ON LOCAL HUDETZ g-ENTROPY

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Abstract. In this paper, a local approach to the concept of Hudetz g-entropy is presented. The introduced concept is stated in terms of Hudetz g-entropy. This representation is based on the concept of g-ergodic decomposition which is a result of the Choquet's representation Theorem for compact convex metrizable subsets of locally convex spaces.

1. Introduction

The concept of g-entropy of a dynamical system [11, 16, 19] is a generalization of the fuzzy entropy of a system [1, 2, 3, 4, 7, 8, 9, 10, 11, 15, 16, 18, 19], where $g : [0, \infty] \to [0, \infty]$ is an increasing bijective map such that $g(0) = 0$ and $g(1) = 1$. A local approach to the concept of g-entropy is given in [14]. It is based on the framework presented in [12]. The case $g(x) = x$ results in the entropy in the sense of Dumitrescu. The Dumitrescu entropy has the following defect: If the σ-algebra of the fuzzy sets contains all constant functions then the entropy equals to infinity. To eliminate this defect, the Hudetz entropy, as a correction to the concept of entropy, is introduced [5, 6]. The general case, Hudetz correction of g-entropy, called *Hudetz g-entropy*, is discussed in [17].

This paper is an attempt to present a local approach to the Hudetz g-entropy, applying the g-ergodic decomposition, discussed in [14].

Section 2 is devoted to recall Hudetz g-entropy and g-ergodic decomposition. In section 3, we introduce a new type of Hudetz g-entropy via a local approach. The main theorem of the paper represents this new quantity in terms of the classical Hudetz g-entropy [17].

2. Hudetz g-entropy and g-ergodic Decomposition

A family $\mathcal{F} \subset [0, 1]^X$ of fuzzy subsets of a set X is said to be a fuzzy $\sigma-$algebra, if the following axioms are satisfied:

(i) $1_X \in \mathcal{F}$.
(ii) If $f, g \in \mathcal{F}$ then $f.g \in \mathcal{F}$ and $(f-g)^+ \in \mathcal{F}$ where $(f-g)^+(x) := max\{(f-g)(x), 0\}$.
(iii) If $\{f_n\}_{n\geq 1} \subset \mathcal{F}$ then $\bigvee_{n=1}^{\infty} f_n \in \mathcal{F}$ where $\bigvee_{n=1}^{\infty} f_n := min\{\sum_{n=1}^{\infty} f_n, 1\}$.

A function $m : \mathcal{F} \to [0, \infty)$ is called a fuzzy measure, if

(i) $m(0_X) = 0$.

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(ii) $m(\bigvee_{n=1}^{\infty} f_n) = \sum_{n=1}^{\infty} m(f_n)$, whenever $f_n \in \mathcal{F}$ and $\sum_{n=1}^{\infty} f_n \leq 1$.

Let $g : [0,1] \to [0,1]$ be an increasing map such that $g(0) = 0$ and $g(1) = 1$. A family $\xi = \{f_1, f_2, ..., f_k\}$ of members of $\mathcal{F}$ is a g-fuzzy partition of X, if $\sum_{i=1}^{k} g \circ f_i = 1$ on X. When $g(x) = x$, a g-fuzzy partition is nothing but a fuzzy partition, i.e., a family $\xi = \{f_1, f_2, ..., f_k\}$ such that $\sum_{i=1}^{k} f_i = 1$ on X.

For $a, b \in [0,1]$ the following operations are defined:

$$a \oplus b := g^{-1}(g(a) + g(b)) \tag{1}$$

$$a \odot b := g^{-1}(g(a).g(b)) \tag{2}$$

and

$$a \ominus b := g^{-1}(g(a) - g(b)) \tag{3}$$

whenever $b \leq a$.

Note that, in (1), $\oplus$ is a partial operation on $[0,1]$, i.e., $a \oplus b$ is defined if $g(a) + g(b) \leq 1$.

Let $m : \mathcal{F} \to [0,1]$ be a g-decomposable measure on a fuzzy σ-algebra $\mathcal{F}$, i.e., $m(1_X) = 1$, $m(0_X) = 0$ and

$$m\left(g^{-1}(\sum_{n=1}^{\infty} g \circ f_n)\right) = g^{-1}\left(\sum_{n=1}^{\infty} g(m(f_n))\right)$$

whenever $f_n \in \mathcal{F}$ $(n = 1, 2, 3, ...)$ are such that $\sum_{n=1}^{\infty} g \circ f_n \leq 1$. Then $m^* := g \circ m \circ g^{-1}$ is a fuzzy measure on $\mathcal{F}$. Also, if $\mathfrak{B} := \{A \subset X : \chi_A \in \mathcal{F}\}$ and $\mu_{m^*} : \mathfrak{B} \to \mathbb{R}$ is defined by $\mu_{m^*}(A) := m^*(\chi_A)$, then μ_{m^*} is a measure on the σ-algebra $\mathfrak{B}$ such that $m^*(f) = \int_X f d\mu_{m^*}$. So, there is a correspondence $m \longleftrightarrow m^* \longleftrightarrow \mu_{m^*}$ between the g-decomposable measures, fuzzy measures and probability measures on X.

For a g-fuzzy partition $\xi = \{f_1, f_2, ..., f_k\}$, the entropy $H_{m,g}(\xi)$ is defined by

$$H_{m,g}(\xi) := \bigoplus_{i=1}^{k} \Phi(m(f_i)) \tag{4}$$

where $\Phi = g^{-1} \circ \phi \circ g$ and $\phi(x) = -x \log x$ for $x > 0$, $\phi(0) = 0$.

One may write (4) in detail as follows:

$$\begin{aligned} H_{m,g}(\xi) &= g^{-1}\left(\sum_{i=1}^{k} g\left(g^{-1} \circ \phi \circ g\right)(m(f_i))\right) \\ &= g^{-1}\left(\sum_{i=1}^{k} \phi(g(m(f_i)))\right) \\ &= g^{-1}\left(-\sum_{i=1}^{k} g(m(f_i)) \log g(m(f_i))\right) \end{aligned}$$

where we set $0 \times \infty = 0$ if $m(f_i) = 0$.

We also have the following quantity:

$$F_{m,g}(\xi) := m(\bigoplus_{i=1}^{k} \Phi(f_i)) \tag{5}$$

Similarly, if one writes (5) in detail then

$$\begin{aligned} F_{m,g}(\xi) &= m\left(g^{-1}\left(\sum_{i=1}^{k} g(\Phi(f_i))\right)\right) \\ &= m\left(g^{-1}\left(\sum_{i=1}^{k} g(g^{-1}\circ\phi\circ g)(f_i)\right)\right) \\ &= m\left(g^{-1}\left(\sum_{i=1}^{k} \phi\circ g\circ f_i\right)\right) \\ &= m\left(g^{-1}\left(-\sum_{i=1}^{k}(g\circ f_i)\log(g\circ f_i)\right)\right) \end{aligned}$$

Now, set

$$H^b_{m,g}(\xi) := H_{m,g}(\xi) \ominus F_{m,g}(\xi).$$

If $T : X \to X$ is a dynamical system preserving μ_{m^*} then

$$h^b_{m,g}(T,\xi) := \lim_{n\to\infty} g^{-1}\left(\frac{1}{n}\right) \odot H^b_{m,g}\left(\bigvee_{i=0}^{n-1} T^{-i}(\xi)\right).$$

Finally, the *Hudetz g-entropy of* T is defined as follows:

$$h^b_{m,g}(T) := \sup_{\xi} h^b_{m,g}(T,\xi)$$

where the supremum is taken over all g-fuzzy partitions.

Suppose that $T : X \to X$ is a continuous map on a compact metric space X, $\mathcal{F} \subset [0,1]^X$ is the family of all Borel measurable maps $f : X \to [0,1]$. Then the corresponding σ-algebra $\mathfrak{B} = \{A \subset X : \chi_A \in \mathcal{F}\}$ is indeed the σ-algebra of Borel sets of X. Let $M^*(X)$ be the set of all fuzzy set measures $m : \mathcal{F} \to [0,\infty]$ satisfying $m(1_X) = 1$ and $m(0_X) = 0$. The set of g-invariant measures of T is defined as the set

$$M^*_g(X,T) := \{m \in M^*(X); \quad m(g^{-1}\circ f\circ T) = m(g^{-1}\circ f), \quad f \in \mathcal{F}\}$$

and the set of g-ergodic measures of T is defined as the set

$$E^*_g(X,T) := \{m \in M^*_g(X,T); \quad f\circ T = f \Rightarrow m(g^{-1}\circ f) \in \{0,1\}\}.$$

In the following $M^*(X)$ is equipped to a topology.

Definition 2.1. The w*-topology on $M^*(X)$ is the smallest topology making each of the maps $m^* \mapsto \int_X f d\mu_{m^*}$ $(f \in C(X))$ continuous. A basis is given by the collection of all sets of the form

$$V_{m_0^*}(f_1,...,f_k;\epsilon) = \{m^* \in M^*(X) : \left|\int_X f_i d\mu_{m^*} - \int_X f_i d\mu_{m_0^*}\right| < \epsilon, \ 1 \le i \le k\}$$

where $m_0^* \in M^*(X)$, $k \ge 1$, $f_i \in C(X)$ and $\epsilon > 0$.

In [14], it is shown that, with the previous w^*-topology, $M_g^*(X,T)$ is a compact convex subset of $M^*(X)$ and $ext(M_g^*(X,T)) = E_g^*(X,T)$. We also have the following corollary which is a generalized form of the decomposition applied in [12, 13].

Corollary 2.2. *([14] Corollary 3.6) For any $m \in M_g^*(X,T)$ there exists a unique probability measure τ on the σ-algebra of all Borel subsets of the compact metrizable space $M_g^*(X,T)$ such that $\tau(E_g^*(X,T)) = 1$ and*

$$\int_X f(x)d\mu_{m^*}(x) = \int_{E_g^*(X,T)} \left(\int_X f(x)d\mu_{\nu^*}(x)\right) d\tau(\nu)$$

for every bounded measurable function $f : X \to \mathbb{R}$.

In particular, if $f \in \mathcal{F}$ then

$$m(f) = g^{-1}\left(\int_{E_g^*(X,T)} g(\nu(f))d\tau(\nu)\right).$$

Under the assumptions of Corollary 2.2 we write $m = \int_{E_g^*(X,T)} \nu d\tau(\nu)$ and it is called the *g-ergodic decomposition* of m.

3. Local Hudetz g-entropy

In this section, we assume that $T : X \to X$ is a continuous map on a compact metric space X. Let $\mathcal{F} \subset [0,1]^X$ be the family of all Borel measurable maps $f : X \to [0,1]$. Then the corresponding $\sigma-$algebra $\mathfrak{B} := \{A \subset X : \chi_A \in \mathcal{F}\}$ is indeed the $\sigma-$algebra of Borel sets of X. For any $x \in X$ and $f \in \mathcal{F}$ define

$$\omega_g(T,x,f) := g^{-1}\left(\liminf_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1}((g \circ f) \circ T^i)(x)\right).$$

We write $\omega(T,x,f)$ instead of $\omega_g(T,x,f)$ when $g(x) = x$.

Definition 3.1. If $\xi = \{h_1, h_2, ..., h_m\}$ is a g-fuzzy partition, then we define

$$\Omega_g^*(T,x,\xi) := \omega_g(T,x,\bigoplus_{i=1}^{k}\Phi(h_i)) = \omega_g(T,x,g^{-1}(\sum_{i=1}^{k}\phi(g(h_i)))).$$

We also define

$$\Omega_g(T,x,\xi) := g^{-1}\left(\sum_{i=1}^{k} g(\Phi(\omega_g(T,x,h_i)))\right) = g^{-1}\left(\sum_{i=1}^{k} \phi(g(\omega_g(T,x,h_i)))\right).$$

We write $\Omega^*(T,x,\xi)$ and $\Omega(T,x,\xi)$ instead of $\Omega_g^*(T,x,\xi)$ and $\Omega_g(T,x,\xi)$ respectively when $g(x) = x$. A direct application of Definition 3.1 will result the following lemma.

Lemma 3.2. *For $x \in X$ and g-fuzzy partition ξ we have:*

(i) $\Omega_g^*(T,x,\xi) = g^{-1}(\Omega^*(T,x,g(\xi)))$;
(ii) $\Omega_g(T,x,\xi) = g^{-1}(\Omega(T,x,g(\xi)))$.

Definition 3.3. If ξ is a g-fuzzy partition, define

$$\begin{aligned}\Omega_g^b(T,x,\xi) &:= \Omega_g(T,x,\xi)\ominus\Omega_g^*(T,x,\xi)\\ &= g^{-1}\left(g(\Omega_g(T,x,\xi))-g(\Omega_g^*(T,x,\xi))\right).\end{aligned}$$

Again we write $\Omega^b(T,x,\xi)$ instead of $\Omega_g^b(T,x,\xi)$ when $g(x)=x$.

To guarantee that Definition 3.3 is meaningful, we prove the following lemma:

Lemma 3.4. *Under the previous assumptions* $\Omega_g^*(T,x,\xi)\le\Omega_g(T,x,\xi)$.

Proof. Let $\xi=\{h_1,h_2,...,h_m\}$. Since g is increasing, by Lemma 3.2, it is enough to show that $\Omega^*(T,x,\xi)\le\Omega(T,x,\xi)$. We have

$$\begin{aligned}\Omega^*(T,x,\xi) &= \omega(T,x,\sum_{i=1}^m\phi(g\circ h_i))\\ &= \liminf_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\left(\sum_{i=1}^m\left(\phi(g\circ h_i)\circ T^k\right)(x)\right)\\ &= \liminf_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\sum_{i=1}^m\phi(g(h_i(T^k(x))))\\ &= \sum_{i=1}^m\liminf_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\phi(g(h_i(T^k(x))))\\ &= \sum_{i=1}^m\liminf_{n\to\infty}\sum_{k=0}^{n-1}\frac{1}{n}\phi(g(h_i(T^k(x))))\\ &\le \sum_{i=1}^m\liminf_{n\to\infty}\phi\left(\frac{1}{n}\sum_{k=0}^{n-1}((g\circ h_i)\circ T^k)(x)\right)\\ &\le \sum_{i=1}^m\phi\left(\liminf_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}((g\circ h_i)\circ T^k)(x)\right)\\ &= \sum_{i=1}^m\phi(\omega(T,x,g\circ h_i))\\ &= \Omega(T,x,g(\xi)).\end{aligned}$$

We used concavity and continuity of the function ϕ. □

Lemma 3.5. *For $x\in X$ and g-fuzzy partition ξ we have*

$$\Omega_g^b(T,x,\xi)=g^{-1}(\Omega^b(T,x,g(\xi))).$$

Proof. Apply Definition 3.3 and Lemma 3.2. □

Definition 3.6. For $x\in X$ and g-fuzzy partition ξ, the local Hudetz g-entropy of T with respect to ξ is defined as follows:

$$\begin{aligned}\mathcal{H}_g^b(T,x,\xi) &:= \liminf_{n\to\infty}g^{-1}(\frac{1}{n})\odot\Omega_g^b(T,x,\vee_{i=0}^{n-1}T^{-i}\xi)\\ &= \liminf_{n\to\infty}g^{-1}\left(\frac{1}{n}g\left(\Omega_g^b(T,x,\vee_{i=0}^{n-1}T^{-i}\xi)\right)\right).\end{aligned}$$

Definition 3.7. For $m \in M_g^*(X,T)$, the average Hudetz g-entropy of T with respect to ξ is defined as follows:

$$h_{m,g}^{*b}(T,\xi) := \int_X \mathcal{H}_g^b(T,x,\xi)d\mu_m(x).$$

Finally, the average Hudetz g-entropy of T is defined as

$$h_{m,g}^{*b}(T) := \sup_{\xi} h_{m,g}^{*b}(T,\xi)$$

where the supremum is taken over all g-fuzzy partitions.

The following theorem states the average Hudetz g-entropy in terms of the Hudetz g-entropy.

Theorem 3.8. *Suppose that $T : X \to X$ is a continuous map on a compact metric space X and $\mathcal{F} \subset [0,1]^X$ is the σ-algebra of Borel measurable maps $f : X \to [0,1]$. If $m \in M_g^*(X,T)$ and $m = \int_{E_g^*(X,T)} \nu d\tau(\nu)$ is the g-ergodic decomposition of m, then the following properties are satisfied:*

(i) *if ξ is a g-fuzzy partition then*

$$h_{m,g}^{*b}(T,\xi) = \int_{E_g^*(X,T)} h_{\nu,g}^b(T,\xi)d\tau(\nu);$$

(ii) *if $E_g^*(X,T)$ is countable then*

$$h_{m,g}^{*b}(T) = \int_{E_g^*(X,T)} h_{\nu,g}^b(T)d\tau(\nu).$$

Proof. Let $\xi = \{f_1, f_2, ..., f_k\}$. First let $m \in E_g^*(X,T)$. Then $\mu_{m^*} \in E(X,T)$ and by Birkhoff ergodic Theorem we obtain

$$\begin{aligned}
\Omega_g^*(T,x,\xi) &= \omega_g(T,x,\bigoplus_{i=1}^k \Phi(f_i)) \\
&= \omega_g\left(T,x,g^{-1}\left(\sum_{i=1}^k g(\Phi(f_i))\right)\right) \\
&= \omega_g\left(T,x,g^{-1}\left(\sum_{i=1}^k \phi(g\circ f_i)\right)\right) \\
&= g^{-1}\left(\liminf_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1} g\circ g^{-1}(\sum_{i=1}^k \phi(g\circ f_i)\circ T^j(x))\right) \\
&= g^{-1}\left(\liminf_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\left(\sum_{i=1}^k \phi(g\circ f_i)\circ T^j(x)\right)\right) \\
&= g^{-1}\left(\int_X \sum_{i=1}^k \phi(g\circ f_i)d\mu_{m^*}\right) \qquad \mu_{m^*}.a.e. \\
&= g^{-1}\left(m^*(\sum_{i=1}^k \phi\circ g\circ f_i)\right) \\
&= m\left(g^{-1}\left(\sum_{i=1}^k g(\Phi(f_i))\right)\right) \\
&= m\left(\bigoplus_{i=1}^k \Phi(f_i)\right) \\
&= F_{m,g}(\xi).
\end{aligned}$$

A similar justification will show that $\Omega_g(T,x,\xi) = H_{m,g}(\xi) \quad \mu_{m^*}.a.e.$ Therefore,

$$\begin{aligned}
\Omega_g^b(T,x,\xi) &= g^{-1}\left(g\left(\Omega_g(T,x,\xi)\right) - g\left(\Omega_g^*(T,x,\xi)\right)\right) \\
&= g^{-1}\left(g\left(H_{m,g}(\xi)\right) - g\left(F_{m,g}(\xi)\right)\right) \qquad \mu_{m^*}.a.e. \\
&= H_{m,g}(\xi).
\end{aligned}$$

Now, it is easy to see that

$$\mathcal{H}_g^b(T,x,\xi) = h_{m,g}^b(T,\xi) \quad \mu_{m^*}.a.e.$$

and so,

$$h_{m,g}^{*b}(T,\xi) = \int_X \mathcal{H}_g^b(T,x,\xi) d\mu_{m^*}(x) = h_{m,g}^b(T,\xi).$$

Since the previous relation holds for any given g-fuzzy partition ξ, we have

$$h_{m,g}^{*b}(T) = h_{m,g}^b(T).$$

Now, let in general, $m \in M_g^*(X,T)$. Define $f_n := \min\{\mathcal{H}_g^b(T,\cdot,\xi), n\}$, $n = 1,2,....$ Then the sequence $\{f_n\}_{n\geq 1}$ is an increasing sequence of bounded measurable functions such that $f_n \nearrow \mathcal{H}_g^b(T,\cdot,\xi)$. Applying Monotone Convergence Theorem we will have

$$\begin{aligned}
h_{m,g}^{*b}(T,\xi) &= \int_X \mathcal{H}_g^b(T,x,\xi) g\mu_{m^*}(x) \\
&= \lim_{n\to\infty} \int_X f_n(x) d\mu_{m^*}(x) \\
&= \lim_{n\to\infty} \int_{E_g^*(X,T)} \left(\int_X f_n(x) d\mu_{\nu^*}(x)\right) d\tau_(\nu) \\
&= \int_{E_g^*(X,T)} \left(\int_X \mathcal{H}_g^b(T,x,\xi) d\mu_{\nu^*}(x)\right) d\tau_(\nu) \\
&= \int_{E_g^*(X,T)} h_{\nu,g}^{*b}(T,\xi) d\tau(\nu) \\
&= \int_{E_g^*(X,T)} h_{\nu,g}^{b}(T,\xi) d\tau(\nu)
\end{aligned}$$

Now, let $E_g^*(X,T) = \{\nu_1, \nu_2, ..., \nu_n, ...\}$. Put $K_n := \{\nu_1, \nu_2, ..., \nu_n\}$, $n = 1,2,....$ Then it is easy to see that the sequence $\{K_n\}_{n\geq 1}$ is an increasing sequence of finite sets such that $E_g^*(X,T) = \bigcup_{n=1}^{\infty} K_n$.

For any given g-fuzzy partition ξ we have

$$h_{m,g}^{*b}(T,\xi) = \int_{E_g^*(X,T)} h_{\nu,g}^b(T,\xi) d\tau(\nu) \leq \int_{E_g^*(X,T)} h_{\nu,g}^b(T) d\tau(\nu).$$

Taking supremum over all g-fuzzy partitions we will have

$$h_{m,g}^{*b}(T) \leq \int_{E_g^*(X,T)} h_{\nu,g}^b(T) d\tau(\nu).$$

On the other hand, for any g-fuzzy partition ξ and $n \in \mathbb{N}$ we have

$$\begin{aligned} h^{*b}_{m,g}(T) &\geq h^{*b}_{m,g}(T,\xi) \\ &= \int_{E^*_g(X,T)} h^b_{\nu,g}(T,\xi)d\tau(\nu) \\ &\geq \int_{K_n} h^b_{\nu,g}(T,\xi)d\tau(\nu) \\ &= \sum_{j=1}^{n} h^b_{\nu_j,g}(T,\xi)\tau(\nu_j) \end{aligned}$$

Since all of the terms in the previous summation are positive, taking supremum over all g-fuzzy partitions ξ we will have

$$\begin{aligned} h^{*b}_{m,g}(T) &\geq \sup_{\xi} \sum_{j=1}^{n} h^b_{\nu_j,g}(T,\xi)\tau(\nu_j) \\ &= \sum_{j=1}^{n} \sup_{\xi} h^b_{\nu_j,g}(T,\xi)\tau(\nu_j) \\ &= \sum_{j=1}^{n} h^b_{\nu_j,g}(T)\tau(\nu_j) \\ &= \int_{K_n} h^b_{\nu,g}(T)d\tau(\nu) \end{aligned}$$

for all $n \in \mathbb{N}$. Finally,

$$\int_{E^*_g(X,T)} h^b_{\nu,g}(T)d\tau(\nu) = \int_{\bigcup_{n=1}^{\infty} K_n} h^b_{\nu,g}(T)d\tau(\nu) = \lim_{n\to\infty} \int_{K_n} h^b_{\nu,g}(T)d\tau(\nu) \leq h^{*b}_{m,g}(T).$$

It completes the proof. □

4. **Summary and Conclusions**

This paper is an attempt to present a local study of the Hudetz g-entropy of a dynamical system. Applying the framework constructed in [14], which resulted in g-ergodic decomposition, we introduced a new type of Hudetz g-entropy via a local approach and represented it in terms of the known Hudetz g-entropy.

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