Iranian Journal of Fuzzy Systems **Vol. 8, No. 5,** (2011) pp. 131-139

GRADUAL NORMED LINEAR SPACE

I. SADEQI AND F. Y. AZARI

Abstract. In this paper, the gradual real numbers are considered and the notion of the gradual normed linear space is given. Also some topological properties of such spaces are studied, and it is shown that the gradual normed linear space is a locally convex space, in classical sense. So the results in locally convex spaces can be translated in gradual normed linear spaces. Finally, we give an example of a gradual normed linear space which is not normable in classical analysis.

1. Introduction

The term "fuzzy numbers" are often applied instead of "fuzzy intervals", especially if the core of fuzzy interval is a point (like; triangular fuzzy number). But such fuzzy numbers also generalize intervals, not numbers. Also fuzzy arithmetics inherit algebraic properties of interval arithmetic, not of numbers. Hence the name "fuzzy number", used by many authors is debatable. To avoid this confusion, the authors in [2] introduce a new concept in fuzzy set theory as "gradual real numbers". A gradual number in general can not be considered as a fuzzy set of real numbers because the mapping from the unit interval to the real line is not necessarily one to one. However gradual real numbers are equipped with the same algebraic structures as real numbers (addition is a group, etc.). For more details on gradual real numbers and fuzzy intervals we refer the reader to [2, 4, 5, 6, 7, 8].

In this paper, we show that the set of gradual real numbers along with gradual addition and gradual scalar multiplication is a real linear space. Also we introduce the gradual normed linear space and study some basic topological properties of it. Finally we show that a gradual normed linear space is a locally convex space (LSC) in classical case. Therefore, all results of locally convex spaces can be translated in gradual normed linear spaces such that most of fundamental theorems of locally convex case do hold true in gradual normed linear spaces. Further, by an example we point out that the spectrum of gradual normed linear space is broader than the classical case.

2. Preliminary and Some Results

Definition 2.1. [2] A gradual real number $\tilde{r}$ is defined by an assignment function $A_{\tilde{r}}$ from $(0, 1]$(half real interval) to the set of real numbers $\mathbb{R}$. From now on, we denote the set of all gradual real numbers by $G(\mathbb{R})$.

Received: January 2010; Revised: August 2010; Accepted: September 2010
Key words and phrases: Fuzzy interval, Gradual real number, Locally convex space.

It is clear that any real number $r \in \mathbb{R}$ can be viewed as an constant assignment function $A_{\tilde{r}}$ such that $A_{\tilde{r}}(\lambda) = r$, $0 < \lambda \leq 1$.

Definition 2.2. A gradual real number $\tilde{r}$ is called non-negative if for all $\lambda \in (0, 1]$, $A_{\tilde{r}}(\lambda) \geq 0$. The set of all non-negative gradual real numbers is denoted by $G^*(\mathbb{R})$.

Let * be any operation in real numbers. In [3] the gradual operations are defined as follows: let $\tilde{r}_1$ and $\tilde{r}_2$, be two gradual numbers with assignment functions $A_{\tilde{r}_1}$ and $A_{\tilde{r}_2}$, the gradual number $\tilde{r}_1 * \tilde{r}_2$ has assignment function $A_{\tilde{r}_1 * \tilde{r}_2}$ such that:

$$\forall \lambda \in (0, 1], \quad A_{\tilde{r}_1 * \tilde{r}_2}(\lambda) = A_{\tilde{r}_1}(\lambda) * A_{\tilde{r}_2}(\lambda).$$

Let A_1 and A_2 be two assignment functions in $(0, 1]$, then the additive gradual real numbers $\tilde{r}_1$, $\tilde{r}_2$ defined as $A_{1+2}(\lambda) = A_1(\lambda) + A_2(\lambda)$ for all $\lambda \in (0, 1]$. Given $\lambda \in (0, 1]$, the additive identities in $G(\mathbb{R})$ is $A_{\tilde{0}}(\lambda) = 0$ and the regular inverse is defined as $(A_{-\tilde{r}}(\lambda) = -A_{\tilde{r}}(\lambda))$ [3]. For $c \in \mathbb{R}$ and $\lambda \in (0, 1]$, we define the gradual scalar multiplication, $c\tilde{r}$ as $A_{c\tilde{r}}(\lambda) = cA_{\tilde{r}}(\lambda)$.

Remark 2.3. By the above definitions, It can be easily checked that $G(\mathbb{R})$, the set of gradual real numbers, with the gradual addition and gradual scalar multiplication, is a real linear space.

A partial order relation on gradual real numbers can be defined as follows.

Definition 2.4. [3] Let $\tilde{r}, \tilde{s} \in \mathbb{R}$, the partial relation " $\leq$ " in $G(\mathbb{R})$ defined as; $\tilde{r} \leq \tilde{s}$ iff $A_{\tilde{r}}(\lambda) \leq A_{\tilde{s}}(\lambda)$ for all $\lambda \in (0, 1]$.

Theorem 2.5. *Let $\tilde{r}, \tilde{s}, \tilde{t} \in G(\mathbb{R})$. Then:*
(a) *if $\tilde{r} \leq \tilde{s}$ then $\tilde{r} - \tilde{t} \leq \tilde{s} - \tilde{t}$;*
(b) *if $\tilde{r} \leq \tilde{s}$ and $\tilde{0} \leq \tilde{t}$ then (i) $\tilde{r}.\tilde{t} \leq \tilde{s}.\tilde{t}$ and (ii) $\tilde{r}/\tilde{t} \leq \tilde{s}/\tilde{t}, \tilde{t} \neq \tilde{0}$;*
(c) $(\tilde{r}.\tilde{s})/\tilde{t} = \tilde{r}.(\tilde{s}/\tilde{t}), \tilde{t} \neq \tilde{0}$.

Proof. a) Let $\tilde{r} \leq \tilde{s}$, then $A_{\tilde{r}}(\lambda) \leq A_{\tilde{s}}(\lambda)$ for all $\lambda \in (0, 1]$. Hence $A_{\tilde{r}}(\lambda) - A_{\tilde{t}}(\lambda) \leq A_{\tilde{s}}(\lambda) - A_{\tilde{t}}(\lambda)$, thus by the additivity in $G(\mathbb{R})$, $A_{\tilde{r}-\tilde{t}}(\lambda) \leq A_{\tilde{s}-\tilde{t}}(\lambda)$, and so $\tilde{r} - \tilde{t} \leq \tilde{s} - \tilde{t}$.

b) For (i), let $\tilde{r} \leq \tilde{s}$, it is obvious that $A_{\tilde{r}}(\lambda) \leq A_{\tilde{s}}(\lambda)$. If $\tilde{0} \leq \tilde{t}$, then $A_{\tilde{0}}(\lambda) \leq A_{\tilde{t}}(\lambda)$, so $A_{\tilde{r}}(\lambda).A_{\tilde{t}}(\lambda) \leq A_{\tilde{s}}(\lambda).A_{\tilde{t}}(\lambda)$ and $A_{\tilde{r}.\tilde{t}}(\lambda) \leq A_{\tilde{s}.\tilde{t}}(\lambda)$. The proof of (ii) is easy.

c) For all $\lambda \in (0, 1]$, we have:

$$\begin{aligned} A_{\frac{\tilde{r}.\tilde{s}}{\tilde{t}}}(\lambda) &= \frac{A_{\tilde{r}.\tilde{s}}(\lambda)}{A_{\tilde{t}}(\lambda)} \\ &= \frac{A_{\tilde{r}}(\lambda).A_{\tilde{s}}(\lambda)}{A_{\tilde{t}}(\lambda)} \\ &= A_{\tilde{r}}(\lambda).A_{\frac{\tilde{s}}{\tilde{t}}}(\lambda) \\ &= A_{\tilde{r}.(\frac{\tilde{s}}{\tilde{t}})}(\lambda), \end{aligned}$$

hence $(\tilde{r}.\tilde{s})/\tilde{t} = \tilde{r}.(\tilde{s}/\tilde{t})$. $\square$

3. Gradual Normed Linear Space

In this section we introduce the concept of the gradual norm on a gradual linear space. Then we point out that, in general, a gradual normed linear space is a locally convex space.

Definition 3.1. Let X be a real vector space and $\|.\|_G$ be a mapping from X to $G^*(\mathbb{R})$. The pair $(X, \|.\|_G)$ is called a gradual normed linear space (GNLS) and $\|.\|_G$ a gradual norm iff

(G1) $\|x\|_G = 0$ iff $x = 0$;
(G2) $\|rx\|_G = |r|\|x\|_G, \forall r \in \mathbb{R}$;
(G3) $\|x + y\|_G \leq \|x\|_G + \|y\|_G$ for all $x, y \in X$.

Remark 3.2. We rewrite the above definition in assignment function language. Let $x, y \in X$, $r \in \mathbb{R}$ and $\lambda \in (0, 1]$, then $(X, \|.\|_G)$ is called a gradual normed linear space iff

(G1) $A_{\|x\|_G}(\lambda) = A_{\tilde{0}}(\lambda)$ iff $x = 0$;
(G2) $A_{\|rx\|_G}(\lambda) = |r|A_{\|x\|_G}(\lambda)$;
(G3) $A_{\|x+y\|_G}(\lambda) \leq A_{\|x\|_G}(\lambda) + A_{\|y\|_G}(\lambda)$.

Example 3.3. Suppose $X = \mathbb{R}^n$ and for $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$, $\lambda \in (0, 1]$, define $\|.\|_G$ by $A_{\|x\|_G}(\lambda) = e^\lambda \sum\limits_{i=1}^{n} |x_i|$. Then $\|.\|_G$ is a gradual norm on $\mathbb{R}^n$ and $(\mathbb{R}^n, \|.\|_G)$ is a gradual normed linear space.

Example 3.4. Let us consider the linear space $X = C[0, 1]$, the space of all continuous real valued function on $[0, 1]$ with the usual linear operation. Consider the two norms on $C[0, 1]$ defined as $\|f\|_0 = (\int_0^1 |f(t)|^2 dt)^{\frac{1}{2}}$ and $\|f\|_1 = \max\limits_{0\leq t\leq 1}\{|f(t)|\}$. Now, we define the function $\|.\|_G : C[0, 1] \to G^*(\mathbb{R})$ by

$$A_{\|f\|_G}(\lambda) = \begin{cases} \|f\|_0 & 0 < \lambda \leq \frac{1}{2} \\ \|f\|_1 & \frac{1}{2} < \lambda \leq 1. \end{cases}$$

Then it can be verified that $\|.\|_G$ is a gradual norm on X.

Definition 3.5. Let $(X, \|.\|_G)$ be a gradual normed linear space. We say that a sequence $\{x_n\} \in X$ convergent to $x \in X$, iff $\lim\limits_{n\to\infty} \|x_n - x\|_G = 0$, i.e., $\lim\limits_{n\to\infty} A_{\|x_n - x\|_G}(\lambda) = A_{\tilde{0}}(\lambda)$ for $\lambda \in (0, 1]$. Also a sequence $\{x_n\} \in X$ in a gradual normed linear space $(X, \|.\|_G)$ is called Cauchy iff $\lim\limits_{n\to\infty} \|x_m - x_n\|_G = 0$, i.e., $\lim\limits_{n\to\infty} A_{\|x_m - x_n\|_G}(\lambda) = A_{\tilde{0}}(\lambda)$ for $\lambda \in (0, 1]$. The pair $(X, \|.\|_G)$ is said to be complete if any Cauchy sequence in $(X, \|.\|_G)$ is convergent.

Theorem 3.6. *Let $(X, \|.\|_G)$ be a gradual normed linear space, then every convergent sequence in X is also a Cauchy sequence.*

Proof. Let the sequence $\{x_n\} \in X$ convergent to $x \in X$. Then $\lim\limits_{n\to\infty} A_{\|x_n - x\|_G}(\lambda) = 0$ for all $\lambda \in (0, 1]$, i.e., given $\epsilon > 0$ and $\lambda \in (0, 1]$ there exists $N(\lambda) \in \mathbb{N}$ such that

for all $n \geq N(\lambda)$, $A_{\|x_n - x\|_G}(\lambda) < \frac{1}{2}\epsilon$.
Now, for every $m, n \geq N(\lambda)$,

$$\begin{aligned} A_{\|x_m - x_n\|_G}(\lambda) &= A_{\|x_m - x + x - x_n\|}(\lambda) \\ &\leq A_{\|x_m - x\|_G}(\lambda) + A_{\|x_n - x\|_G}(\lambda) \\ &\leq \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon. \end{aligned}$$

Therefore, $\{x_n\}$ is a Cauchy sequence in X. □

Lemma 3.7. *Let $(X, \|.\|_G)$ be a gradual normed linear space, $\alpha \in (0,1]$, $\epsilon > 0$ and $N(\epsilon, \alpha) = \{x \ : \ A_{\|x\|_G}(\alpha) < \epsilon\}$. Then:*
i) *$N(\epsilon, \alpha) = \epsilon N(1, \alpha)$;*
ii) *If $\epsilon_1 \leq \epsilon_2$ then $N(\epsilon_1, \alpha) \leq N(\epsilon_2, \alpha)$;*
iii) *for all $x \in X$, if $A_{\|x\|_G}$ is a decreasing function and $\alpha_1 \leq \alpha_2$, then $N(\epsilon, \alpha_1) \subseteq N(\epsilon, \alpha_2)$.*

Proof. i) Let $\alpha \in (0,1]$ and $\epsilon > 0$, then we have:

$$\begin{aligned} N(\epsilon, \alpha) &= \{x \ : A_{\|x\|_G}(\alpha) < \epsilon\} = \{x \ : \ \frac{1}{\epsilon} A_{\|x\|_G}(\alpha) < 1\} \\ &= \{x \ : \ A_{\|\frac{x}{\epsilon}\|}(\alpha) < 1\} = \{y\epsilon \ : \ A_{\|y\|_G}(\alpha) < 1\} \\ &= \epsilon\{y \ : \ A_{\|y\|_G}(\alpha) < 1\} \\ &= \epsilon N(1, \alpha). \end{aligned}$$

ii) If $x \in N(\epsilon_1, \alpha)$, then $A_{\|x\|_G}(\alpha) \leq \epsilon_1$, and since $\epsilon_1 \leq \epsilon_2$, so $A_{\|x\|_G}(\alpha) \leq \epsilon_2$. Hence $x \in N(\epsilon_2, \alpha)$.

iii) If $x \in N(\epsilon, \alpha_1)$ then $A_{\|x\|_G}(\alpha_1) < \epsilon$. Since $A_{\|x\|_G}$ is a decreasing function, so if $\alpha_1 \leq \alpha_2$ then $A_{\|x\|_G}(\alpha_2) \leq A_{\|x\|_G}(\alpha_1)$. Thus $A_{\|x\|_G}(\alpha_2) \leq \epsilon$, hence $x \in N(\epsilon, \alpha_2)$. □

To attain the main result, the provision of the following definitions seems necessary.

Let $(X, \|.\|_G)$ be a gradual normed linear space and $A \subseteq X$. The point $x_0 \in X$ is called an interior point of A if there exists $N(\epsilon_0, \alpha_0)$ such that $x_0 + N(\epsilon_0, \alpha_0) \subseteq A$; $IntA$ denotes the set of all interior points of A; A is called a gradual open set if $IntA = A$. Also x_0 is called a closure point of A if $\{x_0 + N(\alpha, \alpha)\} \bigcap A \neq \phi$ for every $\alpha \in (0,1]$; $\bar{A}$ denotes the set of all closure point of A; A is called gradual closed set if $\bar{A} = A$. The set A is called gradual bounded if for each $\alpha \in (0,1]$ there exists $M = M(\alpha) > 0$ such that $A \subseteq N(M, \alpha)$.

Theorem 3.8. *Let $(X, \|.\|_G)$ be a gradual normed linear space and for every $x \in X$, $A_{\|x\|_G}$ be a decreasing function. Then $N(\epsilon, \alpha)$ is gradual open.*

Proof. Since $A_{\|.\|_G}$ is a gradual norm so for $x_0, y \in X$ and $\alpha \in (0,1]$, $A_{\|x_0+y\|_G}(\alpha) \leq A_{\|x_0\|_G}(\alpha) + A_{\|y\|_G}(\alpha)$. By the assumption, $A_{\|y\|_G}$ is a decreasing function, therefore, for $\alpha_0 \in (0, \alpha]$, $A_{\|x_0+y\|_G}(\alpha) \leq A_{\|x_0\|_G}(\alpha) + A_{\|y\|_G}(\alpha_0)$. Suppose that $x_0 \in N(\epsilon, \alpha)$

then for $\alpha \in (0,1]$ and $\epsilon > 0$, $A_{\|x_0\|_G}(\alpha) < \epsilon$, so there exists $\epsilon_0 > 0$ such that $A_{\|x_0\|_G}(\alpha) = \epsilon - \epsilon_0$. Therefore, for every $y \in N(\epsilon_0, \alpha_0)$ we have:

$$\begin{aligned} A_{\|x_0+y\|_G}(\alpha) &\leq A_{\|x_0\|_G}(\alpha) + A_{\|y\|_G}(\alpha_0) \\ &\leq \epsilon - \epsilon_0 + \epsilon_0 = \epsilon. \end{aligned}$$

This shows that $x_0 + N(\epsilon_0, \alpha_0) \subseteq N(\epsilon, \alpha)$,i.e., $x_0 \in IntN(\epsilon, \alpha)$, and hence $N(\epsilon, \alpha)$ is gradual open. $\square$

Theorem 3.9. *Let $(X, \|.\|_G)$ be a gradual normed linear space and for every $x \in X$, $A_{\|x\|_G}$ be a decreasing function. Then for a given subset $B \subseteq X$, we have:*

i) *$IntB = X \setminus \overline{X \setminus B}$;*
ii) *B is gradual open iff $X \setminus B$ is gradual closed;*
iii) *$\bar{B}$ is gradual closed;*
iv) *$IntB$ is gradual open.*

Proof. i) If $x_0 \in IntB$, then there exists $N(\epsilon_0, \alpha_0)$ such that $\{x_0 + N(\epsilon_0, \alpha_0)\} \subseteq B$; equivalently, $\{x_0 + N(\epsilon_0, \alpha_0)\} \bigcap X \setminus B = \phi$. Now let $\alpha = \min\{\alpha_0, \epsilon_0\}$, then by Lemma 3.7, $N(\alpha, \alpha) \subseteq N(\epsilon_0, \alpha_0)$, so $\{x_0 + N(\alpha, \alpha)\} \bigcap X \setminus B = \phi$; i.e., $x_0 \notin \overline{X \setminus B}$. Hence $x_0 \in X \setminus \overline{X \setminus B}$. The converse is routine.

ii) Let $X \setminus B$ be a gradual closed set, choose $x_0 \in B$, then $x_0 \notin X \setminus B$ and x_0 is not a closure point of $X \setminus B$. So for any $\alpha \in (0,1]$, $\{x_0 + N(\alpha, \alpha)\} \bigcap X \setminus B = \phi$,i.e., $\{x_0 + N(\alpha, \alpha)\} \subset B$. Then x_0 is an interior point of B, hence B is a gradual open set. Conversely, let B be a gradual open set and x_0 be a closure point of $X \setminus B$. So $\{x_0 + N(\alpha, \alpha)\} \bigcap X \setminus B = \phi$ and then x_0 is not an interior point of B. Since B is an open set so $x_0 \in X \setminus B$.

iii) It is clear that for given $x_0, y \in X$ and $\alpha \in (0,1]$, $A_{\|x_0+y\|_G}(\alpha) \leq A_{\|x_0\|_G}(\alpha) + A_{\|y\|_G}(\alpha)$. Since $A_{\|x\|_G}$ is a decreasing function so $A_{\|x_0+y\|_G}(\alpha) \leq A_{\|x_0\|_G}(\alpha_0) + A_{\|y\|_G}(\alpha_0)$ for all $\alpha_0 \in (0, \alpha]$. Thus for $x_0 \in \bar{\bar{B}}$ and every $N(\epsilon, \alpha)$ there exists $N(\frac{\epsilon}{2}, \alpha_0)$ such that $N(\frac{\epsilon}{2}, \alpha_0) + N(\frac{\epsilon}{2}, \alpha_0) \subseteq N(\epsilon, \alpha)$. Since $\{x_0 + N(\frac{\epsilon}{2}, \alpha_0)\} \cap \bar{B} \neq \phi$, there exists $y_0 \in \bar{B}$ and $y_0 \in x_0 + N(\frac{\epsilon}{2}, \alpha_0)$. Then $y_0 + N(\frac{\epsilon}{2}, \alpha_0) \subseteq x_0 + N(\frac{\epsilon}{2}, \alpha_0) + N(\frac{\epsilon}{2}, \alpha_0) \subseteq x_0 + N(\epsilon, \alpha)$. Also since $y_0 \in \bar{B}$ and $\{x_0 + N(\epsilon, \alpha)\} \cap B \supseteq \{y_0 + N(\frac{\epsilon}{2}, \alpha_0)\} \cap B$, so $\{x_0 + N(\epsilon, \alpha)\} \cap B \neq \phi$. Therefore, $x_0 \in \bar{B}$.

(iv) Using (i) and (iii), the proof is straightforward. $\square$

Theorem 3.10. *Let $(X, \|.\|_G)$ be a gradual normed linear space and B be a subset of X. Then:*

i) *$x \in \bar{B}$ iff there exists $\{x_n\}_{n=1}^{\infty} \subseteq B$ such that for any $\alpha \in (0,1]$, $\lim\limits_{n \to \infty} A_{\|x_n\|_G}(\alpha) = A_{\|x\|_G}(\alpha)$;*

ii) *The set B is a gradual bounded iff $\lim\limits_{n \to \infty} \frac{x_n}{n} = 0$, for arbitrary $\{x_n\}_{n=1}^{\infty}$ in X.*

Proof. i) Suppose that for any $\alpha \in (0,1]$, $\lim\limits_{n \to \infty} A_{\|x_n\|_G}(\alpha) = A_{\|x\|_G}(\alpha)$. Then every neighborhood $\{x + N(\alpha, \alpha)\}$ of x contains a point of B, so $x \in \bar{B}$. Conversely, suppose that $B \subseteq X$ and $x \in B$. For each positive integer n and $\alpha \in (0,1]$, take the neighborhood $\{x + N(\alpha, \frac{1}{n})\}$, and choose x_n to be a point of its intersection with B. It means for given $n \in \mathbb{N}$, there exists x_n such that $x_n - x \in N(\alpha, \frac{1}{n})$. For $\epsilon > 0$, if we choose N such that $N\epsilon > 1$, then for all $n > N$, $A_{\|x_n - x\|_G}(\alpha) < \epsilon$.

ii) Let B be a gradual bounded and $\{x_n\} \subseteq B$. Then for $\alpha \in (0,1]$ there exists $M = M(\alpha) > 0$ such that $B \subseteq N(M, \alpha)$; thus $A_{\|\frac{x_n}{x}\|_G}(\alpha) \leq \frac{M}{n}$; hence $\lim_{n\to\infty} A_{\|\frac{x_n}{x}\|_G}(\alpha) = 0$. Conversely if B is not gradual bounded, then there is an $\alpha_0 \in (0,1]$ such that $x_n \in B$ and $x_n \notin N(n, \alpha_0)$ for all $n \in \mathbb{Z}^+$; since $N(n, \alpha_0) = nN(1, \alpha_0)$ this shows that $\frac{x_n}{n} \notin N(1, \alpha_0)$. Therefore, there exists $\{x_n\} \subseteq B$ such that $\lim_{n\to\infty} \frac{x_n}{n} \neq 0$. □

By the following important result, it is shown that gradual normed linear space is also a topological vector space, in classical analysis case.

Theorem 3.11. *Let $(X, \|.\|_G)$ be a gradual normed linear space, for every $x \in X$, $A_{\|.\|_G}$ be a decreasing function. Then $(X, \|.\|_G)$ is a Hausdorff topological vector space, whose neighborhood base of origin 0 is $\{N(\epsilon, \alpha) \ : \ \epsilon > 0 \ , \ \alpha \in (0,1]\}$.*

Proof. For $W_1 = N(\epsilon_1, \alpha_1)$ and $W_2 = N(\epsilon_2, \alpha_2)$, take $\alpha_0 = min\{\alpha_1, \alpha_2\}$ and $\epsilon_0 = min\{\epsilon_1, \epsilon_2\}$, it is clear that $W_0 = N(\epsilon_0, \alpha_0) \subseteq W_1 \cap W_2$. So the family $\{N(\epsilon, \alpha) \ : \ \epsilon > 0, \alpha > 0\}$, is the base of the toplogy on X. Now we show that X is a topological vector space. It needs to verify the following conditions:

(a) the topological space X is a Hausdorff space.
(b) the vector space operations are continuous.

For (a), if $x \in X$, $x \neq 0$; i.e., $\|x\|_G \neq 0$, then there is $\alpha_0 \in (0,1]$ with $A_{\|x\|_G}(\alpha_0) \neq 0$. Hence, there exists $\epsilon_0 > 0$ such that $A_{\|x\|_G}(\alpha_0) > \epsilon_0$; i.e., $x \notin N(\epsilon_0, \alpha_0)$, so X is a Hausdorff space.
For (b), we prove that the vector space operations (the addition and scaler multiplication) are continuous. Let $W = N(\epsilon, \alpha)$, fix $\alpha_0 < \alpha$, we show that $W_0 + W_0 \subseteq W$, where $W_0 = N(\frac{\epsilon}{2}, \alpha_0)$.
Let $x, y \in W_0$ then $A_{\|x\|_G}(\alpha_0) < \frac{\epsilon}{2}$, $A_{\|y\|_G}(\alpha_0) < \frac{\epsilon}{2}$. Since $(X, \|.\|_G)$ is a gradual normed linear space and $A_{\|x\|_G}$ is a decreasing function so:

$$\begin{aligned} A_{\|x+y\|_G}(\alpha) &\leq A_{\|x\|_G}(\alpha) + A_{\|y\|_G}(\alpha) \\ &\leq A_{\|x\|_G}(\alpha_0) + A_{\|y\|_G}(\alpha_0) \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Hence $x+y \in W$, which shows that $W_0 + W_0 \subseteq W$. To show that scalar multiplication is continuous, we should find $N(\epsilon_0, \alpha_0)$ and $N_\lambda(r)$ such that $N_\lambda(r).N(\epsilon_0, \alpha_0) \subseteq N(\epsilon, \alpha)$. Let $b \in N_\lambda(r)$ and $x \in N(\epsilon_0, \alpha_0)$, then

$$\begin{aligned} A_{\|bx\|_G}(\alpha) &= |b|A_{\|x\|_G}(\alpha) = |b - r + r|A_{\|x\|_G}(\alpha) \\ &< (|b-r| + |r|)A_{\|x\|_G}(\alpha) < (\lambda + |r|)A_{\|x\|_G}(\alpha) \\ &< \epsilon, \end{aligned}$$

when $\alpha_0 = \alpha$ and $\epsilon_0 = \frac{\epsilon}{2|r|}$ and $\lambda = |r|$. Therefore, $(X, \|.\|_G)$ is a topological vector space. $\square$

The function μ_α from X to $[0, +\infty)$ is defined by $\mu_\alpha(x) = \inf\{\lambda > 0 \ : \ x \in N(\lambda, \alpha)\}$.

Theorem 3.12. *Let $(X, \|.\|_G)$ be a gradual normed linear space. Then, for any $x \in X$ and $\alpha \in (0, 1]$:*

i) *$N(\epsilon, \alpha)$ is convex;*
ii) *$\mu_\alpha(x) = A_{\|x\|_G}(\alpha)$.*

Proof. i) For all $x, y \in N(\epsilon, \alpha)$, $\lambda \in [0, 1]$ and $\alpha \in (0, 1]$ we have

$$\begin{aligned} A_{\|\lambda x + (1-\lambda)y\|_G}(\alpha) &\leq A_{\|\lambda x\|_G}(\alpha) + A_{\|(1-\lambda)y\|_G}(\alpha) \\ &= \lambda A_{\|x\|_G}(\alpha) + (1-\lambda)A_{\|y\|_G}(\alpha). \end{aligned}$$

Since $x, y \in N(\epsilon, \alpha)$ so $A_{\|x\|_G}(\alpha) \leq \epsilon$ and $A_{\|y\|_G}(\alpha) \leq \epsilon$. Then $\lambda A_{\|x\|_G}(\alpha) + (1 - \lambda)A_{\|y\|_G}(\alpha) \leq \lambda\epsilon + (1-\lambda)\epsilon = \epsilon$. Namely $\lambda x + (1-\lambda)y \in N(\epsilon, \alpha)$, hence $N(\epsilon, \alpha)$ is convex.

ii) If $\lambda > 0$, $x \in N(\lambda, \alpha)$, i.e., $A_{\|x\|_G}(\alpha) < \lambda$, then $A_{\|x\|_G}(\alpha) < \inf \lambda = \mu_\alpha(x)$. Conversely, given $\epsilon > 0$, let $\lambda_0 = A_{\|x\|_G}(\alpha) + \epsilon$, then $x \in N(\lambda_0, \alpha)$, $\mu_\alpha(x) \leq \lambda_0$; since ϵ is arbitrary, $\mu_\alpha(x) \leq A_{\|x\|_G}(\alpha)$. $\square$

Definition 3.13. [1] If X is a vector space over $\mathbb{F}$, a seminorm is a function $p : X \to [0, \infty)$ having the properties:

(a) $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$;
(b) $p(\alpha x) = |\alpha| p(x)$,$\forall \alpha \in \mathbb{F}$ and $x \in X$

Corollary 3.14. *Let $(X, \|.\|_G)$ be a gradual normed linear space, then for any $\alpha \in (0, 1]$, μ_α is a seminorm on X.*

Proof. By Theorem 3.12, for every $x \in X$ and $\alpha \in (0, 1]$, $\mu_\alpha(x) = A_{\|x\|_G}(\alpha)$. Since $(X, \|.\|_G)$ is a gradual normed linear space, so $\mu_\alpha(tx) = |t|\mu_\alpha(x)$, $\mu_\alpha(x + y) \leq \mu_\alpha(x) + \mu_\alpha(y)$. Therefore, μ_α is a seminorm on X. $\square$

Definition 3.15. [1] A locally convex space is a topological vector space whose topology is defined by a family of seminorms P such that $\bigcap\limits_{p \in \mathrm{P}} \{x \ : \ p(x) = 0\} = \{0\}$.

Remark 3.16. If for any $\alpha \in (0, 1]$, $A_{\|x\|_G}(\alpha) = 0$ then $x = 0$; therefore, $\bigcap\limits_{\alpha \in (0,1]} \{x \ : \ \mu_\alpha(x) = 0\} = \{0\}$. As cited in Corollary 3.13, since μ_α is a seminorm, so the induced topology by the family of seminorms μ_α on X, is a toplogical vector space. Hence, by the above definition, X is a locally convex space. For more details we refer the reader to [1]. Therefore, the gradual normed linear space, in general, is a locally convex space and so transition of results from locally convex spaces to gradual normed linear spaces can be derived. Particularly, there are many important well known theorems in locally convex spaces, among them: Hahn-Banach theorem, the uniform boundedness theorem, the open mapping theorem and the closed graph theorem, which remain true for gradual normed linear spaces.

By the above discussion we have the following main result:

Theorem 3.17. *Any gradual normed linear space is a locally convex space.*

What follows is an example of a gradual normed linear space, which is not normable, in classical sense. That is why we claim that the spectrum of gradual normed linear space is broader than classical analysis.

Example 3.18. The space of real continuous function, $C((0,1])$, is a gradual normed linear space, but it is not classical normable.
Define $\|.\|_G : C((0,1]) \to G^*(\mathbb{R})$ as $\|f\|_G(\lambda) = |f(\lambda)|$, we show that $\|.\|_G$ is a gradual norm on $C((0,1])$.

(G1) $\|f\|_G = 0 \Leftrightarrow \|f\|_G(\lambda) = 0$ for all $\lambda \in (0,1] \Leftrightarrow |f(\lambda)| = 0$ for all $\lambda \in (0,1]$ $\Leftrightarrow f = 0$;

(G2) $\|rf\|_G(\lambda) = |rf(\lambda)| = |r||f(\lambda)| = |r|\|f\|_G(\lambda)$ for all $\lambda \in (0,1]$ then $\|rf\|_G = |r|\|f\|_G$;

(G3) $\|f+g\|_G(\lambda) = |(f+g)(\lambda)| \leq |f(\lambda)| + |g(\lambda)| \leq \|f\|_G(\lambda) + \|g\|_G(\lambda)$ for all $\lambda \in (0,1]$, then $\|f+g\|_G \leq \|f\|_G + \|g\|_G$.

Therefore, $(C((0,1]), \|.\|_G)$ is a gradual normed linear space. But, as it is known in classical analysis, since $(0,1]$ is not compact so the space $C((0,1])$ is not normable.

4. Conclusion

The purpose of this paper is to study the gradual real numbers and the notion of gradual normed linear spaces. Also, it is shown that a gradual normed linear space is a locally convex space in general, and by an example it is pointed out that the spectrum of the category of gradual normed linear space is broader than classical case.

References

[1] J. B. Conway, *A course in functional analysis*, Springer-Verlag, New York, 1990.
[2] D. Dubois and H. Prade, *Gradual elements in a fuzzy set*, Soft Computing-A Fusion of Foundations, Methodologies and Applications, **12** (2007), 165-175.
[3] J. Fortin, D. Dubois and H. Prade, *Gradual numbers and their application to fuzzy interval analysis*, IEEE Transactions on Fuzzy Systems, **16** (2008), 388-402.
[4] U. Hoehle, *Fuzzy real numbers as Dedekind cuts with respect to a multiple-valued logic*, Fuzzy Sets and Systems, **24** (1987), 263-278.
[5] E. Kaucher, *Interval analysis in the extended interval space* $\mathbb{R}$, Computing Supplement, **2** (1980), 33-49.
[6] R. Lowen, *Fuzzy set theory*, Ch. 5 : Fuzzy Real Numbers, Kluwer, Dordrecht, 1996.
[7] S. E. Rodabaugh, *Fuzzy addition in the L-fuzzy real line*, Fuzzy Sets and Systems, **8** (1982), 39-51.
[8] I. Sadeqi and M. Salehi, *Fuzzy compact operators and topological degree theory*, Fuzzy Sets and Systems, **160** (2009), 1277-1285.

I. Sadeqi*, Department of Mathematics, Sahand university of technology, Tabriz-Iran
E-mail address: esadeqi@sut.ac.ir

F. Y. Azari, Department of Mathematics, Sahand university of technology, Tabriz-Iran
E-mail address: fyaqubazari@gmail.com

*Corresponding author