# A TAUBERIAN THEOREM FOR $(C,1,1)$ SUMMABLE DOUBLE SEQUENCES OF FUZZY NUMBERS

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Abstract. In this paper, we determine necessary and sufficient Tauberian conditions under which convergence in Pringsheim's sense of a double sequence of fuzzy numbers follows from its $(C,1,1)$ summability. These conditions are satisfied if the double sequence of fuzzy numbers is slowly oscillating in different senses. We also construct some interesting double sequences of fuzzy numbers.

## 1. Introduction

Since the concept of fuzzy sets and fuzzy set operations were introduced by Zadeh [22], several authors discussed various aspects of the theory and applications of fuzzy sets such as fuzzy topological spaces and fuzzy mathematical programming. Matloka [8] introduced bounded and convergent sequences of fuzzy numbers and showed that every convergent sequence is bounded. In [11], Nanda proved that every Cauchy sequence of fuzzy numbers is convergent. Subrahmanyam [13] defined the Cesàro summability of sequences of fuzzy numbers and proved fuzzy analogues of some Tauberian theorems for sequences of fuzzy numbers. Ordinary convergence of a single sequence of real numbers or fuzzy numbers implies Cesàro convergence, but the converse is not true in general. Móricz [10] obtained necessary and sufficient Tauberian conditions under which ordinary convergence follows from Cesàro convergence of single sequences of real numbers. Recently, Talo and Çakan [16] proved fuzzy analogues of the results in [10]. For other interesting results related to Tauberian type theorems for Cesàro summability method of sequences of fuzzy numbers, we refer the papers [2], [3], [4], [5], and [15].

It is well-known that every $P$-convergent and bounded double sequence of real numbers is $(C,1,1)$ summable to its $P$-convergence, but the converse of this implication is not true in general. Móricz [9] obtained necessary and sufficient conditions under which convergence in Pringsheim's sense of a double sequence $(u_{mn})$ of real numbers follows from $(C,1,1)$ summability of $(u_{mn})$.

Applying the concept of fuzzy real numbers, fuzzy real-valued single and double sequences were studied in [18, 19, 20, 21].

Our main goal in this paper is to give fuzzy analogues of Tauberian theorems in [9] for $(C,1,1)$ summability method of fuzzy numbers. In Section 2, we recall some

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notations, basic definitions and theorems for the fuzzy numbers. In Section 3, we prove a Tauberian theorem for $(C,1,1)$ summability of double sequences of fuzzy numbers. In the last section, we define the slow oscillation of a double sequence of fuzzy numbers in different senses and prove that the slow oscillation in some sense is a Tauberian condition for $(C,1,1)$ summability method. We also give a classical Tauberian theorem in Landau's type for $(C,1,1)$ summability method.

## 2. **Preliminaries**

First, we procure some notations and basic definitions which are used throughout this paper. Goetschel and Voxman [7] introduced the concept of fuzzy numbers as follows:

A fuzzy number is a fuzzy set on $\mathbb{R}$, i.e. a mapping $u:\mathbb{R}\to[0,1]$ with the following properties:

(i) $u$ is normal; i.e. there is a unique element $x_0\in\mathbb{R}$ such that $u(x_0)=1$.

(ii) $u$ is fuzzy convex; i.e. for any $x,y\in\mathbb{R}$ and for any $\lambda\in[0,1]$, $u(\lambda x+(1-\lambda)y)\geq\min\{u(x),u(y)\}$.

(iii) $u$ is upper semicontinuous.

(iv) The support of $u$, $[u]_0 := \overline{\{x\in\mathbb{R}:u(x)>0\}}$ is compact, where $\overline{\{x\in\mathbb{R}:u(x)>0\}}$ denotes the closure of the set $\{x\in\mathbb{R}:u(x)>0\}$ in the usual topology of $\mathbb{R}$.

We denote the set of all fuzzy numbers on $\mathbb{R}$ by $E^1$ and call it the space of fuzzy numbers.

For $u\in E^1$, the $\alpha$-level set of $u$ is defined by

$$[u]_\alpha := \begin{cases}\{x\in\mathbb{R}:u(x)\geq\alpha\}, & 0<\alpha\leq 1\\ \overline{\{x\in\mathbb{R}:u(x)>\alpha\}}, & \alpha=0.\end{cases}$$

It is well known (see [6]) that $[u]_\alpha$ is a closed, bounded, and nonempty interval for each $\alpha\in[0,1]$ with $[u]_\beta\subset[u]_\alpha$ if $\alpha<\beta$. The set $[u]_\alpha$ is also defined by $[u]_\alpha:=[u^-(\alpha),u^+(\alpha)]$.

In [7], Goetschel and Voxman presented another representation of a fuzzy number as a pair of functions that satisfy some properties.

**Theorem 2.1.** *(Representation Theorem)* [7] *Let* $u\in E^1$ *and let* $[u]_\alpha=[u^-(\alpha),u^+(\alpha)]$. *Then the functions* $u^-,u^+:[0,1]\to\mathbb{R}$, *defining the end-points of the* $\alpha-$*level sets, satisfy following conditions:*

(i) $u^-(\alpha)\in\mathbb{R}$ *is a bounded, non-decreasing and left continuous function on* $(0,1]$.

(ii) $u^+(\alpha)\in\mathbb{R}$ *is a bounded, non-increasing and left continuous function on* $(0,1]$.

(iii) *The functions* $u^-(\alpha)$ *and* $u^+(\alpha)$ *are right continuous at* $\alpha=0$.

(iv) $u^-(1)\leq u^+(1)$.

Conversely, if the pair of functions $f$ and $g$ satisfies the above conditions (i)-(iv), then there exists a unique fuzzy number $u$ such that $[u]_\alpha:=[f(\alpha),g(\alpha)]$ for each $\alpha\in[0,1]$ and $u(x):=\sup\limits_{\alpha\in[0,1]}\{\alpha:f(\alpha)\leq x\leq g(\alpha)\}$.

The following lemma deals with the algebraic properties of fuzzy numbers.

**Lemma 2.2.** [1]

(i) *The addition of fuzzy numbers is associative and commutative, i.e.,* $u + v = v + u$ *and* $u + (v + w) = (u + v) + w$*, for all* $u, v, w \in E^1$.

(ii) $\bar{0} \in E^1$ *is neutral element with respect to* $+$*, i.e.,* $u + \bar{0} = \bar{0} + u = u$*, for any* $u \in E^1$.

(iii) *With respect to* $+$*, none of* $u \in E^1 \setminus \mathbb{R}$ *has opposite in* $E^1$.

(iv) *For any* $a, b \in \mathbb{R}$ *with* $ab \geq 0$ *and any* $u \in E^1$*, we have* $(a + b)u = au + bu$. *For general* $a, b \in \mathbb{R}$ *this property does not hold.*

(v) *For any* $a \in \mathbb{R}$ *and* $u, v \in E^1$*, we have* $a(u + v) = au + av$.

(vi) *For any* $a, b \in \mathbb{R}$ *and any* $u \in E^1$*, we have* $(ab)u = a(bu)$.

As a conclusion we obtain by Lemma 2.2 that the space of fuzzy numbers is not a linear space, but a metric space discussed as follows.

The most well known, and also the most employed metric in the space of fuzzy numbers is the Hausdorff distance. The Hausdorff fuzzy distance for fuzzy numbers is based on the classical Hausdorff distance between compact convex subsets of $\mathbb{R}^n$. Let us recall the definition of the Hausdorff distance in the case when $A = [A^-, A^+]$, $B = [B^-, B^+]$ are two intervals. Let $W$ be the set of all closed and bounded intervals. The Hausdorf distance on $W$ is defined by

$$d(A, B) := \max \left\{ \left|A^- - B^-\right|, \left|A^+ - B^+\right| \right\}.$$

It is known that with respect to the Haussdorf distance, $W$ is a complete separable metric space (cf. Nanda [11]). Now, we may define the metric $D$ on the space of fuzzy numbers by means of the Hausdorff metric $d$.

**Definition 2.3.** [1] Let $D : E^1 \times E^1 \to \mathbb{R}_+$,

$$\begin{aligned} D(u, v) &:= \sup_{\alpha \in [0,1]} \max \left\{ \left|u^-(\alpha) - v^-(\alpha)\right|, \left|u^+(\alpha) - v^+(\alpha)\right| \right\} \\ &:= \sup_{\alpha \in [0,1]} d([u]_\alpha, [v]_\alpha). \end{aligned}$$

Then $D$ is called Hausdorff distance between fuzzy numbers $u$ and $v$.

It is easy to see that

$$D(u, \bar{0}) = \sup_{\alpha \in [0,1]} \max \left\{ \left|u^-(\alpha)\right|, \left|u^+(\alpha)\right| \right\} = \max \left\{ \left|u^-(0)\right|, \left|u^+(0)\right| \right\}.$$

The additive identity and multiplicative identity in $E^1$ are denoted by $\bar{0}$ and $\bar{1}$, respectively.

**Proposition 2.4.** [1] *Let* $u, v, w, z \in E^1$ *and* $k \in \mathbb{R}$*. Then the following statements hold true.*

(i) $(E^1, D)$ *is a complete metric space.*

(ii) $D(u + w, v + w) = D(u, v)$*, i.e.,* $D$ *is translation invariant.*

(iii) $D(ku, kv) = |k|\, D(u, v)$.

(iv) $D(u + v, w + z) \leq D(u, w) + D(v, z)$.

(v) $|D(u, \bar{0}) - D(v, \bar{0})| \leq D(u, v) \leq D(u, \bar{0}) + D(v, \bar{0})$.

## 3. Main Results

In [12], Savaş introduced the following definitions which we need in the sequel:

**Definition 3.1.** A double sequence $u = (u_{mn})$ of fuzzy numbers is a function $u$ from $\mathbb{N} \times \mathbb{N}$ ($\mathbb{N}$ is the set of all natural numbers) into the set $E^1$. The fuzzy number $u_{mn}$ denotes the value of the function at a point $(m, n) \in \mathbb{N} \times \mathbb{N}$ and is called the $(m, n)$-term of the double sequence.

By $w^2(F)$, we denote the set of all double sequences of fuzzy numbers.

**Definition 3.2.** A double sequence $u = (u_{mn})$ of fuzzy numbers is said to be convergent in Pringsheim's sense (or P-convergent) to the fuzzy number $\mu_0$, written as $\lim_{m,n\to\infty} u_{mn} = \mu_0$, if for every $\epsilon > 0$ there exists a positive integer $n_0(\epsilon)$ such that $D(u_{mn}, \mu_0) < \epsilon$ whenever $m, n \geq n_0$. The number $\mu_0$ is called the Pringsheim limit of $u$.

More exactly, we say that a double sequence $(u_{mn})$ converges to a fuzzy number $\mu_0$ if $(u_{mn})$ tends to $\mu_0$ as both $m$ and $n$ tend to $\infty$ independently of one another.

We denote the space of all $P$-convergent double sequences of fuzzy numbers by $c^2(F)$.

**Definition 3.3.** A double sequence $u = (u_{mn})$ of fuzzy numbers is bounded if there exists a positive number $M$ such that $D(u_{mn}, \overline{0}) < M$ for all $m$ and $n$, i.e, if

$$||u||_{\infty,2} = \sup_{m,n} D(u_{mn}, \overline{0}) < \infty.$$

We denote the set of all bounded double sequences of fuzzy numbers by $\ell^2_\infty(F)$.

Note that unlike single sequences of fuzzy numbers, every $P$-convergent double sequences of fuzzy numbers need not be bounded.

**Example 3.4.** Consider the double sequence $(u_{mn})$ of fuzzy numbers defined by

$$u_{mn} = \begin{cases} \omega_n & \text{if } m = 0, n \in \mathbb{N} \\ \overline{0} & \text{otherwise} \end{cases}$$

where

$$\omega_n = \sum_{k=0}^{n} \nu_k$$

and

$$\nu_k(t) = \begin{cases} 1 + t\sqrt{k+1} & \text{if } t \in [\frac{-1}{\sqrt{k+1}}, 0] \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $(u_{mn})$ is $P$-convergent to $\overline{0}$. One can check that the endpoints of the $\alpha$-level set of the double sequence $u = (u_{mn})$ of fuzzy numbers are

$$u^-_{mn}(\alpha) = \begin{cases} \omega^-_n(\alpha) & \text{if } m = 0, n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad u^+_{mn}(\alpha) = 0, \text{ for all } m, n \in \mathbb{N}.$$

From this point of view, we can see that

$$\begin{aligned} ||u||_{\infty,2} = \sup_{m,n\in\mathbb{N}} D(u_{mn},\overline{0}) &= \sup_{m,n\in\mathbb{N}}\sup_{\alpha\in[0,1]} \max\left\{|u_{mn}^{-}(\alpha)-0|,|u_{mn}^{+}(\alpha)-0|\right\} \\ &= \sup_{m,n\in\mathbb{N}}\sup_{\alpha\in[0,1]} |u_{mn}^{-}(\alpha)| \\ &= \sup_{m,n\in\mathbb{N}} |u_{mn}^{-}(0)| \\ &= \sup_{n\in\mathbb{N}} |\omega_n^{-}(0)|. \end{aligned}$$

Since the sequence $(\omega_n^{-}(0)) = \left(\sum_{k=0}^{n}\frac{(-1)}{\sqrt{k+1}}\right)$ is divergent as $n\to\infty$, the double sequence $u=(u_{mn})$ of fuzzy numbers is not bounded.

The $(C,1,1)$ means of $(u_{mn})$ are defined by

$$\sigma_{mn} := \frac{1}{(m+1)(n+1)}\sum_{j=0}^{m}\sum_{k=0}^{n} u_{jk} \tag{1}$$

for all nonnegative integers $m$ and $n$ (see [9]).

**Definition 3.5.** A double sequence $(u_{mn})$ of fuzzy numbers is said to be $(C,1,1)$ summable to a finite number $\mu_0$ if

$$\lim_{m,n\to\infty}\sigma_{mn} = \mu_0. \tag{2}$$

Note that throughout this paper we always mean convergence in Pringsheim's sense.

**Theorem 3.6.** *If $u=(u_{mn})\in w^2(F)\cap\ell_\infty^2(F)$ is convergent to $\mu_0\in E^1$, then $(u_{mn})$ is $(C,1,1)$ summable to $\mu_0$.*

*Proof.* Let $(u_{mn})\in c^2(F)$. Then, there exists $\mu_0\in E^1$ such that $\lim_{m,n\to\infty} D(u_{mn},\mu_0)=0$. Hence, by Proposition 2.4 we have

$$\begin{aligned} D(\sigma_{mn},\mu_0) &= D\left(\frac{1}{(m+1)(n+1)}\sum_{j=0}^{m}\sum_{k=0}^{n}u_{jk},\mu_0\right) \\ &\le \frac{1}{(m+1)(n+1)}\sum_{j=0}^{m}\sum_{k=0}^{n} D(u_{jk},\mu_0). \end{aligned}$$

Since $\lim_{m,n\to\infty} D(u_{mn},\mu_0)=0$, by the regularity of $(C,1,1)$ summability and the boundedness $(u_{mn})$ of sequences of real numbers, we obtain

$$\lim_{m,n\to\infty} D(\sigma_{mn},\mu_0)=0. \tag{3}$$

□

For another proof of Theorem 3.6, see [14].

By the following example we show that the converse of Theorem 3.6 does not hold in general.

**Example 3.7.** Consider the double sequence $u = (u_{mn})$ of fuzzy numbers defined by

$$u_{mn} = \begin{cases} u_0 & \text{if } m, n \text{ are even} \\ v_0 & \text{otherwise,} \end{cases}$$

where

$$u_0(t) = \begin{cases} 1-t & \text{if } t \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

and

$$v_0(t) = \begin{cases} 1+t & \text{if } t \in [-1,0] \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $(u_{mn})$ is divergent. On the other hand, one can check that the endpoints of the $\alpha$-level set of the double sequence $(u_{mn})$ of fuzzy numbers are

$$u^-_{mn}(\alpha) = \begin{cases} 0 & \text{if } m, n \text{ are even} \\ \alpha - 1 & \text{otherwise} \end{cases}$$

and

$$u^+_{mn}(\alpha) = \begin{cases} 1-\alpha & \text{if } m, n \text{ are even} \\ 0 & \text{otherwise.} \end{cases}$$

It then follows from the definition of $(C, 1, 1)$ means that

$$\sigma^-_{mn}(\alpha) = \begin{cases} \frac{\alpha-1}{4}\left(\frac{(m+2)n+2m(n+1)}{(m+1)(n+1)}\right) & \text{if } m, n \text{ are even} \\ \frac{\alpha-1}{4}\left(\frac{3m+2}{m+1}\right) & \text{if } m \text{ is even,} n \text{ is odd} \\ \frac{3(\alpha-1)}{4} & \text{if } m, n \text{ are odd} \\ \frac{\alpha-1}{4}\left(\frac{3n+2}{n+1}\right) & \text{if } m \text{ is odd,} n \text{ is even} \end{cases}$$

and

$$\sigma^+_{mn}(\alpha) = \begin{cases} \frac{1-\alpha}{4}\left(\frac{(m+2)(n+2)}{(m+1)(n+1)}\right) & \text{if } m, n \text{ are even} \\ \frac{1-\alpha}{4}\left(\frac{m+2}{m+1}\right) & \text{if } m \text{ is even,} n \text{ is odd} \\ \frac{1-\alpha}{4} & \text{if } m, n \text{ are odd} \\ \frac{1-\alpha}{4}\left(\frac{n+2}{n+1}\right) & \text{if } m \text{ is odd,} n \text{ is even.} \end{cases}$$

Therefore, the double sequences $(\sigma^-_{mn}(\alpha))$ and $(\sigma^+_{mn}(\alpha))$ converge to $3(\alpha-1)/4$ and $(1-\alpha)/4$ as $m, n \to \infty$, respectively. From this point of view, if we take $\omega_0 = (u_0 + 3v_0)/4$, we obtain $\lim_{m,n\to\infty} D(\sigma_{mn}, \omega_0) = 0$. In other words, the double sequence $(\sigma_{mn})$ of fuzzy number is convergent to $\omega_0$ and hence $(u_{mn})$ is $(C, 1, 1)$ summable to $\omega_0$.

In this paper our goal is to find necessary and sufficient conditions such that the converse of Theorem 3.6 holds.

The de la Vallée Poussin means of the double sequence $(u_{mn})$ are defined by

$$\tau^{>}_{mn} := \frac{1}{(\lambda_m - m)(\lambda_n - n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} u_{jk}, \ \text{ if } \lambda > 1$$

and

$$\tau^{<}_{mn} := \frac{1}{(m-\lambda_m)(n-\lambda_n)} \sum_{j=\lambda_m+1}^{m} \sum_{k=\lambda_n+1}^{n} u_{jk}, \ \text{ if } 0 < \lambda < 1$$

for sufficiently large nonnegative integers $m, n$.

Here, by $\lambda_n$ we denote the integral part of the product $\lambda n$, in symbol $\lambda_n := [\lambda n]$.

We need the following Lemmas for the proof of our main result.

**Lemma 3.8.** *i) If $\lambda > 1$, $\lambda_m > m$, and $\lambda_n > n$, then*

$$\begin{aligned} &D\left(\tau^{>}_{mn}, \sigma_{mn}\right) \\ &\leq \frac{(\lambda_m+1)(\lambda_n+1)}{(\lambda_m-m)(\lambda_n-n)} D\left(\sigma_{\lambda_m,\lambda_n}, \sigma_{\lambda_m,n}\right) + \frac{(\lambda_m+1)(\lambda_n+1)}{(\lambda_m-m)(\lambda_n-n)} D\left(\sigma_{mn}, \sigma_{m,\lambda_n}\right) \\ &+ \frac{\lambda_m+1}{\lambda_m-m} D\left(\sigma_{\lambda_m,n}, \sigma_{mn}\right) + \frac{\lambda_n+1}{\lambda_n-n} D\left(\sigma_{m,\lambda_n}, \sigma_{mn}\right). \end{aligned}$$

*ii) If $0 < \lambda < 1$, $\lambda_m < m$, and $\lambda_n < n$, then*

$$\begin{aligned} &D\left(\tau^{<}_{mn}, \sigma_{mn}\right) \\ &\leq \frac{(\lambda_m+1)(\lambda_n+1)}{(m-\lambda_m)(n-\lambda_n)} D\left(\sigma_{\lambda_m,\lambda_n}, \sigma_{\lambda_m,n}\right) + \frac{(\lambda_m+1)(\lambda_n+1)}{(m-\lambda_m)(n-\lambda_n)} D\left(\sigma_{mn}, \sigma_{m,\lambda_n}\right) \\ &+ \frac{\lambda_m+1}{m-\lambda_m} D\left(\sigma_{mn}, \sigma_{\lambda_m,n}\right) + \frac{\lambda_n+1}{n-\lambda_n} D\left(\sigma_{mn}, \sigma_{m,\lambda_n}\right). \end{aligned}$$

*Proof.* i) By definition, we have

$$\begin{aligned} &D\left(\tau^{>}_{mn}, \sigma_{mn}\right) \\ &= D\left(\frac{1}{(\lambda_m-m)(\lambda_n-n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} u_{jk}\right. \\ &+ \frac{1}{(\lambda_m-m)(\lambda_n-n)} \left(\sum_{j=0}^{m} \sum_{k=n+1}^{\lambda_n} + \sum_{j=m+1}^{\lambda_m} \sum_{k=0}^{n} + \sum_{j=0}^{m} \sum_{k=0}^{n}\right) u_{jk}, \\ &\left.\sigma_{mn} + \frac{1}{(\lambda_m-m)(\lambda_n-n)} \left(\sum_{j=0}^{m} \sum_{k=n+1}^{\lambda_n} + \sum_{j=m+1}^{\lambda_m} \sum_{k=0}^{n} + \sum_{j=0}^{m} \sum_{k=0}^{n}\right) u_{jk}\right) \\ &= D\left(\frac{1}{(\lambda_m-m)(\lambda_n-n)} \sum_{j=0}^{\lambda_m} \sum_{k=0}^{\lambda_n} u_{jk} + \frac{1}{(\lambda_m-m)(\lambda_n-n)} \sum_{j=0}^{m} \sum_{k=0}^{n} u_{jk},\right. \\ &\left.\sigma_{mn} + \frac{1}{(\lambda_m-m)(\lambda_n-n)} \sum_{j=0}^{m} \sum_{k=0}^{\lambda_n} u_{jk} + \frac{1}{(\lambda_m-m)(\lambda_n-n)} \sum_{j=0}^{\lambda_m} \sum_{k=0}^{n} u_{jk}\right) \end{aligned}$$

$$= D\left(\frac{(\lambda_m+1)(\lambda_n+1)}{(\lambda_m-m)(\lambda_n-n)}\sigma_{\lambda_m,\lambda_n}+\frac{(m+1)(n+1)}{(\lambda_m-m)(\lambda_n-n)}\sigma_{mn},\right.$$
$$\left.\sigma_{mn}+\frac{(m+1)(\lambda_n+1)}{(\lambda_m-m)(\lambda_n-n)}\sigma_{m,\lambda_n}+\frac{(\lambda_m+1)(n+1)}{(\lambda_m-m)(\lambda_n-n)}\sigma_{\lambda_m,n}\right)$$
$$= D\left(\frac{(\lambda_m+1)(\lambda_n+1)}{(\lambda_m-m)(\lambda_n-n)}\sigma_{\lambda_m,\lambda_n}+\frac{(m+1)(n+1)}{(\lambda_m-m)(\lambda_n-n)}\sigma_{mn}+\frac{\lambda_n+1}{\lambda_n-n}\sigma_{m,\lambda_n}\right.$$
$$\left.+\frac{\lambda_m+1}{\lambda_m-m}\sigma_{\lambda_m,n}\ ,\ \sigma_{mn}+\frac{(\lambda_m+1)(\lambda_n+1)}{(\lambda_m-m)(\lambda_n-n)}\left(\sigma_{m,\lambda_n}+\sigma_{\lambda_m,n}\right)\right)$$
$$\leq \frac{(\lambda_m+1)(\lambda_n+1)}{(\lambda_m-m)(\lambda_n-n)}D(\sigma_{\lambda_m,\lambda_n},\sigma_{\lambda_m,n})$$
$$+D\left(\frac{(m+1)(n+1)}{(\lambda_m-m)(\lambda_n-n)}\sigma_{mn}+\frac{\lambda_n+1}{\lambda_n-n}\sigma_{m,\lambda_n}+\frac{\lambda_m+1}{\lambda_m-m}\sigma_{\lambda_m,n}+\frac{n+1}{\lambda_n-n}\sigma_{mn}\right.$$
$$+\frac{m+1}{\lambda_m-m}\sigma_{mn}+\sigma_{mn}\ ,\ \sigma_{mn}+\frac{(\lambda_m+1)(\lambda_n+1)}{(\lambda_m-m)(\lambda_n-n)}\sigma_{m,\lambda_n}+\frac{n+1}{\lambda_n-n}\sigma_{mn}$$
$$\left.+\frac{m+1}{\lambda_m-m}\sigma_{mn}+\sigma_{mn}\right)$$
$$\leq \frac{(\lambda_m+1)(\lambda_n+1)}{(\lambda_m-m)(\lambda_n-n)}D(\sigma_{\lambda_m,\lambda_n},\sigma_{\lambda_m,n})$$
$$+D\left(\frac{(\lambda_m+1)(\lambda_n+1)}{(\lambda_m-m)(\lambda_n-n)}\sigma_{mn}+\frac{\lambda_n+1}{\lambda_n-n}\sigma_{m,\lambda_n}+\frac{\lambda_m+1}{\lambda_m-m}\sigma_{\lambda_m,n},\right.$$
$$\left.\frac{(\lambda_m+1)(\lambda_n+1)}{(\lambda_m-m)(\lambda_n-n)}\sigma_{m,\lambda_n}+\frac{\lambda_n+1}{\lambda_n-n}\sigma_{mn}+\frac{\lambda_m+1}{\lambda_m-m}\sigma_{mn}\right)$$
$$\leq \frac{(\lambda_m+1)(\lambda_n+1)}{(\lambda_m-m)(\lambda_n-n)}D(\sigma_{\lambda_m,\lambda_n},\sigma_{\lambda_m,n})+\frac{(\lambda_m+1)(\lambda_n+1)}{(\lambda_m-m)(\lambda_n-n)}D(\sigma_{mn},\sigma_{m,\lambda_n})$$
$$+\frac{\lambda_m+1}{\lambda_m-m}D(\sigma_{\lambda_m,n},\sigma_{mn})+\frac{\lambda_n+1}{\lambda_n-n}D(\sigma_{m,\lambda_n},\sigma_{mn}).$$

ii) By definition, we have

$$D\left(\tau^{<}_{mn},\sigma_{mn}\right)$$
$$= D\left(\frac{1}{(m-\lambda_m)(n-\lambda_n)}\sum_{j=\lambda_m+1}^{m}\sum_{k=\lambda_n+1}^{n}u_{jk}\right.$$
$$+\frac{1}{(m-\lambda_m)(n-\lambda_n)}\left(\sum_{j=0}^{\lambda_m}\sum_{k=\lambda_n+1}^{n}+\sum_{j=\lambda_m+1}^{m}\sum_{k=0}^{\lambda_n}+\sum_{j=0}^{\lambda_m}\sum_{k=0}^{\lambda_n}\right)u_{jk},$$
$$\left.\sigma_{mn}+\frac{1}{(m-\lambda_m)(n-\lambda_n)}\left(\sum_{j=0}^{\lambda_m}\sum_{k=\lambda_n+1}^{n}+\sum_{j=\lambda_m+1}^{m}\sum_{k=0}^{\lambda_n}+\sum_{j=0}^{\lambda_m}\sum_{k=0}^{\lambda_n}\right)u_{jk}\right)$$

$$= D\left(\frac{1}{(m-\lambda_m)(n-\lambda_n)}\sum_{j=0}^{m}\sum_{k=0}^{n}u_{jk} + \frac{1}{(m-\lambda_m)(n-\lambda_n)}\sum_{j=0}^{\lambda_m}\sum_{k=0}^{\lambda_n}u_{jk},\right.$$
$$\left.\sigma_{mn} + \frac{1}{(m-\lambda_m)(n-\lambda_n)}\sum_{j=0}^{\lambda_m}\sum_{k=0}^{n}u_{jk} + \frac{1}{(m-\lambda_m)(n-\lambda_n)}\sum_{j=0}^{m}\sum_{k=0}^{\lambda_n}u_{jk}\right)$$
$$= D\left(\frac{(m+1)(n+1)}{(m-\lambda_m)(n-\lambda_n)}\sigma_{mn} + \frac{(\lambda_m+1)(\lambda_n+1)}{(m-\lambda_m)(n-\lambda_n)}\sigma_{\lambda_m,\lambda_n},\right.$$
$$\left.\sigma_{mn} + \frac{(\lambda_m+1)(n+1)}{(m-\lambda_m)(n-\lambda_n)}\sigma_{\lambda_m,n} + \frac{(m+1)(\lambda_n+1)}{(m-\lambda_m)(n-\lambda_n)}\sigma_{m,\lambda_n}\right)$$
$$= D\left(\frac{(m+1)(n+1)}{(m-\lambda_m)(n-\lambda_n)}\sigma_{mn} + \frac{(\lambda_m+1)(\lambda_n+1)}{(m-\lambda_m)(n-\lambda_n)}\sigma_{\lambda_m,\lambda_n},\right.$$
$$\left.\sigma_{mn} + \frac{(\lambda_m+1)(\lambda_n+1)}{(m-\lambda_m)(n-\lambda_n)}\left(\sigma_{\lambda_m,n} + \sigma_{m,\lambda_n}\right) + \frac{\lambda_m+1}{m-\lambda_m}\sigma_{\lambda_m,n} + \frac{\lambda_n+1}{n-\lambda_n}\sigma_{m,\lambda_n}\right)$$
$$\leq \frac{(\lambda_m+1)(\lambda_n+1)}{(m-\lambda_m)(n-\lambda_n)}D(\sigma_{\lambda_m,\lambda_n}, \sigma_{\lambda_m,n})$$
$$+ D\left(\frac{(\lambda_m+1)(\lambda_n+1)}{(m-\lambda_m)(n-\lambda_n)}\sigma_{mn} + \frac{\lambda_m+1}{m-\lambda_m}\sigma_{m,n} + \frac{\lambda_n+1}{n-\lambda_n}\sigma_{m,n} + \sigma_{mn},\right.$$
$$\left.\sigma_{mn} + \frac{(\lambda_m+1)(\lambda_n+1)}{(m-\lambda_m)(n-\lambda_n)}\sigma_{m,\lambda_n} + \frac{\lambda_m+1}{m-\lambda_m}\sigma_{\lambda_m,n} + \frac{\lambda_n+1}{n-\lambda_n}\sigma_{m,\lambda_n}\right)$$
$$\leq \frac{(\lambda_m+1)(\lambda_n+1)}{(m-\lambda_m)(n-\lambda_n)}D\left(\sigma_{\lambda_m,\lambda_n}, \sigma_{\lambda_m,n}\right) + \frac{(\lambda_m+1)(\lambda_n+1)}{(m-\lambda_m)(n-\lambda_n)}D\left(\sigma_{mn}, \sigma_{m,\lambda_n}\right)$$
$$+ \frac{\lambda_m+1}{m-\lambda_m}D\left(\sigma_{mn}, \sigma_{\lambda_m,n}\right) + \frac{\lambda_n+1}{n-\lambda_n}D\left(\sigma_{mn}, \sigma_{m,\lambda_n}\right).$$

□

**Lemma 3.9.** *If $(u_{mn})$ is $(C,1,1)$ summable to a fuzzy number $\mu_0$, then*

$$\lim_{m,n\to\infty}\tau^{>}_{mn} = \mu_0, \tag{4}$$

*and*

$$\lim_{m,n\to\infty}\tau^{<}_{mn} = \mu_0. \tag{5}$$

*Proof.* By Proposition 2.4, we have

$$D\left(\tau^{>}_{mn}, \mu_0\right) = D\left(\tau^{>}_{mn} + \sigma_{mn}, \sigma_{mn} + \mu_0\right)$$
$$\leq D\left(\tau^{>}_{mn}, \sigma_{mn}\right) + D\left(\sigma_{mn}, \mu_0\right). \tag{6}$$

By Lemma 3.8 i) and (6), we get

$$D\left(\tau^{>}_{mn}, \mu_0\right)$$
$$\leq \frac{(\lambda_m+1)(\lambda_n+1)}{(\lambda_m-m)(\lambda_n-n)}D\left(\sigma_{\lambda_m,\lambda_n}, \sigma_{\lambda_m,n}\right) + \frac{(\lambda_m+1)(\lambda_n+1)}{(\lambda_m-m)(\lambda_n-n)}D\left(\sigma_{mn}, \sigma_{m,\lambda_n}\right)$$
$$+ \frac{\lambda_m+1}{\lambda_m-m}D\left(\sigma_{\lambda_m,n}, \sigma_{mn}\right) + \frac{\lambda_n+1}{\lambda_n-n}D\left(\sigma_{m,\lambda_n}, \sigma_{mn}\right) + D\left(\sigma_{mn}, \mu_0\right).$$

For all $\lambda > 1$ and sufficiently large $n$, we obtain

$$\frac{\lambda}{\lambda - 1} < \frac{\lambda_n + 1}{\lambda_n - n} < \frac{2\lambda}{\lambda - 1}. \tag{7}$$

The inequality (7) hold for $\lambda_m$ with sufficiently large $m$. Now, (4) follows from (2) and (7).

The proof of (5) can be established following similar technique. □

**Theorem 3.10.** *If $(u_{mn})$ is $(C,1,1)$ summable to a fuzzy number $\mu_0$, then the limit*

$$\lim_{m,n\to\infty} u_{mn} = \mu_0 \tag{8}$$

*exists if and only if either*

$$\liminf_{\lambda\downarrow 1} \limsup_{m,n\to\infty} D\left(\tau^{>}_{mn}, u_{mn}\right) = 0 \tag{9}$$

*or*

$$\liminf_{\lambda\uparrow 1} \limsup_{m,n\to\infty} D\left(\tau^{<}_{mn}, u_{mn}\right) = 0. \tag{10}$$

*Proof. Necessity.* If $(u_{mn})$ is $(C,1,1)$ summable to $\mu_0$ and (8) is satisfied, then

$$\lim_{m,n\to\infty} D\left(u_{mn}, \sigma_{mn}\right) = 0. \tag{11}$$

Now, (9) (res. (10)) follows from (4) (res. (5)) and (11).

*Sufficiency.* Assume that the condition (9) holds. Then, for any given $\epsilon > 0$ there exists $\lambda_0 > 1$ such that

$$\limsup_{m,n\to\infty} D\left(\tau^{>}_{mn}, u_{mn}\right) \le \epsilon. \tag{12}$$

We also have

$$\begin{aligned} D(u_{mn}, \mu_0) &= D(u_{mn} + \tau^{>}_{mn}, \mu_0 + \tau^{>}_{mn}) \\ &\le D(u_{mn}, \tau^{>}_{mn}) + D(\tau^{>}_{mn}, \mu_0). \end{aligned}$$

Taking into account that $(u_{mn})$ is $(C,1,1)$ summable to $\mu_0$, (4) and (12), we get

$$\limsup_{m,n\to\infty} D\left(u_{mn}, \mu_0\right) \le \epsilon. \tag{13}$$

In a similar way, if (10) holds, for any given $\epsilon > 0$ there exists $0 < \lambda_0 < 1$ such that

$$\limsup_{m,n\to\infty} D\left(\tau^{<}_{mn}, u_{mn}\right) \le \epsilon. \tag{14}$$

We also have

$$\begin{aligned} D(u_{mn}, \mu_0) &= D(u_{mn} + \tau^{<}_{mn}, \mu_0 + \tau^{<}_{mn}) \\ &\le D(u_{mn}, \tau^{<}_{mn}) + D(\tau^{<}_{mn}, \mu_0). \end{aligned}$$

Taking into account that $(u_{mn})$ is $(C,1,1)$ summable to $\mu_0$, (5) and (14), we get

$$\limsup_{m,n\to\infty} D\left(u_{mn}, \mu_0\right) \le \epsilon. \tag{15}$$

Since $\epsilon$ is arbitrary, this completes the proof. □

## 4. **Slow Oscillation**

The slowly oscillating sequences of fuzzy numbers was studied by Tripaty and Baruah [18].

A double sequence $(u_{mn})$ of fuzzy numbers is said to be slowly oscillating in sense $(1, 1)$ if

$$\lim_{\lambda\downarrow 1}\limsup_{m,n\to\infty}\max_{\substack{m+1\le j\le\lambda_m\\ n+1\le k\le\lambda_n}} D\left(u_{jk}, u_{mn}\right)=0. \tag{16}$$

Equivalently, we say that $(u_{mn})$ is slowly oscillating in sense $(1, 1)$ if for each $\epsilon > 0$ there exist $n_1(\epsilon)$ and $\lambda = \lambda(\epsilon) > 1$ such that

$$D\left(u_{jk}, u_{mn}\right)\le\epsilon$$

whenever $n_1 < m < j \le \lambda_m$ and $n_1 < n < k \le \lambda_n$.

It is easily follows from the definition that every $P$-convergent double sequence of fuzzy numbers is slowly oscillating in sense $(1, 1)$, but that the converse is not true in general is given by the following example.

**Example 4.1.** Consider the double sequence $u = (u_{mn})$ of fuzzy numbers defined by

$$u_{mn}=\begin{cases}\omega_m & \text{if } m\ge n\\ \omega_n & \text{otherwise}\end{cases}$$

where

$$\omega_n(t)=\begin{cases}1-\dfrac{t}{1+\log(n+1)} & \text{if } t\in[0,1+\log(n+1)]\\ 0 & \text{otherwise.}\end{cases}$$

We note that the sequence $(\omega_n)$ was defined in [17]. It is clear that $(u_{mn})$ is divergent. On the other hand, one can check that the endpoints of the $\alpha$-level set of the double sequence $u = (u_{mn})$ of fuzzy numbers are

$$u^-_{mn}(\alpha)=0 \text{ for all } m,n\in\mathbb{N} \text{ and } u^+_{mn}(\alpha)=\begin{cases}(1-\alpha)\left(1+\log(m+1)\right) & \text{if } m\ge n\\ (1-\alpha)\left(1+\log(n+1)\right) & \text{otherwise.}\end{cases}$$

From this point of view, we can calculate that

$$\begin{aligned}\max_{\substack{m+1\le j\le\lambda_m\\ n+1\le k\le\lambda_n}} D\left(u_{jk}, u_{mn}\right) &= \max_{\substack{m+1\le j\le\lambda_m\\ n+1\le k\le\lambda_n}}\sup_{\alpha\in[0,1]}\max\left\{|u^-_{jk}(\alpha)-u^-_{mn}(\alpha)|, |u^+_{jk}(\alpha)-u^+_{mn}(\alpha)|\right\}\\ &= \max_{\substack{m+1\le j\le\lambda_m\\ n+1\le k\le\lambda_n}} |u^+_{jk}(0)-u^+_{mn}(0)|\end{aligned}$$

$$= \max_{\substack{m+1\le j\le \lambda_m\\ n+1\le k\le \lambda_n}} \begin{cases} |\log(j+1)-\log(m+1)| & \text{if } j\ge k \text{ and } m\ge n\\ |\log(j+1)-\log(n+1)| & \text{if } j\ge k \text{ and } m< n\\ |\log(k+1)-\log(m+1)| & \text{if } j< k \text{ and } m\ge n\\ |\log(k+1)-\log(n+1)| & \text{if } j< k \text{ and } m< n \end{cases}$$

$$= \max_{\substack{m+1\le j\le \lambda_m\\ n+1\le k\le \lambda_n}} \begin{cases} \log\left(\frac{j+1}{m+1}\right) & \text{if } j\ge k \text{ and } m\ge n\\ \log\left(\frac{j+1}{n+1}\right) & \text{if } j\ge k \text{ and } m< n\\ \log\left(\frac{k+1}{m+1}\right) & \text{if } j< k \text{ and } m\ge n\\ \log\left(\frac{k+1}{n+1}\right) & \text{if } j< k \text{ and } m< n \end{cases}$$

$$= \begin{cases} \log\left(\frac{\lambda_m+1}{m+1}\right) & \text{if } j\ge k \text{ and } m\ge n\\ \log\left(\frac{\lambda_m+1}{n+1}\right) & \text{if } j\ge k \text{ and } m< n\\ \log\left(\frac{\lambda_n+1}{m+1}\right) & \text{if } j< k \text{ and } m\ge n\\ \log\left(\frac{\lambda_n+1}{n+1}\right) & \text{if } j< k \text{ and } m< n. \end{cases} \tag{17}$$

By taking the lim sup of both sides of the equality (17) as $m,n\to\infty$, we obtain that

$$\limsup_{m,n\to\infty} \max_{\substack{m+1\le j\le \lambda_m\\ n+1\le k\le \lambda_n}} D(u_{jk},u_{mn}) = \log\lambda.$$

Immediately after, taking the limit of both sides of the last equality as $\lambda\downarrow 1$, we arrive that

$$\lim_{\lambda\downarrow 1}\limsup_{m,n\to\infty} \max_{\substack{m+1\le j\le \lambda_m\\ n+1\le k\le \lambda_n}} D(u_{jk},u_{mn}) = 0.$$

Therefore, it follows from the definition of the slow oscillation in sense $(1,1)$ we obtain that the double sequence $u=(u_{mn})$ of fuzzy numbers is slowly oscillating in sense $(1,1)$.

A double sequence $(u_{mn})$ of fuzzy numbers is said to be slowly oscillating in sense $(1,0)$ if

$$\lim_{\lambda\downarrow 1}\limsup_{m,n\to\infty} \max_{m+1\le j\le \lambda_m} D(u_{jn},u_{mn}) = 0. \tag{18}$$

We say that $(u_{mn})$ satisfies the two-sided Tauberian condition of Hardy type in sense $(1,0)$ if there exists $n_1$ and $H$ such that

$$jD(u_{jn},u_{j-1,n})\le H \text{ whenever } j,n>n_1. \tag{19}$$

It is clear that if (19) holds, then $(u_{mn})$ is slowly oscillating in sense $(1,0)$.

A double sequence $(u_{mn})$ of fuzzy numbers is said to be slowly oscillating in sense $(0,1)$ if

$$\lim_{\lambda\downarrow 1}\limsup_{m,n\to\infty} \max_{n+1\le k\le \lambda_n} D(u_{mk},u_{mn}) = 0. \tag{20}$$

We say that $(u_{mn})$ satisfies the two-sided Tauberian condition of Hardy type in sense $(0,1)$ if there exists $m_1$ and $C$ such that

$$kD(u_{mk},u_{m,k-1})\le C \text{ whenever } k,m>m_1. \tag{21}$$

It is clear that if (21) holds, then $(u_{mn})$ is slowly oscillating in sense $(0, 1)$.

We note that by the same reasoning as in Example 4.1, one easily construct examples of non-convergent double sequences of fuzzy numbers which are slowly oscillating in senses $(1, 0)$ or $(0, 1)$.

**Corollary 4.2.** *If $(u_{mn})$ is slowly oscillating in sense $(1, 1)$ and $(C, 1, 1)$ summable to a fuzzy number to $\mu_0$, then $(u_{mn})$ is convergent to $\mu_0$.*

*Proof.* Assume that $(u_{mn})$ is slowly oscillating. Since

$$\begin{aligned}
&D\left(\tau^{>}_{mn}, u_{mn}\right)\\
&= D\left(\frac{1}{(\lambda_m - m)(\lambda_n - n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} u_{jk}, \frac{1}{(\lambda_m - m)(\lambda_n - n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} u_{mn}\right)\\
&\le \frac{1}{(\lambda_m - m)(\lambda_n - n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} D\left(u_{jk}, u_{mn}\right)\\
&\le \max_{\substack{m+1\le j\le\lambda_m\\ n+1\le k\le\lambda_n}} D\left(u_{jk}, u_{mn}\right),
\end{aligned}$$

we obtain that (9) holds. By Theorem 3.10, $(u_{mn})$ is convergent to $\mu_0$. □

**Lemma 4.3.** *If $(u_{mn})$ satisfies two-sided Tauberian conditions of Hardy type in senses $(1, 0)$ and $(0, 1)$, then $(u_{mn})$ is slowly oscillating in sense $(1, 1)$.*

*Proof.* Assume that $(u_{mn})$ satisfies two-sided Tauberian conditions of Hardy type in senses $(1, 0)$ and $(0, 1)$, i.e., there exist $n_1 = n_1(\epsilon)$ and $H$ such that

$$mD(u_{mn}, u_{m-1,n}) \le H,$$

and

$$nD(u_{mn}, u_{m,n-1}) \le H$$

whenever $m, n > n_1$. For all $1 < n_1 < m < j \le \lambda_m$ and $1 < n_1 < n < k \le \lambda_n$, we obtain

$$\begin{aligned}
D\left(u_{jk}, u_{mn}\right) &\le D\left(u_{jk}, u_{mk}\right) + D\left(u_{mk}, u_{mn}\right)\\
&\le \sum_{r=m+1}^{j} D\left(u_{rk}, u_{r-1,k}\right) + \sum_{s=n+1}^{k} D\left(u_{ms}, u_{m,s-1}\right)\\
&\le H\left(\frac{j-m}{r}\right) + H\left(\frac{k-n}{s}\right)\\
&= H\left(\frac{j}{m} - 1\right) + H\left(\frac{k}{n} - 1\right)\\
&\le 2H(\lambda - 1).
\end{aligned}$$

Hence, for each $\epsilon > 0$ and $1 < \lambda \le 1 + \epsilon/2H$, we have $D\left(u_{jk}, u_{mn}\right) \le \epsilon$ for all $1 < n_1 < m < j \le \lambda_m$ and $1 < n_1 < n < k \le \lambda_n$. □

**Corollary 4.4.** *If* $(u_{mn})$ *satisfies two-sided Tauberians condition of Hardy type in senses* $(1,0)$ *and* $(0,1)$ *and it is* $(C,1,1)$ *summable to a fuzzy number to* $\mu_0$*, then* $(u_{mn})$ *is convergent to* $\mu_0$*.*

## 5. Conclusion

In this paper we have given a Tauberian theorem for $(C,1,1)$ double sequences of fuzzy numbers as an extension of a Tauberian theorem for sequences of single fuzzy sequences, due to Talo and Çakan [16]. Conditions under which convergence in Pringsheim's sense of a double sequence of fuzzy numbers follows from its $(C,1,1)$ summability are satisfied if the double sequence of fuzzy numbers is slowly oscillating in different senses. Some interesting double sequences of fuzzy numbers are also constructed. In the forthcoming paper, we are aiming to extend these results for the weighted mean methods of double sequences of fuzzy numbers.

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