Proof. (1) From $\mu \sim \nu$, we have $\mu \simeq_t \nu$. Now let us suppose $\mu \cong_t \nu$, then by Theorem 3.4, there exists a bijection function f from F_μ to F_ν such that for all $\mu_\alpha \in F_\mu$ we have $\mu_\alpha \cong f(\mu_\alpha)$ *and* supp$\mu \cong$ suppν. Moreover since G is a cyclic group, there is only one subgroup of order d such that $d|o(G)$. Therefore, $\mu_\alpha = f(\mu_\alpha)$ for all $\mu_\alpha \in f_\mu$, and also supp$\mu =$ suppν. Hence we will have $\mu \sim \nu$.
Proofs of (2) and (3) are similar to (1). □

Corollary 3.9. *Let G be a finite cyclic group. If* $|\frac{F_1(G)}{\simeq_t}| = t(G)$ *and* $|\frac{F_1(G)}{\simeq_K}| = k(G)$*, then* $I(G) = s(G) = t(G) = 2k(G) - 1$*.*

4. Counting the Number of Distinct Fuzzy Subgroups of a Finite Group

Two fuzzy subgroups $\mu, \nu \in F(G)$ will be called distinct if $\mu \not\sim \nu$. In this section, we determine the number of fuzzy subgroups of finite groups with respect to the equivalence classes $\sim$ on $F_1(G)$.

Theorem 4.1. *Let G be a finite group. The number of distinct fuzzy subgroups of G such that their supports are exactly G, will be* $\frac{s(G)+1}{2}$*.*

Proof. Let
$$U(G) = \{\frac{\mu}{\sim} | \mu \neq \mu^*, \mu \in F_1(G), \mathrm{supp}\mu = G\}$$
and
$$V(G) = \{\frac{\mu}{\sim} | \mu \in F_1(G), \mathrm{supp}\mu \subset G\}$$
Where μ^* is a fuzzy subgroup of G and $\mu^*(x) = 1$ for all $x \in G$. Since G is a finite group, we define $\frac{\mu}{\sim}$ as follows:
$$\frac{\mu}{\sim} = \frac{(\overbrace{1\cdots 1}^{n'_0}\overbrace{\lambda_1\cdots\lambda_1}^{n'_1}\cdots\overbrace{\lambda_r\cdots\lambda_r}^{n'_r})\varphi}{\sim}$$
where,
$$\mathrm{Im}\mu = \{1, \lambda_1, \cdots, \lambda_\mathrm{r}\}\text{ ‘},\ 1 > \lambda_1 > \cdots > \lambda_\mathrm{r} > 0$$
and
$$\begin{aligned}\varphi : G_0 = (e) \subset G_1 \subset \cdots \subset G_{n_0} = \mu_1 \\ \subset G_{n_0+1} \subset \cdots \subset G_{n_1} = \mu_{\lambda_1} \\ \vdots \\ \subset G_{n_{(r-1)}+1} \subset \cdots \subset G_{n_r} = \mu_{\lambda_r} = G\end{aligned}$$
and $n_0 + n_1 + \cdots + n_r = n$, $n'_0 = n_0$, $n'_i = n_i - (n_1 + \cdots + n_{i-1})$.
We define the map
$$f : U(G) \to V(G)$$
such that
$$f(\frac{\mu}{\sim}) = \frac{(\overbrace{1\cdots 1}^{n'_0}\overbrace{\lambda_1\cdots\lambda_1}^{n'_1}\cdots\overbrace{\lambda_{r-1}\cdots\lambda_{r-1}}^{n'_{r-1}}\overbrace{0\cdots 0}^{n'_r})\varphi}{\sim}$$
It is easy to see that f is bijection. So
$$|U(G)| = |V(G)|$$
and
$$s(G) = |U(G)| + |V(G)| + 1$$

therefore

$$s(G) = 2|U(G)| + 1.$$

Thus

$$|U(G)| = |V(G)| = \frac{s(G) - 1}{2}$$

and hence

$$U(G) + 1 = \frac{s(G) + 1}{2}.$$

□

Corollary 4.2. *Let G be a finite group, then the number of the distinct fuzzy subgroups of G which their supports are G, will be equal to $\frac{S(G)-1}{2}$.*

Theorem 4.3. *Let G be a finite group and H be a subgroup of G. Then the number of distinct fuzzy subgroups of G which their supports are exactly H will be equal to $\frac{s(H)+1}{2}$.*

Proof. We can easily see that the number of distinct fuzzy subgroups of the group G which their supports are exactly H is equal to the number of distinct fuzzy subgroups of H which their supports are exactly H. According to the Theorem 4.1, this number is equal to $\frac{s(H)+1}{2}$. □

Notation. we denote $\frac{s(H)+1}{2}$ with $s'(H)$, when $H \leq G$.

Example 4.4. Let G be a Kleins 4-group, then $s(G) = 15$ and $r(G) = 8$.

We know that the group G has the following flags:

$$\varphi_1 : \{e\} \subset \{e, a\} \subset \{e, a, b, ab\} = G,$$

$$\varphi_2 : \{e\} \subset \{e, b\} \subset \{e, a, b, ab\} = G,$$

and

$$\varphi_3 : \{e\} \subset \{e, ab\} \subset \{e, a, b, ab\} = G.$$

Also we have $\frac{s(G)-1}{2} = 1 + s'(\{e, a\}) + s'(\{e, b\}) + s'(\{e, ab\}) = 1 + 2 + 2 + 2$, hence, $s(G) = 15$ and $r(G) = \frac{15+1}{2} = 8$.

Example 4.5. Let Cp be the cyclic group of order p, where p is a prime number. Then for $G = C_2 \times C_2 \times C_2$ we have $s(G) = 143$ and $r(G) = 72$.

Assume that $G = C_2 \times C_2 \times C_2$ is a group of direct product of three cyclic groups with generator a, b and c respectively. Since the group G has entirely 16 subgroups, we can show them as

$$H_1 =< a >, H_2 =< ab >, H_3 =< bc >, H_4 =< abc >, H_5 =< ac >, H_6 =< b >,$$

$$H_7 =< c >, K_1 =< a, b >, K_2 =< ab, c >, K_3 =< a, bc >, K_4 =< ab, bc >,$$

$$K_5 =< ac, b >, K_6 =< b, c >, K_7 =< a, c > .$$

By Corollary 4.2, we will have

$$\frac{s(G)-1}{2}=1+\sum_{i=1}^{7}s'(H_i)+\sum_{i=1}^{7}s'(K_i)$$

and for $1\leq i\leq 7$, $s(H_i)=3$, and $s(K_i)=15$ we will have:

$$\frac{s(G)-1}{2}=\sum_{i=1}^{7}\frac{s(H_i)+1}{2}+\sum_{i=1}^{7}\frac{s(K_i)+1}{2}+1.$$

Thus $s(G)=143$ and $r(G)=72$.

Theorem 4.6. *Let G be a finite group of order p^n. Then we have $s(G)\geq 2^{n+1}-1$ and $r(G)\geq 2^n$.*

Proof. We prove by induction on n. For $n=1$, it is obvious. Suppose that for all numbers $m<n$, the theorem holds. Since $|G|=p^n$, G is a p-group, so we will have the following series for G:

$$G_0=(e)\subset G_1\subset G_2\cdots\subset G_{n-1}\subset G_n=G$$

and $|G_i|=p^i$, for $i=1,2,\cdots,n-1$. According to the induction's hypothesis we will have $s(G_i)\geq 2^{i+1}-1$. Now the number of fuzzy subgroups of G which their supports are exactly G_i, is equal to $\frac{s(G_i)+1}{2}$. Also the number of distinct fuzzy subgroups of G which their supports are a proper subset of G will be equal to $\frac{s(G)-1}{2}$. Then

$$\frac{s(G)-1}{2}\geq 1+2+2^2+\cdots+2^{n-1}=2^n-1.$$

So $s(G)\geq 2^{n+1}-1$ and the proof is completed. □

Theorem 4.7. [15] *Let G be a p-group and $|G|=p^n$, then G is a cyclic group if and only if:*

$$|\frac{F(G)}{\approx}|=2^n.$$

.

Corollary 4.8. *Let G be a cyclic group of order p^n, then we have $r(G)=2^n$, $s(G)=2^{n+1}-1$ and $I(G)=2^{n+1}-1$.*

Proof. It is straight forward. □

Theorem 4.9. [11] *Let G be a cyclic group and $|G|=p^nq^m$, then:*

$$|\frac{F(G)}{\sim}|=2^{m+n+1}\sum_{r=0}^{m}2^{-r}\binom{n}{n-r}\binom{m}{r}-1,\ (m\leq n)$$

Corollary 4.10. *Let $G=Z_{p^nq^m}$, where p and q are primes numbers and $p\neq q$, then we will have*

(1) $I(G)=s(G)=2^{m+n+1}\sum_{r=0}^{m}2^{-r}\binom{n}{n-r}\binom{m}{r}-1$; *for* $m\leq n$,
(2) $I(G)=s(G)=2^{m+n+1}\sum_{r=0}^{n}2^{-r}\binom{m}{m-r}\binom{n}{r}-1$; *for* $m\geq n$.

Proof. It is straight forward. □

5. Some Probability Problems in Fuzzy Subgroups

Suppose G is a finite group of order n and H is a subgroup of G. We define $F_1^\star(G)$ as the set of all non-isomorphic fuzzy subgroups such that their images under μ are A_r or B_s, where,

$$A_r = \{1, \frac{1}{2}, \cdots, \frac{1}{r}\},\ 1 \leq r \leq n$$

and

$$B_s = \{0, 1, \frac{1}{2}, \cdots, \frac{1}{s-1}\},\ 2 \leq s \leq n-1.$$

Then the number

$$\frac{|\{(\mu,\nu) \in F_1^\star(G) \times F_1^\star(G) | \mu \circ \nu = \nu \circ \mu, \text{supp}\mu = \text{supp}\nu = H\}|}{|F_1^\star(G) \times F_1^\star(G)|}$$

is called the probability of commuting of two non-isomorphic fuzzy subgroups of a group G, which their supports are exactly H and we denote it by $P_F(G_H)$.
Now if $H = G$, then we show this probability with $P_F(G)$ instead of $P_F(G_H)$. Let $\widetilde{F_1}(G)$ be the set of the distinct fuzzy subgroups of G such that their images under μ are A_r or B_s , then the number

$$\frac{|\{(\mu,\nu) \in \widetilde{F_1}(G) \times \widetilde{F_1}(G) | \mu \circ \nu = \nu \circ \mu, supp\mu = supp\nu = G\}|}{|\widetilde{F_1}(G) \times \widetilde{F_1}(G)|}$$

is called the probability of commuting of two distinct fuzzy subgroups of a group G and we denote it by $\widetilde{P_F}(G)$

Theorem 5.1. [9] *Let G be an abelian group, and μ and ν are two fuzzy subgroups of G. Then $\mu \circ \nu = \nu \circ \mu$.*

Theorem 5.2. *Let G be a finite p-group of order p^n, then:*

$$\widetilde{P_F}(G) \leq \frac{4^n}{4^{n+1} - 2^{n+2} + 1}.$$

Proof. We know that
$\{(\mu,\nu) \in \widetilde{F_1}(G) \times \widetilde{F_1}(G) | \mu \circ \nu = \nu \circ \mu, supp\mu = supp\nu = G\}$ is a subset of $\{(\mu,\nu) \in \widetilde{F_1}(G) \times \widetilde{F_1}(G) | supp\mu = supp\nu = G\}$.
Thus we can easily see that

$$\widetilde{P_F}(G) \leq \frac{1}{4}(1 + \frac{1}{s(G)})^2$$

and by Theorem 4.6, $s(G) \geq 2^{n+1} - 1$, then

$$\widetilde{P_F}(G) \leq \frac{4^n}{4^{n+1} - 2^{n+2} + 1}.$$

□

Theorem 5.3. *Let G be a finite cyclic group and H be a subgroup of G. Then the probability of commuting of two non-isomorphic fuzzy subgroups which their supports are exactly H, will be equal to $(\frac{I(H)+1}{2I(G)})^2$.*

Proof. Since G is a finite cyclic group, then by Theorems 3.8 and 5.1, $pF(G_H)$ will be equal to:

$$\frac{|\{(\frac{\mu}{\sim}, \frac{\nu}{\sim}) \in \frac{F_1(G)}{\sim} \times \frac{F_1(G)}{\sim} | \mathrm{supp}\mu = \mathrm{supp}\nu = H\}|}{|\frac{F_1(G)}{\sim} \times \frac{F_1(G)}{\sim}|}$$

and by Theorem 4.3, it will be equal to:

$$pF(G_H) = \frac{(\frac{s(H)+1}{2})^2}{(I(G))^2} = (\frac{I(H)+1}{2I(G)})^2.$$

□

Theorem 5.4. *If G be a finite abelian group, then:*

$$\widetilde{P_F}(G) = (\frac{s(G)+1}{2s(G)})^2$$

.

Example 5.5. Let G be a Kleins 4-group, then: $\widetilde{P_F}(G) = \frac{64}{225}$.

According to example, 4.4 we have $s(G) = 15$, and Theorem 5.4, gives us $\widetilde{P_F}(G) = \frac{64}{225}$.

Example 5.6. Let $G = C_2 \times C_2 \times C_2$, then we will have $\widetilde{P_F}(G) = \frac{5184}{20449}$.

By Example 4.5 , we have $s(G) = 143$, and from Theorem 5.4, we obtain $\widetilde{P_F}(G) = \frac{5184}{20449}$.

Theorem 5.7. *Let G be a finite p-group of order p^2, then $\widetilde{P_F}(G) = \frac{16}{49}$ or $(\frac{2p+4}{4p+7})^2$.*

Proof. We know that any finite p-group of order p^2 is either isomorphic to C_{p^2} or to $C_p \times C_p$. If $G \simeq C_{p^2}$, then by Corollary 4.8 , we have $s(G) = 7$ and if $G \simeq C_p \times C_p$, then according to the Proposition 3.6, of [10], we will have $s(G) = 4p + 7$. So by Theorem 5.4, the proof is completed. □

Theorem 5.8. *Let G' be a commutator subgroup of the dihedral group of order $2p^n$, then*

$$P_F(G') = \widetilde{P_F}(G') = \begin{cases} (\frac{2^{n-1}}{2^n-1})^2 & P = 2, \\ (\frac{2^n}{2^{n+1}-1})^2 & P \neq 2, \end{cases}$$

Proof. If $p = 2$, then according to [5], we have $G' \simeq Z_{2^{n-1}}$ and if $p \neq 2$, then $G' \simeq Z_{p^n}$, now we obtain the desired result applying Theorems 5.3 and 5.4. □

Corollary 5.9. *Let G be a finite p-group of order p^3, then $\widetilde{P_F}(G') = 1$ or $\frac{4}{9}$.*

Proof. If G is an abelian group, then $G' = 1$ and $\widetilde{P_F}(G') = 1$. Otherwise, by [2], G' would be a cyclic group of order p, hence $\widetilde{P_F}(G') = \frac{4}{9}$. □

Example 5.10. Let G be a finite abelian p-group of order 21, then $\widetilde{P_F}(G) = \frac{36}{121}$.

Suppose that m_1, m_2 and n_1, n_2 are natural numbers, such that $m_1 \geq m_2$ and $n_1 \geq n_2$. Then we denote the number

$$(\frac{2^{n_1+n_2} \sum_{r=0}^{n_2} 2^{-r} \binom{n_1}{n_1-r} \binom{n_2}{r}}{2^{m_1+m_2+1} \sum_{r=0}^{m_2} 2^{-r} \binom{m_1}{m_1-r} \binom{m_2}{r} - 1})^2$$

by $p(n_1, n_2, m_1, m_2)$.

Corollary 5.11. *Suppose that p, q are prime numbers, $m, n, i, j \in \mathbb{N}$ and $G = Z_{p^n q^m}$, then the probability of commuting two non-isomorphic fuzzy subgroups, which their supports are subgroups of order $p^i q^j$ in G, will be equal to*

(1) $p(n, m, j, i)$ *if* $m \leq n, i \leq j$,
(2) $p(n, m, i, j)$ *if* $m \leq n, i \geq j$,
(3) $p(m, n, j, i)$ *if* $n \leq m, i \leq j$, *and*
(4) $p(m, n, i, j)$ *if* $n \leq m, i \geq j$.

Proof. It is obvious using Theorem 5.3 and Corollaries 9.9 and 4.10. □

Theorem 5.12. *Let G be a p-group of order p^n and suppose there is only one subgroup of order k for any $k|p^n$. Then the probability of commuting two non-isomorphic fuzzy subgroups, which their supports are subgroups of order p^i in G, will be equal to*

$$(\frac{2^i}{2^{n+1} - 1})^2.$$

Proof. According to [1], G is a cyclic group and it would lead to the desired result using Theorem 5.3 and Corollary 4.8. □

Acknowledgment. The authors would like to thank the referee for evaluation and improving some results.

References

[1] P. B. Bhattacharya, S. K. Jain and S. R. Nagpaul, *Basic abstarct algebra*, Cambridge University, 1986.
[2] C. Degang and J. Jiashang, *Some notes on equivalence fuzzy sets and fuzzy subgroups*, Fuzzy Sets and Systems, **152** (2005), 403-409.
[3] V. N. Dixit, R. Kumar and N. Ajmal, *Level subgroups and union of fuzzy subgroups*, Fuzzy Sets and Systems, **37** (1990), 359-371.
[4] W. H. Gustafson, *What is the probability that two group elements commute?*, Amer. Math. Monthly, **80** (1973), 1031-1034.
[5] T. W. Hungerford, *Algebra*, Springer, New york, 1974.
[6] A. Jain, *Fuzzy subgroup and certain equivalence relations*, Iranian Journal of Fuzzy Systems, **3(2)** (2006), 75-91.
[7] M. Mashinchi and M. Mukaidonon, *A classification of fuzzy subgroups*, Ninth Fuzzy System Symposium, Sapporo, Japan, (1992), 649-652.
[8] M. Mashinchi and M. Mukaidonon, *On fuzzy subgroups classification*, Reserch Report of Meiji University Japan, **9(65)** (1993), 31-36.
[9] J. N. Mordeson, K. R. Bhutani and A. Rosenfeld, *Fuzzy group theory*, Springer-verlag, 2005.

[10] V. Murali and B. B. Makamba, *On an equivalence of fuzzy subgroups I*, Fuzzy Sets and Systems, **123** (2001), 259-264.
[11] V. Murali and B. B. Makamba, *Counting the number of fuzzy subgroups of an abelian group of order $p^n q^m$*, Fuzzy Sets and Systems, **144** (2004), 459-470.
[12] S. Ray, *Isomorphism fuzzy groups*, Fuzzy Sets and Systems, **50** (1992), 201-207.
[13] G. J. Sherman, *What is the probability, an automorphism fixes a group element*, American Mathematics Montly, **82** (1975), 261-264.
[14] M. Tărnăuceanu and L. Bentea, *On the number of fuzzy subgroups of finite abelian groups*, Fuzzy Sets and Systems, **159** (2008), 1084-1096.
[15] Y. Zhang and K. Zou, *A note on an equivalence relation on fuzzy subgroups*, Fuzzy Sets and Systems, **95** (1998), 243-247.

Ali Iranmanesh*, Department of Mathematics, Tarbiat Modares University, P.O. Box: 14115-137, Tehran, Iran
E-mail address: iranmana@modares.ac.ir

Hossein Naraghi, Tarbiat Modares University, P.O. Box: 14115-137, Tehran, Iran
E-mail address: h.naraghi56@gmail.com

*Corresponding author