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ON FELBIN'S-TYPE FUZZY NORMED LINEAR SPACES AND FUZZY BOUNDED OPERATORS

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Abstract. In this note, we aim to present some properties of the space of all weakly fuzzy bounded linear operators, with the Bag and Samanta's operator norm on Felbin's-type fuzzy normed spaces. In particular, the completeness of this space is studied. By some counterexamples, it is shown that the inverse mapping theorem and the Banach-Steinhaus's theorem, are not valid for this fuzzy setting. Also finite dimensional normed fuzzy spaces are considered briefly. Next, a Hahn-Banach theorem for weakly fuzzy bounded linear functional with some of its applications are established.

1. Introduction

An idea of fuzzy norm on a linear space first introduced by Katsaras [11]. Felbin [8] defined a fuzzy norm (the induced fuzzy metric of which is of Kaleva and Seikkala's type [10]), by assigning a non-negative fuzzy real number to each element of a linear space. A further developments along this line of inquiry took place in 1994, when Cheng and Mordeson [6] evolved the definition of a further type of fuzzy norm having a corresponding metric of the Kramosil and Michalek's type [12].

In [5], Bag and Samanta considered a fuzzy norm slightly different from the one defined by Cheng and Mordeson and for which a suitable decomposition theorem was proved. Based on this theorem it has been possible to establish four fundamental theorems of functional analysis in [4], the Hahn-Banach theorem, the open mapping theorem, the closed graph theorem and the uniform boundedness principle. Also best approximation in this space is studied in [16].

Fuzzy bounded linear operators in Felbin's-type fuzzy normed spaces were introduced by M. Itoh and M. Chō in [13]. They introduced a fuzzy norm for fuzzy bounded operators. In [9] Felbin introduced an idea of fuzzy bounded operators and defined a fuzzy norm for such an operator which was erroneous as shown in Example 3.1 of [3]. Xiao and Zhu [17], [18], studied various properties of Felbin's-type fuzzy normed linear spaces and a new definition for norm of bounded operators was discussed.

A different definition of a fuzzy bounded linear operator and a "fuzzy norm" for such an operator was introduced by Bag and Samanta [3]. The dual of a fuzzy normed space and a Hahn-Banach's theorem for fuzzy strongly bounded linear functional were established in this work.

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A comparative study among several types of fuzzy norms on a linear space defined by various authors, has been made by Bag and Samanta [2]. They classified these norms into two types, one of which is Katsaras's type, and the other is Felbin's type. Also one can see [15] for intuitionistic fuzzy bounded linear operators.
In this paper we consider the recent definition of Bag and Samanta for fuzzy bounded linear operators and its new norm.
In section 2, some preliminaries and essential concepts for the study are stated.
In section 3, first some elementary properties of $B(X,Y)$, the space of all weakly bounded linear operators with this new norm, are stated and proved. Then by some counterexamples, it is shown that the uniform boundedness principle, inverse mapping theorem and the Banach-Steinhaus's theorem are not valid in this fuzzy setting. Next a simpler proof for characterization of finite dimensional fuzzy normed linear space is established.
In section 4, a Hahn-Banach's theorem for fuzzy weakly bounded operators is considered and some of its results are stated.

2. **Preliminaries**

According to Mizumoto and Tanaka [14], a fuzzy number is a mapping $x : \mathbb{R} \to [0,1]$ over the set $\mathbb{R}$ of all reals.
x is called convex if $x(t) \geq \min(x(s), x(r))$ where $s \leq t \leq r$.
If there exists a $t_0 \in \mathbb{R}$ such that $x(t_0) = 1$, then x is called normal. For $0 < \alpha \leq 1$, α-level set of an upper semicontinuous convex normal fuzzy set x of $\mathbb{R}$ (denoted by $[\eta]_\alpha$) is a closed interval $[a_\alpha, b_\alpha]$, where $a_\alpha = -\infty$ and $b_\alpha = +\infty$ are admissible. When $a_\alpha = -\infty$, for instance, then $[a_\alpha, b_\alpha]$ means the interval $(-\infty, b_\alpha]$. Similar is the case when $b_\alpha = +\infty$.
x is called non-negative if for all $t < 0$, $x(t) = 0$. Kaleva and Seikkala [10] (Felbin [8]) denoted the set of all convex, normal, upper semi-continuous fuzzy real numbers by $E(R(I))$ and the set of all non-negative, convex, normal, upper semi-continuous fuzzy real numbers by $G(R^*(I))$.
As α-level sets of a convex fuzzy number is an interval, there is a debate in the nomenclature of fuzzy numbers/fuzzy real numbers. In [7], Dubois and Prade suggested to call this as fuzzy interval. They developed a different notion of a fuzzy real number by considering it as a fuzzy element of the real line, each α-cut of this number is an interval real numbers. From now on "fuzzy real numbers" are renamed as "fuzzy intervals". While refereing to previous results involving fuzzy real number, the term fuzzy interval is written within brackets after fuzzy real number to avoid any confusion; otherwise the new nomenclature i.e. fuzzy interval is used.

In this paper we consider the concept of fuzzy real numbers (fuzzy intervals) in the sense of Xiao and Zhu [17] which is defined below:
A mapping $\eta : \mathbb{R} \to [0,1]$, whose α-level set is denoted by $[\eta]_\alpha := \{t : \eta(t) \geq \alpha\}$, is called a fuzzy real number (or fuzzy interval) if it satisfies two axioms:

(N_1) There exists $t_0 \in \mathbb{R}$ such that $\eta(t_0) = 1$.

(N_2) For each $\alpha \in (0,1]$; $[\eta]_\alpha = [\eta^1_\alpha, \eta^2_\alpha]$, where $-\infty < \eta^1_\alpha \leq \eta^2_\alpha < +\infty$.

The set of all fuzzy real numbers (fuzzy intervals) is denoted by $\mathcal{F}$. For each $r \in \mathbb{R}$,

let $\overline{r} \in \mathcal{F}$ be defined by $\overline{r}(t) = 1$, if $t = r$ and $\overline{r}(t) = 0$, if $t \neq r$, so $\overline{r}$ is a fuzzy interval and $\mathbb{R}$ can be embedded in $\mathcal{F}$.
Let $\eta \in \mathcal{F}$, η is called positive fuzzy real number if for all $t < 0$, $\eta(t) = 0$. The set of all positive fuzzy real numbers (fuzzy interval) is denoted by $\mathcal{F}^+$.

A partial order $\preceq$ in $\mathcal{F}$ is defined as follows, $\eta \preceq \delta$ if and only if for all $\alpha \in (0, 1]$, $\eta^1_\alpha \leq \delta^1_\alpha$ and $\eta^2_\alpha \leq \delta^2_\alpha$ where, $[\eta]_\alpha = [\eta^1_\alpha, \eta^2_\alpha]$ and $[\delta]_\alpha = [\delta^1_\alpha, \delta^2_\alpha]$. The strict inequality in $\mathcal{F}$ is defined by $\eta \prec \delta$ if and only if for all $\alpha \in (0, 1]$, $\eta^1_\alpha < \delta^1_\alpha$ and $\eta^2_\alpha < \delta^2_\alpha$.

Kaleva and Seikkala, in [10] proved a sufficient condition for a family of intervals to represent the α-level sets of a fuzzy real number. In fact, let $[a_\alpha, b_\alpha]$, $0 < \alpha \leq 1$, be a given family of nonempty intervals. If:

(i) for all $0 < \alpha_1 \leq \alpha_2$, $[a_{\alpha_1}, b_{\alpha_1}] \supseteq [a_{\alpha_2}, b_{\alpha_2}]$.

(ii) $[\lim_{k\to\infty} a_{\alpha_k}, \lim_{k\to\infty} b_{\alpha_k}] = [a_\alpha, b_\alpha]$, whenever $\{\alpha_k\}$ is an increasing sequence in $(0, 1]$ converging to α,

then the family $[a_\alpha, b_\alpha]$ represents the α-level sets of a fuzzy real number (fuzzy interval). Conversely, if $[a_\alpha, b_\alpha]$, $0 < \alpha \leq 1$, are the α-level sets of a fuzzy number then the condition (i) and (ii) are satisfied.

According to Mizumoto and Tanaka [14], the arithmetic operations $\oplus$, $\ominus$, $\odot$ on $\mathcal{F} \times \mathcal{F}$ are defined by

$(x \oplus y)(t) = \sup_{s\in R} \min\{x(s), y(t - s)\}$, $t \in \mathbb{R}$,
$(x \ominus y)(t) = \sup_{s\in R} \min\{x(s), y(s - t)\}$, $t \in \mathbb{R}$,
$(x \odot y)(t) = \sup_{0\neq s\in R} \min\{x(s), y(\frac{t}{s})\}$, $t \in \mathbb{R}$.

We also consider an operation $\oslash$ on $\eta \in \mathcal{F}$ and $\delta(\succ 0) \in \mathcal{F}^+$ as follows
$(\eta \oslash \delta)(t) = \sup_{s\in R} \min\{\eta(st), \delta(s)\}, t \in \mathbb{R}$.
As in [8] we know that, for $\eta, \delta \in \mathcal{F}$, if $[\eta]_\alpha = [\eta^1_\alpha, \eta^2_\alpha]$, $[\delta]_\alpha = [\delta^1_\alpha, \delta^2_\alpha]$, $\alpha \in (0, 1]$, then

$[\eta \oplus \delta]_\alpha = [\eta^1_\alpha + \delta^1_\alpha, \eta^2_\alpha + \delta^2_\alpha]$,
$[\eta \ominus \delta]_\alpha = [\eta^1_\alpha - \delta^2_\alpha, \eta^2_\alpha - \delta^1_\alpha]$,
furthermore if η , $\delta \in \mathcal{F}^+$, then $[\eta \odot \delta]_\alpha = [\eta^1_\alpha.\delta^1_\alpha, \eta^2_\alpha.\delta^2_\alpha]$,
and when $\delta \succ \overline{0}$, $[\overline{1} \oslash \delta]_\alpha = [\frac{1}{\delta^2_\alpha}, \frac{1}{\delta^1_\alpha}]$.

Now one can see that for $\delta \succ \overline{0}$ and $\eta \in \mathcal{F}^+$,

$[\eta \oslash \delta]_\alpha = [\frac{\eta^1_\alpha}{\delta^2_\alpha}, \frac{\eta^2_\alpha}{\delta^1_\alpha}]$.

A definition of fuzzy norm on a linear space was introduced by Felbin [8]. Bag and Samanta [3], changed slightly this definition to define a fuzzy norm on a linear space as given below and as is done in [3].

Definition 2.1. [3] Let X be a linear space over $\mathbb{R}$. Suppose $\parallel . \parallel: X \to \mathcal{F}^+$ is a mapping satisfying

(i) $\parallel x \parallel = \overline{0}$ if and only if $x = 0$,
(ii) $\parallel rx \parallel = | r | \parallel x \parallel, x \in X, r \in \mathbb{R}$,
(iii) for all $x, y \in X$, $\parallel x + y \parallel \preceq \parallel x \parallel \oplus \parallel y \parallel$

and

(A'): $x \neq 0 \Rightarrow \parallel x \parallel (t) = 0, \forall t \leq 0$.

$(X, \parallel . \parallel)$ is called a fuzzy normed linear space and $\parallel . \parallel$ is called a fuzzy norm on X.

In the rest of this paper we use the previous definition of fuzzy norm. We note that

(i) condition $(\acute{A})$ in Definition 2.1 is equivalent to the condition

(A''): For all $x(\neq 0) \in X$ and each $\alpha \in (0,1]$, $\parallel x \parallel_\alpha^1 > 0$, where $[\parallel x \parallel]_\alpha = [\parallel x \parallel_\alpha^1, \parallel x \parallel_\alpha^2]$ and

(ii) $\parallel x \parallel_\alpha^i$, $i = 1, 2$, are crisp norms on X.

Let $(X, \parallel . \parallel)$ be a fuzzy normed linear space. A sequence $\{x_n\}$ in X is said to be convergent to $x \in X$ if and only if for each $\alpha \in (0,1]$, $\lim_{n\to\infty} \parallel x_n - x \parallel_\alpha^2 = 0$. In this case we write $\lim_{n\to\infty} x_n = x$. Also a sequence $\{x_n\}$ is called a Cauchy sequence if for each $\alpha \in (0,1]$, $\lim_{n,m\to\infty} \parallel x_n - x_m \parallel_\alpha^2 = 0$. A fuzzy normed linear space $(X, \parallel . \parallel)$ is said to be complete if every Cauchy sequence in X converges in X (see [8] or [17]).

Proposition 2.2. [1] *Let $\{[a_\alpha, b_\alpha] : \alpha \in (0,1]\}$, be a family of nested bounded closed intervals and $\eta : \mathbb{R} \to [0,1]$ be a function defined by $\eta(t) = \bigvee\{\alpha \in (0,1] : t \in [a_\alpha, b_\alpha]\}$. Then η is a fuzzy real number (fuzzy interval).*

The fuzzy real number (fuzzy interval) η which is constructed in Proposition 2.2 is called the fuzzy real number (fuzzy interval) generated by the family of nested bounded closed intervals $\{[a_\alpha, b_\alpha] : \alpha \in (0,1]\}$. In this case, for $\beta < \alpha$, $[\eta]_\alpha = [\eta_\alpha^1, \eta_\alpha^2] \subseteq [a_\beta, b_\beta]$ (see [1]).
Now if $\beta > \alpha$, then $[a_\beta, b_\beta] \subseteq [\eta_\alpha^1, \eta_\alpha^2]$. In fact for $t \in [a_\beta, b_\beta]$, $\eta(t) = \bigvee\{\gamma \in (0,1] : t \in [a_\gamma, b_\gamma]\} \geq \beta > \alpha$, which implies that $t \in [\eta_\alpha^1, \eta_\alpha^2]$.

Proposition 2.3. [1] *If η_i, $i = 1, 2$, are the fuzzy real numbers (fuzzy intervals) generated by the family of nested bounded closed intervals $\{[a_\alpha^i, b_\alpha^i] : \alpha \in (0,1]\}$, $i = 1, 2$, and for each $\alpha \in (0,1]$, $a_\alpha^1 \leq a_\alpha^2, b_\alpha^1 \leq b_\alpha^2$, then $\eta_1 \preceq \eta_2$.*

From [3] we know that if η is a fuzzy real number (fuzzy interval) with $[\eta]_\alpha = [\eta_\alpha^1, \eta_\alpha^2]$, and η^* is the fuzzy number (fuzzy interval) generated by the family of nested bounded closed intervals $[\eta_\alpha^1, \eta_\alpha^2]$, $0 < \alpha \leq 1$, then $\eta = \eta^*$.

Let $(X, \parallel . \parallel)$ and $(Y, \parallel . \parallel^\sim)$ be two fuzzy normed linear spaces. A function $T : X \to Y$ is said to be weakly fuzzy continuous at $x_0 \in X$ if for a given $\epsilon > 0$, $\exists \delta \in \mathcal{F}^+, \delta \succ 0$, such that

$$\parallel Tx - Tx_0 \parallel_\alpha^{\sim 1} < \epsilon \ \text{ whenever } \ \parallel x - x_0 \parallel_\alpha^2 < \delta_\alpha^2,$$
$$\parallel Tx - Tx_0 \parallel_\alpha^{\sim 2} < \epsilon \ \text{ whenever } \ \parallel x - x_0 \parallel_\alpha^1 < \delta_\alpha^1,$$

where for $\alpha \in (0,1]$, $[\delta]_\alpha = [\delta_\alpha^1, \delta_\alpha^2]$.
Also a linear mapping $T : X \to Y$ is called weakly fuzzy bounded if there exists a fuzzy interval $\eta \in \mathcal{F}^+, \eta \succ 0$, such that for each $x(\neq 0) \in X$, $\parallel Tx \parallel^\sim \oslash \parallel x \parallel \preceq \eta$. In this case the set of all weakly fuzzy bounded operators defined from X to Y is denoted by $B(X,Y)$ (see [3]). In the sequel, we simply apply "fuzzy continuous" and

"fuzzy bounded" instead of "weakly fuzzy continuous" and "weakly fuzzy bounded", respectively.

From [3] we know that a linear mapping $T : X \to Y$ is fuzzy continuous if and only if it is fuzzy bounded. Also the set $B(X,Y)$ is a linear space with respect to usual operations.

The following result of Bag and Samanta [3] is essential in this paper.

Let $(X, \| \,.\, \|)$ and $(Y, \| \,.\, \|^{\sim})$ be two fuzzy normed linear spaces and $T \in B(X,Y)$. By definition $\exists \eta \in \mathcal{F}^+, \eta \succ 0$, such that for all $x(\neq 0) \in X$,

$$\| Tx \|^{\sim} \oslash \| x \| \preceq \eta.$$

If $[\eta]_\alpha = [\eta^1_\alpha, \eta^2_\alpha], 0 < \alpha \leq 1$, we get

$$\| Tx \|^{\sim 1}_\alpha \leq \eta^1_\alpha . \| x \|^2_\alpha \;\text{ and }\; \| Tx \|^{\sim 2}_\alpha \leq \eta^2_\alpha . \| x \|^1_\alpha \,.$$

Define

$$\| T \|^{*1}_\alpha = \sup_{0 \neq x \in X} \{ \frac{\| Tx \|^{\sim 1}_\alpha}{\| x \|^2_\alpha} \} \leq (\eta^1_\alpha),$$

and

$$\| T \|^{*2}_\alpha = \sup_{0 \neq x \in X} \{ \frac{\| Tx \|^{\sim 2}_\alpha}{\| x \|^1_\alpha} \} \leq (\eta^2_\alpha).$$

Then $\{\| \,.\, \|^{*2}_\alpha : \alpha \in (0,1]\}$ and $\{\| \,.\, \|^{*1}_\alpha : \alpha \in (0,1]\}$ are descending and ascending family of norms, respectively. Thus $\{[\| T \|^{*1}_\alpha, \| T \|^{*2}_\alpha] : \alpha \in (0,1]\}$ is a family of nested bounded closed intervals in $\mathbb{R}$. Define the function $\| T \|^* : \mathbb{R} \to [0,1]$ by

$$\| T \|^* (t) = \bigvee \{\alpha \in (0,1] : t \in [\| T \|^{*1}_\alpha, \| T \|^{*2}_\alpha]\}.$$

$\| T \|^*$ is called the fuzzy norm of T. $(B(X,Y), \|.\|^*)$ is a fuzzy normed space.

3. Some Properties of $B(X,Y)$

In this section, first we present some properties of the space $(B(X,Y), \|.\|^*)$. Then using the recent results of Bag and Samanta, some consequences of fuzzy linear spaces analogous to the ordinary normed spaces are established. Despite our expectation, the fuzzy version of some well-known theorems in functional analysis, such as uniform boundedness principle, inverse mapping theorem and the Banach-Stienhaus's theorem is not valid in this fuzzy setting. These will be shown with some counterexamples. Next, finite dimensional normed spaces are considered. The concept of equivalent norms is defined and it is proved that every two fuzzy norms on a finite dimensional vector space are equivalent.

First we prove a memorable result for $B(X,Y)$, which has a famous analogous in functional analysis. A similar result on $B(X,\mathbb{C})$ is proved in [3] Theorem 6.2.

Theorem 3.1. *Let $(X, \| \,.\, \|)$ be a fuzzy normed space and $(Y, \| \,.\, \|^{\sim})$ be a complete fuzzy normed space, then $(B(X,Y), \| \,.\, \|^*)$ is a complete fuzzy normed linear space.*

Proof. Let $\{T_n\}$ be a Cauchy sequence in $(B(X,Y), \| \,.\, \|^*)$. So for all $\alpha \in (0,1]$,

$$\lim_{n,m\to\infty} \| T_n - T_m \|_\alpha^{*1} = \lim_{n,m\to\infty} \| T_n - T_m \|_\alpha^{*2} = 0.$$

From $\lim_{n,m\to\infty} \| T_n - T_m \|_\alpha^{*1} = 0$ we have

$$\lim_{m,n\to\infty} \sup_{0\neq x\in X} \frac{\| T_n(x) - T_m(x) \|_\alpha^{\sim 1}}{\| x \|_\alpha^2} = 0,$$

which implies that for each $\alpha \in (0,1]$ and $x \in X$,

$$\lim_{n,m\to\infty} \| T_n(x) - T_m(x) \|_\alpha^{\sim 1} = \lim_{n,m\to\infty} \| T_n(x) - T_m(x) \|_\alpha^{\sim 2} = 0.$$

From the fact that Y is a complete fuzzy normed space, for some y_x, $\lim_{n\to\infty} T_n(x) = y_x$ exists in Y.
Let for each $x \in X$, $\lim_{n\to\infty} T_n(x) = T(x)$. Trivially T is linear. Now for every $\alpha \in (0,1]$,

$$|\| T_n \|_\alpha^{*1} - \| T_m \|_\alpha^{*1}| \leq \| T_n - T_m \|_\alpha^{*1} \to 0 \text{ as } n, m \to \infty.$$

This implies that for all $\alpha \in (0,1]$,

$$|\| T_n \|_\alpha^{*1} - \| T_m \|_\alpha^{*1}| \to 0 \text{ as } n, m \to \infty.$$

Thus for each $\alpha \in (0,1]$, $\{\| T_n \|_\alpha^{*1}\}$ is a Cauchy sequence of non-negative real number and so is convergent. Let for all $\alpha \in (0,1]$, $\lim_{n\to\infty} \| T_n \|_\alpha^{*1} = a_\alpha$.
Similarly $\{\| T_n \|_\alpha^{*2}\}$ is a Cauchy sequence of non-negative real number for each $\alpha \in (0,1]$, so is convergent. Put $\lim_{n\to\infty} \| T_n \|_\alpha^{*2} = b_\alpha,\ \alpha \in (0,1]$. One can easily verify that $\{[a_\alpha, b_\alpha] : \alpha \in (0,1]\}$ is a family of nested bounded closed interval of real numbers. Hence by Proposition 2.2, it generates a fuzzy real number, say η.
Now from

$$\| T(x) \|_\alpha^{\sim 1} = \lim_{n\to\infty} \| T_n(x) \|_\alpha^{\sim 1} \leq \lim_{n\to\infty} (\| T_n \|_\alpha^{*1} \,.\, \| x \|_\alpha^2) = a_\alpha . \| x \|_\alpha^2,$$

we have

$$\frac{\| T(x) \|_\alpha^{\sim 1}}{\| x \|_\alpha^2} \leq a_\alpha,\ \forall x (\neq 0) \in X,\ \forall \alpha \in (0,1]. \tag{1}$$

Similarly

$$\frac{\| T(x) \|_\alpha^{\sim 2}}{\| x \|_\alpha^1} \leq b_\alpha,\ \forall x (\neq 0) \in X,\ \forall \alpha \in (0,1]. \tag{2}$$

Using Proposition 2.2 and 2.3, we know $\{[\frac{\|T(x)\|_\alpha^{\sim 1}}{\|x\|_\alpha^2}, \frac{\|T(x)\|_\alpha^{\sim 2}}{\|x\|_\alpha^1}] :\ \alpha \in (0,1]\}$ generates the fuzzy interval $\| T(x) \|^\sim \oslash \| x \| \preceq \eta, \forall x (\neq 0) \in X$. This implies that $T \in B(X,Y)$.
We claim that $\| T_n - T \|^* \to 0$ as $n \to \infty$. For a given $\epsilon > 0$, and each $\alpha \in (0,1]$, there exists positive integer $N(\epsilon, \alpha)$ such that for all $n, m > N(\epsilon, \alpha)$,

$$\| T_n - T_m \|_\alpha^{*1} < \epsilon.$$

Thus for any $\alpha \in (0,1]$ and $n, m \geq N(\epsilon, \alpha)$,

$$\| T_n(x) - T_m(x) \|_\alpha^{*1} \leq \| T_n - T_m \|_\alpha^{*1} . \| x \|_\alpha^2 \leq \epsilon . \| x \|_\alpha^2 .$$

So when $m \to \infty$, for every $\alpha \in (0,1]$ and $n \geq N(\epsilon, \alpha)$, we have

$$\| T_n(x) - T(x) \|_\alpha^{*1} \leq \epsilon . \| x \|_\alpha^2,$$

which implies that

$$\bigvee_{0 \neq x \in X} \{ \frac{\| T_n(x) - T(x) \|_\alpha^{*1}}{\| x \|_\alpha^2} \} \leq \epsilon, \forall n \geq N(\epsilon, \alpha).$$

Hence for $n \geq N(\epsilon, \alpha)$ and $\alpha \in (0,1]$,

$$\| T_n - T \|_\alpha^{*1} \leq \epsilon,$$

i.e. as $n \to \infty$ and $\alpha \in (0,1]$

$$\| T_n - T \|_\alpha^{*1} \to 0.$$

Similarly we have $\| T_n - T \|_\alpha^{*2} \to 0$ as $n \to \infty$ and $\alpha \in (0,1]$. This follows that $\| T_n - T \|^* \to 0$ as $n \to \infty$, which implies that $(B(X,Y), \| . \|^*)$ is a complete fuzzy normed space. □

In spite of our expectation the following example shows that for $S, T \in B(X)$, none of the relations $\|SoT\|^* \preceq \|S\|^* \odot \|T\|^*$, $\|SoT\|^* \oslash \|S\|^* \preceq \|T\|^*$ and $\|SoT\|^* \oslash \|T\|^* \preceq \|S\|^*$, are valid.

Example 3.2. Let $(X, \|.\|)$ be an arbitrary Banach space, define

$$|||x|||(t) = \begin{cases} \frac{\|x\|}{t}, & t \geq \|x\|, \\ 0, & \text{otherwise.} \end{cases}$$

So for every $\alpha \in (0,1]$, $[|||x|||]_\alpha = [\|x\|, \frac{\|x\|}{\alpha}]$. Now let $S = T = I$. Trivially, for any $\alpha \in (0,1]$,

$$\|I\|_\alpha^{*1} = \sup_{0 \neq x \in X} \frac{\|x\|}{\|x\|/\alpha} = \alpha \text{ and } \|I\|_\alpha^{*2} = \sup_{0 \neq x \in X} \frac{\|x\|/\alpha}{\|x\|} = \frac{1}{\alpha}.$$

Suppose δ is the fuzzy real number generated by $\{[\alpha, \frac{1}{\alpha}] : \alpha \in (0,1]\}$. So

$$\|SoT\|^* = \|S\|^* = \|T\|^* = \delta.$$

On the other hand, $[\delta \odot \delta]_\alpha = [\alpha^2, \frac{1}{\alpha^2}]$, and $[\delta \oslash \delta]_\alpha = [\alpha^2, \frac{1}{\alpha^2}]$, $\alpha \in (0,1]$. But non of the relations $\alpha \leq \alpha^2$ and $\frac{1}{\alpha^2} \leq \frac{1}{\alpha}$ are valid. This shows that the relations $\delta \preceq \delta \odot \delta$ and $\delta \oslash \delta \preceq \delta$ are not correct.

Remark 3.3. If we define a fuzzy Banach algebra A, to be a fuzzy Banach space with a multiplication, with the ordinary properties of a product in Banach algebra, and with the property $\|ab\| \oslash \|a\| \preceq \|b\|$, $a, b \in A$, then the previous example shows that $B(X)$ is not a fuzzy Banach algebra, when X is a fuzzy Banach space. Also if we replace the condition $\|ab\| \oslash \|a\| \preceq \|b\|$ with $\|ab\| \preceq \|a\| \odot \|b\|$, then $B(X)$ does not satisfy this relation.

The following useful theorem provide a sufficient and necessary condition for fuzzy boundedness of an operator.

Theorem 3.4. *Let $(X, \|.\|)$ and $(Y, \|.\|^{\sim})$ be two fuzzy normed spaces. Then $T \in B(X, Y)$ if and only if for every $\alpha \in (0, 1]$, $T \in B((X, \|.\|^1_\alpha), (Y, \|.\|^{\sim 2}_\alpha))$ and $T \in B((X, \|.\|^2_\alpha), (Y, \|.\|^{\sim 1}_\alpha))$.*

Proof. Without loss of generality T is supposed to be non-null. Let for each $\alpha \in (0, 1]$, $T \in B((X, \|.\|^1_\alpha), (Y, \|.\|^{\sim 2}_\alpha)) \cap B((X, \|.\|^2_\alpha), (Y, \|.\|^{\sim 1}_\alpha))$. We show that $T \in B(X, Y)$. For $\alpha \in (0, 1]$, there exist $\delta^2_\alpha, \delta^1_\alpha > 0$ such that for all $0 \neq x \in X$,

$$\frac{\|T(x)\|^{\sim 2}_\alpha}{\|x\|^1_\alpha} \leq \delta^2_\alpha, \text{ and } \frac{\|T(x)\|^{\sim 1}_\alpha}{\|x\|^2_\alpha} \leq \delta^1_\alpha,$$

since $T \in B((X, \|.\|^1_\alpha), (Y, \|.\|^{\sim 2}_\alpha)) \cap B((X, \|.\|^2_\alpha), (Y, \|.\|^{\sim 1}_\alpha))$.
For each $\alpha \in (0, 1]$, define

$$\delta^{*2}_\alpha = \inf\{\delta^2_\alpha : \frac{\|T(x)\|^{\sim 2}_\alpha}{\|x\|^1_\alpha} \leq \delta^2_\alpha, \forall x (\neq 0) \in X\},$$

and

$$\delta^{*1}_\alpha = \inf\{\delta^1_\alpha : \frac{\|T(x)\|^{\sim 1}_\alpha}{\|x\|^2_\alpha} \leq \delta^1_\alpha, \forall x (\neq 0) \in X\}.$$

So for all $0 \neq x \in X$, by definition of infimum, $\frac{\|T(x)\|^{\sim 1}_\alpha}{\|x\|^2_\alpha} \leq \delta^{*1}_\alpha$ and $\frac{\|T(x)\|^{\sim 2}_\alpha}{\|x\|^1_\alpha} \leq \delta^{*2}_\alpha$. We know $\{[\delta^{*1}_\alpha, \delta^{*2}_\alpha] : \alpha \in (0, 1]\}$ is a family of nested bounded closed intervals of real numbers, so using Proposition 2.2, this family generates a positive fuzzy interval, $\eta \succ 0$, and by Proposition 2.3,

$$\|T(x)\|^{\sim} \oslash \|x\| \preceq \eta.$$

This means that $T \in B(X, Y)$. The converse is trivial. □

This theorem has the following useful corollary (see also [3] Theorem4.3) which shows that every linear operator from a finite dimensional fuzzy linear space is fuzzy continuous. For its proof it is enough to use the previous theorem and the ordinary version of this corollary in functional analysis.

Corollary 3.5. *Let $T : X \to Y$ be a linear operator where $(X, \|.\|)$ and $(Y, \|.\|^{\sim})$ are fuzzy normed linear spaces. If X has finite dimension, then T is fuzzy bounded (so is fuzzy continuous).*

In the sequel we present some counterexamples. The first is a counterexample for the uniform boundedness principle in this fuzzy structure. Before stating this example we need the following definition.

Definition 3.6. Let $(X, \| \,.\, \|)$ and $(Y, \| \,.\, \|^{\sim})$ be two fuzzy normed linear spaces. A family $\{T_n\} \subseteq B(X, Y)$ is called point-wise bounded if for every $x(\neq 0) \in X$, there exists fuzzy number $\delta_x \in \mathcal{F}^+, \delta_x \succ 0$, such that for all $n > 0$,

$$\| T_n(x) \|^{\sim} \oslash \| x \| \preceq \delta_x,$$

and is said to be uniformly bounded if there exists fuzzy number $\delta \in \mathcal{F}^+, \delta \succ 0$, such that for each $n > 0$ and $x(\neq 0) \in X$,

$$\| T_n(x) \|^{\sim} \oslash \| x \| \preceq \delta.$$

For a sequence $\{T_n\}$ of fuzzy bounded operators from a fuzzy Banach space X to a fuzzy Banach space Y, it is expected that the point-wise and uniformly boundedness of this sequence are equivalent (a fuzzy version of uniform boundedness principle), but the following example shows that this is not true in general.

Example 3.7. Consider $X = l^1$, the set of all real valued sequences whose series is absolutely convergent, we know X with $\|.\|_1$ is Banach but with $\|.\|_\infty$ is not Banach and $\|.\|_\infty \le \|.\|_1$. Now suppose $\|x\|$ is the fuzzy norm generated by the nested family $\{[\alpha\|x\|_\infty, \frac{1}{\alpha}\|x\|_1] : \alpha \in (0, 1]\}$ of intervals. Let $Y = \mathbb{R}$ with the fuzzy norm generated by the family $\{[|x|, |x|] : \alpha \in (0, 1]\}$. One can see that $(X, \|.\|)$ and $(Y, |.|)$ are two fuzzy Banach spaces.
Define $T_n : X \to Y$ by

$$T_n((x_k)_k) = \sum_{k=1}^{n} x_k.$$

From Theorem 3.4, $T_n \in B(X, Y)$, $n \in \mathbb{N}$. We are going to show that $\{T_n\}$ is point-wise bounded but is not uniformly bounded. Let $0 \neq x = (x_k)_{k\in\mathbb{N}} \in X$. If δ_x is the fuzzy number generated by the family $\{[\alpha, \frac{\|x\|_1}{\alpha\|x\|_\infty}] : \alpha \in (0, 1]\}$, then for any $n > 0$,

$$|T_n(x)| \oslash \|x\| \preceq \delta_x.$$

Now in contrary suppose $\{T_n\}$ is uniformly bounded, so there exists $\delta \in \mathcal{F}^+$ such that for each $0 \neq x \in X$, and $n > 0$,

$$|T_n(x)| \oslash \|x\| \preceq \delta.$$

Thus for every $\alpha \in (0, 1]$ and $x = (x_k)_{k\in\mathbb{N}}$, with $x_1 = ... = x_n = 1$ and $x_k = 0,\ k > n$,

$$\frac{|T_n(x)|^2_\alpha}{\|x\|^1_\alpha} \le \delta^2_\alpha,$$

this implies that for each $n \in \mathbb{N}$, $\frac{n}{1} \le \delta^2_\alpha$. Hence $\delta^2_\alpha = \infty$, which contradicts the definition of fuzzy real number.

From inverse mapping theorem in ordinary case, we know if X, Y are ordinary Banach spaces and $T \in B(X, Y)$ is a bijection map, then its inverse belongs to $B(Y, X)$. The following example shows that this is not valid in this fuzzy setting.

Example 3.8. Suppose $\|.\|_1$ and $\|.\|_2$ are two ordinary norm on a vector space X, such that $(X, \|.\|_1)$ is not Banach space and $(X, \|.\|_2)$ is Banach space, for which

$$\|x\|_1 \leq \|x\|_2, \ \forall x \in X,$$

(For example let $X = C[0,1]$, with $\|.\|_1$ and $\|.\|_\infty$). For each x, trivially $\{[\|x\|_1, \frac{\|x\|_2}{\alpha}] : \alpha \in (0,1]\}$ is a family of nested closed intervals which generates a fuzzy interval $|||x|||$. Also consider the fuzzy norm $|||x|||^\sim$ on X generated by the family $\{[\|x\|_2, \|x\|_2] : \alpha \in (0,1]\}$. One can see that $(X, |||.|||)$ and $(X, |||.|||^\sim)$ are fuzzy Banach spaces. The identity mapping $I : (X, |||.|||^\sim) \to (X, |||.|||)$ is a fuzzy bounded linear operator, since if δ is the fuzzy number generated by the family $\{[1, \frac{1}{\alpha}] : \alpha \in (0,1]\}$, then

$$|||x||| \oslash |||x|||^\sim \preceq \delta.$$

Now we show that I^{-1} is not bounded. Otherwise, so there exists $\eta \in \mathcal{F}^+$ such that for each $x \in X$,

$$r(x) := |||x|||^\sim \oslash |||x||| \preceq \eta.$$

It means that for each $\alpha \in (0,1]$, if $[\eta]_\alpha = [\eta^1_\alpha, \eta^2_\alpha]$, and $[r(x)]_\alpha = [r(x)^1_\alpha, r(x)^2_\alpha]$, then $r(x)^1_\alpha \leq \eta^1_\alpha$ and $r(x)^2_\alpha \leq \eta^2_\alpha$. Now let $\alpha, \beta \in (0,1]$ with $\beta > \alpha$. From the assertion before Proposition 2.3, for each $x \in X$,

$$[\frac{\|x\|_2}{\frac{1}{\beta}\|x\|_2}, \frac{\|x\|_2}{\|x\|_1}] \subseteq [r(x)^1_\alpha, r(x)^2_\alpha].$$

Hence

$$\frac{\|x\|_2}{\|x\|_1} \leq r(x)^2_\alpha \leq \eta^2_\alpha.$$

This shows that $\|x\|_2 \leq \eta^2_\alpha \|x\|_1$, on the other hand by our hypothesis $\|x\|_1 \leq \|x\|_2$. This means that $\|.\|_1$ and $\|.\|_2$ are equivalent which is a contradiction.

Remark 3.9. Suppose for the fuzzy Banach spaces X and Y, and each $\alpha \in (0,1]$, $(X, \|.\|^1_\alpha)$, $(X, \|.\|^2_\alpha)$, $(Y, \|.\|^{\sim 1}_\alpha)$ and $(Y, \|.\|^{\sim 2}_\alpha)$ are Banach spaces, then using the elementary functional analysis, one can prove the fuzzy version of uniform boundedness principle, the inverse mapping theorem and the Banach-Steinhaus's theorem.

Definition 3.10. Let $\| . \|_1$ and $\| . \|_2$ be two fuzzy norm on X. $\| . \|_1$ and $\| . \|_2$ are said to be semi-equivalent if there exists fuzzy numbers $\eta_1 \succ 0$ and $\eta_2 \succ 0$, such that for each nonzero $x \in X$,

$$\| x \|_1 \oslash \| x \|_2 \preceq \eta_1 \text{ and } \| x \|_2 \oslash \| x \|_1 \preceq \eta_2.$$

Remark 3.11. 1) One can easily see that if $\|.\|_1$ and $\|.\|_2$ are semi-equivalent then $\|.\|_2$ and $\|.\|_1$ are also semi-equivalent. Also if $\|.\|_1$ is semi-equivalent to $\|.\|_2$ and $\|.\|_2$ is semi-equivalent to $\|.\|_3$, then $\|.\|_1$ and $\|.\|_3$ are semi-equivalent. Indeed if

$$\| x \|_1 \oslash \| x \|_2 \preceq \eta_1 \text{ and } \| x \|_2 \oslash \| x \|_1 \preceq \eta_2$$

and

$$\| x \|_2 \oslash \| x \|_3 \preceq \beta_1 \text{ and } \| x \|_3 \oslash \| x \|_2 \preceq \beta_2$$

then

$$\| x \|_1 \oslash \| x \|_3 \preceq \beta_1 \odot \eta_1 \text{ and } \| x \|_1 \oslash \| x \|_1 \preceq \beta_1 \odot \eta_1.$$

Hence semi-equivalence relation is symmetric and transitive but is not reflexive in general.

2) If $\|.\|_1$ and $\|.\|_2$ are semi-equivalent and $(X, \|.\|_1)$ is complete then $(X, \|.\|_2)$ is also complete. In fact if

$$\| x \|_1 \oslash \| x \|_2 \preceq \eta_1 \text{ and } \| x \|_2 \oslash \| x \|_1 \preceq \eta_2,$$

then for $x, y \in X$, and every $\alpha \in (0, 1]$, the relations

$$\|x - y\|_{1\alpha}^1 \leq \eta_{1\alpha}^1 \|x - y\|_{2\alpha}^2 \ , \ \|x - y\|_{1\alpha}^2 \leq \eta_{1\alpha}^2 \|x - y\|_{2\alpha}^1,$$
$$\|x - y\|_{2\alpha}^1 \leq \eta_{2\alpha}^1 \|x - y\|_{1\alpha}^2 \text{ and } \|x - y\|_{2\alpha}^2 \leq \eta_{2\alpha}^2 \|x - y\|_{1\alpha}^1$$

prove our claim.

Theorem 3.12. *Every two fuzzy norms on a finite dimensional vector space X are semi-equivalent.*

Proof. Let $\| \,.\, \|_1$ and $\| \,.\, \|_2$ be two fuzzy norms on a finite dimensional vector space X and $\alpha \in (0, 1]$. Define $I : (X, \| \,.\, \|_{1\alpha}^2) \to (X, \| \,.\, \|_{2\alpha}^1)$ and $I : (X, \| \,.\, \|_{1\alpha}^1) \to (X, \| \,.\, \|_{2\alpha}^2)$ by $I(x) = x$. Thus for all $\alpha \in (0, 1]$ there exist positive real numbers δ_α^1 and δ_α^2 such that

$$\frac{\|x\|_{2\alpha}^1}{\|x\|_{1\alpha}^2} \leq \delta_\alpha^1, \forall x (\neq 0) \in X,$$

and

$$\frac{\|x\|_{2\alpha}^2}{\|x\|_{1\alpha}^1} \leq \delta_\alpha^2, \forall x (\neq 0) \in X.$$

For each $\alpha \in (0, 1]$, define

$$\delta_\alpha^{*1} = \inf\{\delta_\alpha^1 : \ \frac{\|x\|_{2\alpha}^1}{\|x\|_{1\alpha}^2} \leq \delta_\alpha^1, \forall x (\neq 0) \in X\},$$

and

$$\delta_\alpha^{*2} = \inf\{\delta_\alpha^2 : \ \frac{\|x\|_{2\alpha}^2}{\|x\|_{1\alpha}^1} \leq \delta_\alpha^2, \forall x (\neq 0) \in X\}.$$

Now from the fact that $\{[\delta_\alpha^{*1}, \delta_\alpha^{*2}] : \alpha \in (0, 1]\}$ is a family of nested bounded closed interval of real numbers, using Proposition 2.2, this family generates a positive fuzzy interval δ. By Proposition 2.3, $\delta \succ 0$ and

$$\|x\|_2 \oslash \|x\|_1 \preceq \delta, \forall x (\neq 0) \in X.$$

Similarly there exists positive fuzzy interval η such that for all $x(\neq 0) \in X$, $\|x\|_1 \oslash \|x\|_2 \preceq \eta$, which implies that $\|.\|_1$ and $\|.\|_2$ are semi-equivalent. □

Corollary 3.13. *Every finite dimensional fuzzy normed space is complete.*

Proof. Let $(X, \|.\|_1)$ be a finite dimensional fuzzy normed space. We know that there exits a linear one-to-one correspondence ϕ from X onto $\mathbb{C}^n$. Consider the fuzzy norm on $\mathbb{C}^n$ generated by the crisp Euclidean norm. Trivially $\mathbb{C}^n$ with this fuzzy norm is complete. Now for any $x \in X$, define $\|x\|_2 := \|\phi(x)\|$. This define a fuzzy norm on X which is semi-equivalent to $\|.\|_1$, by the previous theorem. Now part 2 of Remark 3.11 complets the proof. □

4. **Hahn-Banach Theorem**

Bag and Samanta in [3] established a Hahn-Bancah theorem for the strong dual of fuzzy normed linear space. Here we prove it for the weak dual of fuzzy linear spaces. Also some interesting consequences of this theorem are established.
We recall that a fuzzy bounded linear operator from a fuzzy normed space X into $\mathbb{R}$, with the fuzzy norm $\|.\|^{\sim}$ generated by the family $\{[|x|, |x|] : \alpha \in (0, 1]\}$ is called a fuzzy bounded functional, and the set of all such a functional is denoted by X^*.

Theorem 4.1. *Let X be a fuzzy normed linear space and Z be a subspace of X. If f is a fuzzy bounded linear functional on $(Z, \|.\|)$, then for each $\alpha \in (0, 1]$, there exists a pair of linear functionals f^1_α and f^2_α over $(X, \|.\|^2_\alpha)$ and $(X, \|.\|^1_\alpha)$, respectively, such that for all $x \in Z$, $f^1_\alpha(x) = f^2_\alpha(x) = f(x)$ and for each $\alpha \in (0, 1]$, $\|f^1_\alpha\|^{*1}_\alpha = \|f\|^{*1}_\alpha$, $\|f^2_\alpha\|^{*2}_\alpha = \|f\|^{*2}_\alpha$.*

Proof. By definition, there exists $\delta \in \mathcal{F}^+$, $\delta \succ \overline{0}$, such that for all $(0 \neq)x \in Z$,

$$\|f(x)\|^{\sim} \oslash \|x\| \preceq \delta.$$

So for each $\alpha \in (0, 1]$,

$$\frac{|f(x)|}{\|x\|^2_\alpha} \leq \delta^1_\alpha, \text{ and } \frac{|f(x)|}{\|x\|^1_\alpha} \leq \delta^2_\alpha, \ .$$

Thus

$$\|f\|^{*1}_\alpha = \sup\{\frac{|f(x)|}{\|x\|^2_\alpha} : (0 \neq)x \in Z\} \leq \delta^1_\alpha$$

and

$$\|f\|^{*2}_\alpha = \sup\{\frac{|f(x)|}{\|x\|^1_\alpha} : (0 \neq)x \in Z\} \leq \delta^2_\alpha.$$

This means that f is in the dual of $(Z, \|.\|^1_\alpha)$ and $(Z, \|.\|^2_\alpha)$, with the norms $\|.\|^{*2}_\alpha$ and $\|.\|^{*1}_\alpha$, respectively. Now using ordinary Hahn-Banach theorem, there exist f^1_α and f^2_α in the dual of $(X, \|.\|^2_\alpha)$ and $(X, \|.\|^1_\alpha)$, respectively, such that

$$\|f^1_\alpha\|^{*1}_\alpha = \|f\|^{*1}_\alpha \text{ and } \|f^2_\alpha\|^{*2}_\alpha = \|f\|^{*2}_\alpha,$$

and for all $x \in Z$, $f^1_\alpha(x) = f^2_\alpha(x) = f(x)$. □

Suppose $(X, \|.\|)$ is a fuzzy normed space. If Z is a subspace of X, then for each $\alpha \in (0,1]$, from the fact that $(X, \|.\|_\alpha^1)$ and $(X, \|.\|_\alpha^2)$ are normed space and from a corollary of ordinary Hahn-Banach theorem, there exists $f_\alpha^i \in X_\alpha^{i*}$, $i = 1, 2$, such that $f_\alpha^i(x) \neq 0$, where X_α^{i*} is denoted for the ordinary dual of $(X, \|.\|_\alpha^i)$, $i = 1, 2$.

Suppose Z is a subset of a fuzzy normed linear space X, we say $x \in \overline{Z}$, if for each $\delta \in \mathcal{F}^+$, $\delta \succ \overline{0}$, there exists $z \in Z$ such that $\|x - z\| \prec \delta$. In this case one can see that there exists a sequence $(x_n)_{n\in\mathbb{N}} \in Z$, such that $x_n \to x$, as $n \to \infty$.
Now we have the following corollary.

Corollary 4.2. *Suppose $(X, \|.\|)$ is a fuzzy normed space and Z is a subspace of X.*
(a) If x is not in $\overline{Z}$, then for each $\alpha \in (0,1]$ there exists $f_\alpha^i \in X_\alpha^{i*}$, $i = 1, 2$, such that $Z \subseteq \ker f_\alpha^i$ and $f_\alpha^i(x) \neq 0$.

(b) Let $M_i = \bigcap_{\alpha\in(0,1]}\{\ker f_\alpha^i : f_\alpha^i \in X_\alpha^{i*}$ and $Z \subseteq \ker f_\alpha^i\}$, $i = 1, 2$, then

$$\overline{Z} = M_1 \cap M_2.$$

Proof. (a) Suppose x is not in $\overline{Z}$, then there is a $\delta \in \mathcal{F}^+$, $\delta \succ \overline{0}$, such that for any $z \in Z$, $\|z - x\| \succeq \delta$. Let $M = \{z + rx : z \in Z,\ r \in \mathbb{R}\}$ and define $f(z + rx) = r$. Since

$$|r|\delta \preceq |r|\|x + r^{-1}z\| = \|rx + z\|,$$

the fact that $\delta \succ \overline{0}$, implies that

$$\|f\|^* \preceq 1 \oslash \delta.$$

So from the previous theorem for each $\alpha \in (0,1]$, there exists a pair of linear functionals f_α^1 and f_α^2 over $(X, \|.\|_\alpha^2)$ and $(X, \|.\|_\alpha^1)$, respectively, such that for all $x \in Z$, $f_\alpha^1(x) = f_\alpha^2(x) = f(x)$ and for each $\alpha \in (0,1]$, $\|f_\alpha^1\|_\alpha^{*1} = \|f\|_\alpha^{*1}$, $\|f_\alpha^2\|_\alpha^{*2} = \|f\|_\alpha^{*2}$. Obviously $Z \subseteq \ker f_\alpha^i$, i=1,2, which completes the proof.
(b) Trivially $Z \subseteq M_i$, $i = 1, 2$. Also $M_1 \cap M_2$ is fuzzy closed, since if $(0 \neq)x \in \overline{M_1 \cap M_2}$, then there exists a sequence $(x_n)_{n\in\mathbb{N}} \in M_1 \cap M_2$, such that $x_n \to x$, in fuzzy norm. Thus for each $\alpha \in (0,1]$, $\lim_{n\to\infty} \|x_n - x\|_\alpha^i = 0$, $i = 1, 2$. Now if $\alpha \in (0,1]$ and $f_\alpha^i \in X_\alpha^{i*}$, with $Z \subseteq \ker f_\alpha^i$, then $x_n \in \ker f_\alpha^i$. Thus $f_\alpha^i(x_n) = 0$, $n \in \mathbb{N}$. From continuity of f_α^i, $x \in \ker f_\alpha^i$. It means that $x \in M_1 \cap M_2$. This shows that $\overline{Z} \subseteq M_1 \cap M_2$. Now if x is not in $\overline{Z}$, from part (a), for each $\alpha \in (0,1]$ there exists $f_\alpha^i \in X_\alpha^{i*}$, $i = 1, 2$, such that $Z \subseteq \ker f_\alpha^i$ and $f_\alpha^i(x) \neq 0$. This implies that x is not in $M_1 \cap M_2$. □

5. **Conclusion**

In this note, using the definition of Bag and Samanta for fuzzy weakly bounded linear operator and its fuzzy norm, we were able to establish fuzzy version of some well-known theorems in functional analysis and simpler proofs for some theorems which have already been proved, although the most famous theorems in functional analysis was disapproved by some counterexamples.
A definition of fuzzy Banach algebra was represented. It can be considered as the start point of the theory of fuzzy Banach algebra.

A Hahn-Banach theorem for weakly fuzzy bounded operators and some of its interesting consequences are proved.

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