

ψ -WEAK CONTRACTIONS IN FUZZY METRIC SPACES

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ABSTRACT. In this paper, the notion of ψ -weak contraction [18] is extended to fuzzy metric spaces. The existence of common fixed points for two mappings is established where one mapping is ψ -weak contraction with respect to another mapping on a fuzzy metric space. Our result generalizes a result of Gregori and Sapena [9].

1. Introduction and Preliminaries

The origin of fuzzy mathematics can be solely attributed to the introduction of fuzzy sets in the pioneering paper of Zadeh [20] in 1965, which provides a new way to represent the vagueness in every day life. The study of fuzzy sets initiated an extensive fuzzification of several mathematical concepts and has applications to various branches of applied sciences namely: neural networking theory, image processing, control theory, modeling theory and many others(see for example, [4]). In the course of this fuzzification process, like other concepts, the concept of a fuzzy metric was introduced in many ways ([3]). George and Veeramani [7, 8] modified the concept of a fuzzy metric space introduced by Kramosil and Michalek [12] which has been exploited by many authors to prove a multitude of results of varied kind. Banach contraction principle is indeed a classical result of modern analysis. This principle has been extended and generalized in different directions in metric spaces. For a comprehensive description of such work, we refer to [13] and references mentioned therein. Grabiec [5] initiated the study of fixed point theory in fuzzy metric space (see also [6], [15] [17]). Recently, Gregori and Sapena [9] introduced new kind of contractive mappings in modified fuzzy metric spaces and proved a fuzzy version of Banach contraction principle.

In 2001, Rhoades [18] established a fixed point theorem for ψ -weak contraction in the frame work of complete metric spaces. Actually, in [1], the authors defined such contractions for single valued maps on Hilbert spaces and proved an existence result on fixed points.

The aim of this paper is to study ψ -weak contraction mapping on a fuzzy metric space and to prove the existence of fixed points for such mapping. Our results extend and generalize several comparable and related results in the existing literature (see [9, 18] and some references mentioned therein).

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For sake of completeness, we recall some relevant definitions and known results in fuzzy metric spaces.

Definition 1.1. [20] A fuzzy set A in a nonempty set X is a function with domain X and values in $[0, 1]$.

Definition 1.2. [16] A binary operation $* : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is a continuous t -norm if it satisfies the following conditions:

- (1) $*$ is associative and commutative,
- (2) $*$ is continuous,
- (3) $a * 1 = a$ for every $a \in [0, 1]$,
- (4) $a * b \leq c * d$ if $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Definition 1.3. [7] The triplet $(X, M, *)$ is a fuzzy metric space (in the sense of George and Veeramani) if X is an arbitrary set, $*$ is a continuous t -norm and M is a fuzzy set in $X \times X \times (0, \infty)$ satisfying the following conditions:

- (i) $M(x, y, t) > 0$,
- (ii) $M(x, y, t) = 1$ iff $x = y$,
- (iii) $M(x, y, t) = M(y, x, t)$,
- (iv) $M(x, y, t) * M(y, z, t) \leq M(x, z, t + s)$,
- (v) $M(x, y, \cdot) : (0, \infty) \rightarrow [0, 1]$ is continuous

for each $x, y, z \in X$ and $s, t > 0$.

Note that, $M(x, y, t)$ can be realized as the measure of nearness between x and y with respect to t . It is known that $M(x, y, \cdot)$ is nondecreasing for all $x, y \in X$.

Throughout the paper \mathbb{N} denotes the set of positive integers.

George and Veeramani ([7, 8]) proved that every fuzzy metric M on X induces a Hausdorff first countable topology τ_M whose base is the family of open balls $\{B(x, r, t) : x \in X, 0 < r < 1, t > 0\}$, where

$$B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}.$$

A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is said to be a Cauchy sequence if for every $0 < \varepsilon < 1$ and for every $t > 0$, there is $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for every $n, m \geq n_0$. Also, a sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is said to be a G -Cauchy sequence (i.e. Cauchy sequence in the sense of Grabiec [5]) if $M(x_n, x_{n+p}, t) \rightarrow 1$ as $n \rightarrow \infty$, for every $p \in \mathbb{N}$ and for every $t > 0$. Hence, a fuzzy metric space $(X, M, *)$ is called complete (respectively G -complete) if every Cauchy sequence (respectively G -Cauchy sequence) is convergent. Vasuki and Veeramani suggested that the definition of a G -Cauchy sequence is weaker than the one contained in [19].

Remark 1.4. [8] If d is a metric on X , $a * b = ab$ for all $a, b \in [0, 1]$ and $M(x, y, t) = \frac{t}{t + d(x, y)}$ for every $(x, y, t) \in X \times X \times (0, \infty)$, then $(X, M, *)$ is a fuzzy metric space. We call this M as the standard fuzzy metric induced by d . Even if we define $a * b = \min\{a, b\}$, $(X, M, *)$ will be a fuzzy metric space.

Definition 1.5. [14] Let $(X, M, *)$ be a fuzzy metric space. A sequence $\{x_n\}$ in X is said to be pointwise convergent to $x \in X$ (we write $x_n \rightarrow_p x$) if there exists $t(x) > 0$ such that

$$\lim_{n \rightarrow \infty} M(x_n, x, t) = 1.$$

It has been established in [14] that there exists sequences which are pointwise convergent but not convergent.

We introduce the notion of ψ -weak contractivity of a mapping T with respect to a self map f on a fuzzy metric spaces X as follows:

Definition 1.6. Let $(X, M, *)$ be a fuzzy metric space and $f : X \rightarrow X$ be a map. The map $T : X \rightarrow X$ is called a ψ -weak contraction with respect to f if there exists a function $\psi : [0, \infty) \rightarrow [0, \infty)$ with $\psi(r) > 0$ for $r > 0$ and $\psi(0) = 0$ such that

$$\frac{1}{M(Tx, Ty, t)} - 1 \leq \left(\frac{1}{M(fx, fy, t)} - 1 \right) - \psi \left(\frac{1}{M(fx, fy, t)} - 1 \right) \quad (1)$$

for every $x, y \in X$ and each $t > 0$. If $f = I_X$ (an identity mapping on X) then T is called a ψ -weak contraction.

Definition 1.7. Let $(X, M, *)$ be a fuzzy metric space and f, T be two self mappings on X . A point x in X is called a coincidence point (common fixed point) of f and T if $fx = Tx$ ($fx = Tx = x$). Also the pair of mappings $f, T : X \rightarrow X$ are said to be weakly compatible if they commute on the set of coincidence points.

2. Main Results

Our first result utilizes the idea of pointwise convergence due to Mihet [14] to prove the existence of fixed point for ψ -weak contraction mappings.

Theorem 2.1. Let (X, M, T) be a fuzzy metric space under a t -norm T satisfying $\sup_{a < 1} T(a, a) = 1$ and $f : X \rightarrow X$ be a ψ -weak contraction. If for some x_0 in X , the sequence $\{x_n\}$ defined by $x_{n+1} = fx_n$, $n \in \mathbb{N}$ has a p -convergent subsequence, then f has a unique fixed point.

Proof. Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ which is p -convergent to $x \in X$. Consequently, there exists $t > 0$ such that

$$\lim_{k \rightarrow \infty} M(x_{n_k}, x, t) = 1. \quad (2)$$

Following arguments similar to those given in the proof of Theorem 1.3 ([2]), we obtain

$$\lim_{i \rightarrow \infty} M(x_{n_i}, x_{n_{i+1}}, t) = 1. \quad (3)$$

For $\delta > 0$, there exist $k_1, k_2 \in \mathbb{N}$ such that for all $k' > k_1$, $k'' > k_2$, followings hold:

$$M(x_{n_{k'}}, x, t) > 1 - \delta \text{ and } M(x_{n_{k''}}, x_{n_{k''+1}}, t) > 1 - \delta.$$

Take $k_0 = \max\{k', k''\}$. Note that,

$$M(x_{n_j}, x, t) > 1 - \delta \text{ and } M(x_{n_j}, x_{n_j+1}, t) > 1 - \delta \quad (4)$$

for all $j > k_0$. Thus

$$M(x_{n_j+1}, x, 2t) \geq T(M(x_{n_j+1}, x_{n_j}, t), M(x_{n_j}, x, t)) \geq T((1 - \delta), (1 - \delta)).$$

Since $\sup_{a < 1} T(a, a) = 1$, therefore for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$T((1 - \delta), (1 - \delta)) > 1 - \varepsilon.$$

Using (4), we arrive at

$$M(x_{n_j+1}, x, 2t) > 1 - \varepsilon$$

for all $j > k_0$. So,

$$\lim_{j \rightarrow \infty} M(x_{n_j+1}, x, 2t) = 1. \quad (5)$$

Now, from (5) it is possible to find a positive integer N_1 such that

$$M(x_{n_i+1}, x, 2t) > 0.$$

for $i > N_1$. Therefore for all $i > N_1$

$$\begin{aligned} \frac{1}{M(x_{n_i+1}, fx, 2t)} - 1 &\leq \frac{1}{M(fx_{n_i}, fx, 2t)} - 1 \\ &\leq \left(\frac{1}{M(x_{n_i}, x, 2t)} - 1 \right) - \psi \left(\frac{1}{M(x_{n_i}, x, 2t)} - 1 \right), \end{aligned}$$

which on taking limit as $i \rightarrow \infty$ and using (2) give

$$M(x_{n_i+1}, fx, 2t) \rightarrow 1.$$

Thus $x_{n_i+1} \rightarrow_p fx$ as $i \rightarrow \infty$. Since the pointwise convergence is Frechet (see page 4 of Mihet [14]), we deduce that $fx = x$. \square

Now we prove a common fixed point theorem for ψ -weak contractions involving a pair of self mappings.

Theorem 2.2. *Let $(X, M, *)$ be a fuzzy metric space and $T : X \rightarrow X$ be a ψ -weak contraction with respect to self mapping f on X . If the range of f contains the range of T and $f(X)$ is a G -complete subspace of X , then f and T have coincidence point in X provided that ψ is a continuous mapping.*

Proof. Let x_0 be an arbitrary point in X . Choose a point x_1 in X such that $Tx_0 = fx_1$. This can be done since the range of f contains the range of T . Continuing this process indefinitely, for every x_n in X one can find x_{n+1} such that $y_n = Tx_n = fx_{n+1}$. Without loss of generality, one may assume that $y_{n+1} \neq y_n$ for all $n \in N$,

otherwise f and T have a coincidence point and there is nothing to prove. In case $y_{n+1} \neq y_n$, using (1), we have

$$\begin{aligned} & \frac{1}{M(y_n, y_{n+1}, t)} - 1 \\ &= \left(\frac{1}{M(fx_{n+1}, fx_{n+2}, t)} - 1 \right) \\ &= \left(\frac{1}{M(Tx_n, Tx_{n+1}, t)} - 1 \right) \\ &\leq \left(\frac{1}{M(fx_n, fx_{n+1}, t)} - 1 \right) - \psi \left(\frac{1}{M(fx_n, fx_{n+1}, t)} - 1 \right) \\ &\leq \left(\frac{1}{M(y_{n-1}, y_n, t)} - 1 \right) \end{aligned} \quad (6)$$

which implies that $M(y_n, y_{n+1}, t) \geq M(y_{n-1}, y_n, t)$ for all n and hence $M(y_{n-1}, y_n, t)$ is an increasing sequence of positive real numbers in $(0, 1]$. Let $S(t) = \lim_{n \rightarrow \infty} M(y_{n-1}, y_n, t)$. Now, we show that $S(t) = 1$ for all $t > 0$. Otherwise, there must exist some $t > 0$ such that $S(t) < 1$. Taking $n \rightarrow \infty$ in (6), we obtain

$$\frac{1}{S(t)} - 1 \leq \left(\frac{1}{S(t)} - 1 \right) - \psi \left(\frac{1}{S(t)} - 1 \right)$$

which is a contradiction. Therefore $M(y_n, y_{n+1}, t) \rightarrow 1$ as $n \rightarrow \infty$. Note that, for each positive integer p ,

$$\begin{aligned} & M(y_n, y_{n+p}, t) \\ &\geq M(y_n, y_{n+1}, t/p) * M(y_{n+1}, y_{n+2}, t/p) * \dots * M(y_{n+p-1}, y_{n+p}, t/p). \end{aligned}$$

This implies that

$$\lim_{n \rightarrow \infty} M(y_n, y_{n+p}, t) \geq 1 * 1 * \dots * 1 = 1.$$

Therefore $\{y_n\}$ is a G -Cauchy sequence. Since $f(X)$ is G -complete, there exists $q \in f(X)$ such that $y_n \rightarrow q$ as $n \rightarrow \infty$. Consequently we obtain a point p in X such that $fp = q$. Next we show that p is a coincidence point of f and T . To accomplish this; using (1), we get

$$\begin{aligned} & \frac{1}{M(Tp, fx_{n+1}, t)} - 1 \\ &= \frac{1}{M(Tp, Tx_n, t)} - 1 \\ &\leq \left(\frac{1}{M(fp, fx_n, t)} - 1 \right) - \psi \left(\frac{1}{M(fp, fx_n, t)} - 1 \right) \end{aligned}$$

for every $t > 0$. Taking limit as $n \rightarrow \infty$, we obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} M(Tp, fx_{n+1}, t) &= \lim_{n \rightarrow \infty} M(Tp, Tx_n, t) \\ &= M(Tp, fp, t) = 1. \end{aligned}$$

That is, $fp = Tp$. □

The following example demonstrates Theorem 2.2.

Example 2.3. Consider $X = [0, 1]$ equipped with a minimum norm $*$. Let M be a fuzzy metric defined by

$$M(x, y, t) = \frac{t}{t + d(x, y)}, \text{ for all } x, y \in X, t > 0.$$

Define $\psi : [0, \infty) \rightarrow [0, \infty)$ as $\psi(t) = \frac{1}{c}t$, $Tx = ax$, $a \neq 0$ and $fx = b + cx$, $c > 0$, $b \neq 0$, $1, a \neq b$ and $c - 1 \geq a$. Now

$$\begin{aligned} \left(\frac{1}{M(fx, fy, t)} - 1\right) - \psi\left(\frac{1}{M(fx, fy, t)} - 1\right) &= (c - 1)\frac{|x - y|}{t} \\ &\geq a\left(\frac{|x - y|}{t}\right) \\ &= \left(\frac{1}{M(Tx, Ty, t)} - 1\right) \end{aligned}$$

Therefore T satisfies all the conditions of Theorem 2.2. Moreover f and T have a coincidence point which is not a common fixed point. Notice that this example also demonstrate the necessity of weak compatibility to ensure the existence of common fixed point.

Corollary 2.4. *Let $(X, M, *)$ be a G -complete fuzzy metric space and $f : X \rightarrow X$ be a ψ -weak contraction. If ψ is continuous then f has a unique fixed point.*

If $\psi(r) = (1 - k)r$, ($r > 0$) in Corollary 2.4 above, then we obtain the following result in [9] as a corollary.

Corollary 2.5. [9] *Let $(X, M, *)$ be a fuzzy metric space. $f : X \rightarrow X$ be a mapping satisfying*

$$\frac{1}{M(fx, fy, t)} - 1 \leq k \left(\frac{1}{M(x, y, t)} - 1 \right)$$

for each $x, y \in X$, $t > 0$ and $k \in (0, 1)$. Then f has a unique fixed point.

Theorem 2.6. *Let $(X, M, *)$ be a fuzzy metric space and $T : X \rightarrow X$ be a ψ -weak contraction with respect to self mapping f on X . If the range of f contains the range of T and $f(X)$ is a complete subspace of X , then f and T have a common fixed point in X provided that ψ is a continuous and the pair of mappings (T, f) is weakly compatible.*

Proof. By Theorem 2.2, we obtain a point p in X such that $Tp = fp = q$ (say) which further implies $fTp = Tfp$. Obviously, $Tq = fq$. Now we show that $fq = q$.

If not, then

$$\begin{aligned} \frac{1}{M(fq, q, t)} - 1 &= \frac{1}{M(Tq, Tp, t)} - 1 \\ &\leq \left(\frac{1}{M(fq, fp, t)} - 1 \right) - \psi \left(\frac{1}{M(fq, fp, t)} - 1 \right) \\ &= \left(\frac{1}{M(fq, q, t)} - 1 \right) - \psi \left(\frac{1}{M(fq, q, t)} - 1 \right), \end{aligned}$$

a contradiction which proves the result. \square

Example 2.7. Let $X = [0, 1]$ and $a * b = \min\{a, b\}$. Let M be the standard fuzzy metric induced by d , where $d(x, y) = |x - y|$ for $x, y \in X$. Then $(X, M, *)$ is a complete fuzzy metric space. Let

$$fx = \begin{cases} \frac{1}{2}(1-x), & x \in [0, \frac{1}{2}) \cup (\frac{1}{2}, 1] \\ \frac{3}{4}, & x = \frac{1}{2} \end{cases}$$

and $Tx = \frac{1}{3}$ for all $x \in [0, 1]$.

The pair $\{f, T\}$ satisfies all the conditions of Theorem 2.6. Moreover f and T have $1/3$ as a point of coincidence which also turns out to be their common fixed point.

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