# A NEW WAY TO FUZZY $h$-IDEALS OF HEMIRINGS

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Abstract. By means of a kind of new idea, we consider the $(\in, \in\vee q)$-fuzzy $h$-ideals of a hemiring. First, the concepts of $(\in, \in\vee q)$-fuzzy left(right) $h$-ideals of a hemiring are provided and some related properties are investigated. Then, a kind of quotient hemiring of a hemiring by an $(\in, \in\vee q)$-fuzzy $h$-ideal is presented and studied. Moreover, the notions of generalized $\varphi$-compatible $(\in, \in\vee q)$-fuzzy left(right) $h$-ideals of a hemiring are introduced and some properties of them are provided. Finally, the relationships among $(\in, \in\vee q)$-fuzzy $h$-ideals, quotient hemirings and homomorphisms are explored and several homomorphism theorems are provided.

## 1. Introduction

Semirings which are regarded as a generalization of rings and bounded distributive lattice have been found useful in solving problems in different disciplines of applied mathematics and information sciences, since the structure of a semiring provides an algebraic framework for modelling and studying the key factors in these applied areas. By a hemiring, we mean a special semiring with a zero and with a commutative addition. In applications, hemirings are useful in automata and formal languages [4, 13].

It is well known that ideals of semirings play a central role in the structure theory and are useful for many purposes. However, they do not in general coincide with the usual ring ideals and, for this reason, their use is somewhat limited in trying to obtain analogues of ring theorems for semirings. Indeed, many results in rings apparently have no analogues in semirings using only ideals. In order to overcome this deficiency, Henriksen [5] defined a more restricted class of ideals in semirings, which is called the class of $k$-ideals, with the property that if the semiring $S$ is a ring then a complex in $S$ is a $k$-ideal if and only if it is a ring ideal. A still more restricted class of ideals in hemirings has been given by Iizuka [7]. According to Iizula's definition, an ideal in any additively commutative semiring $S$ can be given which coincides with a ring ideal provided $S$ is a hemiring, and it is called $h$-ideal. The properties of $h$-ideals and $k$-ideals of hemirings were thoroughly investigated by La Torre [9] and by using the $h$-ideals and $k$-ideals, La Torre established some analogous ring theorems for hemirings. The general properties of fuzzy $h$-ideals have been considered by Huang, Jun, Zhan, Yin and others. The reader is refereed to [6, 8, 10, 14, 16].

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After the introduction of fuzzy groups by Rosenfeld [12], there have been a number of generalizations of this fundamental concept. Using the notion "belongingness (∈)" and "quasi-coincidence (q)" of fuzzy points with fuzzy sets, the concept of $(\alpha, \beta)$-fuzzy subgroups where $\alpha, \beta$ are any two of $\{\in, q, \in \vee q, \in \wedge q\}$ with $\alpha \neq \in \wedge q$ was introduced by Bhakat and Das [1] in 1992, in which the $(\in, \in \vee q)$-fuzzy subgroup is an important and useful generalization of Rosenfeld's fuzzy subgroup. It is now natural to investigate similar types of generalizations of the existing fuzzy subsystems with other algebraic structures. Davvaz and Corsini [2] introduced the notion of $(\alpha, \beta)$-fuzzy $H_v$-ideals of $H_v$-rings and investigated especially the $(\in, \in \vee q)$-fuzzy $H_v$-ideals of $H_v$-rings. Zhan et al. [17] introduced and studied the notions of $(\in, \in \vee q)$-fuzzy $p$-ideals, $(\in, \in \vee q)$-fuzzy $q$-ideals and $(\in, \in \vee q)$-fuzzy $a$-ideals in BCI-algebras . Recently, Ma et al. [11] introduced the concept of an $(\in, \in \vee q)$-fuzzy $h$-bi-ideal (resp., $h$-quasi-ideal) of a hemiring and investigated some of its properties. Dudek et al. [3] studied $(\in, \in \vee q)$-prime and semiprime ideals of hemirings. Yin et al. [15] presented and investigated the concept of $(\in, \in \vee q)$-fuzzy $h$-ideals and $(\in, \in \vee q)$-fuzzy $h$-interior ideals of a hemiring by a new ideal. In particular, they considered the characterization of $h$-semisimple hemirings in terms of these ideals.

The rest of this paper is organized as follows. In Section 2, we summarize some basic concepts which will be used throughout the paper. In Section 3, by a new idea, we redefine the concept of left(right) $(\in, \in \vee q)$-fuzzy $h$-ideals of hemirings and investigate some of their properties. Two kinds of quotient hemirings of a hemiring by an $(\in, \in \vee q)$-fuzzy $h$-ideal are also introduced and studied. In Section 4, the notion of generalized $\varphi$-compatible $(\in, \in \vee q)$-fuzzy left(right) $h$-ideals of a hemiring is introduced and the relationships among $(\in, \in \vee q)$-fuzzy $h$-ideals, quotient hemirings and homomorphisms are explored. Several homomorphism theorems are provided. Some conclusions are given in the last Section.

## 2. **Preliminaries**

A semiring is an algebraic system $(S, +, \cdot)$ consisting of a non-empty set $S$ together with two binary operations on $S$ called addition and multiplication (denoted in the usual manner) such that $(S, +)$ and $(S, \cdot)$ are semigroups and the following distributive laws

$$a \cdot (b + c) = a \cdot b + b \cdot c \text{ and } (a + b) \cdot c = a \cdot c + b \cdot c$$

are satisfied for all $a, b, c \in S$.

By zero of a semiring $(S, +, \cdot)$ we mean an element $0 \in S$ such that $0 \cdot x = x \cdot 0 = 0$ and $0 + x = x + 0 = x$ for all $x \in S$. A semiring $(S, +, \cdot)$ with zero is called a hemiring if $(S, +)$ is commutative. A hemiring is called additive cancellable if $x + z = y + z$ implies $x = y$ for all $x, y, z \in S$. For the sake of simplicity, we shall omit the symbol " · ", writing $ab$ for $a \cdot b \, (a, b \in S)$.

A subset $A$ in a hemiring $S$ is called a left(resp., right) ideal of $S$ if $A$ is closed under addition and $SA \subseteq A$(resp., $AS \subseteq A$). Further, $A$ is called an ideal of $S$ if it is both a left ideal and a right ideal of $S$.

A left ideal $A$ of $S$ is called a left $h$-ideal if $x, z \in S, a, b \in A$, and $x+a+z = b+z$ implies $x \in A$. Right $h$-ideals, h-ideals are defined similarly.

We now recall some fuzzy logic concepts. A fuzzy subset $\mu$ in a hemiring $S$ is defined as a mapping from $S$ to $[0,1]$, where $[0,1]$ is the usual interval of real numbers. We denote by $\mathscr{F}(S)$ the set of all fuzzy subsets in $S$. A fuzzy subset $\mu$ in $S$ of the form

$$\mu(y) = \begin{cases} r(\neq 0) & \text{if } y = x, \\ 0 & \text{otherwise} \end{cases}$$

is said to be a fuzzy point with support $x$ and value $r$ and is denoted by $x_r$. A fuzzy point $x_r$ is said to belong to (resp., be quasi-coincident with) a fuzzy set $\mu$, written as $x_r \in \mu$ (resp., $x_r \, q \, \mu$) if $\mu(x) \geq r$ (resp., $\mu(x) + r > 1$). If $\mu(x) \geq r$ or $\mu(x) + r > 1$, then we write $x_r \in\!\vee q \ \mu$.

For $\mu \in \mathscr{F}(S)$ and $r \in (0,1]$. The sets $\mu_r = \{x \in S | \mu(x) \geq r\}$ and $[\mu]_r = \{x \in S | x_r \in\!\vee q \mu\}$ are called a level subset of $\mu$ and an $\in\!\vee q$-level subset of $\mu$, respectively. And $\mu$ is said to have the sup-property if for any non-empty subset $A$ in $S$, there exists $x \in A$ such that $\mu(x) = \bigvee_{y \in A} \mu(y)$.

Next we define a new ordering relation " $\subseteq \vee q$" on $\mathscr{F}(S)$, which is called the fuzzy inclusion or quasi-coincidence relation, as follows:

For any $\mu, \nu \in \mathscr{F}(S)$, $\mu \subseteq \vee q \, \nu$ if and only if $x_r \in \mu$ implies $x_r \in\!\vee q \ \nu$ for all $x \in S$ and $r \in (0,1]$.

And we define a relation " $\approx$ " on $\mathscr{F}(S)$ as follows:

For any $\mu, \nu \in \mathscr{F}(S)$, $\mu \approx \nu$ if and only if $\mu \subseteq \vee q \ \nu$ and $\nu \subseteq \vee q \ \mu$.

In the sequel, unless otherwise stated, $M(r_1, r_2, \cdots, r_n)$, where $n$ is a positive integer, will denote $r_1 \wedge r_2 \wedge \cdots \wedge r_n$ for all $r_1, r_2, \cdots, r_n \in [0,1]$, $\overline{\in\!\vee q}$ means $\in\!\vee q$ does not hold and $\overline{\subseteq \vee q}$ implies $\subseteq \vee q$ is not true.

**Lemma 2.1.** [15] *Let $\mu, \nu \in \mathscr{F}(S)$. Then $\mu \subseteq \vee q \, \nu$ if and only if $\nu(x) \geq M(\mu(x), 0.5)$ for all $x \in S$.*

**Lemma 2.2.** [15] *Let $\mu, \nu, \omega \in \mathscr{F}(S)$ be such that $\mu \subseteq \vee q \ \nu$ and $\nu \subseteq \vee q \ \omega$. Then $\mu \subseteq \vee q \ \omega$.*

Then Lemma 2.1 implies that $\mu \approx \nu$ if and only if $M(\mu(x), 0.5) = M(\nu(x), 0.5)$ for all $x \in S$ and $\mu, \nu \in \mathscr{F}(S)$, and it follows from Lemmas 2.1 and 2.2 that " $\approx$ " is an equivalence relation on $\mathscr{F}(S)$.

Next, we will give the definition of the product and sum of two fuzzy subsets in a hemiring $S$.

**Definition 2.3.** Let $S$ be a hemiring and $\mu, \nu \in \mathscr{F}(S)$. We define the sum and product, denoted by $\mu + \nu$ and $\mu \circ \nu$, of $\mu$ and $\nu$ by

$$(\mu + \nu)(x) = \sup_{x = a+b} M(\mu(a), \nu(b))$$

and

$$(\mu \circ \nu)(x) = \begin{cases} \sup\limits_{x=ab} M(\mu(a), \nu(b)) & \textit{if there exist } a, b \in S \textit{ such that } x = ab, \\ 0 & \textit{otherwise,} \end{cases}$$

respectively, for all $x \in S$. For any $x \in S$, define $x + \mu$ by

$$(x + \mu)(y) = \begin{cases} \sup\limits_{y=x+a} \mu(a) & \textit{if there exists } a \in S \textit{ such that } y = x + a, \\ 0 & \textit{otherwise,} \end{cases}$$

for all $y \in S$.

Note that if $x \in S$ and $\mu \in \mathscr{F}(S)$, then $x + \mu = x_1 + \mu$. The $h$-sum of two fuzzy subsets in a hemiring $S$ is given as follows.

**Definition 2.4.** [15] Let $\mu$ and $\nu$ be fuzzy subsets in a hemiring $S$. Then the $h$-sum of $\mu$ and $\nu$ is defined by

$$(\mu +_h \nu)(x) = \sup_{x+a_1+b_1+z=a_2+b_2+z} M(\mu(a_1), \mu(a_2), \nu(b_1), \nu(b_2))$$

if there exist $a_1, a_2, b_1, b_2, z \in S$ such that $x + a_1 + b_1 + z = a_2 + b_2 + z$. Otherwise, $(\mu +_h \nu)(x) = 0$.

**Example 2.5.** Let $S = \{0, a, b\}$ be a set with an addition operation $(+)$ and a multiplication operation $(\cdot)$ as follows:

| $+$ | $0$ | $a$ | $b$ |
|---|---|---|---|
| $0$ | $0$ | $a$ | $b$ |
| $a$ | $a$ | $0$ | $b$ |
| $b$ | $b$ | $b$ | $0$ |

and

| $\cdot$ | $0$ | $a$ | $b$ |
|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ |
| $a$ | $0$ | $0$ | $0$ |
| $b$ | $0$ | $0$ | $b$ |

Then $S$ is a hemiring. Define two fuzzy subsets $\mu$ and $\nu$ in $S$ by
$\mu(0) = 0.2,\ \mu(a) = 0.5,\ \mu(b) = 0.6$ and $\nu(0) = 0.2,\ \nu(a) = 0.2,\ \nu(b) = 0.6$, respectively. Then routine verification gives that

$$(\mu + \nu)(0) = 0.6,\ (\mu + \nu)(a) = 0.2,\ (\mu + \nu)(b) = 0.5$$

and

$$(\mu +_h \nu)(0) = 0.6,\ (\mu +_h \nu)(a) = 0.6,\ (\mu +_h \nu)(b) = 0.2.$$

Example 2.5 indicates that there is not necessary connection between $\mu + \nu$ and $\mu +_h \nu$. However, in Section 3 we will see that $\mu + \nu \subseteq \vee q \mu +_h \nu$ under certain conditions.

**Lemma 2.6.** *Let $S$ be a hemiring and $\mu_1, \mu_2, \nu_1, \nu_2 \in \mathscr{F}(S)$ such that $\mu_1 \subseteq \vee q\ \mu_2$ and $\nu_1 \subseteq \vee q\ \nu_2$. Then*

(1) $\mu_1 + \nu_1 \subseteq \vee q\ \mu_2 + \nu_2$, $\mu_1 +_h \nu_1 \subseteq \vee q\ \mu_2 +_h \nu_2$ *and* $\mu_1 \circ \nu_1 \subseteq \vee q\ \mu_2 \circ \nu_2$.
(2) $\mu_1 \cap \nu_1 \subseteq \vee q\ \mu_2 \cap \nu_2$.

*Proof.* We only prove $\mu_1 + \nu_1 \subseteq\vee q\ \mu_2 + \nu_2$. The other properties can be similarly proved. For any $x \in S$, since $\mu_1 \subseteq\vee q\ \mu_2$ and $\nu_1 \subseteq\vee q\ \nu_2$, we have

$$\begin{aligned}(\mu_2 + \nu_2)(x) = \sup_{x=a+b} M(\mu_2(a), \nu_2(b)) &\geq \sup_{x=a+b} M(M(\mu_1(a), 0.5), M(\nu_1(b), 0.5)) \\ =M\left(\sup_{x=a+b} M(\mu_1(a), \nu_1(b), 0.5\right) &= M((\mu_1 + \nu_1)(x), 0.5).\end{aligned}$$

Hence $\mu_1 + \nu_1 \subseteq\vee q\ \mu_2 + \nu_2$ by Lemma 2.1. □

Lemma 2.6 indicates that the equivalence relation " $\approx$ " is a congruence relation on $(\mathscr{F}(S), +)$, $(\mathscr{F}(S), +_h)$ and $(\mathscr{F}(S), \circ)$.

**Lemma 2.7.** *Let $S$ be a hemiring and $\mu, \nu, \omega \in \mathscr{F}(S)$. Then*

(1) $\mu + (\nu \cup \omega) = \mu + \nu \cup \mu + \omega$, $(\mu \cup \nu) + \omega = \mu + \omega \cup \nu + \omega$.
(2) $\mu + (\nu \cap \omega) \subseteq \vee q\, \mu + \nu \cap \mu + \omega$, $(\mu \cap \nu) + \omega \subseteq \vee q\, \mu + \omega \cap \nu + \omega$.
(3) $\mu +_h (\nu \cup \omega) = \mu +_h \nu \cup \mu +_h \omega$, $(\mu \cup \nu) +_h \omega = \mu +_h \omega \cup \nu +_h \omega$.
(4) $\mu +_h (\nu \cap \omega) \subseteq \vee q\, \mu +_h \nu \cap \mu +_h \omega$, $(\mu \cap \nu) +_h \omega \subseteq \vee q\, \mu +_h \omega \cap \nu +_h \omega$.
(5) $\mu \circ (\nu \cup \omega) = \mu \circ \nu \cup \mu \circ \omega$, $(\mu \cup \nu) \circ \omega = \mu \circ \omega \cup \nu \circ \omega$.
(6) $\mu \circ (\nu \cap \omega) \subseteq \vee q\, \mu \circ \nu \cap \mu \circ \omega$, $(\mu \cap \nu) \circ \omega \subseteq \vee q\, \mu \circ \omega \cap \nu \circ \omega$.

*Proof.* The proof is similar to that of Lemma 2.6. □

## 3. $(\in, \in\vee q)$-fuzzy $h$-ideals of Hemirings

It is well known that ideal theory plays a fundamental role in the development of hemirings. In this section, using the new ordering relation on $\mathscr{F}(S)$, we define and investigate $(\in, \in \vee q)$-fuzzy left(right) $h$-ideals of hemirings in a different manner compared with [15].

**Definition 3.1.** A fuzzy subset $\mu$ in a hemiring $S$ is called an $(\in, \in \vee q)$-fuzzy left(resp., right) $h$-ideal if it satisfies:

(F1a) $\mu + \mu \subseteq\vee q\ \mu$,
(F2a) $\chi_S \circ \mu \subseteq\vee q\ \mu$(resp., $\mu \circ \chi_S \subseteq\vee q\ \mu$),
(F3a) $x + a + z = b + z, a_r, b_s \in \mu \Rightarrow x_{M(r,s)} \in \vee q\ \mu$ for all $a, b, x, z \in S$ and $r, s \in (0, 1]$.

A fuzzy subset in a hemiring $S$ is called an $(\in, \in\vee q)$-*fuzzy $h$-ideal* of $S$ if it is both an $(\in, \in\vee q)$-fuzzy right $h$-ideal and an $(\in, \in\vee q)$-fuzzy left $h$-ideal of $S$.

Let us first provide some auxiliary lemmas as follows.

**Lemma 3.2.** *Let $S$ be a hemiring and $\mu \in \mathscr{F}(S)$. Then* (F1a) *holds if and only if one of the following conditions holds:*

(F1b) $x_r, y_s \in \mu \Rightarrow (x + y)_{M(r,s)} \in \vee q\, \mu$ *for all $x, y \in S$ and $r, s \in (0, 1]$.*
(F1c) $\mu(x + y) \geq M(\mu(x), \mu(y), 0.5)$ *for all $x, y \in S$.*

*Proof.* (F1a)⇒(F1b) Let $x, y \in S$ and $r, s \in (0, 1]$ be such that $x_r, y_s \in \mu$. Then

$$(\mu+\mu)(x+y)=\bigvee_{x+y=a+b} M(\mu(a),\mu(b))\geq M(\mu(x),\mu(y))\geq M(r,s).$$

Hence $(x+y)_{M(r,s)}\in\mu+\mu$ and so $(x+y)_{M(r,s)}\in\vee q\,\mu$ by (F1a).

(F1b)⇒(F1c) If possible, let $\mu(x+y)<r=M(\mu(x),\mu(y),0.5)$ for some $x,y\in S$. Then $x_r,y_r\in\mu$ and $\mu(x+y)+r<r+r\leq 1$, that is, $(x+y)_r\overline{\in\vee q}\,\mu$, a contradiction. Hence (F1c) is valid.

(F1c)⇒(F1a) For $x_r\in\mu+\mu$, if possible, let $x_r\overline{\in\vee q}\,\mu$. Then $\mu(x)<r$ and $\mu(x)+r\leq 1$, which gives that $\mu(x)<0.5$. If $x=y+z$ for some $y,z\in S$, by (F1c), we have $0.5>\mu(x)\geq M(\mu(y),\mu(z),0.5)$, which implies $\mu(x)\geq M(\mu(y),\mu(z))$. Hence we have

$$r\leq(\mu+\mu)(x)=\sup_{x=a+b} M(\mu(a),\mu(b))\leq\sup_{x=a+b}\mu(x)=\mu(x),$$

a contradiction. Hence (F1a) is satisfied. □

**Lemma 3.3.** *Let $S$ be a hemiring and $\mu\in\mathscr{F}(S)$. Then* (F2a) *holds if and only if one of the following conditions holds:*

(F2b) $x_r\in\mu\Rightarrow(yx)_r\in\vee q\,\mu$ *(resp., $x_r\in\mu\Rightarrow(xy)_r\in\vee q\,\mu$) for all $x,y\in S$ and $r\in(0,1]$.*

(F2c) $\mu(xy)\geq M(\mu(y),0.5)$ *(resp., $\mu(xy)\geq M(\mu(x),0.5)$) for all $x,y\in S$.*

*Proof.* The proof is similar to that of Lemma 3.2. □

**Lemma 3.4.** *Let $S$ be a hemiring and $\mu\in\mathscr{F}(S)$. Then* (F3a) *holds if and only if the following condition holds:*

(F3c) $x+a+z=b+z\Rightarrow\mu(x)\geq M(\mu(a),\mu(b),0.5)$ *for all $a,b,x,z\in S$.*

*Proof.* (F3a)⇒(F3c) If there exist $x,z,a,b\in S$ such that $x+a+z=b+z$ but $\mu(x)<r=M(\mu(a),\mu(a),0.5)$, then $a_r,b_r\in\mu$ and $\mu(x)+r<r+r\leq 1$, that is, $x_r\overline{\in\vee q}\,\mu$, a contradiction. Hence (F3c) is valid.

(F3c)⇒(F3a) Let $x,z,a,b\in S$ be such that $x+a+z=b+z$ and $a_r,b_s\in\mu$. Then $\mu(a)\geq r$ and $\mu(b)\geq s$, If $x_{M(r,s)}\overline{\in\vee q}\,\mu$, then $\mu(x)<M(r,s)$ and $\mu(x)+M(r,s)\leq 1$, which gives that $\mu(x)<0.5$ and $\mu(x)<M(r,s)\leq M(\mu(a),\mu(b))$. Thus $\mu(x)<M(\mu(a),\mu(b),0.5)$, a contradiction. Hence (F3a) is satisfied. □

By Lemma 3.4, if $\mu$ is an $(\in,\in\vee q)$-fuzzy left(right) $h$-ideal, then $\mu(0)\geq M(\mu(x),0.5)$ for all $x\in S$ since $0+x+0=x+0$.

The next theorem provides the relationships between $(\in,\in\vee q)$-fuzzy left(right) $h$-ideals of hemirings and crisp left(right) $h$-ideals of hemirings.

**Theorem 3.5.** *Let $S$ be a hemiring and $\mu\in\mathscr{F}(S)$. Then*

(1) *$\mu$ is an $(\in,\in\vee q)$-fuzzy left(resp., right) $h$-ideal of $S$ if and only if $\mu_r(\mu_r\neq\emptyset)$ is a left(resp., right) $h$-ideal of $S$ for all $r\in(0,0.5]$.*

(2) *$\mu$ is an $(\in,\in\vee q)$-fuzzy left(resp., right) $h$-ideal of $S$ if and only if $[\mu]_r([\mu]_r\neq\emptyset)$ is a left(resp., right) $h$-ideal of $S$ for all $r\in(0,0.5]$.*

*Proof.* The proof is straightforward. □

**Theorem 3.6.** *Let $\mu$ and $\nu$ be any two $(\in, \in\vee q)$-fuzzy left(resp., right) $h$-ideals of a hemiring $S$. Then so are $\mu\cap\nu$ and $\mu +_h \nu$.*

*Proof.* Let $\mu$ and $\nu$ be any two $(\in, \in\vee q)$-fuzzy left $h$-ideals of $S$. It is clear that $\mu\cap\nu$ is also an $(\in, \in\vee q)$-fuzzy left $h$-ideal of $S$. Now we show that $\mu +_h \nu$ is an $(\in, \in\vee q)$-fuzzy left $h$-ideal of $S$. In fact:

(1) For any $x, y \in S$, we have

$$\begin{aligned}(\mu +_h \nu)(x+y) =& \sup_{x+y+a_1+b_1+z=a_2+b_2+z} M(\mu(a_1), \mu(a_2), \nu(b_1), \nu(b_2))\\ \geq& M\left(\sup_{x+c_1+d_1+z_1=c_2+d_2+z_1} M(\mu(c_1), \mu(c_2), \nu(d_1), \nu(d_2)),\right.\\ &\left.\sup_{y+e_1+f_1+z_2=e_2+f_2+z_2} M(\mu(e_1), \mu(e_2), \nu(f_1), \nu(f_2)), 0.5\right)\\ =& M((\mu +_h \nu)(x), (\mu +_h \nu)(y), 0.5).\end{aligned}$$

(2) For any $x, y \in S$, we have

$$\begin{aligned}M((\mu +_h \nu)(y), 0.5) =& M\left(\sup_{y+a_1+b_1+z=a_2+b_2+z} M(\mu(a_1), \mu(a_2), \nu(b_1), \nu(b_2)), 0.5\right)\\ \leq& \sup_{xy+xa_1+xb_1+xz=xa_2+xb_2+xz} M(\mu(xa_1), \mu(xa_2), \nu(xb_1), \nu(xb_2))\\ \leq& \sup_{xy+c_1+d_1+z_1=c_2+d_2+z_1} M(\mu(c_1), \mu(c_2), \nu(d_1), \nu(d_2))\\ =& (\mu +_h \nu)(xy).\end{aligned}$$

(3) Let $a, b, x$ and $z_1$ be any elements of $S$ such that $x + a + z_1 = b + z_1$. If there exist $c_1, c_2, d_1, d_2, e_1, e_2, f_1, f_2, z_2, z_3 \in S$ such that

$$a + c_1 + d_1 + z_2 = c_2 + d_2 + z_2 \quad \text{and} \quad b + e_1 + f_1 + z_3 = e_2 + f_2 + z_3,$$

then we have

$$x + c_2 + d_2 + e_1 + f_1 + z_4 = c_1 + d_1 + e_2 + f_2 + z_4,$$

where $z_4 = z_1 + z_2 + z_3$, and so

$$\begin{aligned}&(\mu +_h \nu)(x)\\ =& \sup_{x+a_1+b_1+z=a_2+b_2+z} M(\mu(a_1), \mu(a_2), \nu(b_1), \nu(b_2))\\ \geq& M(\mu(c_2 + e_1), \mu(c_1 + e_2), \nu(d_2 + f_1), \nu(d_1 + f_2))\\ \geq& M(M(\mu(c_2), \mu(e_1), 0.5), M(\mu(c_1), \mu(e_2), 0.5), M(\nu(d_2), \nu(f_1), 0.5),\\ &M(\nu(d_1), \nu(f_2), 0.5))\\ =& M(M(\mu(c_1), \mu(c_2), \nu(d_1), \nu(d_2)), M(\mu(e_1), \mu(e_2), \nu(f_1), \nu(f_2)), 0.5),\end{aligned}$$

this gives

$$
\begin{aligned}
(\mu +_h \nu)(x) \geq & M\left( \sup_{a+c_1+d_1+z_2=c_2+d_2+z_2} M(\mu(c_1), \mu(c_2), \nu(d_1), \nu(d_2)), \right. \\
& \left. \sup_{b+e_1+f_1+z_3=e_2+f_2+z_3} M(\mu(e_1), \mu(e_2), \nu(f_1), \nu(f_2)), 0.5 \right) \\
= & M((\mu +_h \nu)(a), (\mu +_h \nu)(b), 0.5).
\end{aligned}
$$

Otherwise, we have $(\mu +_h \nu)(a) = 0$ or $(\mu +_h \nu)(b) = 0$, and so

$$(\mu +_h \nu)(x) \geq 0 = M((\mu +_h \nu)(a), (\mu +_h \nu)(b), 0.5).$$

Summing up the above statements, $\mu +_h \nu$ is an $(\in, \in\vee q)$-fuzzy left $h$-ideal of $S$. The case for $(\in, \in\vee q)$-fuzzy right $h$-ideals can be similarly proved. $\square$

Now denote by $GFI(S)$ the set of all $(\in, \in \vee q)$-fuzzy left(right) $h$-ideals of a hemiring $S$ with the same tip $t$, that is, $\mu(0) = \nu(0)$ for all $\mu, \nu \in GFI(S)$. Then we have the following result.

**Theorem 3.7.** *Let $S$ be a hemiring. Then $(GFI(S), +_h, \cap)$ is a bounded complete lattice under the relation "$\subseteq \vee q$".*

*Proof.* Let $\mu, \nu \in GFI(S)$. It follows from Theorem 3.6 that $\mu \cap \nu \in GFI(S)$ and $\mu +_h \nu \in GFI(S)$. It is clear that $\mu \cap \nu$ is the greatest lower bound of $\mu$ and $\nu$. We now show that $\mu +_h \nu$ is the least upper bound of $\mu$ and $\nu$. Since $\mu(0) = \nu(0)$, for any $x \in S$, we have

$$
\begin{aligned}
(\mu +_h \nu)(x) = & \bigvee_{x+a_1+b_1+z=a_2+b_2+z} M(\mu(a_1), \mu(a_2), \nu(b_1), \nu(b_2)) \\
\geq & M(\mu(0), \mu(x), \nu(0), \nu(0)) = M(\mu(0), \mu(x)) \geq M(\mu(x), 0.5),
\end{aligned}
$$

hence $\mu \subseteq\vee q\, \mu +_h \nu$. Similarly, we have $\nu \subseteq\vee q\, \mu +_h \nu$. Now, let $\omega \in GFI(S)$ be such that $\mu, \nu \subseteq\vee q\, \omega$. Then, we have $\mu +_h \nu \subseteq\vee q\, \omega +_h \omega \subseteq\vee q\, \omega$. Hence $\mu \vee \nu = \mu +_h \nu$. There is no difficulty in replacing the $\{\mu, \nu\}$ with an arbitrary family of $GFI(S)$ and so $(GFI(S), +_h, \cap)$ is a complete lattice under the relation "$\subseteq \vee q$". It is easy to see that $\emptyset$ and $\chi_S$ are the minimal and the maximal elements in $(GFI(S), +_h, \cap)$, respectively. Therefore, $(GFI(S), +_h, \cap)$ is a bounded complete lattice. $\square$

Next, we will construct the quotient hemirings of a hemiring $S$ by an $(\in, \in \vee q)$-fuzzy $h$-ideal of $S$. Before proceeding, let us provide an auxiliary lemma.

**Lemma 3.8.** *Let $S$ be a hemiring and $\mu, \nu, \omega \in \mathscr{F}(S)$. Then*

(1) $\mu \circ (\nu + \omega) \subseteq \mu \circ \nu + \mu \circ \omega$, $(\nu + \omega) \circ \mu \subseteq \nu \circ \mu + \omega \circ \mu$.

(2) *If $\mu$ and $\nu$ are $(\in, \in \vee q)$-fuzzy left(right) $h$-ideals of $S$, then $\mu + \nu \subseteq\vee q\, \mu +_h \nu$ and $\mu + \mu \approx \mu +_h \mu \approx \mu$.*

(3) *If $\mu$ is an $(\in, \in \vee q)$-fuzzy left(right) $h$-ideal of $S$ such that $0.5 \in Im(\mu)$, then $\mu(0) \geq 0.5$ and $x_1 \in\vee q\, x + \mu$ for all $x \in S$.*

*Proof.* (1) Let $x \in S$. If $x$ can not be expressed as $x = ab$ for all $a, b \in S$, then $(\mu \circ (\nu + \omega))(x) = 0$ and so $(\mu \circ (\nu + \omega))(x) = 0 \leq (\mu \circ \nu + \mu \circ \omega)(x)$. Otherwise, we have

$$\begin{aligned}(\mu \circ (\nu + \omega))(x) &= \bigvee_{x=ab} M(\mu(a), (\nu + \omega)(b)) = \bigvee_{x=ab} M\left(\mu(a), \bigvee_{b=c+d} M(\nu(c), \omega(d))\right) \\ &= \bigvee_{x=ab, b=c+d} M(\mu(a), \nu(c), \mu(a), \omega(d)) \\ &\leq \bigvee_{x=ac+ad} M((\mu \circ \nu)(ac), (\mu \circ \omega)(ad)) \\ &\leq \bigvee_{x=a'+b'} M((\mu \circ \nu)(a'), (\mu \circ \omega)(b')) \\ &= (\mu \circ \nu + \mu \circ \omega)(x).\end{aligned}$$

Hence $\mu \circ (\nu + \omega) \subseteq \mu \circ \nu + \mu \circ \omega$. In a similar way, we have $(\nu + \omega) \circ \mu \subseteq \nu \circ \mu + \omega \circ \mu$.

(2) Let $\mu$ and $\nu$ be $(\in, \in \vee q)$-fuzzy left $h$-ideals of $S$, and let $x, c, d \in S$ be such that $x = c + d$. Then

$$\begin{aligned}(\mu +_h \nu)(x) &= \bigvee_{x+a_1+b_1+z=a_2+b_2+z} M(\mu(a_1), \mu(a_2), \nu(b_1), \nu(b_2)) \\ &\geq M(\mu(c), \mu(0), \nu(d), \nu(0)) \geq M(\mu(c), \nu(d), 0.5),\end{aligned}$$

and so

$$(\mu +_h \nu)(x) \geq \bigvee_{x=a+b} M(\mu(a), \nu(b), 0.5) = M((\mu + \nu)(x), 0.5).$$

This implies $\mu + \nu \subseteq \vee q\, \mu +_h \nu$. In particular, we have $\mu + \mu \subseteq \vee q\, \mu +_h \mu$. On the other hand, we have

$$(\mu + \mu)(x) = \bigvee_{x=a+b} M(\mu(a), \mu(b)) \geq M(\mu(x), \mu(0)) \geq M(\mu(x), 0.5).$$

This gives $\mu \subseteq \vee q\, \mu + \mu$.

In the following, we show $\mu +_h \mu \subseteq \vee q\, \mu$. For $x_r \in \mu +_h \mu$, if possible, let $x_r \overline{\in \vee q}\, \mu$. Then $\mu(x) < r$ and $\mu(x) + r \leq 1$, which gives that $\mu(x) < 0.5$. If there exist $a_1, a_2, b_1, b_2, x, z \in S$ with $x + a_1 + b_1 + z = a_2 + b_2 + z$, then by Lemmas 3.2 and 3.4, we have

$$\begin{aligned}0.5 > \mu(x) &\geq M(\mu(a_1 + b_1), \mu(a_2 + b_2), 0.5) \geq M(M(\mu(a_1), \mu(b_1), 0.5), M(\mu(a_2), \mu(b_2), 0.5), 0.5) \\ &= M(\mu(a_1), \mu(a_2), \mu(b_1), \mu(b_2), 0.5),\end{aligned}$$

which implies $\mu(x) \geq M(\mu(a_1), \mu(a_2), \mu(b_1), \mu(b_2))$. Hence we have

$$\begin{aligned}r \leq (\mu +_h \mu)(x) &= \sup_{x+a_1+b_1+z=a_2+b_2+z} M(\mu(a_1), \mu(a_2), \mu(b_1), \mu(b_2)) \\ &\leq \sup_{x+a_1+b_1+z=a_2+b_2+z} \mu(x) = \mu(x),\end{aligned}$$

a contradiction. Hence $\mu +_h \mu \subseteq \vee q\, \mu$. Thus, we have

$$\mu + \mu \subseteq \vee q\, \mu +_h \mu \subseteq \vee q\, \mu \subseteq \vee q\, \mu + \mu,$$

and so $\mu + \mu \approx \mu +_h \mu \approx \mu$.

The case for $(\in, \in \vee q)$-fuzzy right $h$-ideals can be similarly proved.

(3) It is straightforward. □

Let $S$ be a hemiring and $\mu$ an $(\in, \in \vee q)$-fuzzy $h$-ideal of $S$. We denote by $S/\mu$ the set of all $x + \mu$, where $x \in S$.

**Theorem 3.9.** *Let $S$ be a hemiring and $\mu$ an $(\in, \in \vee q)$-fuzzy $h$-ideal of $S$ with $0.5 \in Im(\mu)$. Then $(S/\mu, \oplus, \otimes)$ is a hemiring, called the quotient hemiring of $S$ by $\mu$, under the relation "$\approx$" with respect to the operations "$\oplus$" and "$\otimes$" defined by*

$$x + \mu \oplus y\mu = (x + y) + \mu \quad \text{and} \quad x + \mu \otimes y + \mu = xy + \mu,$$

*respectively.*

*Proof.* We shall first show that the given operations are well-defined. Let $x, y, x', y' \in S$ be such that $x + \mu \approx x' + \mu$ and $y + \mu \approx y' + \mu$. We need to show that $x + \mu \oplus y + \mu \approx x' + \mu \oplus y' + \mu$. In fact, since $\mu$ is an $(\in, \in \vee q)$-fuzzy $h$-ideal of $S$ and $0.5 \in Im(\mu)$, it follows from Lemma 3.8 that $\mu + \mu \approx \mu$ and $x_1 \in \vee q\, x + \mu$. Thus we have

$$\begin{aligned} x + \mu \oplus y + \mu =& (x + y) + \mu = (x_1 + y_1) +_h \mu \subseteq \vee q\, ((x + \mu) + (y + \mu)) + \mu \\ \approx& ((x' + \mu) + (y' + \mu)) + \mu = ((x' + y') + \mu + \mu) + \mu \\ \subseteq& \vee q\, ((x' + y') + \mu + \mu) +_h \mu \subseteq \vee q\, (x' + y') + \mu \\ =& x' + \mu \oplus y' + \mu. \end{aligned}$$

This implies $x+\mu\oplus y+\mu \subseteq \vee q\, x'+\mu\oplus y'+\mu$. In a similar way, we have $x'+\mu\oplus y'+\mu \subseteq \vee q\, x + \mu \oplus y + \mu$. Hence "$\oplus$" is well defined. On the other hand, we have

$$\begin{aligned} x + \mu \otimes y + \mu = xy +_h \mu &= (x_1 \circ y_1) + \mu \subseteq \vee q\, ((x + \mu) \circ (y + \mu)) + \mu \\ &\approx ((x' + \mu) \circ (y' + \mu)) + \mu \subseteq (x'y' + x' \circ \mu + \mu \circ y' + \mu \circ \mu) + \mu \\ &\subseteq \vee q\, (x'y' + \mu +_h \mu + \mu) + \mu \subseteq \vee q\, x'y' + \mu = x' + \mu \otimes y' + \mu. \end{aligned}$$

Hence $x + \mu \otimes y + \mu \subseteq \vee q\, x' + \mu \otimes y' + \mu$. In a similar way, we have $x' + \mu \otimes y' + \mu \subseteq \vee q\, x + \mu \otimes y + \mu$. Therefore, $x + \mu \otimes y + \mu \approx x' + \mu \otimes y' + \mu$ and so "$\otimes$" is well defined. Now it is easy to verify that $(S/\mu, \oplus, \otimes)$ is a hemring. □

**Theorem 3.10.** *Let $S$ be a hemiring, $\mu$ an $(\in, \in \vee q)$-fuzzy $h$-ideal of $S$ with $0.5 \in Im(\mu)$ and $\nu \in \mathscr{L}(S)$. Define a fuzzy subset $\nu/\mu$ in $S/\mu$ by*

$$\nu/\mu(x + \mu) = \bigvee_{x+\mu\approx y+\mu} \nu(y)$$

*for all $x \in S$. If $\nu$ is an $(\in, \in \vee q)$-fuzzy left(resp., right) $h$-ideal of $S$, then $\nu/\mu$ is also an $(\in, \in \vee q)$-fuzzy left(resp., right) $h$-ideal of $(S/\mu, \oplus, \otimes)$.*

*Proof.* Let $\nu$ be an $(\in, \in \vee q)$-fuzzy left $h$-ideal of $S$. We show that $\nu/\mu$ is also an $(\in, \in \vee q)$-fuzzy left $h$-ideal of $(S/\mu, \oplus, \otimes)$. The case for $(\in, \in \vee q)$-fuzzy right $h$-ideals can be similarly proved. Let $x, x', y, y' \in S$ be such that $x + \mu \approx x' + \mu$ and $y + \mu \approx y' + \mu$. Then

$$\begin{aligned}\nu/\mu(x+\mu \oplus y+\mu) =& \nu/\mu((x+y)+\mu) = \bigvee_{(x+y)+\mu\approx z+\mu} \nu(z) \\ \geq& \nu(x'+y') \geq M(\nu(x'), \nu(y'), 0.5).\end{aligned}$$

Hence

$$\begin{aligned}\nu/\mu(x+\mu \oplus y+\mu) \geq& \bigvee_{x+\mu\approx a+\mu} \bigvee_{y+\mu\approx b+\mu} M(\nu(a), \nu(b), 0.5) \\ =& M\left(\bigvee_{x+\mu\approx a+\mu} \nu(a), \bigvee_{y+\mu\approx b+\mu} \nu(b), 0.5\right) \\ =& M(\nu/\mu(x+\mu), \nu/\mu(y+\mu), 0.5).\end{aligned}$$

In a similar way, we may show that $\nu/\mu(x+\mu \otimes y+\mu) \geq M(\nu/\mu(y+\mu), 0.5)$ and that $x+\mu \oplus a+\mu \oplus z+\mu \approx b+\mu \oplus z+\mu$ implies $\nu/\mu(x+\mu) \geq M(\nu/\mu(a+\mu), \nu/\mu(b+\mu), 0.5)$ for all $x, y, a, b, z \in S$. This completes the proof. □

## 4. **The Homomorphism of Hemirings in the Framework of Fuzzy Setting**

Let $(S, +, \cdot)$ and $(S', +', \cdot')$ be two hemirings. Recall that a mapping $\varphi : S \to S'$ is said to be a homomorphism of $(S, +, \cdot)$ to $(S', +', \cdot')$ if

$$\varphi(x+y) = \varphi(x) +' \varphi(y) \text{ and } \varphi(x \cdot y) = \varphi(x) \cdot' \varphi(y)$$

for all $x, y \in S$. If such a homomorphism is surjective, injective or bijective, it is called an epimorphism, a monomorphism or an isomorphism.

**Definition 4.1.** Let $(S, +, \cdot)$ and $(S', +', \cdot')$ be two hemirings and $\varphi : S \to S'$ a homomorphism of hemirings. A left(right) $h$-ideal $A$ of $S$ is called $\varphi$-compatible if, for all $x, z \in S, a, b \in A$, $\varphi(x+a+z) = \varphi(b+z)$ implies $x \in A$. An $(\in, \in\vee q)$-fuzzy left(right) $h$-ideal $\mu$ of $S$ is called generalized $\varphi$-compatible if, for all $x, a, b, z \in S$, $\varphi(x+a+z) = \varphi(b+z)$ implies $\mu(x) \geq M(\mu(a), \mu(b), 0.5)$.

Clearly, if $\varphi$ is a monomorphism, then every $((\in, \in\vee q)$-fuzzy) left(right) $h$-ideal of $S$ is (generalized) $\varphi$-compatible.

**Example 4.2.** Let $(\mathbb{N}_0, +, \cdot)$ be the hemiring of all non-negative integers and $(\mathbb{N}_0/_{<10>}, +, \cdot)$ the residue class hemiring of $(\mathbb{N}_0, +, \cdot)$ modulo 10. Define a mapping $\varphi : \mathbb{N}_0 \to \mathbb{N}_0/_{<10>}$ by $\varphi(x) = [x]_{<10>}$. Then $\varphi$ is an epimorphism of $(\mathbb{N}_0, +, \cdot)$ onto $(\mathbb{N}_0/_{<10>}, +, \cdot)$. Now define a fuzzy subset $\mu$ in $\mathbb{N}_0$ by

$$\mu(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0.6 & \text{if } x \in <10> \backslash \{0\}, \\ 0.2 & \text{otherwise,} \end{cases}$$

for all $x \in \mathbb{N}_0$. Then $\mu$ is a generalized $\varphi$-compatible $(\in, \in\vee q)$-fuzzy $h$-ideal of $\mathbb{N}_0$.

Let $(S, +, \cdot)$ and $(S', +', \cdot')$ be two hemirings, $\mu$ a fuzzy subset in $X$ and $\mu'$ a fuzzy subset in $X'$. Then the inverse image $\varphi^{-1}(\mu')$ of $\mu'$ is the fuzzy subset in $S$ defined by $\varphi^{-1}(\mu')(x) = \mu'(\varphi(x))$ for all $x \in S$. The image $\varphi(\mu)$ of $\mu$ is the fuzzy subset in $S'$ defined by

$$\varphi(\mu)(x') = \begin{cases} \bigvee_{x\in\varphi^{-1}(x')} \mu(x) & \text{if } \varphi^{-1}(x') \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

for all $x' \in S'$. The following basic assertions hold:

(1) For any $\mu \in \mathscr{L}(S)$, $\mu \subseteq \varphi^{-1}(\varphi(\mu))$. If $\varphi$ is injective, then $\mu = \varphi^{-1}(\varphi(\mu))$.
(2) For any $\mu \in \mathscr{L}(S')$, $\varphi(\varphi^{-1}(\mu')) \subseteq \mu'$. If $\varphi$ is surjective, then $\mu' = \varphi(\varphi^{-1}(\mu'))$.

**Lemma 4.3.** *Let $(S, +, \cdot)$ and $(S', +', \cdot')$ be hemirings and $\varphi : S \to S'$ a homomorphism of hemirings and let $\mu, \nu \in \mathscr{L}(S), \mu', \nu' \in \mathscr{L}(S')$. Then:*

(1) *If $\mu \subseteq\vee q\, \nu$, then $\varphi(\mu) \subseteq\vee q\, \varphi(\nu)$.*
(2) *If $\mu' \subseteq\vee q\, \nu'$, then $\varphi^{-1}(\mu') \subseteq\vee q\, \varphi^{-1}(\nu')$.*
(3) *$\varphi(\mu +_h \nu) \subseteq \varphi(\mu) +'_h \varphi(\nu)$. If $\varphi$ is injective, then $\varphi(\mu +_h \nu) = \varphi(\mu) +'_h \varphi(\nu)$.*
(4) *$\varphi(\mu + \nu) = \varphi(\mu) +' \varphi(\nu)$ and $\varphi(\mu \circ \nu) = \varphi(\mu) \circ' \varphi(\nu)$.*

*Proof.* The proof of (1) and (2) is straightforward by Lemma 2.1. We prove (3). The proof of (4) is similar. Let $x' \in S'$. If $\varphi^{-1}(x') = \emptyset$, it is clear that $\varphi(\mu +_h \nu)(x') = 0 \leq (\varphi(\mu) +'_h \varphi(\nu))(x')$. Otherwise, we have

$$\begin{aligned}
&(\varphi(\mu) +'_h \varphi(\nu))(x') \\
&= \bigvee_{x'+'a'_1+'b'_1+'z'=a'_2+'b'_2+'z'} M(\varphi(\mu)(a'_1), \varphi(\mu)(a'_2), \varphi(\nu)(b'_1), \varphi(\nu)(b'_2)) \\
&= \bigvee_{a'_1,a'_2,b'_1,b'_2\in Im(\varphi), x'+'a'_1+'b'_1+'z'=a'_2+'b'_2+'z'} M(\varphi(\mu)(a'_1), \varphi(\mu)(a'_2), \varphi(\nu)(b'_1), \varphi(\nu)(b'_2)) \\
&= \bigvee_{a'_1,a'_2,b'_1,b'_2\in Im(\varphi), x'+'a'_1+b'_1+'z'=a'_2+b'_2+'z'} M\Bigg(\bigvee_{\varphi(a_1)=a'_1} \mu(a_1), \bigvee_{\varphi(a_2)=a'_2} \mu(a_2), \\
&\quad \bigvee_{\varphi(b_1)=b'_1} \nu(b_1), \bigvee_{\varphi(b_2)=b'_2} \nu(b_2)\Bigg) \\
&= \bigvee_{x'+'\varphi(a_1)+'\varphi(b_1)+'z'=\varphi(a_2)+'\varphi(b_2)+'z'} M(\mu(a_1), \mu(a_2), \nu(b_1), \nu(b_2)) \\
&= \bigvee_{\varphi(x)=x', \varphi(x+a_1+b_1+z)=\varphi(a_2+b_2+z)} M(\mu(a_1), \mu(a_2), \nu(b_1), \nu(b_2)) \\
&\geq \bigvee_{\varphi(x)=x', x+a_1+b_1+z=a_2+b_2+z} M(\mu(a_1), \mu(a_2), \nu(b_1), \nu(b_2)) \\
&= \bigvee_{\varphi(x)=x'} (\mu +_h \nu)(x) = \varphi(\mu +_h \nu)(x').
\end{aligned}$$

This implies $\varphi(\mu +_h \nu) \subseteq \varphi(\mu) +'_h \varphi(\nu)$. If $\varphi$ is injective, then the inequality " $\geq$ " can be replaced by an equal " $=$ ". Hence $\varphi(\mu +_h \nu) = \varphi(\mu) +'_h \varphi(\nu)$. □

**Lemma 4.4.** *Let $(S, +, \cdot)$ and $(S', +', \cdot')$ be hemirings, $\varphi : S \to S'$ a homomorphism of hemirings and $\mu$ an $(\in, \in\vee q)$-fuzzy left(right) $h$-ideal of $S$. Then $\mu$ is generalized $\varphi$-compatible if and only if non-empty subset $\mu_r (\mu_{[r]})$ is $\varphi$-compatible for all $r \in (0, 0.5]$.*

*Proof.* The proof is straightforward. □

**Theorem 4.5.** *Let $(S, +, \cdot)$ and $(S', +', \cdot')$ be hemirings, $\varphi : S \to S'$ an epimorphism of hemirings. If $\mu$ is a generalized $\varphi$-compatible $(\in, \in \vee q)$-fuzzy left(resp., right) $h$-ideal of $S$, then $\varphi(\mu)$ is an $(\in, \in\vee q)$-fuzzy (resp., right) $h$-ideal of $S'$.*

*Proof.* Let $\mu$ be a generalized $\varphi$-compatible $(\in, \in \vee q)$-fuzzy left $h$-ideal of $S$. We show that $\varphi(\mu)$ is an $(\in, \in\vee q)$-fuzzy left $h$-ideal of $S'$. The case for $(\in, \in\vee q)$-fuzzy right $h$-ideals can be similarly proved.

(1) Let $x', y' \in S'$. Then

$$\begin{aligned}\varphi(\mu)(x' +' y') &= \bigvee_{\varphi(a)=x'+'y'} \mu(a) \geq \bigvee_{\varphi(x)=x',\varphi(y)=y'} \mu(x + y)\\ &\geq \bigvee_{\varphi(x)=x',\varphi(y)=y'} M(\mu(x), \mu(y), 0.5) = M\left(\bigvee_{\varphi(x)=x'} \mu(x), \bigvee_{\varphi(y)=y'} \mu(y), 0.5\right)\\ &= M(\varphi(\mu)(x'), \varphi(\mu)(y'), 0.5).\end{aligned}$$

(2) Let $x', y' \in S'$. In a similar way we have $\varphi(\mu)(x' \cdot' y') \geq M(\varphi(\mu)(y'), 0.5)$.

(3) Let $x, a, b, z \in S$ and $x', a', b', z' \in S'$ be such that $x' +' a' +' z' = b' +' z'$, $\varphi(x) = x', \varphi(a) = a', \varphi(b) = b'$ and $\varphi(z) = z'$. Then $\varphi(x + a + z) = \varphi(b + z)$. Since $\mu$ is generalized $\varphi$-compatible, we have $\mu(x) \geq M(\mu(a), \mu(b), 0.5)$. Thus we have

$$\varphi(\mu)(x') = \bigvee_{\varphi(x)=x'} \mu(x) \geq M(\mu(a), \mu(b), 0.5),$$

and so

$$\begin{aligned}\varphi(\mu)(x') &\geq \bigvee_{\varphi(a)=a',\varphi(b)=b'} M(\mu(a), \mu(b), 0.5) = M\left(\bigvee_{\varphi(a)=a'} \mu(a), \bigvee_{\varphi(b)=b'} \mu(b), 0.5\right)\\ &= M(\varphi(\mu)(a'), \varphi(\mu)(b'), 0.5)\end{aligned}$$

Summing up the above arguments, $\varphi(\mu)$ is an $(\in, \in\vee q)$-fuzzy left $h$-ideal of $S'$. □

It is worth noting that if $\mu$ is not generalized $\varphi$-compatible, then Theorem 4.5 may not be true in general as shown in the following example.

**Example 4.6.** We use the hemiring $(\mathbb{N}_0, +, \cdot)$ to define operations "$+'$" and "$\cdot'$" on the subset $S' = \{0, 1, 2, 3\} \subseteq \mathbb{N}_0$ by $x +' y = \min\{x + y, 3\}$ and $x \cdot' y = \min\{x \cdot y, 3\}$. Then $(S', +', \cdot')$ is a hemiring. We further define a mapping $\varphi : \mathbb{N}_0 \to S'$ by

$\varphi(x) = \min\{x, 3\}$ for all $x \in \mathbb{N}_0$. Routine calculation gives that $\varphi$ is an epimorphism of hemirings. Now define a fuzzy subset $\mu$ in $\mathbb{N}_0$ by

$$\mu(x) = \begin{cases} 0.6 & \textit{if } x \in < 2 >, \\ 0.2 & \textit{otherwise.} \end{cases}$$

Then $\mu$ is an $(\in, \in \vee q)$-fuzzy $h$-ideal of $\mathbb{N}_0$ but it is not generalized $\varphi$-compatible since $\varphi(3+2+1) = \varphi(4+1)$ but $\mu(3) = 0.2 < 0.5 = M(\mu(2), \mu(4), 0.5)$, and routine verification gives that $\varphi(\mu)$ is a fuzzy subset in $S'$ such that

$$\varphi(\mu)(x) = \begin{cases} 0.6 & \textit{if } x \in \{0, 2, 3\}, \\ 0.2 & \textit{otherwise.} \end{cases}$$

Clearly, $\varphi(\mu)$ is not an $(\in, \in \vee q)$-fuzzy $h$-ideal of $S'$ since $1 +' 2 +' 3 = 2 +' 3$ but $\varphi(\mu)(1) = 0.2 < 0.5 = M(\varphi(\mu)(2), 0.5)$.

**Theorem 4.7.** *Let $(S, +, \cdot)$ and $(S', +', \cdot')$ be hemirings, $\varphi : S \to S'$ a homomorphism of hemirings. If $\mu'$ is an $(\in, \in \vee q)$-fuzzy left(resp., right) $h$-ideal of $S'$, then $\varphi^{-1}(\mu)$ is a generalized $\varphi$-compatible $(\in, \in \vee q)$-fuzzy left(resp., right) $h$-ideal of $S$.*

*Proof.* Let $\mu'$ be an $(\in, \in \vee q)$-fuzzy left $h$-ideal of $S'$. We show that $\varphi^{-1}(\mu)$ is a generalized $\varphi$-compatible $(\in, \in \vee q)$-fuzzy left $h$-ideal of $S$. The case for $(\in, \in \vee q)$-fuzzy right $h$-ideals can be similarly proved.

(1) Let $x, y \in S$. Then

$$\begin{aligned} \varphi^{-1}(\mu')(x+y) =& \mu'(\varphi(x+y)) = \mu'(\varphi(x) +' \varphi(y)) \geq M(\mu'(\varphi(x)), \mu'(\varphi(y)), 0.5) \\ =& M(\varphi^{-1}(\mu')(x), \varphi^{-1}(\mu')(y), 0.5). \end{aligned}$$

(2) Let $x, y \in S$. In a similar way, we have $\varphi^{-1}(\mu')(xy) \geq M(\varphi^{-1}(\mu')(y), 0.5)$.

(3) Let $x, a, b, z \in S$ be such that $\varphi(x + a + z) = \varphi(b + z)$. Then $\varphi(x) +' \varphi(a) +' \varphi(z) = \varphi(b) +' \varphi(z)$. Since $\mu'$ is an $(\in, \in \vee q)$-fuzzy left $h$-ideal of $S'$, we have $\mu'(\varphi(x)) \geq M(\mu'(\varphi(a)), \mu'(\varphi(b)), 0.5)$, that is, $\varphi^{-1}(\mu')(x) \geq M(\varphi^{-1}(\mu')(a), \varphi^{-1}(\mu')(b), 0.5)$.

Summing up the above arguments, $\varphi^{-1}(\mu')$ is a generalized $\varphi$-compatible $(\in, \in \vee q)$-fuzzy left $h$-ideal of $S$. □

Combining Theorems 4.5 and 4.7, we have the following result.

**Theorem 4.8.** *Let $(S, +, \cdot)$ and $(S', +', \cdot')$ be hemirings, $\varphi : S \to S'$ an epimorphism of hemirings. Then the mapping $\psi : \mu \to \varphi(\mu)$ defines a one-to-one correspondence between the set of all generalized $\varphi$-compatible $(\in, \in \vee q)$-fuzzy left(resp., right) $h$-ideals of $S$ and the set of all $(\in, \in \vee q)$-fuzzy left(resp., right) $h$-ideals of $S'$.*

Next, we will establish the generalized fuzzy homomorphism theorems of hemirings. In the sequel, unless otherwise stated, the quotient hemirings are II-quotient hemirings. Before proceeding, let us give an auxiliary lemma.

**Lemma 4.9.** *Let $S$ be a hemiring and $\mu$ an $(\in, \in \vee q)$-fuzzy left(right) $h$-ideal of $S$. Then*

(1) *$M(\mu(0), 0.5) \geq M((x + \mu)(y), 0.5)$ for all $x, y \in S$.*

$x+\mu \approx y+\mu \Leftrightarrow M(\mu(0),0.5) = M((y+\mu)(x),0.5) = M((x+\mu)(y),0.5)$. *In particular, if* $0.5 \in Im(\mu)$, *then*

$$x+\mu \approx y+\mu \Leftrightarrow M((y+\mu)(x),(x+\mu)(y)) \geq 0.5.$$

(2) *Define* $\mu_S = \{x \in S | M(\mu(0),0.5) = M(\mu(x),0.5)\}$, *then* $\mu_S$ *is a left(right) $h$-ideal of $S$. If $\mu$ has the sup-property, then*

$$x+\mu \approx y+\mu \Leftrightarrow x+\mu_S = y+\mu_S.$$

*In particular, if* $0.5 \in Im(\mu)$, *then* $\mu_S = \{x \in S | \mu(x) \geq 0.5\}$.

*Proof.* Let $\mu$ be an $(\in, \in \vee q)$-fuzzy left $h$-ideal of $S$. The proof of (1) is straightforward. We show (2) and (3). The case for $(\in, \in \vee q)$-fuzzy right $h$-ideals can be similarly proved.

(2) Let $x, y \in S$ be such that $x+\mu \approx y+\mu$. Then

$$M((y+\mu)(x),0.5) = M((x+\mu)(x),0.5) = M\left(\bigvee_{x=x+a} \mu(a), 0.5\right) = M(\mu(0),0.5).$$

In a similar way, we have $M(\mu(0),0.5) = M((x+\mu)(y),0.5)$.

Now, assume that $M(\mu(0),0.5) = M((y+\mu)(x),0.5) = M((x+\mu)(y),0.5)$. Let $z$ be any element of $S$. Then, by (1), we have

$$\begin{aligned}
&M((x+\mu)(z),0.5)\\
&= M\left(M((x+\mu)(z),0.5), M(\mu(0),0.5)\right) = M\left(M((x+\mu)(z),0.5), M((y+\mu)(x),0.5)\right)\\
&= M\left(M\left(\bigvee_{z=x+a}\mu(a),0.5\right), M\left(\bigvee_{x=y+b}\mu(b),0.5\right)\right)\\
&= M\left(\bigvee_{z=x+a,x=y+b} M(\mu(a),\mu(b)),0.5\right)\\
&= M\left(\bigvee_{z=y+a+b} M(\mu(a),\mu(b)),0.5\right) = \bigvee_{z=y+a+b} M(\mu(a),\mu(b),0.5)\\
&\leq \bigvee_{z=y+a+b}\mu(a+b) \leq \bigvee_{z=y+c}\mu(c) = (y+\mu)(z).
\end{aligned}$$

This implies $x+\mu \subseteq \vee q\, y+\mu$. In a similar way, we have $y+\mu \subseteq \vee q\, x+\mu$. Hence $x+\mu \approx y+\mu$.

If $0.5 \in Im(\mu)$, then $\mu(0) \geq 0.5$ and so

$$x+\mu \approx y+\mu \Leftrightarrow M((y+\mu)(x),(x+\mu)(y)) \geq 0.5.$$

(3) We first show that $\mu_S$ is a left $h$-ideal of $S$. Let $x, y \in \mu_S$. Then

$$M(\mu(x+y),0.5) \geq M(M(\mu(x),\mu(y),0.5),0.5) = M(\mu(0),0.5) \geq M(\mu(x+y),0.5),$$

this implies $M(\mu(x+y), 0.5) = M(\mu(0), 0.5)$ and so $x+y \in \mu_S$. Similarly, we may show that $xy \in \mu_S$ for all $x \in S$ and $y \in \mu_S$ and that $x+a+z = b+z$ implies $x \in \mu_S$ for all $a, b \in \mu_S$ and $x, z \in S$. Hence $\mu_S$ is a left $h$-ideal of $S$.

Now assume that $\mu$ has the sup-property. Let $x, y \in S$ be such that $x+\mu \approx y+\mu$. By (2), we have $M(\mu(0), 0.5) = M((y+\mu)(x), 0.5) = M((x+\mu)(y), 0.5)$, that is, $M(\mu(0), 0.5) = M\left(\bigvee_{y=x+a} \mu(a), 0.5\right)$. Since $\mu$ has the sup-property, there exists $z \in S$ such that $y = x+z$ and $M(\mu(0), 0.5) = M(\mu(z), 0.5)$, that is, $z \in \mu_S$ and so $y + \mu_S = x + z + \mu_S \subseteq x + \mu_S$. In a similar way, we have $x + \mu_S \subseteq y + \mu_S$. Hence $x + \mu_S = y + \mu_S$. Conversely, assume that $x + \mu_S = y + \mu_S$. Then there exists $z \in \mu_S$ such that $y = x + z$. Thus we have

$$M(\mu(0), 0.5) \geq M((x+\mu)(y), 0.5) \geq M(\mu(z), 0.5) = M(\mu(0), 0.5).$$

This implies $M(\mu(0), 0.5) = M((x+\mu)(y), 0.5)$. In a similar way, we have $M(\mu(0), 0.5) = M((y+\mu)(x), 0.5)$. By (1), we have $x+\mu \approx y+\mu$. This completes the proof. □

**Theorem 4.10.** *Let $(S, +, \cdot)$ be a hemiring and $\mu$ an $(\in, \in\vee q)$-fuzzy $h$-ideal with $0.5 \in Im(\mu)$. Define $\varphi(x) = x + \mu$ for all $x \in S$. Then $\varphi$ is an epimorphism from $(S, +, \cdot)$ onto $(S/\mu, \boxplus, \boxdot)$ with $Ker(\varphi) = \mu_S$ under the relation "$\approx$". Moreover, if $\mu$ has the sup-property, then $S/\mu_S \cong S/\mu$.*

*Proof.* It is clear that $\varphi$ is surjective. Let $x, y \in S$. Then

$$\varphi(x+y) = x+y+\mu = (x+\mu) \boxplus (y+\mu) = \varphi(x) \boxplus \varphi(y)$$

and

$$\varphi(xy) = xy + \mu = (x+\mu) \boxdot (y+\mu) = \varphi(x) \boxdot \varphi(y).$$

Hence $\varphi$ is a homomorphism. Finally, by Lemma 4.9(2), $x \in Ker(\varphi) \Leftrightarrow \varphi(x) = x + \mu \approx \mu \Leftrightarrow \mu(x) \geq 0.5 \Leftrightarrow x \in \mu_S$. Now assume that $\mu$ has the sup-property. To show $S/\mu_S \cong S/\mu$, it suffices to show that $x+\mu \approx y+\mu$ implies $x+\mu_S = y+\mu_S$. In fact, if $x+\mu \approx y+\mu$ for some $x, y \in S$, it follows from 4.9(3) that $x+\mu_S = y+\mu_S$. This completes the proof. □

**Lemma 4.11.** *Let $(S, +, \cdot)$ and $(S', +', \cdot')$ be hemirings, $\varphi : S \to S'$ a homomorphism of hemirings. If $S + a = S$ for all $a \in H$ and $S'$ is additive cancellable, then $\varphi(x) = \varphi(y)$ implies $x + Ker(\varphi) = y + Ker(\varphi)$ for all $x, y \in S$.*

*Proof.* Assume that $S + a = S$ for all $a \in S$ and that $S'$ is additive cancellable. Let $\varphi(x) = \varphi(y)$ for some $x, y \in S$. Then $y = x + z$ for some $z \in S$ since $S + a = S$ for all $a \in S$. Thus $\varphi(x) = \varphi(y) = \varphi(x+z) = \varphi(x) + \varphi(z)$ and so it follows from additive cancellativity of $S'$ that $\varphi(z) = 0'$, that is, $z \in Ker(\varphi)$. Hence $y + Ker(\varphi) = x + z + Ker(\varphi) \subseteq x + Ker(\varphi)$. In a similar way, we have $x + Ker(\varphi) \subseteq y + Ker(\varphi)$. Therefore, $x + Ker(\varphi) = y + Ker(\varphi)$. □

**Theorem 4.12.** *Let $(S, +, \cdot)$, $(S', +', \cdot')$ be hemirings, and $\varphi : S \to S'$ an epimorphism of hemirings. Let $\mu$ be an $(\in, \in\vee q)$-fuzzy $h$-ideal of $S$ such that $0.5 \in$*

*$Im(\mu), \mu_S \subseteq Ker(\varphi)$ and that $\mu$ has the sup-property. Then there exists an unique epimorphism $\psi$ from $(S/\mu, \boxplus, \boxdot)$ onto $(S', +', \cdot')$ such that $\varphi = \psi \circ \eta$ under the relation " $\approx$ ", where $\eta(x) = x + \mu$ for all $x \in S$. Moreover, if $S + a = S$ for all $a \in S$, $\mu_S = Ker(\varphi)$ and $S'$ is additive cancellable, then $\psi$ is an isomorphism.*

*Proof.* Define a mapping $\psi : S/\mu \to S'$ by $\psi(x+\mu) = \varphi(x)$ for all $x \in S$. Then $\psi$ is well defined. In fact, if $x+\mu \approx y+\mu$ for some $x, y \in S$, then it follows from Lemma 4.9 that $x+\mu_S = y+\mu_S$. Since $\mu_S \subseteq Ker(\varphi)$, we have $x+Ker(\varphi) = y+Ker(\varphi)$ and so $\varphi(x) = \varphi(y)$ by Lemma 4.11. Now it is easy to check that $\psi$ is a homomorphism.

Further, since $\varphi$ is onto, $\psi$ is also onto. On the other hand, $\varphi(x) = \psi(x + \mu) = \psi(\eta(x)) = (\psi \circ \eta)(x)$ for all $x \in S$. Finally, we show that $\psi$ is unique. If there exists another epimorphism $\phi$ from $(S/\mu, \boxplus, \boxdot)$ onto $(S', +', \cdot')$ such that $\varphi = \phi \circ \eta$. Then $\psi(x + \mu) = \varphi(x) = (\phi \circ \eta)(x) = \phi(x + \mu)$ for all $x \in S$. This implies $\psi = \phi$.

Suppose $S + a = S$ for all $a \in S$, $\mu_S = Ker(\varphi)$ and $S'$ is additive cancellable. To show that $\psi$ is an isomorphism, it remains to show that $\psi$ is injective. In fact, let $x, y \in S$ be such that $\varphi(x) = \varphi(y)$. By Lemma 4.11, we have $x + Ker(\varphi) = y + Ker(\varphi)$. Since $\mu_S = Ker(\varphi)$, we have $x + \mu_S = y + \mu_S$ and so $x + \mu \approx y + \mu$ by Lemma 4.9. This completes the proof. □

**Theorem 4.13.** *Let $(S, +, \cdot)$, $(S', +', \cdot')$ be hemirings, and $\varphi : S \to S'$ a homomorphism of hemirings and $\mu$ and $\mu'$ $(\in, \in\vee q)$-fuzzy $h$-ideals of $S$ and $S'$, respectively, such that $\varphi(\mu) \subseteq \vee q\, \mu'$ and $0.5 \in Im(\mu)$. Then there exists a homomorphism $\psi : S/\mu \to S'/\mu'$ such that the diagram*

$$\begin{array}{ccc} S & \xrightarrow{\ \varphi\ } & S' \\ \downarrow & & \downarrow \\ S/\mu & \xrightarrow[\ \psi\ ]{} & S'/\mu' \end{array}$$

*is commutative. Moreover, if $\varphi$ is an isomorphism and $\mu' \approx \varphi(\mu)$, then $\psi$ is also an isomorphism.*

*Proof.* Define a mapping $\psi : S/\mu \to S'/\mu'$ by $\psi(x + \mu) = \varphi(x) + \mu'$ for all $x \in S$. We first show that $\psi$ is well defined. In fact, if $x + \mu \approx y + \mu$ for some $x, y \in S$, since $0.5 \in \text{Im}(\mu)$, it follows from Lemma 4.9 that $M((x + \mu)(y), (y + \mu)(x)) \geq 0.5$ and $0.5 \in \text{Im}(\mu')$. To show $\varphi(x) + \mu' \approx \varphi(y) + \mu'$, it needs only to show $M((\varphi(x) +' \mu')(\varphi(y)), (\varphi(y) +' \mu')(\varphi(x))) \geq 0.5$ by Lemma 4.9 since $0.5 \in \text{Im}(\mu')$. Now, by the assumption, $\varphi(\mu) \subseteq\vee q\, \mu'$, we have $\varphi(x) +' \varphi(\mu) \subseteq\vee q\, \varphi(x) +' \mu'$ and so

$$\begin{aligned}(\varphi(x) +' \mu')(\varphi(y)) \geq& M((\varphi(x) +' \varphi(\mu))(\varphi(y)), 0.5) = M(\varphi(x + \mu)(\varphi(y)), 0.5) \\ \geq& M((x + \mu)(y), 0.5) \geq 0.5.\end{aligned}$$

In a similar way, we have $(\varphi(y) +' \mu')(\varphi(x)) \geq 0.5$. Hence $\psi$ is well defined. Now it is easy to check that $\psi$ is a homomorphism and that the diagram is commutative.

Next assume that $\varphi$ is an isomorphism and $\mu' \approx \varphi(\mu)$. Since $\varphi$ is onto, it is clear that $\psi$ is also onto. Let $x, y \in S$ be such that $\varphi(x) + \mu' \approx \varphi(y) + \mu'$. Then

$\varphi(x) + \varphi(\mu) \approx \varphi(y) + \varphi(\mu)$ and so

$$M((\varphi(x) + \varphi(\mu))(\varphi(y)), (\varphi(y) + \varphi(\mu))(\varphi(x))) \geq 0.5.$$

Since $\varphi$ is injective, we have

$$\begin{aligned}(\varphi(x) + \varphi(\mu))(\varphi(y)) &= \bigvee_{\varphi(y)=\varphi(x)+a'} \varphi(\mu)(a') = \bigvee_{\varphi(y)=\varphi(x+a)} \varphi(\mu)(\varphi(a)) \\ &= \bigvee_{y=x+a} \mu(a) = (x + \mu)(y) \geq 0.5.\end{aligned}$$

In a similar way, we have $(\varphi(y) + \varphi(\mu))(\varphi(x))) \geq 0.5$. Hence $x + \mu \approx y + \mu$ by Lemma 4.9, and so $\psi$ is injective. Therefore, $\psi$ is an isomorphism. This completes the proof. □

As two special cases of Theorem 4.13, we have the following results.

**Theorem 4.14.** *Let $(S, +, \cdot)$ and $(S', +', \cdot')$ be hemirings, $\varphi : S \to S'$ an epimorphism of hemirings and $\mu$ a generalized $\varphi$-compatible $(\in, \in \vee q)$-fuzzy $h$-ideal of $S$ such that $0.5 \in Im(\mu)$. Then there exists an epimorphism from $S/\mu$ onto $S'/\varphi(\mu)$. Moreover, if $\varphi$ is an isomorphism, then $S/\mu \cong S'/\varphi(\mu)$.*

*Proof.* Since $\mu$ is a generalized $\varphi$-compatible $(\in, \in\vee q)$-fuzzy $h$-ideal of $S$, $\varphi(\mu)$ is an $(\in, \in\vee q)$-fuzzy $h$-ideal of $S'$ by Theorem 4.5. Define a mapping $\psi : S/\mu \to S'/\varphi(\mu)$ by $\psi(x + \mu) = \varphi(x) +' \varphi(\mu)$ for all $x \in S$. By the proof of Theorem 4.13, $\psi$ is a homomorphism. Since $\varphi$ is onto, it is clear that $\psi$ is also onto. If $\varphi$ is an isomorphism, then so is $\psi$ by Theorem 4.13. Hence $S/\mu \cong S'/\varphi(\mu)$. □

**Theorem 4.15.** *Let $(S, +, \cdot)$ and $(S', +', \cdot')$ be hemirings, $\varphi : S \to S'$ a homomorphism of hemirings and $\mu'$ an $(\in, \in\vee q)$-fuzzy $h$-ideal of $S'$ such that $0.5 \in Im(\mu')$. Then there exists a homomorphism from $S/\varphi^{-1}(\mu')$ to $S'/\mu'$. Moreover, if $\varphi$ is an isomorphism, then $S/\varphi^{-1}(\mu') \cong S'/\mu'$.*

*Proof.* Since $\mu'$ is an $(\in, \in \vee q)$-fuzzy $h$-ideal of $S'$, $\varphi^{-1}(\mu')$ is a generalized $\varphi$-compatible $(\in, \in \vee q)$-fuzzy $h$-ideal of $S$ by Theorem 4.7. Define a mapping $\psi : S/\varphi^{-1}(\mu') \to S'/\mu'$ by $\psi(x+\varphi^{-1}(\mu')) = \varphi(x)+'\mu'$ for all $x \in S$. Since $\varphi(\varphi^{-1}(\mu')) \subseteq \mu'$, that is, $\varphi(\varphi^{-1}(\mu')) \subseteq\vee q\mu'$, then $\psi$ is a homomorphism by the proof of Theorem 4.13. If $\varphi$ is an isomorphism, then so is $\psi$ by Theorem 4.13. Hence $S/\varphi^{-1}(\mu') \cong S'/\mu'$. □

**Theorem 4.16.** *Let $(S, +, \cdot)$ be a hemiring, $\mu$ and $\nu$ two $(\in, \in\vee q)$-fuzzy $h$-ideals of $S$ such that $0.5 \in Im(\mu) \cap Im(\nu)$. Then $\mu_S/\mu \cap \nu \cong (\mu_S + \nu_S)/\nu$.*

*Proof.* It is clear that both $\mu_S$ and $\mu_S + \nu_S$ are hemirings, and that $\mu \cap \nu$ and $\nu$ are $(\in, \in\vee q)$-fuzzy $h$-ideals of $\mu_S$ and $\mu_S + \nu_S$, respectively. Thus it follows from Theorem 3.12 that both $(\mu_S/\mu, \boxplus, \boxdot)$ and $((\mu_S + \nu_S)/\nu, \boxplus, \boxdot)$ are hemirings.

Define a mapping $\varphi : \mu_S/\mu \cap \nu \to (\mu_S + \nu_S)/\nu$ by $\varphi(x + \mu \cap \nu) = x + \nu$ for all $x \in \mu_S$. We first show that $\varphi$ is well defined. In fact, let $x, y \in \mu_S$ be such that $x + \mu \cap \nu \approx y + \mu \cap \nu$. Since $0.5 \in Im(\mu) \cap Im(\nu)$, it follows from Lemma 4.9 that

$M((x+\mu\cap\nu)(y),(y+\mu\cap\nu)(x))\geq 0.5$. Thus we have $M((x+\nu)(y),(y+\nu)(x))\geq M((x+\mu\cap\nu)(y),(y+\mu\cap\nu)(x))\geq 0.5$ and so $x+\nu\approx y+\nu$ by Lemma 4.9. Hence $\varphi$ is well defined. Then it is easy to check that $\varphi$ is a homomorphism.

Now let $x$ be any element of $\mu_S+\nu_S$. Then there exist $y\in\mu_S, z\in\nu_S$ such that $x=y+z$. Since $z\in\nu_S$, we have $\nu(z)\geq 0.5$ and $(z+\nu)(0)=\bigvee_{0=z+a}\nu(a)\geq\bigvee_{0=z+a}M(\nu(0),\nu(z),0.5)\geq 0.5$, hence $z+\nu\approx\nu$. Thus $x+\nu=y+z+\nu\approx y+\nu=\varphi(y+\mu\cap\nu)$. This implies $\varphi$ is onto. To show that $\varphi$ is injective, let $x',y'\in\mu_S+\nu_S$ be such that $x'+\nu\approx y'+\nu$. Then it follows from the above proof that there exist $x,y\in\mu_S$ such that $x+\nu\approx x'+\nu\approx y'+\nu\approx y+\nu$. Thus $M((x+\nu)(y),(y+\nu)(x))\geq 0.5$. On the other hand, since $x,y\in\mu_S$, we have

$$\begin{aligned}(x+\mu\cap\nu)(y)&=\bigvee_{y=x+a}M(\mu(a),\nu(a))\geq\bigvee_{y=x+a}M(M(\mu(x),\mu(y),0.5),\nu(a))\\&=\bigvee_{y=x+a}M(0.5,\nu(a))=M\left(\bigvee_{y=x+a}\nu(a),0.5\right)\\&=M((x+\nu)(y),0.5)=0.5.\end{aligned}$$

In a similar way, we have $(y+\mu\cap\nu)(x)=0.5$. Thus $x+\mu\cap\nu\approx y+\mu\cap\nu$ by Lemma 4.9. This completes the proof. □

**Theorem 4.17.** *Let $(S,+,\cdot)$ be a hemiring and $\mu$ and $\nu$ two $(\in,\in\vee q)$-fuzzy $h$-ideals of $S$ such that $\mu\subseteq\vee q\,\nu$ and $0.5\in Im(\mu)\cap Im(\nu)$. Then $(S/\mu)/(\nu_S/\mu)\cong S/\nu$.*

*Proof.* By Lemma 4.9, $\nu_S/\mu$ is an $h$-ideal of $S/\mu$. Define a mapping $\varphi: S/\mu\to S/\nu$ by $\varphi(x+\mu)=x+\nu$ for all $x\in S$. We first show that $\varphi$ is well defined. In fact, if $x+\mu\approx y+\mu$ for some $x,y\in S$, then $M((x+\mu)(y),(y+\mu)(x))\geq 0.5$ by Lemma 4.9. Since $\mu\subseteq\vee q\,\nu$, we have $M((x+\nu)(y),(y+\nu)(x))\geq M((x+\mu)(y),(y+\mu)(x),0.5)\geq 0.5$ and so $x+\nu\approx y+\nu$ by Lemma 4.9. Hence $\varphi$ is well defined.

Further, it is clear that $\varphi$ is onto, and so $(S/\mu)/Ker(\varphi)\cong S/\nu$. Now we show $Ker(\varphi)=\nu_S/\mu$. Before proceeding, we first show that $x+\mu\approx y+\mu$ for some $x\in\nu_S, y\in S$ implies $y\in\nu_S$. In fact, it follows from $x+\mu\approx y+\mu$ that $(x+\mu)(y)\geq 0.5$ by Lemma 4.9. Now, if $y=x+a$ for some $a\in S$, since $x\in\nu_S$, that is, $M(\nu(0),0.5)=M(\nu(x),0.5)$, we have

$$\nu(y)\geq M(\nu(x),\nu(a),0.5)=M(\nu(0),\nu(a),0.5)\geq M(\nu(a),0.5)\geq M(\mu(a),0.5)$$

and so

$$\nu(y)\geq\bigvee_{y=x+a}M(\mu(a),0.5)=M((x+\mu)(y),0.5)\geq 0.5.$$

This implies $y\in\nu_S$. Thus $x\in Ker(\varphi)\Leftrightarrow x+\nu\approx\nu\Leftrightarrow\nu(x)\geq 0.5\Leftrightarrow x\in\nu_S\Leftrightarrow x+\mu\in\nu_S/\mu$. Hence $(S/\mu)/(\nu_S/\mu)=(S/\mu)/Ker(\varphi)\cong S/\nu$. □

## 5. Conclusions

In this paper, our aim is to promote research and the development of fuzzy technology by studying the fuzzy hemirings. The goal is to explain new methodological development in fuzzy hemirings which will also be of growing importance in the future. The obtained results can be applied to the other algebraic structures. Our future work on this topic will focus on studying the relationships among hemirings, BL-algebras and IS-algebras.

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