SOME TOPOLOGICAL PROPERTIES OF SPECTRUM OF FUZZY SUBMODULES

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Abstract. Let R be a commutative ring with identity and M be an R-module. Let $FSpec(M)$ denotes the collection of all prime fuzzy submodules of M. In this regards some basic properties of Zariski topology on $FSpec(M)$ are investigated. In particular, we prove some equivalent conditions for irreducible subsets of this topological space and it is shown under certain conditions $FSpec(M)$ is a T_0-space or Hausdorff.

1. Introduction

The study of prime spectrum or topological space obtained by introducing Zariski topology on the set of prime submodules of a unitary module M, over a commutative ring with identity R, plays an important role in the field of commutative algebra, algebraic geometry and lattice theory (for example see [12-15]). In the last few years a considerable amount of works has been done on fuzzy submodules in general and prime fuzzy modules in particular (for example see [2-4] , [16], [20], [22]). Therefore, it is natural to attempt to introduce an appropriate topology on the set of prime fuzzy submodules and study its topological properties. The authors in [2] introduced and studied the notion of prime L-submodules of a module M over a commutative ring with identity R, where L is a complete lattice, and in [3] they introduced and studied Zariski topology on $L-Spec(M)$, the set of all prime L-submodules of M, which is called prime L-spectrum of M.

Now in this paper we follow [2] and [3] and find more topological properties of Zariski topology of $X = FSpec(M)$, the collection of all prime fuzzy submodules of M, such as irreduciblelity and seperation properties.

In this regards, we extend the results on Zariski topology of prime submodules to prime fuzzy submodules, and obtain some basic results of this topological space.

Throughout the paper R denotes a commutative ring with identity and all related modules are unitary modules over R.

In Section 2, some definitions and results to be used in the sequel are given. In Section 3, the irreducible subsets of $X = FSpec(M)$ are studied. In particular, it is shown every variety, $V(P)$ of X is irreducible closed subset of X for every prime fuzzy submodule P of M.

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Finally, in Section 4, the separation properties of $FSpec(M)$ are investigated. In particular, by using the natural mapping some equivalent conditions that X being a T_0 or Hausdorff are given. Finally, it is proved that X is homeomorphic to the topological space $Spec(M) \times [0, 1)$.

2. Preliminaries

Throughout this paper by R we mean a commutative ring with identity, and M is a unitary R-module. L is *regular* if for all $a, b \in L$ such that $a \neq 0, b \neq 0$, then $a \wedge b \neq 0$. By an *L-subset* μ of a non-empty set X, we mean a function μ from X to L. If $L = [0, 1]$, then μ is called a *fuzzy* subset of X. L^X denotes the set of all L-subsets of X. Let $a \in L$ and $y \in M$. Let A be a subset of X and $y \in L$. Define $y_A \in F^X$ as follows:

$$y_A(x) = \begin{cases} y & x \in A; \\ 0 & otherwise \end{cases}$$

In particular if $A = \{a\}$ we denote $y_{\{a\}}$ by y_a, and it is called a fuzzy point of X.

For $\mu, \nu \in F^X$ we say that μ is contained in ν and we write $\mu \subseteq \nu$ if for all $x \in X$, $\mu(x) \leq \nu(x)$.

For $\mu, \nu \in F^X$, the intersection and union, $\mu \cup \nu$, $\mu \cap \nu \in F^X$, are defined by

$$(\mu \cup \nu)(x) = \mu(x) \vee \nu(x) \ \ and \ \ (\mu \cap \nu)(x) = \mu(x) \wedge \nu(x),$$

for all $x \in X$.

We recall some definitions and theorems from [16], [2] and [3], which are needed for the development of our paper.

Definition 2.1. Let $\mu \in L^R$. Then μ is called a fuzzy ideal of R if for every $x, y \in R$ the following conditions are satisfied:

1)$\mu(x - y) \geq \mu(x) \wedge \mu(y)$;

2)$\mu(xy) \geq \mu(x) \vee \mu(y)$.

The set of all fuzzy ideals of R is denoted by $FI(R)$.

Definition 2.2. $\zeta \in FI(R)$ is called a prime fuzzy ideal of R if ζ is non-constant and for every $\mu, \nu \in FI(R)$, $\mu\nu \subseteq \zeta$ implies that $\mu \subseteq \zeta$ or $\nu \subseteq \zeta$. Moreover $\mu\nu(x) = \bigvee\{\mu(y) \wedge \nu(z) | y, z \in R, x = yz\}$ $\forall x \in R$, and in [15] it has been proved that $\mu\nu \in FI(R)$.

By $FSpec(R)$ we mean the set of all prime fuzzy ideals of R.

Definition 2.3. Let $\zeta \in F^R$ and $\mu \in F^M$. Define $\zeta \cdot \mu \in F^M$ as follows:

$$(\zeta \cdot \mu)(x) = \bigvee\{\zeta(r) \wedge \mu(y) | r \in R, y \in M, ry = x\} \ \ for \ \ all \ \ x \in M.$$

Definition 2.4. A fuzzy subset $\mu \in F^M$ is a fuzzy submodule of M if:

(1) $\mu(0) = 1$;

(2) $\mu(rx) \geq \mu(x)$ for all $r \in R$ and $x \in M$;

(3) $\mu(x + y) \geq \mu(x) \wedge \mu(y)$ for all $x, y \in M$.

The set of all fuzzy submodules of M is denoted by $F(M)$.

For $\mu, \nu \in F^M, \mu : \nu \in F^R$ and is defined as follows:

$$\mu : \nu = \bigcup \{\eta | \eta \in F^R, \eta \cdot \nu \subseteq \mu\}$$

In [15] it has been proved that if $\mu \in F(M)$, $\nu \in F^M$, then $\mu : \nu = \bigcup\{\eta|\eta \in FI(R), \eta \cdot \nu \subseteq \mu\}$. Also it has been proved that if $\mu \in F(M)$ and $\nu \in F^M$, then $\mu : \nu \in FI(R)$.

In [15] it is proved that if $c \in L$ and N is a submodule of M, then

$$(1_N \cup c_M) : 1_M = 1_{N:M} \cup c_R.$$

Recall that a non-constant fuzzy submodule μ of M is said to be *prime* if for $\zeta \in FI(R)$ and $\nu \in F(M)$ such that $\zeta.\nu \subseteq \mu$ then either $\nu \subseteq \mu$ or $\zeta \subseteq \mu : 1_M$.(Ref. [1])

In the sequel FSpec(M) denotes the set of all prime fuzzy submodules of M.

In [1] we obtained the following results:

Theorem 2.5. [2] *$\mu \in FSpec(M)$ if and only if $\mu = 1_{\mu_*} \cup c_M$ such that $\mu_* = \{x \in M|\mu(x) = 1\}$ is a prime submodule of M and $c \in [0, 1]$.*

Theorem 2.6. [2] *If $\mu \in FSpec(M)$, then $\mu : 1_M$ is a prime fuzzy ideal of R.*
For $\mu \in F^M$ set

$$V^*(\mu) = \{P \in FSpec(M)|\mu \subseteq P\}$$

and

$$V(\mu) = \{P \in FSpec(M)|\mu : 1_M \subseteq P : 1_M\}.$$

Now, we put

$$F\zeta^*(M) = \{V^*(\mu)|\mu \in F(M)\};$$
$$F\zeta'(M) = \{V^*(\eta.1_M)|\eta \in FI(R)\};$$
$$F\zeta(M) = \{V(\mu)|\mu \in F(M)\}.$$

We consider the topologies of $FSpec(M)$ induced, respectively, by these three sets. In [3], it is shown that there exists a topology τ^* say, on $FSpec(M)$ having $F\zeta^*(M)$ as the collection of all closed sets if and only if $F\zeta^*(M)$ is closed under finite union. In this case, we call the topology τ^* the quasi-Zariski topology on $FSpec(M)$. Following [2], a module M is called a fuzzy top module, if $F\zeta^*(M)$ induces the topology τ^* on $FSpec(M)$.

For $p \in FSpec(R)$, by $FSpec_p(M)$ we mean the set of all $\mu \in F(M)$ such that $\mu : 1_M = p$. In other words

$$FSpec_p(M) = \{\mu \in FSpec(M)|\mu : 1_M = p\}.$$

Let μ be a prime fuzzy submodule of M. Then by Theorem 2.6, we have that $(\mu : 1_M)$ is a prime fuzzy ideal of R. Define $\overline{(\mu : 1_M)} \in F^{R/Ann(M)}$as follows:

$$\overline{(\mu : 1_M)}([x]) = \bigvee\{(\mu : 1_M)(z)|z \in [x]\}$$

and the map $\psi : FSpec(M) \longrightarrow FSpec(R/Ann(M))$ by

$$\psi(\mu) = \overline{(\mu : 1_M)} \quad for \quad \mu \in FSpec(M),$$

is called the *natural map.*

For any $R-$module M, we consider the set $\mathbb{B} = \{D(x_\beta.1_M)|x \in R, \beta \in (0,1]\}$ such that $D(x_\beta.1_M) = X \setminus V(x_\beta.1_M)$. In [2] it is proved that $\mathbb{B}$ forms a base for Zariski topology on $X = FSpec(M)$.

Example 2.7. [3] (1) Consider the ring of integers $M = \mathbb{Z}$ as $\mathbb{Z}$-module and let L be an arbitrary lattice. Suppose that $p \in \mathbb{Z}$ is prime. For every prime element $t \in L$, define $P(t) \in L(\mathbb{Z})$ by

$$P(t)(x) = \begin{cases} 1 \quad \text{if} \quad x \in < p >; \\ t \ \text{if}\ x \in \mathbb{Z}\setminus < p > \end{cases}$$

Then by Theorem 2.5, $P(t)$ is a prime L-submodule of M. Thus $L - Spec(M) = \{P(t)|t \text{ is a prime element of L and p is a prime idael of } \mathbb{Z}\}$, while $L-Spec(M) = \{P(t)|t \in [0,1] \quad \text{and p is prime ideal of } \ \mathbb{Z}\}$ for $L = [0,1]$.

(2) Consider $M = \mathbb{R}[x]$ as $\mathbb{R}[x] - module$, where $\mathbb{R}$ is the field of real numbers. For every $P \in \mathbb{R}[x]$ and every $t \in L$, define the fuzzy subset $P(t)$ of $\mathbb{R}[x]$ by

$$P(t)(x) = \begin{cases} 1 \quad x \in < p >; \\ t \quad \text{otherwise} \end{cases}$$

Then by Theorem 2.5, $P(t)$ is a prime L-submodule of M if and only if P is irreducible and t is a prime element of L. Moreover, for $L = [0,1]$, we have $L - Spec(M) = \{P(t)| \ \ p \ \text{ is irreducible in } \ \mathbb{R}[x], t \in [0,1]\}$.

(3) Suppose M is an arbitrary R-module and P is a prime submodule of M. For every $t \in L$, define

$$P(t)(x) = \begin{cases} 1 \quad x \in P; \\ t \quad \text{otherwise} \end{cases}$$

Then by Theorem 2.5, $P(t)$ is a prime L-submodule of M if and only if t is a prime element of L. If $Spec(L)$ denote the set of all prime elements of L, then $L - Spec(M) = \{P(t)| \ \ t \in Spec(L) \ \text{ and P is a prime submodule of M } \}$.

(4) If we consider $M = \mathbb{R}[x]$ as $\mathbb{R} - module$. Then every proper submodule P of M, Which is denoted by $P < M$, is prime. Then by part (3) $L - Spec(M) = \{P(t)| \ \ t \in Spec(L) \ \text{ and } \ P < M\}$.

(5) Suppose that $L = \{0, a, b, 1\}$ is a lattice which is not a chain, that is, a and b are not comparable. Then $L - Spec(M) = \emptyset$, for every R-module M, since L has no any prime element. This example shows that $L - Spec(M) = \emptyset$, but $Spec(M)$ may be not empty.

3. Irreducible Subsets of FSpec(M)

In the sequel we assume that M is an R-module and $X = FSpec(M)$. For $Y \subseteq X$ we denote the intersection of all elements in Y by $\Gamma(Y)$, and closure of Y in X for this topology by $\overline{Y}$.

Proposition 3.1. *Let $Y \subseteq X$. Then $V(\Gamma(Y)) = \overline{Y}$. Hence Y is closed if and only if $V(\Gamma(Y)) = Y$.*

Proof. Let $P \in Y$, then $\Gamma(Y) \subseteq P$, so $P \in V(\Gamma(Y))$, and hence $Y \subseteq V(\Gamma(Y))$. Let $V(\mu)$ be any closed subset of X such that $Y \subseteq V(\mu)$. Now for every $P \in Y, P \in V(\mu)$ we have

$$\mu : 1_M \subseteq P : 1_M \Longrightarrow \mu : 1_M \subseteq \bigcap_{P \in Y} (P : 1_M) = (\bigcap_{P \in Y} P) : 1_M = \Gamma(Y) : 1_M.$$

Now let $Q \in V(\Gamma(Y))$, then $\Gamma(Y) : 1_M \subseteq Q : 1_M$, but $\mu : 1_M \subseteq \Gamma(Y) : 1_M \subseteq Q : 1_M$, so $Q \in V(\mu)$. Then we conclude that $V(\Gamma(Y)) \subseteq V(\mu)$ and then $\overline{Y} = V(\Gamma(Y))$, and hence it is easy to see that Y is a closed subset if and only if $V(\Gamma(Y)) = Y$. $\square$

Lemma 3.2. *Every fuzzy ideal of R is contained in a maximal fuzzy ideal.*

Proof. Let η be a fuzzy ideal of R. Put $\eta_* = \{x \in R | \eta(x) = 1\}$. Since η_* is an ideal of R, then there exists a maximal ideal m of R such that $\eta_* \subseteq m$. Letting $a = \bigvee_{t \in \eta(R)} t$. Put $\nu = 1_m \cup a_R$. So ν is a maximal fuzzy ideal such that $\eta \subseteq \nu$. On the other words, there exists a maximal fuzzy ideal ν of R such that $\eta \subseteq \nu$. $\square$

Proposition 3.3. *Let ν be a maximal fuzzy ideal of R. Then $\nu.1_M$ is a prime fuzzy submodule of M.*

Proof. Let ν be maximal, then $\nu = 1_{\nu_*} \cup c_R$, such that ν_* is a maximal ideal of R and $c \in [0, 1)$. Then $\nu.1_M = 1_{\nu_* M} \cup c_M$. But since ν_* is maximal, then $\nu_* M$ is a prime submodule of M, [11, Proposition 2]. Therefore by Theorem 2.5, $\nu.1_M$ is a prime fuzzy submodule of M. $\square$

Definition 3.4. A fuzzy subset μ of M is called *maximal prime fuzzy submodule* if $\mu \in FSpec(M)$ and there is not any $\nu \in FSpec(M)$, such that $\mu \subsetneqq \nu$.

Lemma 3.5. *If $\mu \in FSpec(M)$ is maximal prime, then $\mu : 1_M$ is a maximal fuzzy ideal of R.*

Proof. Let $\mu \in FSpec(M)$ be maximal prime. Let for $\eta \in FI(R), \mu : 1_M \subseteq \eta$ (1). Then by Lemma 3.2 there exists a maximal fuzzy ideal m of R such that $\eta \subseteq m$. Since $\mu : 1_M \subseteq \eta$, then $\mu \subseteq \eta.1_M \subseteq m.1_M$ and by Proposition 3.3, $m.1_M$ is a fuzzy prime submodule of M, then $\mu = m.1_M$, since μ is maximal prime and so $\mu = \eta.1_M$. Thus $\eta \subseteq \mu : 1_M$ (2). Now by (1), (2) we have $\mu : 1_M = \eta$ and thus $\mu : 1_M$ is a maximal fuzzy ideal of R. $\square$

Proposition 3.6. *For any element P of X, the following statements are satisfied:*

(i) $\overline{\{P\}} = V(P)$.

(ii) For any $Q \in X, Q \in \overline{\{P\}}$ if and only if $P : 1_M \subseteq Q : 1_M$ if and only if $V(Q) \subseteq V(P)$.

(iii) The set $\{P\}$ is closed if and only if

(a) P is a maximal prime fuzzy submodule of M and,

(b) $FSpec_p(M) = \{P\}$, such that $P : 1_M = p$.

Proof. (i) It is an immediate consequence of Proposition 3.1.

(ii) Follows from (i).

(iii) Let $\{P\}$ be closed. Then $\{P\} = \overline{\{P\}} = V(P)$. Suppose that $\mu \in FSpec(M)$ and $P \subseteq \mu$, then $P : 1_M \subseteq \mu : 1_M$, and hence $\mu \in V(P) = \{P\}$. Thus $\mu = P$. This means that P is a maximal prime fuzzy submodule of M. Now suppose that $\mu \in FSpec_p(M)$, then $\mu : 1_M = p = P : 1_M$. So $\mu \in V(P) = \{P\}$, and hence $\mu = P$. Conversely, suppose that (a) and (b) are satisfied. Let $\mu \in V(P)$, then $P : 1_M \subseteq \mu : 1_M$. Since P is maximal prime, then by Lemma 3.5 it is concluded, that $P : 1_M = p$ is a maximal fuzzy ideal of R. Then $p = P : 1_M = \mu : 1_M$. This means that $\mu \in FSpec_p(M) = \{P\}$. Thus $\mu = P$, and hence $V(P) = \{P\}$. But $\overline{\{P\}} = V(P) = \{P\}$. It means that $\{P\}$ is closed. □

Remark 3.7. The last proposition gives the conditions, such that X is a T_1 space. In fact, it says that X is T_1 space if and only if every prime fuzzy submodule of M is maximal prime and $|FSpec_p(M)| \leq 1$ for all $p \in FSpec(R)$.

Recall that a topological space A is said to be irreducible if for any decomposition $A = A_1 \cup A_2$, of closed subsets A_1 and A_2 of A implies that $A_1 = A$ or $A_2 = A$. A subspace A' of A is irreducible if it is irreducible as a subspace of A.

An irreducible component of a topological space A is a maximal irreducible subset of A.

Theorem 3.8. *$V(P)$ is an irreducible closed subset of $FSpec(M)$ for every prime fuzzy submodule P of M.*

Proof. By Proposition 3.6(i), $V(P) = \overline{\{P\}}$. Let $V(P) = A_1 \cup A_2$ for closed sets A_1 and A_2, so $\overline{\{P\}} = A_1 \cup A_2$. But $P \in \overline{\{P\}}$, then $P \in A_1$ or $P \in A_2$. Let $P \in A_1$, then $P \in A_1 \in \overline{\{P\}}$, which is a contradiction. Thus we must have $A_1 = \overline{\{P\}}$ and this means that $V(P)$ is irreducible. □

Corollary 3.9. *Let Y be a subset of $FSpec(M)$. If $\Gamma(Y)$ is a prime fuzzy submodule of M, then Y is irreducible.*

Proof. Let $\Gamma(Y) = P$ be a prime fuzzy submodule of M. By Proposition 3.1 $V(P) = V(\Gamma(Y)) = \overline{Y}$ is irreducible. Let $Y = A_1 \cup A_2$ (1) for closed subsets A_1 and A_2. Then $\overline{Y} = \overline{A_1 \cup A_2} = \overline{A_1} \cup \overline{A_2} = A_1 \cup A_2$. Since $\overline{Y}$ is irreducible, then $\overline{Y} = A_1$ or $\overline{Y} = A_2$. Without loose of generality suppose that $\overline{Y} = A_1$. Then $Y \subseteq A_1$ and (1) implies that $A_1 \subseteq Y$ and hence $Y = A_1$. This means that Y is irreducible. □

Corollary 3.10. *Let $P^* = \bigcap_{P \in X} P$. If P^* is a prime fuzzy submodule of M, then X is irreducible.*

Proof. Immediately follows from Corollary 3.8. □

Corollary 3.11. *For an $R-$module M the following hold:*

i) If $Y = \{P_i \in X | i \in I\}$ is linearly ordered by the set inclusion, then Y is irreducible in X;

(ii) $FSpec_p(M)$ is irreducible for $p \in FSpec(R)$;

(iii) If p is a maximal fuzzy ideal of R, then $Fspec_p(M)$ is an irreducible closed subset of X.

Proof. (i) Since the elements of Y are linearly ordered by the set inclusion, $\Gamma(Y)$ is a prime fuzzy submodule of M. Then by Corollary 3.9, Y is irreducible.

(ii) We show that $\Gamma(FSpec_p(M))$ is a prime fuzzy submodule of M. For this we have $\bigcap_{P\in FSpec_p(M))} P : 1_M = \bigcap_{P\in FSpec_p(M)} (P : 1_M) = p$. Thus $\Gamma(FSpec_p(M)) : 1_M = p$. Now suppose that $\alpha \in FI(R)$, $\nu \in F(M)$ and $\alpha.\nu \subseteq \Gamma(FSpec_p(M))$ such that $\nu \nsubseteq \Gamma(FSpec_p(M))$. Then there exists $P' \in FSpec_p(M)$ such that $\nu \nsubseteq P'$. Therefore $\alpha \subseteq P' : 1_M = p = \Gamma(FSpec_p(M)) : 1_M$. This means that $\Gamma(FSpec_p(M))$ is a prime fuzzy submodule. Then $FSpec_p(M)$ is irreducible by Corollary 3.9.

(iii) Suppose that p is a maximal fuzzy ideal of R. By (2), $FSpec_p(M)$ is irreducible. But since p is maximal, then $(p.1_M) : 1_M = p$. Also for $Q \in V(p.1_M)$ we have $p = (p.1_M) : 1_M \subseteq Q : 1_M$ and since p is maximal then

$$Q : 1_M = p \Longrightarrow Q \in FSpec_p(M) \Longrightarrow V(p.1_M) \subseteq FSpec_p(M) \tag{1}$$

but for $P \in FSpec_p(M)$ it is concluded that $P : 1_M = p = (p.1_M) : 1_M$. Thus

$$P \in V(p.1_M) \Longrightarrow Fspec_p(M) \subseteq V(p.1_M) \tag{2}$$

From (1), (2) we obtain $V(p.1_M) = Fspec_p(M)$. Therefore, $FSpec_p(M)$ is closed as desired. □

Corollary 3.12. *Let Y be a subset of X and $\Gamma(Y) : 1_M = p$ be a prime fuzzy ideal of R. If $FSpec_p(M) \neq \emptyset$ then Y is irreducible.*

Proof. Let $P \in FSpec_p(M)$. Then $P : 1_M = \Gamma(Y) : 1_M = p$. Thus $V(\Gamma(Y)) = V(P)$, by Proposition 3.3 of [2]. But by Proposition 3.1 we have $V(\Gamma(Y)) = \overline{Y}$, hence $V(P) = \overline{Y}$. Then by Theorem 3.8, $V(P)$ is irreducible. Therefore, $\overline{Y}$ and hence Y are irreducible. □

4. **Separation Properties of FSpec(M)**

Theorem 4.1. *For X the following statements are equivalent:*
$(i)X$ is T_0 space;
(ii) the natural map ψ is injective;
(iii) if $V(P) = V(Q)$, then $P = Q$ for any $P, Q \in X$;
$(iv)|FSpec_p(M)| \leq 1$ for every $p \in FSpec(R)$.

Proof. By Proposition 4.4 of [3] $(ii), (iii)$ and (iv) are equivalent. Only it reminds to prove $(i) \Longleftrightarrow (iii)$. It is well-known that a topological space is T_0 if and only if the closures of distinct points are distinct. Now suppose that X is a T_0 space and let $V(P) = V(Q)$ for $P, Q \in X$. If $P \neq Q$, then we have $\overline{\{P\}} \neq \overline{\{Q\}}$, but by Proposition 3.6 (i), we have $V(P) \neq V(Q)$, a contradiction. Thus $P = Q$. For the converse, suppose that $P, Q \in X$ and $P \neq Q$. By (2), $V(P) \neq V(Q)$, again by Proposition 3.6 (i), $\overline{\{P\}} \neq \overline{\{Q\}}$. This means that X is a T_0 space. □

Corollary 4.2. *If M is a fuzzy top module, then $Fspec(M)$ is a T_0 space for Zariski topology and τ^*.*

Proof. Let $P, Q \in X$ and $P \neq Q$. Then $P \not\subseteq Q$ or $Q \not\subseteq P$. Let $P \not\subseteq Q$, then $Q \notin V^*(P)$, so $Q \in D^*(P)$, but $P \notin D^*(P)$ and $D^*(P)$ is an open set in topology τ^*. This shows that X is T_0 space for topology τ^*. Then from $\tau \leq \tau^*$, it conclude that $FSpec(M)$ is T_0. □

Let $n = \{P : 1_M | P \in FSpec(M)\}$ and $n^* = \{p_* | p \in n\}$.

Lemma 4.3. $D(x_\beta.1_M) = \emptyset$ *if and only if* $x \in \bigcap n^*$.

Proof. Let $D(x_\beta.1_M) = \emptyset$, then $V(x_\beta.1_M) = X$. Let P be a prime submodule of M and set $\mu = \chi_p$. Then $\mu \in FSpec(M)$. Let $p = \mu : 1_M$. Then $(x_\beta.1_M) : 1_M \subseteq \mu : 1_M = p$, but $x_\beta \subseteq (x_\beta.1_M) : 1_M$, and hence $x_\beta \subseteq p$. Thus $\beta \leq p(x) = 1$. This shows that $x \in p_*$, and so $x \in \bigcap n^*$. Conversely, suppose that $x \in \bigcap n^*$ and $P \in Fspec(M)$. If $p = P : 1_M$, then $x \in p_*$. So

$$\begin{aligned} p(x) = 1 &\implies (P : 1_M)(x) = 1 \\ &\implies x_\beta \subseteq P : 1_M \\ &\implies x_\beta.1_M \subseteq P \\ &\implies (x_\beta.1_M) : 1_M \subseteq P : 1_M. \end{aligned}$$

Therefore, $P \in V(x_\beta.1_M)$ and hence $V(x_\beta.1_M) = X$. Thus $D(x_\beta.1_M) = \emptyset$. □

Let $X = FSpec(M)$ and $\alpha \in [0, 1)$. We denote the subspace $\{\mu \in X | Im\mu = \{1, \alpha\}\}$ of X by A_α.

Lemma 4.4. *If the natural map ψ is injective and every prime ideal of R is maximal then the subspace A_α of X is Hausdorff.*

Proof. Let $\mu, \nu \in A_\alpha$ be any two distinct elements of A_α. Then $\mu = 1_{\mu^*} \cup \alpha_M$ and $\nu = 1_{\nu^*} \cup \alpha_M$. Since ψ is injective then $\overline{\mu : 1_M} \neq \overline{\nu : 1_M}$ and then $\mu : 1_M \neq \nu : 1_M$. But, $\mu : 1_M = 1_{\mu_* : M} \cup c_R$ and $\nu : 1_M = 1_{\nu_* : M} \cup c_R$. Also there exists $x \in R$ such that $x \in \mu_* : M$ and $x \notin \nu_* : M$. Then $(\mu : 1_M)(x) = 1$ and $(\nu : 1_M)(x) = \alpha$. Let β be a real number such that $\alpha < \beta < 1$. Therefore, $x_\beta = (x_\beta.1_M) : 1_M \not\subseteq \nu : 1_M$. Thus

$$\nu \notin V(x_\beta.1_M) \Longrightarrow \nu \in D(x_\beta.1_M).$$

Since $\nu_* : M$ is a prime ideal of R and $x \notin \nu_* : M$, then x is not a nilpotent element in R, and hence $< x + \eta_R >$ is idempotent where η_R denotes the nilradical of R. Thus there exists $a \in R$, such that $x(1 - ax) \in \eta_R$, and hence $x(1 - ax)$ is nilpotent. For $x \in \mu_* : M$ by hypothesis and the fact $\mu_* : M$ is prime we have $\mu_* : M$ is maximal and so $1 - ax \notin \mu_* : M$. Thus $(\mu : 1_M)(1 - ax) = \alpha$. But $(1 - ax)_\beta = [(1 - ax)_\beta.1_M] : 1_M \not\subseteq \mu : 1_M$. Therefore,

$$\mu \notin V((1 - ax)_\beta.1_M) \Longrightarrow \mu \in D((1 - ax)_\beta.1_M).$$

On the other hands, we have $D(x_\beta.1_M) \cap D((1 - ax)_\beta.1_M) = D((x(1 - ax))_\beta.1_M)$, by Proposition 5.4 of [2]. Also $x(1 - ax)$ is nilpotent, then by Lemma 3.2, $D((x(1 - ax))_\beta.1_M) = \emptyset$, that is, A_α is Hausdorff. □

In the next example we show that the space X is not Hausdorff even if every prime ideal of R is maximal and the natural map ψ is injective.

Example 4.5. Let M be an arbitrary $R-$module and let P be any prime submodule of M. Consider the prime fuzzy submodules μ and ν of M as follows:

$$\mu(x) = \begin{cases} 1 \text{ if } & x \in P; \\ 0.1 & \text{otherwise} \end{cases}$$

$$\nu(x) = \begin{cases} 1 & \text{if } x \in P; \\ 0.1 & \text{otherwise} \end{cases}$$

Let $D(x_\beta.1_M)$ and $D(y_{\beta'}.1_M)$ be two basic open sets, such that $\mu \in D(x_\beta.1_M)$ and $\nu \in D(y_{\beta'}.1_M)$. So $x_\beta = (x_\beta.1_M) : 1_M \not\subseteq \mu : 1_M$ and $y_{\beta'} = (y_{\beta'}.1_M) : 1_M \not\subseteq \nu : 1_M$. $\mu : 1_M = 1_{P:M} \cup (0.1)_R$ and $\nu : 1_M = 1_{P:M} \cup (0.2)_R$, by Theorem 2.5, and $x \notin P : M$ and $y \notin P : M$. But $P : M$ is a prime ideal of R, therefore $xy \notin P : M$, and hence $xy \notin n^*$. Thus $D((xy)_{\beta\wedge\beta'}.1_M) \neq \emptyset$ and we obtain that $D(x_\beta.1_M) \cap D(y_{\beta'}.1_M) = D((xy)_{\beta\wedge\beta'}.1_M)$. This shows that X is not Hausdorff.

Proposition 4.6. *The subspace A_α of X is homeomorphic to $Spec(M)$.*

Proof. Define the mapping $\varphi : A_\alpha \longrightarrow Spec(M)$ by $\varphi(\mu) = \mu_*$. Let D_r be a basic open set in $Spec(M)$. Then

$$D_r = X \setminus V(rM) = \{P \in X | rM \not\subseteq P\} = \{P \in X | ry \notin P \ \ for \ \ some \ \ y \in M\},$$

and hence

$$D(r_1.1_M) \cap A_\alpha = \{\mu \in X | \mu(x) = \alpha \ \ for \ \ some \ \ x, y \in M \ \ such \ \ that \ \ ry = x\}.$$

Thus $\varphi^{-1}(D_r) = D(r_1.1_M) \cap A_\alpha$ and since $D(r_1.1_M) \cap A_\alpha$ is an open set in A_α, then φ is continuous.
Now we define $\psi : Spec(M) \longrightarrow A_\alpha$ as follows:

$$for \ \ each \ \ P \in Spec(M); \ \ \psi(P)(x) = \left\{ \begin{cases} 1 & x \in P, \\ \alpha & x \notin P. \end{cases} \right\}$$

Suppose that $D(r_\beta.1_M) \cap A_\alpha$ is a basic open set in A_α. Then

$$D(r_\beta.1_M) \cap A_\alpha = \{\mu \in X | \mu(x) = \alpha \ \ for \ \ some \ \ x, y \in M \ \ such \ \ that \ \ ry = x\}.$$

It is easy to verify that $\psi^{-1}(D(r_\beta.1_M) \cap A_\alpha) = D_r$ and since D_r is an open set in $Spec(M)$, then ψ is continuous. Clearly φ and ψ are inverse of each other. Then A_α and $Spec(M)$ are homeomorphic. $\square$

Proposition 4.7. *The spectrum $FSpec(M)$ is homeomorphic to the space $Spec(M) \times [0, 1)$.*

Proof. Define the mapping $\varphi : FSpec(M) \longrightarrow Spec(M) \times [0, 1)$ as follows:
Let $\mu \in FSpec(M)$, such that $Im(\mu) = \{1, \alpha\}$, then $\varphi(\mu) = (\mu_*, \alpha)$. Suppose that $D_r \times [0, \alpha)$ is a basic open set in $Spec(M) \times [0, 1)$. Then

$$\begin{aligned} \varphi^{-1}(D_r \times [0, 1)) &= \{\mu \in X | \mu(x) \in [0, \alpha) \ \ such \ \ that \ \ x = ry \ \ for \ \ some \ \ x, y \in M\} \\ &= \bigcup \{D(r_\beta.1_M) | \beta \in [0, \alpha)\}, \end{aligned}$$

which is an open set in $FSpec(M)$. So φ is continuous. Now we define $\psi : Spec(M) \times [0,1) \longrightarrow FSpec(M)$ as follows:
for $(P,\alpha) \in Spec(M) \times [0,1); \psi_{(P,\alpha)} = \mu_{(P,\alpha)}$ where,

$$\mu_{(P,\alpha)}(x) = \begin{cases} 1 & \text{if } x \in P; \\ \alpha & \text{if } x \notin P \end{cases}$$

Let $D(r_\beta.1_M)$ be a basic open set in $FSpec(M)$, then we can show that $\psi^{-1}(D(r_\beta.1_M)) = D_r \times [0,\beta)$, which is an open set in $Spec(M) \times [0,1)$. So ψ is continuous and φ and ψ are inverse of each other. Then $FSpec(M)$ and $Spec(M) \times [0,1)$ are homeomorphic. □

5. Conclusion

In this paper we have constituted a topology on the collection of all prime fuzzy submodules of a module M over a commutative ring with identity say R, which is called Zariski topology, and then the basic topological properties of this topological space has been studied. In this regard by finding many results it has been shown that this topological spaces is enough rich in the view point of topological properties. Also, we have tried in this paper to bring the first stones of fuzzy spectral theory based on prime fuzzy submodules, and hence we hope that this paper encourage researchers in the field of fuzzy algebra and fuzzy topology to continue this way for finding further and deep results.

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