FUZZY REFLEXIVITY OF FELBIN'S TYPE FUZZY NORMED LINEAR SPACES AND FIXED POINT THEOREMS IN SUCH SPACES

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Abstract. An idea of fuzzy reflexivity of Felbin's type fuzzy normed linear spaces is introduced and its properties are studied. Concept of fuzzy uniform normal structure is given and using the geometric properties of this concept fixed point theorems are proved in fuzzy normed linear spaces.

1. **Introduction**

In 1984, Katsaras [14] first introduced a definition of fuzzy norm while he was studying fuzzy topological vector spaces. After that, Felbin [10] in 1992, framed an alternative definition of a fuzzy norm on a linear space with an associated metric of the Kaleva & Seikkala type [13]. A further development in this direction took place when in 1994, S.C.Cheng & J.N.Mordeson [8] evolved the definition of another type of fuzzy norm having a corresponding metric of the Kramosil & Michalek type [16]. In 2003, the present authors [1] offered another definition of fuzzy norm with a view to exploring the possibilities of arriving at yet another definition of the norm that might prove to be capable of more effective application in appropriate fields. By using this definition we have established many results of functional analysis in fuzzy setting (for references please see [2, 3, 4, 6, 18]). On the other hand many authors studied Felbin's type fuzzy normed linear spaces and established some results (for references please see [11, 12]). Actually after that the research in fuzzy functional analysis has gained a great momentum and it has become a highly potential field of research in fuzzy mathematics.

In crisp theory, reflexivity of a normed linear space is an important property based on which various results of functional analysis are developed. But not much work have been found towards the fuzzification of this concept. Very recently we have attempted to introduce fuzzy reflexivity of (B-S)-type fuzzy normed linear spaces [7].

In this paper, fuzzy reflexivity of Felbin's type fuzzy normed linear spaces is introduced. In doing so, firstly a relation is established between Felbin's type fuzzy norm and (B-S)-type fuzzy norm. In fact, it is shown that a Felbin type fuzzy norm can be extended as a pair of (B-S)-type fuzzy norms. Fuzzy reflexivity of Felbin

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type spaces is then introduced as a conjuction of reflexivities of (B-S)-type spaces. A concept of "fuzzy uniform normal structure" is introduced and some results on reflexivity of a fuzzy normed linear space are derived from this structure. It has been possible to extend the celebrated Kirk Theorem in fuzzy setting.

The organization of the paper is as follows:

Section 2 provides some preliminary results which are used in this paper. In Section 3, an idea of fuzzy reflexive space is developed and some properties of such spaces are studied. In the last Section 4, a concept of "fuzzy uniform normal structure" is introduced in a Felbin's type space and extend Kirk's fixed point theorem in such spaces.

2. **Preliminary Results**

A mapping $x : R \to [0 , 1]$ over the set R of all reals is called a fuzzy real number.
x is called non-negative if $x(t) = 0, \forall t < 0$.
For any real number r, $\bar{r}$ is defined by $\bar{r}(t) = 1$ if $t = r$ and $\bar{r}(t) = 0$ if $t \neq r$.
According to Felbin [10], we denote the set of all convex(i.e. $x(t) \geq min\,(x(s)\,,\,x(r))$ where $s \leq t \leq r$), normal (i.e. $x(t_0) = 1$ for some $t_0 \in R$), upper semicontinuous fuzzy real numbers by R(I) and the set of all non-negative, convex, normal, upper semicontinuous fuzzy real numbers by $R^*(I)$.

A partial ordering " $\preceq$ " in R(I) is defined by $\eta \preceq \delta$ if and only if $a^1_\alpha \leq a^2_\alpha$ and $b^1_\alpha \leq b^2_\alpha$ for all $\alpha \in (0 , 1]$ where $[\eta]_\alpha = [a^1_\alpha , b^1_\alpha]$ and $[\delta]_\alpha = [a^2_\alpha , b^2_\alpha]$. The strict inequality in R(I) is defined by $\eta \prec \delta$ if and only if $a^1_\alpha < a^2_\alpha$ and $b^1_\alpha < b^2_\alpha$ for each $\alpha \in (0 , 1]$.

Arithmetic operations of fuzzy numbers are defined according to Mizumoto and Tanaka [17].

Taking in particular, Upper and Lower functions as "Max" and "Min", the definition of Felbin's type fuzzy norm is as follows:

Definition 2.1. [10] Let X be a vector space over R and $|| \; || : X \to R^*(I)$ be a mapping.
Let $[||x||]_\alpha = [||x||^1_\alpha \, , \, ||x||^2_\alpha]$ for $x \in X,\ 0 < \alpha \leq 1$ and suppose for all $x \in X,\ x \neq \underline{0}$, there exists $\alpha_0 \in (0 , 1]$ independent of x such that for all $\alpha \leq \alpha_0$,

(A) $||x||^2_\alpha < \infty$,

(B) $\inf ||x||^1_\alpha > 0$.

Then $(X \, , \, || \; ||)$ is called a fuzzy normed linear space and $|| \; ||$ is a fuzzy norm if

(i) $||x|| = \bar{0}$ if and only if $x = \underline{0}$ (the null vector),
(ii)$||rx|| = |r|||x||,\ x \in X,\ r \in R$,
(iii) for all $x, y \in X$,

$||x + y|| \preceq ||x|| \oplus ||y||$ where according to Mizumoto and Tanaka [16], $\oplus$ is defined by
$(x \oplus y)(t) = Sup_{s \in R} min\,\{x(s)\,,\,y(t - s)\},\ t \in R.$

Definition 2.2. [10] Let $(X\ ,\ ||\ ||)$ be a fuzzy normed linear space. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is said to converge to $x \in X$ denoted by $\lim_{n\to\infty} x_n = x$ if and only if $\lim_{n\to\infty} ||x_n - x|| = \bar{0}$

i.e. $\lim_{n\to\infty} ||x_n - x||_\alpha^1 = \lim_{n\to\infty} ||x_n - x||_\alpha^2 = 0,\ \forall\alpha \in (0\ ,\ 1]$.

$\{x_n\}$ is called a Cauchy sequence if $\lim_{m,n\to\infty} ||x_n - x_m|| = \bar{0}$.

i.e. if $\lim_{m,n\to\infty} ||x_n - x_m||_\alpha^1 = \lim_{m,n\to\infty} ||x_n - x_m||_\alpha^2 = 0,\ \forall\alpha \in (0\ ,\ 1]$.

A fuzzy normed linear space $(X\ ,\ ||\ ||)$ is said to be complete if every Cauchy sequence in X converges in X.

Proposition 2.3. [5] *Let* $(X\ ,\ ||\ ||)$ *be a Felbin-fuzzy normed linear space and* $[||x||]_\alpha = [||x||_\alpha^1\ ,\ ||x||_\alpha^2],\ \alpha \in (0,1]$.

Define

$$N_1(x,t) = \begin{cases} \bigvee\{\alpha \in (0,1] :\ ||x||_\alpha^1 \le t\} & \text{when } (x,t) \ne (\underline{0},0) \\ 0 & \text{when } (x,t) = (\underline{0},0). \end{cases} \tag{1}$$

Then N_1 *is a B-S-fuzzy norm satisfying (N6).*

Again, if we define

$$||x||'_\alpha = \bigwedge\{t > 0 :\ N_1(x\ ,\ t) \ge \alpha\},\ \alpha \in (0,1], \tag{2}$$

then $||x||'_\alpha$ *is a norm on X and* $||x||_\alpha^1 = ||x||'_\alpha\ \forall\alpha \in (0,1]$.

Definition 2.4. [1] Let X be a linear space over F(field of real/complex numbers).A fuzzy subset N of $X \times R$(R-the set of all real numbers) will be called a *B-S-fuzzy norm* on X if and only if $\forall x, u \in U$ and $c \in F$

(N1) $\forall t \in R$ with $t \le 0,\ N(x,t) = 0$,

(N2) $(\forall t \in R,\ t > 0,\ N(x,t) = 1)$ iff $x = \underline{0}$,

(N3) $\forall t \in R,\ t > 0,\ N(cx,t) = N(x, \frac{t}{|c|})$ if $c \ne 0$,

(N4) $\forall s,t \in R,\ x,u \in X$

$N(x+u, s+t) \ge \min\{N(x,s), N(u,t)\}$,

(N5)$N(x,.)$ is a non-decreasing function of R and $\lim_{t\to\infty} N(x,t) = 1$.

The pair (X,N) will be referred to as a *B-S-fuzzy normed linear space.*

Theorem 2.5. [1] *Let*(X,N) *be a B-S-fuzzy normed linear space.*
Assume further that

(**N6**) $N(x,t) > 0\ \forall t > 0$ *implies* $x = \underline{0}$.

Define $||x||_\alpha = \bigwedge\{t :\ N(x,t) \ge \alpha\},\ \alpha \in (0,1)$.

Then $\{||\ ||_\alpha : \alpha \in (0,1)\}$ *is an ascending family of norms on X. We call these norms as* α*-norms on X corresponding to the B-S-fuzzy norm N on X.*

Definition 2.6. [3] Let $(U\ ,\ N)$ be a fuzzy normed linear space. A subset A of U is said to be level fuzzy bounded (l-fuzzy) if for any $\alpha \in (0\ ,\ 1)$, $\exists t(\alpha) > 0$ such that $N(x\ ,\ t(\alpha)) \ge \alpha\ \forall x \in A$.

Proposition 2.7. [3] *Let $(U\ ,\ N)$ be a fuzzy normed linear space satisfying (N6). Then a subset A of U is l-fuzzy bounded iff A is bounded with respect to $||\ ||_\alpha$, for all $\alpha \in (0\ ,\ 1)$, where $||\ ||_\alpha$ denotes the α-norm of N.*

Definition 2.8. [3] Let $(U\ ,\ N)$ be a fuzzy normed linear space. A subset F of U is said to be l-fuzzy closed if for each $\alpha \in (0\ ,\ 1)$ and for any sequence $\{x_n\}$ in F and $x \in U, (\underline{lim}_{n\to\infty} N(x_n - x\ ,\ t) \geq \alpha\ \forall t > 0) \Rightarrow x \in F$.

Proposition 2.9. [3] *Let $(U\ ,\ N)$ be a fuzzy normed linear space satisfying (N6) and $F \subset U$. Then F is l-fuzzy closed iff F is closed w.r.t. $||\ ||_\alpha$ ($\alpha-$norm of N) for each $\alpha \in (0\ ,\ 1)$.*

Definition 2.10. [7] Let $(U\ ,\ N)$ be a fuzzy normed linear space. U is said to have fuzzy uniform normal structure if $\exists\ k \in (0\ ,\ 1)$ such that

$$\bigwedge_{\alpha\in(0,1)} \bigvee_{\beta\geq\alpha} [\bigvee_{D\in\mathcal{D}} \{\frac{\bigwedge_{u\in D}\{\bigwedge\{t > 0\ : N(u - v\ ,\ t) \geq \beta,\ \forall v \in D\}\}}{\bigwedge\{t > 0\ : N(u - v\ ,\ t) \geq \beta,\ \forall u, v \in D\}}\}] < k$$

where $\mathcal{D}$ is the collection of all nonempty, convex, l-fuzzy closed and l-fuzzy bounded subsets of U.

Theorem 2.11. [7] *Let $(U\ ,\ N)$ be a fuzzy normed linear space satisfying (N6). If U has fuzzy uniform normal structure, then $\exists \alpha \in (0\ ,\ 1)$ such that U has uniform normal structure w.r.t. $||\ ||_\beta\ \forall \beta \geq \alpha$, where $||\ ||_\alpha$ denotes the $\alpha-$norm of N.*

Theorem 2.12. [7] *Let $(U\ ,\ N)$ be α-complete fuzzy normed linear space for each $\alpha \in (0\ ,\ 1)$ satisfying (N6). If U has fuzzy uniform normal structure then it is fuzzy reflexive.*

Definition 2.13. [3] Let $T\ :\ X \to X$ be a mapping where $(X\ ,\ ||\ ||)$ is a fuzzy normed linear space. T is said to be fuzzy nonexpansive if $||Tx - Ty|| \preceq ||x - y||\ \forall x, y \in X$.

Proposition 2.14. [3] *Let $(X\ ,\ ||\ ||)$ be a fuzzy normed linear space. A mapping $T\ :\ X \to X$ is fuzzy nonexpansive iff T is nonexpansive w.r.t.$||\ ||^1_\alpha$ and $||\ ||^2_\alpha\ \forall \alpha \in (0\ ,\ 1]$.*

Theorem 2.15. [15] *Let K be a nonempty, weakly compact, convex subset of a Banach space and supose K has normal structure. Then every nonexpansive mapping $T : K \to K$ has a fixed point.*

3. **Fuzzy Reflexive Space**

In this section ideas of left fuzzy reflexive space and right fuzzy reflexive space are introduced and some relevant properties are studied.

Definition 3.1. Let $(X\ ,\ ||\ ||)$ be a fuzzy normed linear space and $\alpha \in (0\ ,\ 1]$.

(i) A sequence $\{x_n\}$ is said to be left α-convergent and converges to x if $||x_n - x||_\alpha^1 \to 0$ as $n \to \infty$.

(ii) A sequence $\{x_n\}$ is said to be right α-convergent and converges to x if $||x_n - x||_\alpha^2 \to 0$ as $n \to \infty$.

(iii) A sequence $\{x_n\}$ is said to be left α-Cauchy $||x_n - x_m||_\alpha^1 \to 0$ as $m, n \to \infty$.

(iv) A sequence $\{x_n\}$ is said to be right α-Cauchy $||x_n - x_m||_\alpha^2 \to 0$ as $m, n \to \infty$.

Note 3.2. Each right α-convergent sequence is also left α-convergent sequence. Each right α-Cauchy sequence is also left α-Cauchy sequence.

Definition 3.3. Let $(X\ ,\ ||\ ||)$ be a fuzzy normed linear space and $\alpha \in (0\ ,\ 1]$.

(i) X is said to be left α-complete, if every left α-Cauchy sequence in X is left α-convergent to some point in X.

(ii) X is said to be right α-complete, if every right α-Cauchy sequence in X is right α-convergent to some point in X.

(iii) X is said to be α-complete, if it is both right α-complete and left α-complete.

Definition 3.4. Let $(X\ ,\ ||\ ||)$ be a fuzzy normed linear space and $\alpha \in (0\ ,\ 1]$.

(i) A subset F of X is said to be left α-closed if for any sequence $\{x_n\}$ in F, $||x_n - x||_\alpha^1 \to 0$ as $n \to \infty$ implies $x \in F$.

(ii) A subset F of X is said to be right α-closed if for any sequence $\{x_n\}$ in F, $||x_n - x||_\alpha^2 \to 0$ as $n \to \infty$ implies $x \in F$.

Note 3.5. F is right α-closed if it is left α-closed.

Notation 3.6. Let $(X\ ,\ ||\ ||)$ be a fuzzy normed linear space. Then $(X\ ,\ ||\ ||_\alpha^1)$ and $(X\ ,\ ||\ ||_\alpha^2)$ are normed linear spaces for each $\alpha \in (0\ ,\ 1]$. We denote by $(X_\alpha^{*1}\ ,\ ||\ ||_\alpha^{*1})$ and $(X_\alpha^{*2}\ ,\ ||\ ||_\alpha^{*2})$ the first conjugate space of $(X\ ,\ ||\ ||_\alpha^1)$ and $(X\ ,\ ||\ ||_\alpha^2)$ respectively; by X_α^{**1} and X_α^{**2} the second conjugate spaces of $(X\ ,\ ||\ ||_\alpha^1)$ and $(X\ ,\ ||\ ||_\alpha^2)$ respectively.
For a mapping $f : X \to Y$ and for a subset A of X, f/A denotes the restriction of f on A.

Proposition 3.7. *Let* $(X\ ,\ ||\ ||)$ *be a fuzzy normed linear space.*
Then for $\alpha_2 > \alpha_1,\ \alpha_1, \alpha_2 \in (0\ ,\ 1]$,
(i) $X_{\alpha_1}^{*1} \subset X_{\alpha_2}^{*1}$,
(ii) $F \in X_{\alpha_2}^{**1} \Rightarrow F/X_{\alpha_1}^{*1} \in X_{\alpha_1}^{**1}$,
(iii) $X_{\alpha_2}^{*2} \subset X_{\alpha_1}^{*2}$,
(iv) $F \in X_{\alpha_1}^{**2} \Rightarrow F/X_{\alpha_2}^{*2} \in X_{\alpha_2}^{**2}$,
(v) $X_\alpha^{*1} \subset X_\alpha^{*2}\ \forall \alpha \in (0\ ,\ 1]$,
(vi) $F \in X_\alpha^{**2} \Rightarrow F/X_\alpha^{*1} \in X_\alpha^{**1}$.

Proof. Take $\alpha_2 > \alpha_1,\ \alpha_1, \alpha_2 \in (0\ ,\ 1]$.

(i) Let $f \in X^{*1}_{\alpha_1}$.

Then $|f(x)| \leq \ ||f||^{*1}_{\alpha_1}||x||^1_{\alpha_1} \leq ||f||^{*1}_{\alpha_1}||x||^1_{\alpha_2} \ \forall x \in X$.
$\Rightarrow \ f \in X^{*1}_{\alpha_2}$
So $X^{*1}_{\alpha_1} \subset X^{*1}_{\alpha_2}$.

(ii) For $\forall f \in X^{*1}_{\alpha_1}$,

$$||f||^{*1}_{\alpha_2} \ = \bigvee_{x \in X, ||x||^1_{\alpha_2} \neq 0} \frac{|f(x)|}{||x||^1_{\alpha_2}} \ \leq \bigvee_{x \in X, ||x||^1_{\alpha_2} \neq 0} \frac{|f(x)|}{||x||^1_{\alpha_1}} \ = ||f||^{*1}_{\alpha_1}.$$

So $||f||^{*1}_{\alpha_2} \ \leq ||f||^{*1}_{\alpha_1} \ \forall f \in X^{*1}_{\alpha_1}$.
Again,
$F \in X^{**1}_{\alpha_2}$
$\Rightarrow$ F is bounded and linear over $X^{*1}_{\alpha_2}$
$\Rightarrow \ \exists k > 0$ such that $|F(f)| \ \leq k||f||^{*1}_{\alpha_2} \ \forall f \in X^{*1}_{\alpha_2}$
$\Rightarrow |F(f)| \ \leq k||f||^{*1}_{\alpha_1} \ \forall f \in X^{*1}_{\alpha_1}$
(Since $X^{*1}_{\alpha_1} \subset X^{*1}_{\alpha_2}$ and $||f||^{*1}_{\alpha_2} \ \leq ||f||^{*1}_{\alpha_1} \ \forall f \in X^{*1}_{\alpha_1}$).
$\Rightarrow \ F/X^{*1}_{\alpha_1} \in X^{**1}_{\alpha_1}$
So $F \in X^{**1}_{\alpha_2} \Rightarrow F/X^{*1}_{\alpha_1} \in X^{**1}_{\alpha_1}$.

(iii) Can be obtained by approaching similar to the proof of (i).

(iv) is similar to that of (ii).

(v) Let $f \in X^{*1}_{\alpha}$.

Then $|f(x)| \leq \ ||f||^{*1}_{\alpha}||x||^1_{\alpha} \ \forall x \in X$.
i.e. $|f(x)| \leq \ ||f||^{*1}_{\alpha}||x||^2_{\alpha} \ \forall x \in X$.
$\Rightarrow \ f \in X^{*2}_{\alpha}$
So $X^{*1}_{\alpha} \subset X^{*2}_{\alpha} \ \forall \alpha \in (0 \ , \ 1]$.

(vii) For $\forall f \in X^{*2}_{\alpha}$,

$$||f||^{*2}_{\alpha} \ = \bigvee_{x \in X, ||x||^2_{\alpha} \neq 0} \frac{|f(x)|}{||x||^2_{\alpha}} \ \leq \bigvee_{x \in X, ||x||^1_{\alpha} \neq 0} \frac{|f(x)|}{||x||^1_{\alpha}} \ = ||f||^{*1}_{\alpha}.$$

Now,

$F \in X^{**2}_{\alpha}$
$\Rightarrow$ F is bounded and linear over X^{*2}_{α}
$\Rightarrow \ \exists k > 0$ such that $|F(f)| \ \leq k||f||^{*2}_{\alpha} \ \forall f \in X^{*2}_{\alpha}$
$\Rightarrow |F(f)| \ \leq k||f||^{*1}_{\alpha} \ \forall f \in X^{*1}_{\alpha}$
$\Rightarrow \ F/X^{*1}_{\alpha} \in X^{**1}_{\alpha}$
So $F \in X^{**2}_{\alpha} \Rightarrow F/X^{*1}_{\alpha} \in X^{**1}_{\alpha}$. □

Remark 3.8. If $F \in X^{**1}_{\alpha_2}$, then

$$||F/X^{*1}_{\alpha_1}||^{**1}_{\alpha_1} = \bigvee_{f \in X^{*1}_{\alpha_1}, f \neq 0} \frac{|F/X^{*1}_{\alpha_1}(f)|}{||f||^{*1}_{\alpha_1}} \leq \bigvee_{f \in X^{*1}_{\alpha_1}, f \neq 0} \frac{|F(f)|}{||f||^{*1}_{\alpha_2}} \leq \bigvee_{f \in X^{*1}_{\alpha_2}, f \neq 0} \frac{|F(f)|}{||f||^{*1}_{\alpha_2}} = ||F||^{**1}_{\alpha_2}.$$

Thus $||F/X^{*1}_{\alpha_1}||^{**1}_{\alpha_1} \ \leq \ ||F||^{**1}_{\alpha_2} \ \forall F \in X^{**1}_{\alpha_2}$.
Similarly $||F/X^{*2}_{\alpha_2}||^{**2}_{\alpha_2} \ \leq \ ||F||^{**2}_{\alpha_1} \ \forall F \in X^{**2}_{\alpha_1}$.

Remark 3.9. $F_\alpha^{x2}(f) = f(x)\ \forall f \in X_\alpha^{*1} \subset X_\alpha^{*2}$.
i.e. $F_\alpha^{x2}/X_\alpha^{*1} = F_\alpha^{x1}$.

For fixed $x \in X$, define for each $\alpha \in (0\ ,\ 1]$, a functional F_α^{x1} on X_α^{*1} by $F_\alpha^{x1}(f) = f(x)$, $\forall f \in X_\alpha^{*1}$.

Clearly F_α^{x1} is linear over X_α^{*1}.

Also $|F_\alpha^{x1}(f)|\ = |f(x)|\ \leq ||f||_\alpha^{*1}||x||_\alpha^1\ \forall f \in X_\alpha^{*1}$.

Thus F_α^{x1} is a bounded linear functional over X_α^{*1}.

i.e. $F_\alpha^{x1} \in X_\alpha^{**1}$.

For each $\alpha \in (0\ ,\ 1]$, define a mapping $C_\alpha^1 :\ X \to X_\alpha^{**1}$ by $C_\alpha^1(x)\ = F_\alpha^{x1}\ \forall x \in X$.

Now for two scalars a and b, $F_\alpha^{(ax+by)1}(f)\ =\ f(ax+by)$
$=\ af(x) + bf(y)$
$=\ aF_\alpha^{x1}(f)\ +\ bF_\alpha^{y1}(f)$
$=\ (aF_\alpha^{x1}\ +\ bF_\alpha^{y1})(f)\ \forall f \in X_\alpha^{*1}$.

Thus $C_\alpha^1(ax+by)\ =\ F_\alpha^{(ax+by)1}\ =\ aF_\alpha^{x1}\ +\ bF_\alpha^{y1}\ =\ aC_\alpha^1(x) + bC_\alpha^1(y)$.
So C_α^1 is a linear operator.

Further
$||F_\alpha^{x1}||_\alpha^{**1}\ =\ Sup\{\frac{|F_\alpha^{x1}(f)|}{||f||_\alpha^{*1}} :\ f \in X_\alpha^{*1},\ ||f||_\alpha^{*1} \neq 0\}$

$=\ Sup\{\frac{|f(x)|}{||f||_\alpha^{*1}} :\ f \in X_\alpha^{*1},\ ||f||_\alpha^{*1} \neq 0\}$

$=\ ||x||_\alpha^1$.
i.e. $||C_\alpha^1(x)||_\alpha^1\ =\ ||x||_\alpha^1,\ \forall x \in X$.

Thus C_α^1 is isomorphically isometric between the space $(X\ ,\ ||\ ||_\alpha^1)$ and the subspace $C_\alpha^1(X)$ of X_α^{**1} and $||C_\alpha^1||_\alpha^1\ =\ 1,\ \forall \alpha \in (0\ ,\ 1]$.

Remark 3.10. For each $\alpha \in (0\ ,\ 1]$, if we define a mapping $C_\alpha^2 :\ X \to X_\alpha^{**2}$ by $C_\alpha^2(x)\ =\ F_\alpha^{x2}\ \forall x \in X$ then approaching as above, it can be shown that C_α^2 is isomorphically isometric between the space $(X\ ,\ ||\ ||_\alpha^2)$ and the subspace $C_\alpha^2(X)$ of X_α^{**2} and $||C_\alpha^2||_\alpha^2\ =\ 1\ \forall \alpha \in (0\ ,\ 1]$.

Definition 3.11. Let $(X\ ,\ ||\ ||)$ be a fuzzy normed linear space. X is said to be left fuzzy reflexive if $\exists \alpha_1 \in (0\ ,\ 1]$ such that C_α^1 is onto, $\forall \alpha \geq \alpha_1$.
i.e. $C_\alpha^1(X)\ =\ X_\alpha^{**1}\ \forall \alpha \geq \alpha_1$.

Definition 3.12. Let $(X\ ,\ ||\ ||)$ be a fuzzy normed linear space. X is said to be right fuzzy reflexive if $\exists \alpha_2 \in (0\ ,\ 1]$ such that C_α^2 is onto, $\forall \alpha \leq \alpha_2$.
i.e. $C_\alpha^2(X)\ =\ X_\alpha^{**2}\ \forall \alpha \geq \alpha_2$.

Definition 3.13. Let $(X\ ,\ ||\ ||)$ be a fuzzy normed linear space. X is said to be fuzzy reflexive if it is both left and right fuzzy reflexive. i.e. $\exists \beta_1,\ \beta_2 \in (0\ ,\ 1)$ such that

$C_\alpha^1(X)\ =\ X_\alpha^{**1}\ \forall \alpha \geq \beta_1$. and

$C_\alpha^2(X)\ =\ X_\alpha^{**2}\ \forall \alpha \leq \beta_2$.

Proposition 3.14. *Let $(X\ ,\ ||\ ||)$ be a fuzzy normed linear space. If X is of finite dimension then it is fuzzy reflexive.*

Proof. Note that $(X\ ,\ ||\ ||_\alpha^1)$ and $(X\ ,\ ||\ ||_\alpha^2)$ are normed linear spaces $\forall \alpha \in (0\ ,\ 1]$. Since X is of finite dimension thus $(X\ ,\ ||\ ||_\alpha^1)$ and $(X\ ,\ ||\ ||_\alpha^2)$ are reflexive $\forall \alpha \in (0\ ,\ 1]$.
i.e. if C_α^1 and C_α^2 are canonical mappings we have, $C_\alpha^1(X) = X_\alpha^{**1}$ and $C_\alpha^2(X) = X_\alpha^{**2}$ $\forall \alpha \in (0\ ,\ 1]$.
From definition it follows that X is fuzzy reflexive. □

Proposition 3.15. *Let $(X\ ,\ ||\ ||)$ be an α-complete fuzzy normed linear space for each $\alpha \in (0\ ,\ 1]$. Then X is fuzzy reflexive iff $\exists \beta_1,\ \beta_2\ \in (0\ ,\ 1)$ such that X_α^{*1} is reflexive $\forall \alpha \geq \beta_1$ and X_α^{*2} is reflexive $\forall \alpha \leq \beta_2$.*

Proof. First suppose that X is fuzzy reflexive. Given that $(X\ ,\ ||\ ||)$ is an α-complete fuzzy normed linear space for each $\alpha \in (0\ ,\ 1]$.
So $(X\ ,\ ||\ ||_\alpha^1)$ and $(X\ ,\ ||\ ||_\alpha^2)$ are Banach spaces $\forall \alpha \in (0\ ,\ 1]$.
Since X is fuzzy reflexive thus $\exists \beta_1,\ \beta_2\ \in (0\ ,\ 1)$ such that $(X\ ,\ ||\ ||_\alpha^1)$ is reflexive $\forall \alpha \geq \beta_1$ and $(X\ ,\ ||\ ||_\alpha^2)$ is reflexive $\forall \alpha \leq \beta_2$.
Further a Banach space X is reflexive iff X^* is reflexive.
So $\exists \beta_1,\ \beta_2\ \in (0\ ,\ 1)$ such that X_α^{*1} is reflexive $\forall \alpha \geq \beta_1$ and X_α^{*2} is reflexive $\forall \alpha \leq \beta_2$.
Converse part follows easily. □

Proposition 3.16. *Let $(X\ ,\ ||\ ||)$ be a fuzzy normed linear space. If X is fuzzy reflexive and Y is left α-closed subspace of X for each $\alpha \in (0\ ,\ 1]$, then Y is also fuzzy reflexive.*

Proof. Note that $(X\ ,\ ||\ ||_\alpha^1)$ and $(X\ ,\ ||\ ||_\alpha^2)$ are normed linear spaces for each $\alpha \in (0\ ,\ 1]$.
Since X is fuzzy reflexive, $\exists \beta_1,\ \beta_2\ \in (0\ ,\ 1)$ such that X is reflexive w.r.t. the norm $||\ ||_\alpha^1$ $\forall \alpha \geq \beta_1$ and it is also reflexive w.r.t. the norm $||\ ||_\alpha^2$ $\forall \alpha \leq \beta_2$.
Further, Y being left α-closed, it is also closed w.r.t. $||\ ||_\alpha^1$ and $||\ ||_\alpha^2$ $\forall \alpha \in (0\ ,\ 1]$.
Hence Y is reflexive w.r.t. $||\ ||_\alpha^1$ $\forall \alpha \geq \beta_1$ and w.r.t. $||\ ||_\alpha^2$ $\forall \alpha \leq \beta_2$.
So Y is fuzzy reflexive. □

Proposition 3.17. *Let $(X\ ,\ ||\ ||)$ be a fuzzy normed linear space. If X is fuzzy reflexive then X is fuzzy complete.*

Proof. Since X is fuzzy reflexive, $\exists \beta_1,\ \beta_2\ \in (0\ ,\ 1)$ such that $(X\ ,\ ||\ ||_\alpha^1)$ is isometrically isomorphic to $(X_\alpha^{**1}\ ,\ ||\ ||_\alpha^{**1})$ $\forall \alpha \geq \beta_1$ and $(X\ ,\ ||\ ||_\alpha^2)$ is isometrically isomorphic to $(X_\alpha^{**2}\ ,\ ||\ ||_\alpha^{**2})$ $\forall \alpha \leq \beta_2$.
Since $(X_\alpha^{**1}\ ,\ ||\ ||_\alpha^{**1})$ and $(X_\alpha^{**2}\ ,\ ||\ ||_\alpha^{**2})$ are Banach spaces $\forall \alpha \in (0\ ,\ 1]$, it follows that
for $\alpha \geq \beta_1,\ (X\ ,\ ||\ ||\alpha^1)$ is a Banach space and for $\alpha \leq \beta_2,\ (X\ ,\ ||\ ||\alpha^2)$ is so.
Let $\{x_n\}$ be a Cauchy sequence in $(X\ ,\ ||\ ||)$.
Then $\lim\limits_{m,n\to\infty} ||x_n - x_m|| = \bar{0}$.

i.e. $\lim\limits_{m,n\to\infty} ||x_n - x_m||^1_\alpha \ = \lim\limits_{m,n\to\infty} ||x_n - x_m||^2_\alpha \ = 0 \ \forall\alpha \in (0\ ,\ 1]$.
Since $\forall\alpha \geq \beta_1,\ (X\ ,\ ||\ ||^1_\alpha)$ is a Banach space, $\exists x^\alpha$(say) such that
$||x_n - x^\alpha||^1_\alpha \to 0$ as $n \to \infty\ \forall\alpha \geq \beta_1$. (3)

Firstly it will be shown that x^α is independent of α.
In fact for $\beta\ \geq\ \alpha\ \geq \beta_1$,
$||x^\alpha - x^\beta||^1_\alpha \ = ||x^\alpha - x_n + x_n - x^\beta||^1_\alpha$.
$\leq ||x^\alpha - x_n||^1_\alpha\ +\ ||x_n - x^\beta||^1_\alpha$.
$\leq ||x^\alpha - x_n||^1_\alpha\ +\ ||x_n - x^\beta||^1_\beta$.
i.e. $||x^\alpha - x^\beta||^1_\alpha \to 0$ as $n \to \infty$ (by (3)).
i.e. $x^\alpha\ =\ x^\beta\ \forall\beta\ \geq\ \alpha\ \geq \beta_1$.
Thus the limit x^α is independent of α and hence $\exists x$ such that
$||x_n - x||^1_\alpha \to 0$ as $n \to \infty\ \forall\alpha \geq \beta_1$
and hence $||x_n - x||^1_\alpha \to 0$ as $n \to \infty\ \forall\alpha \in (0\ ,\ 1]$. (4)

(Since $||\ ||^1_\alpha$ is increasing w.r.t. α).
Arguing as above it can be shown that, for $\alpha \leq \beta_2,\ \exists y$ (independent of α) such that
$||x_n - y||^2_\alpha \to 0$ as $n \to \infty$ and hence
$||x_n - y||^2_\alpha \to 0$ as $n \to \infty\ \forall\alpha \in (0\ ,\ 1]$. (5)

(Since $||\ ||^2_\alpha$ is decreasing w.r.t. α).
We claim that $x\ = y$.
In fact, $||x - y||^1_\alpha\ \leq ||x - x_n||^1_\alpha\ +\ ||x_n - y||^1_\alpha$
$\leq ||x - x_n||^1_\alpha\ +\ ||x_n - y||^2_\alpha\ \forall\alpha \in (0\ ,\ 1]$.
This implies that $||x - y||^1_\alpha\ = 0$ (by (4) and (5)).
i.e. $x\ =\ y$.
Thus $\exists x \in X$ such that $||x_n - x||^1_\alpha \to 0$ and $||x_n - x||^2_\alpha \to 0$ as $n \to \infty$.
i.e. $\lim\limits_{n\to\infty} ||x_n - x||\ = \bar{0}$.
Hence $(X\ ,\ ||\ ||)$ is complete. □

4. **Fuzzy Uniform Normal Structure**

In this section relations between Felbin's type fuzzy norm and (B-S)-type fuzzy norm are formulated. Also a geometric property viz. uniform normal structure is introduced in fuzzy normed linear spaces and its properties are studied . Using this concept Kirk's type fixed point theorem is proved in fuzzy normed linear spaces.

Theorem 4.1. *Let $(X\ ,\ ||\ ||)$ be a fuzzy normed linear space.*
Let $[||x||]_\alpha\ =\ [||x||^1_\alpha\ ,\ ||x||^2_\alpha],\ \alpha \in (0\ ,\ 1]$.
Assume that,
(*) *for any sequence $\{\alpha_n\}$ in $(0\ ,\ 1)$ with $\alpha_n \downarrow \alpha_0$ where $\alpha_0 \in (0\ ,\ 1)$ implies $||x||^2_{\alpha_n} \to ||x||^2_{\alpha_0}$.*

Let N_1 and N_2 be two functions from $X \times R$ to $[0\ ,\ 1]$ defined by

$$N_1(x,t) = \begin{cases} \bigvee\{\alpha \in (0,1): \ ||x||^1_\alpha \leq t\} & \textit{when } (x,t) \neq (\underline{0},0) \\ 0 & \textit{when } (x,t) = (\underline{0},0). \end{cases} \tag{6}$$

and

$$N_2(x,t) = \begin{cases} \bigvee\{\beta \in (0,1): \ ||x||^2_{1-\beta} \leq t\} & \textit{when } (x,t) \neq (\underline{0},0) \\ 0 & \textit{when } (x,t) = (\underline{0},0). \end{cases} \tag{7}$$

Then N_1 and N_2 are B-S-fuzzy norms satisfying (N6).
If further,

$$||x||'_\alpha \ = \bigwedge\{t \ > 0: \ N_1(x \ , \ t) \ \geq \alpha\}, \ \alpha \in (0,1) \tag{8}$$

and

$$||x||''_\alpha \ = \bigwedge\{t \ > 0: \ N_2(x \ , \ t) \ \geq \alpha\}, \ \alpha \in (0,1) \tag{9}$$

then $||x||^1_\alpha \ = \ ||x||'_\alpha$ *and* $||x||^2_{1-\alpha} \ = \ ||x||''_\alpha \ \forall \alpha \in (0 \ , \ 1)$.

Proof. $\forall t \ > 0, \ N_1(x \ , \ t) \ > 0$
$\Rightarrow \forall t \ > 0, \ \bigvee\{\alpha \in (0,1): \ ||x||^1_\alpha \leq t\} \ > 0$
$\Rightarrow \forall t \ > 0, \ \exists \alpha = \alpha(t) \in (0,1)$ such that $||x||^1_\alpha \leq t$
$\Rightarrow Inf_{\alpha \in (0,1]} ||x||^1_\alpha \ = 0$
$\Rightarrow x = \underline{0}$ (by using the condition (B) of *Felbin's fuzzy* norm).
Hence N_1 satisfies (N6) condition.
Similarly we can show that N_2 satiesfies (N6) condition.

The relation $||x||^1_\alpha \ = \ ||x||'_\alpha \ \forall \alpha \in (0 \ , \ 1)$ follows from the Proposition 2.3.
We shall show that $||x||^2_{1-\alpha} \ = ||x||''_\alpha \ \forall \alpha \in (0 \ , \ 1)$.
If $x = \underline{0}$ then $||x||^2_{1-\alpha} \ = ||x||''_\alpha \ \forall \alpha \in (0 \ , \ 1)$.
So we suppose that $x \neq \underline{0}$.
Choose $\alpha_0 \in (0 \ , \ 1)$ and put $||x||^2_{1-\alpha_0} \ = t_0$. Then $t_0 > 0$.
From (7), it follows that $N_2(x \ , \ t_0) \ \geq \alpha_0$
$\Rightarrow ||x||''_{\alpha_0} \leq t_0$ by (9)

$$\Rightarrow ||x||''_{\alpha_0} \leq ||x||^2_{1-\alpha_0} \tag{10}$$

Next $||x||''_{\alpha_0} \ < r$
$\Rightarrow N_2(x \ , \ r) \ \geq \ \alpha_0$ by (9)

Case I: If $N_2(x \ , \ r) \ > \ \alpha_0$ then $||x||^2_{1-\alpha_0} \leq r$ by (7)

$$\text{So } ||x||''_{\alpha_0} \geq ||x||^2_{1-\alpha_0} \tag{11}$$

Case II: If $N_2(x \ , \ r) \ = \ \alpha_0$.
Then $\exists$ a sequence $\{\alpha_n\}$ in $(0 \ , \ 1)$ such taht $\alpha_n \uparrow \alpha_0$ and $||x||^2_{1-\alpha_n} \leq r$.
Thus $||x||^2_{1-\alpha_0} \leq r$ by (*).

$$\text{So } ||x||''_{\alpha_0} \geq ||x||^2_{1-\alpha_0} \tag{12}$$

From (10), (11) and (12) we get $||x||''_{\alpha_0} \ = \ ||x||^2_{1-\alpha_0}$.

Since $\alpha_0 \in (0 \ , \ 1)$ is arbitrary, we have $||x||''_\alpha \ = \ ||x||^2_{1-\alpha} \ \forall \alpha \in (0 \ , \ 1)$. □

Definition 4.2. Let $(X\ ,\ ||\ ||)$ be a fuzzy normed linear space. Then X is said to have fuzzy uniform normal structure if $\exists k,\ 0 < k < 1$ such that $\gamma_1(\mathcal{D}_1) \bigvee \gamma_2(\mathcal{D}_2)\ <\ k$ where

$$\gamma_1(\mathcal{D}_1) = \bigwedge_{\alpha\in(0,1)} \bigvee_{\beta\geq\alpha} [\bigvee_{D\in\mathcal{D}_1} \{ \frac{\bigwedge_{u\in D} \{\bigwedge\{t > 0\ : N_1(u - v\ ,\ t) \geq \beta,\ \forall v \in D\}\}}{\bigwedge\{t > 0\ : N_1(u - v\ ,\ t) \geq \beta,\ \forall u, v \in D\}} \}]$$

where $\mathcal{D}_1$ is the collection of all nonempty, convex, l-fuzzy bounded and l-fuzzy closed subsets of X and N_1 is defined in as Theorem 4.1 and

$$\gamma_2(\mathcal{D}_2) = \bigwedge_{\alpha\in(0,1)} \bigvee_{\beta\leq\alpha} [\bigvee_{D'\in\mathcal{D}_2} \{ \frac{\bigwedge_{u\in D'} \{\bigwedge\{t > 0\ : N_2(u - v\ ,\ t) \geq \beta,\ \forall v \in D'\}\}}{\bigwedge\{t > 0\ : N_2(u - v\ ,\ t) \geq \beta,\ \forall u, v \in D'\}} \}]$$

where $\mathcal{D}_2$ is the collection of all nonempty, convex, l-fuzzy bounded and l-fuzzy closed subsets of X and N_2 is defined as in Theorem 4.1.

Theorem 4.3. *Let $(X\ ,\ ||\ ||)$ be a fuzzy normed linear space. If X has fuzzy uniform normal structure then $\exists \beta_1,\ \beta_2 \in (0\ ,\ 1)$ such that X has uniform normal structure w.r.t. $||\ ||_\alpha^1\ \forall\alpha \geq \beta_1$ and X has uniform normal structure w.r.t. $||\ ||_\alpha^2\ \forall\alpha \leq \beta_2$.*

Proof. Suppose X has fuzzy uniform normal structure. Thus $\exists k,\ 0 < k < 1$ such that $\gamma_1(\mathcal{D}_1) \bigvee \gamma_2(\mathcal{D}_2)\ <\ k$ where $\gamma_1(\mathcal{D}_1)$ and $\gamma_2(\mathcal{D}_2)$ are defined in Definition 4.2. i.e. $\gamma_1(\mathcal{D}_1)\ < k$ and $\gamma_2(\mathcal{D}_2)\ < k$.
Then X has fuzzy uniform normal structure (Definition 2.10) in the (B-S)-type fuzzy normed linear space $(X\ ,\ N_i),\ i = 1, 2$.
The proof of this theorem follows from the Theorem 2.11. □

Theorem 4.4. *Let $(X\ ,\ ||\ ||)$ be α-complete fuzzy normed linear space for each $\alpha \in (0\ ,\ 1]$. If X has fuzzy uniform normal structure then it is fuzzy reflexive.*

Proof. Since $(X\ ,\ ||\ ||)$ is α-complete for each $\alpha \in (0\ ,\ 1]$, thus X is complete w.r.t. $||\ ||_\alpha^1\ \&||\ ||_\alpha^2$ for each $\alpha \in (0\ ,\ 1]$. Again X has fuzzy uniform normal structure, thus by Theorem 4.3, it follows that $\exists\beta_1,\ \beta_2 \in (0\ ,\ 1)$ such that X has uniform normal structure w.r.t. $||\ ||_\alpha^1\ \forall\alpha \geq \beta_1$ and X has uniform normal structure w.r.t. $||\ ||_\alpha^2\ \forall\alpha \leq \beta_2$.
We know that if a Banach space has uniform normal structure then it is reflexive. Thus X is reflexive w.r.t. $||\ ||_\alpha^1\ \forall\alpha \geq \beta_1$ and w.r.t. $||\ ||_\alpha^2\ \forall\alpha \leq \beta_2$.
Hence X is fuzzy reflexive. □

Theorem 4.5. $(Kirk)$. *Let K be a nonempty convex, l-fuzzy bounded and l-fuzzy closed subset of a fuzzy normed linear space $(X\ ,\ ||\ ||)$ and suppose X has fuzzy uniform normal structure. Then every fuzzy nonexpansive mapping $T:\ K \to K$ has a fixed point.*

Proof. Since X has fuzzy uniform normal structure by Theorem 4.4, it follows that X is fuzy reflexive.

Thus $\exists \beta_1,\ \beta_2 \in (0\ ,\ 1)$ such that
$C^1_\alpha(X)\ =\ X^{**1}_\alpha\ \forall\alpha \geq \beta_1$ and
$C^2_\alpha(X)\ =\ X^{**2}_\alpha\ \forall\alpha \leq \beta_2$.
i.e. $(X\ ,\ ||\ ||^1_\alpha)$ is reflexive $\forall\alpha \geq \beta_1$ and $(X\ ,\ ||\ ||^1_\alpha)$ is reflexive $\forall\alpha \leq \beta_2$.
So X is a Banach space w.r.t. $||\ ||^1_\alpha\ \forall\alpha \geq \beta_1$ and w.r.t. $||\ ||^2_\alpha\ \forall\alpha \leq \beta_2$.
Again K is l-fuzzy bounded and l-fuzzy closed subset of X. So K is bounded and closed w.r.t. $||\ ||^1_\alpha$ and $||\ ||^2_\alpha\ \forall\alpha \in (0\ ,\ 1]$.
Note that every closed convex subset of a reflexive Banach space is weakly compact.
Thus K is weakly compact w.r.t. $||\ ||^1_\alpha\ \forall\alpha \geq \beta_1$ and w.r.t. $||\ ||^2_\alpha\ \forall\alpha \leq \beta_2$.
Now K has fuzzy uniform normal structure. Thus by Theorem 4.3, it follows that $\exists\gamma_1,\ \gamma_2 \in (0\ ,\ 1)$ such that K has uniform normal structure w.r.t. $||\ ||^1_\alpha\ \forall\alpha \geq \gamma_1$ and w.r.t. $||\ ||^2_\alpha\ \forall\alpha \leq \gamma_2$.
Hence K has normal structure w.r.t. $||\ ||^1_\alpha\ \forall\alpha \geq \gamma_1$ and w.r.t. $||\ ||^2_\alpha\ \forall\alpha \leq \gamma_2$.
Let $\alpha_1 =\ max\{\beta_1,\ \gamma_1\}$ and $\alpha_2 =\ min\{\beta_2,\ \gamma_2\}$.
Then from above it follows that K is weakly compact and has normal structure w.r.t. $||\ ||^1_\alpha\ \forall\alpha \geq \alpha_1$ and w.r.t. $||\ ||^2_\alpha\ \forall\alpha \leq \alpha_2$.
Also T is fuzzy nonexpansive $\Rightarrow$ T is nonexpansive w.r.t. $||\ ||^1_\alpha$ and $||\ ||^2_\alpha\ \forall\alpha \in (0\ ,\ 1]$.
Thus by Kirk fixed point Theorem 2.15, it follows that T has a fixed point. □

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