PROBABILISTIC NORMED GROUPS

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Abstract. In this paper, we introduce the probabilistic normed groups. Among other results, we investigate the continuity of inner automorphisms of a group and the continuity of left and right shifts in probabilistic group-norm. We also study midconvex functions defined on probabilistic normed groups and give some results about locally boundedness of such functions.

1. Introduction and Preliminaries

Let G be a group with identity element e. A function $\|\cdot\| : G \to \mathbb{R}$ is called a group-norm if the following hold for all $x, y \in G$:
(i) $\|x\| \geq 0$ and $\|x\| = 0$ iff $x = e$ (Positivity);
(ii) $\|xy\| \leq \|x\| + \|y\|$ (Triangle inequality);
(iii) $\|x\| = \|x^{-1}\|$ (Symmetry).
Then G equipped with a group-norm $\|\cdot\|$ is said to be a normed group.

Group-norms have played a role in topological groups [2, 4, 11]. The Birkhoff-Kakutani's metrization theorem for groups states that each first-countable Hausdorf group has a right invariant metric [3, 6]. The term group-norm probably first appeared in Pettis's paper in 1950 [12]. In fact, the groups equipped with a group-norm are those which carry a right invariant metric. Invariant metrics on groups were studied by V. L. Klee in 1950 [7].

One of the advances in the setting of measurement was the concept of an abstract metric space, introduced by Fréchet in 1906 [5]. The advent of quantum mechanics showed that the uncertainties of measurements are somehow inherent and, in essence, cannot be removed from the measurement process. According to the existence of such inherent uncertainties, Menger [10] proposed a probabilistic generalization of the theory of metric spaces.

In this paper, we aim to introduce a probabilistic counterpart of group-norms according to the idea given by Menger. In fact, we look upon the group-norm concept as a probabilistic rather than determinate one. The idea is to replace nonnegative real numbers by distribution functions. We also follow Šerstnev's approach to the probabilistic normed spaces (see [14]) and use triangle functions in defining probabilistic group-norms. For more details on probabilistic metric spaces and triangle functions the reader is referred to [13].

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This paper is organized as follows. In Section 2 we introduce probabilistic normed groups. We give results on probabilistic group-norms and their relations with invariant probabilistic metrics defined on groups. We will also investigate the continuity, in probabilistic group-norm, of inner automorphisms of a group and the continuity of left and right shifts. In Section 3 we study midconvex functions defined on probabilistic normed groups and give some results about locally boundedness of such functions.

We start with some notions which will be needed in this paper.
A distribution function is a function $F : [-\infty, +\infty] \to [0, 1]$ which is nondecreasing and left-continuous with $F(-\infty) = 0$, $F(+\infty) = 1$. The set of all distribution functions will be denoted by Δ. A subset of Δ containing of all $F \in \Delta$ such that $F(0) = 0$ will be denoted by Δ^+. For $F, G \in \Delta^+$, the relation $F \leq G$ is meant by $F(x) \leq G(x)$, for all $x \in \mathbb{R}$. The maximal element for Δ^+ w.r.t. $\leq$ is the distribution function $\mathcal{H}_0$ where $\mathcal{H}_a$ for all $a \in \mathbb{R}$ is given by

$$\mathcal{H}_a(x) = \begin{cases} 0, & \text{if } x \leq a, \\ 1, & \text{if } x > a. \end{cases}$$

Also $\mathcal{H}_\infty$ is defined by

$$\mathcal{H}_\infty(x) = \begin{cases} 0, & \text{if } x < \infty, \\ 1, & \text{if } x = \infty. \end{cases}$$

We assume that Δ is metrized by the modified Lévy metric d_L (see [13, 15]). Indeed, let F and G be two distribution functions in Δ, $h \in (0, 1]$, and let $(F, G; h)$ denote the condition

$$F(x - h) - h \leq G(x) \leq F(x + h) + h, \ x \in (-\frac{1}{h}, \frac{1}{h}].$$

Modified Lévy metric $d_L(F, G)$ on $\Delta \times \Delta$ is defined by

$$d_L(F, G) = \inf\{h| \text{ both } (F, G; h) \text{ and } (G, F; h) \text{ hold}\}.$$

A triangular norm (briefly t-norm) is a binary function $T : [0, 1] \times [0, 1] \to [0, 1]$ which is associative, commutative, nondecreasing in each place and $T(a, 1) = a$, for all $a \in [0, 1]$. A triangle function is a function $\tau : \Delta^+ \times \Delta^+ \to \Delta^+$ such that the following conditions are satisfied for all $F, G, H \in \Delta^+$:

$$\begin{aligned} \tau(\tau(F, G), H) &= \tau(F, \tau(G, H)), \\ \tau(F, G) &= \tau(G, F), \\ F \leq G &\Rightarrow \tau(F, H) \leq \tau(G, H), \\ \tau(F, \mathcal{H}_0) &= F. \end{aligned}$$

It means that for all $F, G, H \in \Delta^+$, τ is associative, commutative, nondecreasing and $\mathcal{H}_0$ plays as a unit [8].

Let $\{F_n\}$ be a sequence in Δ^+. We say that $\{F_n\}$ converges weakly to a distribution function F if and only if the sequence $\{F_n(x)\}$ converges to $F(x)$ at each continuity point x of F. Then we will write $F_n \xrightarrow{w} F$ (see Definition 4.2.4. in [13]). The continuity of a triangle function τ means that if two sequences $\{F_n\}$ and $\{G_n\}$

in Δ^+ converge weakly to distribution functions $F \in \Delta^+$ and $G \in \Delta^+$, respectively, then $\tau(F_n, G_n) \xrightarrow{w} \tau(F, G)$. An example for a continuous triangle function with a left-continuous t-norm T is τ_T defined by

$$\tau_T(F, G)(x) = \sup_{s+t=x} T(F(s), G(t)), \tag{1}$$

for all $F, G \in \Delta^+$ and every $x, s, t \in \mathbb{R}$. Also τ_M, the maximal triangle function, is defined by

$$\tau_M(F, G)(x) = \min\{F(x), G(x)\},$$

for all $F, G \in \Delta^+$ and every $x \in \mathbb{R}$.

Definition 1.1. [1] Let V be a nonempty set, D be a mapping from $V \times V$ into Δ^+ and τ be a triangle function. Let the value of D at a pair (p, q) be denoted by $D_{p,q}$. If the conditions

(PM1) $D_{p,q} = \mathcal{H}_0$ if, and only if, $p = q$;

(PM2) $D_{p,q} = D_{q,p}$;

(PM3) $D_{p,r} \geq \tau(D_{p,q}, D_{q,r})$,

hold for all $p, q, r \in V$, then the triple (V, D, τ) is called a probabilistic metric (or PM-) space.

Definition 1.2. [14] Let V be a real vector space, τ a continuous triangle function, and let ν be a mapping from V into Δ^+. The triple (V, ν, τ) is called a probabilistic normed space if for all p, q in V the following hold:

(PN1) $\nu_p = \mathcal{H}_0$ if and only if $p = \theta$, where θ is the null vector in V;

(PN2) $\nu_{\lambda p}(x) = \nu_p(\frac{x}{|\lambda|})$, for all $\lambda \neq 0$ and $x \in \mathbb{R}$;

(PN3) $\nu_{p+q} \geq \tau(\nu_p, \nu_q)$, for all $p, q \in V$.

2. Probabilistic Normed Groups

In this section we study the probabilistic group-norms and their relations with invariant probabilistic metrics defined on groups. We start with the following definition.

Definition 2.1. Let G be a group with identity element e, τ a continuous triangle function, and let F be a mapping from G into Δ^+. If the following conditions are satisfied:

(PGN1) $F_x = \mathcal{H}_0$ if and only if $x = e$;

(PGN2) $F_{xy} \geq \tau(F_x, F_y)$, whenever $x, y \in G$;

(PGN3) $F_{x^{-1}} = F_x$, where x^{-1} is the inverse element of x,

then F is called a probabilistic group-norm on G and the triple (G, F, τ) is called a probabilistic normed group.

If (PGN1) and (PGN2) hold we will speak of a group probabilistic pre-norm. We say that a probabilistic group-norm is abelian, if $F_{xy} = F_{yx}$ for each $x, y \in G$.

Every probabilistic group-norm F on a group G induces a probabilistic metric $D : G \times G \to \Delta^+$ via $D_{x,y} = F_{xy^{-1}}$ for all $x, y \in G$.

Let (G, F, τ) be a probabilistic normed group. For each x in G and $\lambda > 0$, the strong λ-neighborhood of x is the set

$$N_x(\lambda) \quad = \quad \{y \in G : F_{xy^{-1}}(\lambda) > 1 - \lambda\},$$

and the strong neighborhood system for G is the union $\bigcup_{x \in G} \mathcal{N}_x$, where $\mathcal{N}_x = \{N_x(\lambda) : \lambda > 0\}$. The strong neighborhood system for G determines a Hausdorff topology for G (see Theorem 12.1.2 in [13]).

The following example shows that an ordinary normed group can also be regarded as a probabilistic normed group.

Example 2.2. Let $(G, \|\cdot\|)$ be a normed group and define $F : G \to \Delta^+$ via

$$F_x = \mathcal{H}_{\|x\|},$$

for each $x \in G$. Let τ be a continuous triangle function such that $\tau(\mathcal{H}_a, \mathcal{H}_b) = \mathcal{H}_{a+b}$, for all $a, b \geq 0$ (e.g. the triangle function (1)). Then triple (G, F, τ) is a probabilistic normed group.

Example 2.3. Consider the probabilistic normed group $(\mathbb{R}, F, \tau_T)$, where $F_x = \mathcal{H}_{|x|}$ for all $x \in \mathbb{R}$ and $\mathbb{R}$ is the additive group. For each $n \in \mathbb{N}$, let $f_n : (\mathbb{R}, F, \tau_T) \to (\mathbb{R}, F, \tau_T)$ be given by $f_n(x) = x^n$. We have $F_{x^n} = F_{nx} = \mathcal{H}_{|nx|}{=}H_{|x^n|}$. Then

$$F_{f_n(x)f_n^{-1}(y)}(t) = \mathcal{H}_{|x^n y^{-n}|}(t) = \mathcal{H}_{n|x-y|}(t) = \begin{cases} 0, & \text{if } t \leq n \mid x - y \mid \\ 1, & \text{if } t > n \mid x - y \mid, \end{cases}$$

for all $x, y, t \in \mathbb{R}$. Choosing $0 < c_g \leq n \leq C_g$, we get

$$F_{xy^{-1}}(\frac{t}{C_g}) \leq F_{xy^{-1}}(\frac{t}{n}) \leq F_{xy^{-1}}(\frac{t}{c_g}),$$

for all $x, y, t \in \mathbb{R}$. It implies that

$$F_{f_n(x)f_n^{-1}(y)}(t) = F_{xy^{-1}}(\frac{t}{n}) = \mathcal{H}_{|x-y|}(\frac{t}{n}) = \begin{cases} 0, & \text{if } \frac{t}{n} \leq \mid x - y \mid \\ 1, & \text{if } \frac{t}{n} > \mid x - y \mid, \end{cases}$$

for all $x, y, t \in \mathbb{R}$.

Example 2.4. Let G be the nonabelian dihedral group D_3. A matrix representation of this group is given by

$$r_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad s_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$r_1 = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \qquad s_1 = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

$$r_2 = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \qquad s_2 = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}.$$

The relations

$$r_i r_j = r_{i+j}$$
$$r_i s_j = s_{i+j}$$
$$s_i r_j = s_{i-j}$$
$$s_i s_j = r_{i-j},$$

hold for integers i and j such that $0 \le i, j \le 2$ and both $i+j$ and $i-j$ are computed modulo 3. Note that r_0 is the identity element, $r_i^{-1} = r_{3-i}$ and $s_i^{-1} = s_i$, for each $0 \le i \le 2$.

Now let $F_A = \mathcal{H}_{2-tr(A)}$ for each $A \in D_3$, where $tr(A)$ denote the trace of matrix A. For every $X, Y \in \Delta^+$ and integers $0 \le i, j \le 2$, let

$$\tau(X,Y) = \begin{cases} \mathcal{H}_3, & \text{if } X = F_{s_i} \text{ and } Y = F_{s_j} \\ \tau_M(X,Y), & \text{otherwise.} \end{cases}$$

Then (G, F, τ) is a nonabelian probabilistic normed group with abelian probabilistic group-norm F. In fact, for each $A \in D_3$ we have $F_A = \mathcal{H}_0$ iff $A = r_0$. Because if $A = r_0$, then $F_A = \mathcal{H}_{2-tr(r_0)} = \mathcal{H}_0$. Now suppose that $F_A = \mathcal{H}_0$. It means $\mathcal{H}_{2-tr(A)} = \mathcal{H}_0$ and so $A = r_0$.
To see (PGN3), we have

$$F_{r_0^{-1}} = F_{r_3} = F_{r_0} = \mathcal{H}_0,$$
$$F_{r_1^{-1}} = F_{r_2} = F_{r_1} = F_{r_2^{-1}} = \mathcal{H}_3,$$

and since $s_i = s_i^{-1}$,

$$F_{s_i^{-1}} = F_{s_i},$$

for $0 \le i \le 2$. Finally, for (PGN2) we have

$$F_{r_i r_j} = F_{r_{i+j}} \ge \tau_M(F_{r_i}, F_{r_j}),$$
$$F_{r_i s_j} = F_{s_{i+j}} \ge \tau_M(F_{r_i}, F_{s_j}),$$
$$F_{s_i r_j} = F_{s_{i-j}} \ge \tau_M(F_{s_i}, F_{r_j}),$$
$$F_{s_i s_j} = F_{r_{i-j}} \ge \tau(F_{s_i}, F_{s_j}) = \mathcal{H}_3.$$

Note that F is abelian because

$$F_{r_i r_j} = F_{r_{i+j}} = F_{r_{j+i}} = F_{r_j r_i},$$
$$F_{r_i s_j} = F_{s_{i+j}} = \mathcal{H}_2 = F_{s_j r_i},$$
$$F_{s_i s_j} = F_{r_{i-j}} = \begin{cases} \mathcal{H}_0, & \text{if } i = j \\ \mathcal{H}_3, & \text{if } i \ne j \end{cases} = F_{s_j s_i}.$$

Definition 2.5. Let G be a semigroup and (G, D, τ) be a PM-space. The probabilistic metric D is called left invariant if $D_{gx,gy} = D_{x,y}$ and right invariant if $D_{xg,yg} = D_{x,y}$ whenever $g, x, y \in G$. And D is invariant if it is both right and left invariant.

Theorem 2.6. *Let (G, F, τ) be a probabilistic normed group. Then $D_{x,y} := F_{xy^{-1}}$, is a right invariant probabilistic metric on G. Equivalently, $\tilde{D}_{x,y} := D_{x^{-1},y^{-1}} = F_{x^{-1}y}$ is called the conjugate left invariant probabilistic metric on G. Conversely, if D is a right invariant probabilistic metric on a group G and τ is a continuous triangle function, then (G, F, τ) is a probabilistic normed group, where $F_x := D_{e,x} = \tilde{D}_{e,x}$ for each $x \in G$.*

Consequently, probabilistic metric D is invariant iff $F_{xy^{-1}} = F_{x^{-1}y} = F_{y^{-1}x}$ for each $x, y \in G$ iff the probabilistic group-norm is abelian.

Proof. Suppose that (G, F, τ) is a probabilistic normed group. Put $D_{x,y} = F_{xy^{-1}}$ for all $x, y \in G$. Then $F_{xy^{-1}} = \mathcal{H}_0$ iff $xy^{-1} = e$. It implies that $x = y$. The symmetry property follows from $D_{x,y} = F_{xy^{-1}} = F_{(xy^{-1})^{-1}} = F_{yx^{-1}} = D_{y,x}$. Finally

$$D_{x,y} = F_{xy^{-1}} = F_{xz^{-1}zy^{-1}} \geq \tau(F_{xz^{-1}}, F_{zy^{-1}}) = \tau(D_{x,z}, D_{z,y}).$$

It is obvious that $D_{x,y} = F_{xy^{-1}} = F_{xgg^{-1}y^{-1}} = D_{xg,yg}$, for all $x, y, g \in G$.

Conversely, suppose that D is a right invariant probabilistic metric on group G and τ is a triangle function. Let $F_x = D_{e,x}$, for all $x \in G$. Now $F_x = D_{e,x} = \mathcal{H}_0$ iff $x = e$. Next $F_{x^{-1}} = D_{e,x^{-1}} = D_{x,e} = F_x$ and

$$F_{xy} = D_{xy,e} = D_{x,y^{-1}} \geq \tau(D_{x,e}, D_{e,y^{-1}}) = \tau(F_x, F_y).$$

Consequently, probabilistic metric D is invariant iff $D_{e,yx^{-1}} = D_{x,y} = D_{e,x^{-1}y}$ iff $F_{yx^{-1}} = F_{x^{-1}y}$ and it yields the abelian property of the probabilistic group-norm. □

Remark 2.7. Suppose that G is a group with left invariant probabilistic metric D. Then D is right invariant iff D is invariant under inversion iff D is invariant under every inner automorphism of G. In fact, if D is right invariant, then

$$D_{x^{-1},y^{-1}} = D_{x^{-1}x,y^{-1}x} = D_{e,(x^{-1}y)^{-1}} = D_{e,x^{-1}y} = D_{x,y},$$

for all $x, y \in G$. This means that D is also invariant under inversion.

Now, let D be invariant under inversion. Then we have

$$D_{gxg^{-1},gyg^{-1}} = D_{xg^{-1},yg^{-1}} = D_{gx^{-1},gy^{-1}} = D_{x^{-1},y^{-1}} = D_{x,y},$$

for every $x, y, g \in G$ and it shows that D is invariant under every inner automorphism of G.

Finally, suppose that D is invariant under every inner automorphism of G. Then

$$D_{xg,yg} = D_{gxgg^{-1},gygg^{-1}} = D_{gx,gy} = D_{x,y},$$

for every $x, y, g \in G$.

Remark 2.8. Suppose that G is a semigroup and (G, D, τ) is a probabilistic metric space. Then invariancy of D implies that

$$D_{ab,xy} \geq \tau(D_{a,x}, D_{b,y}), \text{ whenever } \quad \{a, b, x, y\} \subset G. \tag{2}$$

If G is a group, then the invariancy of D is equivalent to (2). To see this, if D is invariant, then

$$D_{ab,xy} \geq \tau(D_{ab,xb}, D_{xb,xy}) = \tau(D_{a,x}, D_{b,y}).$$

If (2) holds and G is a group, then

$$D_{gu,gv} \geq \tau(D_{g,g}, D_{u,v}) = \tau(\mathcal{H}_0, D_{u,v}) = D_{u,v} = D_{g^{-1}gu,g^{-1}gv} \geq D_{gu,gv}.$$

So D is left invariant. Similarly we can also show that D is right invariant.

For a probabilistic normed group (G, F, τ) the above discussion implies that F is abelian iff

$$F_{ab(xy)^{-1}} \geq \tau(F_{ax^{-1}}, F_{by^{-1}}),$$

for all $x, y, a, b \in G$. Equivalently,

$$F_{uabv} \geq \tau(F_{uv}, F_{ab}),$$

for all $x, y, a, b \in G$. Therefore, according to Remark 2.7, a probabilistic group-norm F is invariant under inner automorphisms iff $F_{uabv} \geq \tau(F_{uv}, F_{ab})$, for all $a, b, u, v \in G$.

Remark 2.9. Let (G, F', τ) be a probabilistic pre-normed group. Then the symmetrization refinement

$$F_x := \min\{F'_x, F'_{x^{-1}}\} \qquad (x \in G),$$

is a probabilistic group-norm on G. Clearly, (PGN1) and (PGN2) hold. Note that, for any μ, ν, υ and η in Δ^+,

$$\tau(\mu, \nu) \geq \tau(\min\{\mu, \upsilon\}, \min\{\nu, \eta\}).$$

Without loss of generality suppose that

$$\min\{\tau(F'_x, F'_y), \tau(F'_{x^{-1}}, F'_{y^{-1}})\} = \tau(F'_x, F'_y).$$

We have

$$\begin{aligned} F_{xy} = \min\{F'_{xy}, F'_{y^{-1}x^{-1}}\} &\geq \min\{\tau(F'_x, F'_y), \tau(F'_{x^{-1}}, F'_{y^{-1}})\} \\ &= \tau(F'_x, F'_y) \\ &\geq \tau(\min\{F'_x, F'_{x^{-1}}\}, \min\{F'_y, F'_{y^{-1}}\}) \\ &= \tau(F_x, F_y). \end{aligned}$$

Definition 2.10. Let (G, D, τ) be a probabilistic metric space and $\pi : G \to G$ be a bijection. The π-probabilistic metric is defined by

$$D^{\pi}_{x,y} := D_{(\pi(x), \pi(y))},$$

for each $x, y \in G$.

If G is a group in the probabilistic metric space (G, D, τ) and $\pi : G \to G$ is a bijection, we will write

$$F^{\pi}_x := D_{(\pi(x), \pi(e))}, \qquad (x \in G),$$

and

$$B^{\pi}_r(x) = \{y : D^{\pi}_{x,y}(r) > 1 - r\} \qquad (r > 0).$$

Note that F^{π}_x is not a probabilistic group-norm in general.

The following bijections on G will be used later:

$$\pi(x) = x^{-1} \quad (x \in G)$$

$$\lambda_g(x) = gx \quad (x \in G, g \in G)$$

$$\rho_g(x) = xg \quad (x \in G, g \in G)$$

and the conjugacy map under $g \in G$ (inner automorphism) is defined by

$$\gamma_g(x) := gxg^{-1} \qquad (x \in G).$$

Definition 2.11. Let (G, D, τ) be a probabilistic metric space, $K \subseteq G$, and $\varepsilon > 0$. Then the ε-swelling of K is defined by

$$B_\varepsilon(K) := \{z : D_{z,k}(\varepsilon) > 1 - \varepsilon, \quad \text{for some } k \in K\}.$$

If G is a group and $A, B \subseteq G$, we denote the subset $\{ab : a \in A, b \in B\}$ by AB.

Remark 2.12. Let (G, F, τ) be a probabilistic normed group, $K \subseteq G$ and $\varepsilon > 0$. Then

$$B_\varepsilon(K) = \{wk : k \in K, w \in G, F_w(\varepsilon) > 1 - \varepsilon\} = B_\varepsilon(e)K.$$

In fact, if D is the right invariant probabilistic metric induced by F, $w \in G$, $D_{x,y} = F_{xy^{-1}}$, and $F_w(\varepsilon) > 1 - \varepsilon$, then $D_{wk,k}(\varepsilon) = D_{w,e}(\varepsilon) = F_w(\varepsilon) > 1 - \varepsilon$, for all $k \in K$. This implies that $wk \in B_\varepsilon(K)$. Conversely, if $D_{z,k} = D_{zk^{-1},e} > 1 - \varepsilon$, then putting $w = zk^{-1}$, we have $z = wk \in B_\varepsilon(K)$.

Definition 2.13. A sequence $\{x_k\}$ in a probabilistic normed group (G, F, τ) is called distributionally convergent (briefly $\mathcal{D}$-convergent) to x and denoted by $x_k \to x$ as $k \to \infty$ if for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for every $n > N$ we have

$$F_{x_k x^{-1}}(\varepsilon) > 1 - \varepsilon.$$

As we have seen in Theorem 2.6 every probabilistic normed group (G, F, τ) gives two probabilistic metrics on G. We denote the right invariant probabilistic metric and the conjugate left invariant probabilistic metric by D^R and D^L, respectively. Indeed, $D^R_{x,y} = F_{xy^{-1}}$ and $D^L_{x,y} = D^R_{x^{-1},y^{-1}} = F_{x^{-1}y}$, for all $x, y \in G$. We write $\to_R$ for convergence under D^R and $\to_L$ for convergence under D^L. Note that both probabilistic metrics define the same balls centered in e. That is, $D^L_{x,e} = D^R_{x^{-1},e} = D^R_{e,x^{-1}} = F_x$, for each $x \in G$. Therefore

$$B^R(e, r) := \{x : D_{e,x}(r) > 1 - r\} = B^L(e, r).$$

Now by Remark 2.12 we have

$$B^R(a, r) = \{ya : y \in G \quad D^R_{a,x}(r) = D^R_{e,y}(r) > 1 - r\} = B_r(e)a,$$

$$B^L(a, r) = \{ay : y \in G \quad D^L_{a,x}(r) = D^L_{e,y}(r) > 1 - r\} = aB_r(e).$$

That is, the balls are right- or left-shifts of the probabilistic group-norm balls at the origin. Therefore the right-shift $\rho_a : x \to xa$ on (G, F, τ) is uniformly D^R-continuous because

$$D^R_{xa,ya} = D^R_{x,y},$$

and also the left-shift $\lambda_a : x \to ax$ is uniformly D^L-continuous as

$$D^L_{ax,ay} = D^L_{x,y},$$

for all $x, y \in G$. Particularly, we have $y \to_R b$ iff $yb^{-1} \to_R e$, as $D^R_{e,yb^{-1}} = D^R_{y,b}$. Likewise, we have $x \to_L a$ iff $a^{-1}x \to_L e$, as $D^L_{e,a^{-1}x} = D^L_{x,a}$.

Lemma 2.14. *Let (G, F, τ) be a probabilistic normed group. Then γ_g is D^R-continuous at any point of G iff it is D^R-continuous at e.*

Proof. Let $a \in G$ and γ_g be D^R-continuous at e. Let $x \to_R a$. Then $xa^{-1} \to_R e$ and therefore $\gamma_g(xa^{-1}) \to_R \gamma_g(e)$. That is, $gxa^{-1}g^{-1} \to_R e$ or $gxg^{-1}ga^{-1}g^{-1} \to_R e$. Hence, $\gamma_g(x) \to_R \gamma_g(a)$. Since a was arbitrary, this implies that γ_g is D^R-continuous at any point of G. The converse is trivial. □

Lemma 2.15. *Let (G, F, τ) be a probabilistic normed group and the inversion mapping is D^R-continuous. Then $x \to_R a$ iff $a^{-1}x \to_R e$.*

Proof. Suppose that $x \to_R a$ and the inversion mapping is D^R-continuous. Let $\varepsilon > 0$ be given. There exists $\delta > 0$ if $D^R_{x,a}(\delta) > 1-\delta$, then $D^R_{x^{-1},a^{-1}}(\varepsilon) = D^R_{e,a^{-1}x}(\varepsilon) > 1-\varepsilon$. Conversely, let $a^{-1}x \to_R e$. Since for each $\varepsilon > 0$,

$$D^R_{a^{-1}x,e}(\varepsilon) = D^R_{x^{-1},a^{-1}}(\varepsilon) > 1-\varepsilon,$$

the D^R-continuity of inversion mapping implies $x \to_R a$. □

Theorem 2.16. *In a probabilistic normed group (G, F, τ) the following are equivalent:*

(i) The inversion mapping is D^R-continuous;

(ii) Each D^L-open set is D^R-open;

(iii) For each element $a \in G$, the conjugacy γ_a is D^R-continuous at e;

(iv) Left-shifts are D^R-continuous.

Proof. (i) $\iff$ (ii): Suppose that (i) holds. Let A be any D^L-open set in G and $a \in A$. There exists $\varepsilon > 0$ such that $B^L(a, \varepsilon) \subseteq A$. Since inversion mapping is D^R-continuous, there is $\delta > 0$ such that $D^R_{x,a}(\delta) > 1-\delta$ implies $D^R_{x^{-1},a^{-1}}(\varepsilon) = D^L_{x,a}(\varepsilon) > 1-\varepsilon$, for all $x \in G$. Therefore

$$B_\delta(e)a = B^R(a, \delta) \subseteq B^L(a, \varepsilon) = aB_\varepsilon(e).$$

This shows that a is an D^R-interior point of A. For the converse, let $\varepsilon > 0$. As $B^L(a, \varepsilon)$ is an D^L-open ball, it is D^R-open and so there is $\delta > 0$ such that

$$B^R(a, \delta) \subseteq B^L(a, \varepsilon).$$

If $D^R_{x,a}(\delta) > 1-\delta$, we have $D^R_{x^{-1},a^{-1}}(\varepsilon) = D^L_{x,a}(\varepsilon) > 1-\varepsilon$. That is the inversion mapping is D^R-continuous.

(ii) $\iff$ (iii): Suppose that each D^L-open set is D^R-open. Let $\varepsilon > 0$ and $a \in G$ be given. There exists $\delta > 0$ such that $B_\delta(e)a \subseteq aB_\varepsilon(e)$ and $B_\delta(e)a^{-1} \subseteq a^{-1}B_\varepsilon(e)$. If $x \in \gamma_a(B_\delta(e)) = aB_\delta(e)a^{-1}$, then $x \in B_\varepsilon(e)$. This implies that $\gamma_a(B_\delta(e)) \subseteq B_\varepsilon(e)$. Hence γ_a is D^R-continuous at e and therefore, by Lemma 2.14, it is D^R-continuous for each $x \in G$. Conversely, let $A \subseteq G$ be a D^L-open set and fix $a \in A$. Then there is $\varepsilon_1 > 0$ such that $B^L(a, \varepsilon_1) \subseteq A$. Suppose that the conjugacy γ_a is D^R-continuous at e. Then for given $\varepsilon_1 > 0$ there exists $\delta > 0$ such that $\gamma_a(B_\delta(e)) \subseteq B_{\varepsilon_1}(e)$. It means that $aB_\delta(e) \subseteq B_{\varepsilon_1}(e)a$. Therefore

$$B^R(a, \delta) = aB_\delta(e) \subseteq B_{\varepsilon_1}(e)a = B^L(a, \varepsilon_1) \subseteq A.$$

Thus, A is a D^R-open set.

(iii) $\iff$ (iv): Since the right-shifts are D^R-continuous and $\rho_a \circ \gamma_a = \lambda_a$, for each $a \in G$, then the conjugacy γ_a is D^R-continuous iff λ_a is D^R-continuous. □

Theorem 2.17. *A probabilistic normed group (G, F, τ) is a topological group under the topology induced by the right invariant probabilistic metric D^R iff each conjugacy γ_a is D^R-continuous at e, where $a \in G$.*

Proof. As seen, in Theorem 2.16, the inversion mapping is D^R-continuous iff each conjugacy γ_g is D^R-continuous at e. Let each conjugacy be D^R-continuous at e. We show that multiplication is jointly D^R-continuous, i.e., if $x_n \to_R a$ and $y_n \to_R b$, then $x_n y_n \to_R ab$. Let $z_n = y_n b^{-1}$ and $u_n = a^{-1} x_n$, where $n \in \mathbb{N}$. Thus by the assumption and Theorem 2.16 (iv), we obtain $z_n \to_R e$ and $u_n \to_R e$. Hence $D^R_{u_n z_n, e} = F_{u_n z_n} \geq \tau(F_{u_n}, F_{z_n}) \to \mathcal{H}_0$, as $n \to \infty$. This shows that multiplication is jointly D^R-continuous. □

Theorem 2.18. *Let (G, F, τ) be a probabilistic normed group and set*

$$D^S_{x,y} := \min\{F_{xy^{-1}}, F_{x^{-1}y}\} = \min\{D^R_{x,y}, D^L_{x,y}\},$$

where $x, y \in G$. Then G is a topological group under the topology induced by the right invariant probabilistic metric D^R iff G is a topological group under the symmetrization refinement probabilistic metric D^S.

Proof. Let G be a topological group under the topology induced by the right invariant probabilistic metric D^R. Then the inversion mapping is D^R-continuous. Let $\to_S$ denote the convergence under D^S. By Theorem 2.16 (ii) and Theorem 2.17, if $x_n \to_R x$, then $x_n \to_L x$. Therefore $x_n \to_S x$. Now let $x_n \to_S x$. Since $D^R \geq D^S$, we have $x_n \to_R x$. Thus, D^S generates a topology on G and G is a topological group under D^S.

Conversely, suppose that G is a topological group under D^S. Then the topology is generated by the D^S-neighborhoods of e. Since $D^S_{x,e} = F_x$, so the D^S-neighborhoods of e are also generated by the probabilistic group-norm. Therefore G is a topological group under probabilistic group-norm topology. □

Recall that the function d_L is a metric on Δ^+ and $d_L(F, F_n) \to 0$ if and only if $F_n \xrightarrow{w} F$. We also mention that the metric space (Δ^+, d_L) is complete [13].

Definition 2.19. Let A and C are subgroups of a probabilistic normed group (G, F, τ) with C invariant under A and (B, F', τ') be a probabilistic normed group. We say that a function $h : C \to B$ is distributional-slowly varying on C over A if

$$F_{x_n} \xrightarrow{w} \mathcal{H}_\infty,$$

implies that

$$F_{h(tx_n)h(x_n)^{-1}} \xrightarrow{w} \mathcal{H}_0,$$

for each $t \in A$ and for each sequence $\{x_n\} \in C$.

Example 2.20. Consider the group $(\mathbb{R}, +)$. Let $A = B = C = \mathbb{R}$. Choosing $\tau = \tau_T$ and $F_x = \mathcal{H}_{|x|}$ for each $x \in \mathbb{R}$, then $(\mathbb{R}, F, \tau_T)$ is a probabilistic normed group. The identity function on $\mathbb{R}$ is not distributional-slowly varying. Because let $t > 0$ and sequence $\{x_n\}$ converge to infinity. Then $F_{x_n} \to \mathcal{H}_\infty$ while $F_{(t+x_n)-x_n} = F_t \nrightarrow \mathcal{H}_0$.

Example 2.21. Consider probabilistic group (B, F, τ_T), where B is the multiplicative group $\mathbb{R}_+$ and $F_h = \mathcal{H}_{|\log(h)|}$ for all $h \in B$. Also, consider probabilistic normed group $(\mathbb{R}, F', \tau_T)$, where $F'_x = \mathcal{H}_{|x|}$, for each $x \in \mathbb{R}$. Let $A = C = \mathbb{R}$. The function $f : C \to B$ defined by

$$f(x) = \begin{cases} |x| & \text{if } x \neq 0 \\ 1 & \text{if } x = 0, \end{cases}$$

is distributional-slowly varying on C over A. For $\{x_n\}$ in C with $F_{x_n} \xrightarrow{w} \mathcal{H}_\infty$, and t in A we have $x_n \to +\infty$ and

$$F_{f(t+x_n)f(x_n)^{-1}} = F_{|\frac{t+x_n}{x_n}|} \to \mathcal{H}_{|\log(1)|} = \mathcal{H}_0.$$

Therefore f is distributional-slowly varying on C over A.

Theorem 2.22. *Let (G, F, τ) be a probabilistic normed group such that F is an abelian probabilistic group norm and let $A, B, C \subseteq G$ and C be invariant under A. A function $h : C \to B$ is distributional-slowly varying on C over A if for some distributional-slowly varying function $g : C \to B$ on C over A and for some $\mu \in B$ we have*

$$F_{h(x_n)g(x_n)^{-1}\mu^{-1}} \xrightarrow{w} \mathcal{H}_0, \text{ as } F_{x_n} \xrightarrow{w} \mathcal{H}_\infty, \tag{3}$$

where $\{x_n\}$ is in C.

Proof. Let (3) hold for some distributional-slowly varying function g on C over A and for some $\mu \in B$. Let $t \in A$ and sequence $\{x_n\}$ in C be given such that $F_{x_n} \xrightarrow{w} \mathcal{H}_\infty$. We have

$$\begin{aligned} F_{h(tx_n)h(x_n)^{-1}} &= F_{h(tx_n)g(tx_n)^{-1}\mu^{-1}\mu g(tx_n)g(x_n)^{-1}\mu^{-1}\mu g(x_n)h(x_n)^{-1}} \\ &\geq \tau(F_{h(tx_n)g(tx_n)^{-1}\mu^{-1}}, F_{\mu g(tx_n)g(x_n)^{-1}\mu^{-1}\mu g(x_n)h(x_n)^{-1}}) \\ &\geq \tau(F_{h(tx_n)g(tx_n)^{-1}\mu^{-1}}, \tau(F_{\mu g(tx_n)g(x_n)^{-1}\mu^{-1}}, F_{\mu g(x_n)h(x_n)^{-1}})) \\ &= \tau(F_{h(tx_n)g(tx_n)^{-1}\mu^{-1}}, \tau(F_{\mu g(tx_n)g(x_n)^{-1}\mu^{-1}}, F_{(\mu g(x_n)h(x_n)^{-1})^{-1}})) \\ &= \tau(F_{h(tx_n)g(tx_n)^{-1}\mu^{-1}}, \tau(F_{\mu g(tx_n)g(x_n)^{-1}\mu^{-1}}, F_{h(x_n)g(x_n)^{-1}\mu^{-1}})) \xrightarrow{w} \mathcal{H}_0. \end{aligned}$$

Note that in the above inequalities, we have used the fact that F is abelian, invariant under inner automorphisms and $F_{\mu g(tx_n)g(x_n)^{-1}\mu^{-1}} = F_{g(tx_n)g(x_n)^{-1}}$ for each $n \geq 1$. Since $\mathcal{H}_0$ is maximal element of Δ^+, we have $F_{h(tx_n)h(x_n)^{-1}} \xrightarrow{w} \mathcal{H}_0$. Hence h is distributional-slowly varying function on C over A. □

3. Midconvexity on Probabilistic Normed Groups

In this section, we give some results about midconvex and bounded functions on probabilistic normed groups. The main idea of the proofs is borrowed from [9].

Definition 3.1. [2] Let G be a group. A subset C of G is called midconvex, if for every $x, y \in C$ there exists an element $z \in C$, denoted by $\sqrt{xy}$, such that $z^2 = xy$. A function $h : G \to \mathbb{R}$ is midconvex on the midconvex subset C if

$$h(\sqrt{xy}) \leq \frac{1}{2}(h(x) + h(y)),$$

for all $x, y \in C$.

Theorem 3.2. *Let (G, F, τ) be a probabilistic normed group and f be a real-valued midconvex function on G. If f is locally bounded above at p, then it is locally bounded below at $p^{-1} \in G$. Moreover, if F is an abelian probabilistic group-norm, then f is locally bounded below at p.*

Proof. Suppose that f is bounded above by M in $B(p, \delta) = \{y \in G : F_{py^{-1}}(\delta) > 1 - \delta\}$. Let $u \in \tilde{B}(p, \delta)$. We have $\tilde{D}_{p,u}(\delta) = F_{u^{-1}p}(\delta) > 1 - \delta$, and

$$D_{t,p}(\delta) = F_{tp^{-1}}(\delta) = F_{u^{-1}p}(\delta) = \tilde{D}_{u,p}(\delta) > 1 - \delta,$$

where $t = u^{-1}p^2$. Then $t \in B(p, \delta)$ and since $p^2 = ut$, we have

$$2f(p) \leq f(u) + f(t) \leq f(u) + M,$$

and so

$$f(u) \geq 2f(p) - M.$$

Therefore $2f(p) - M$ is a lower bound for f on the $\tilde{B}(p, \delta) = B(p^{-1}, \delta)$. This shows that f is locally bounded below at p^{-1}. In addition, if F is an abelian probabilistic group-norm, then $\tilde{B}(p, \delta) = B(p, \delta)$ and f is locally bounded below at p. □

Theorem 3.3. *Let (G, F, τ_M) be a probabilistic normed group such that F is an abelian probabilistic group-norm and suppose that f is a real-valued midconvex function on G. If f is locally bounded above at some point $p \in G$, then f is locally bounded above at all points.*

Proof. Suppose that G is an arbitrary group, (G, F, τ_M) is a probabilistic normed group, and f is bounded above by M in $B(p, \delta)$. Fix $t \in G$ and set $z = p^{-1}t^2$. Consider $\beta_t : G \to G$ by $\beta_t(y) = ytyt^{-1}y^{-2}$, for all $y \in G$. Note that β_t is continuous at each $y \in G$. To see this, let y_n be a sequence in G such that $y_n \xrightarrow{F} y$. Then by Remark 2.8 we have

$$\begin{aligned} F_{\beta_t(y_n)(\beta_t(y))^{-1}} &= F_{y_n t y_n t^{-1} y_n^{-2} y^2 t y^{-1} t^{-1} y^{-1}} \\ &\geq \tau_M(F_{y_n t t^{-1} y^{-1}}, F_{y_n t^{-1} y_n^{-2} y^2 t y^{-1}}) \\ &\geq \tau_M(F_{y_n y^{-1}}, \tau_M(F_{y_n t^{-1} t y^{-1}}, F_{y_n^{-2} y^2}) \\ &\geq \tau_M(F_{y_n y^{-1}}, \tau_M(F_{y_n y^{-1}}, \tau_M(F_{y_n^{-1} y}, F_{y_n^{-1} y})). \end{aligned}$$

It shows that $\beta_t(y_n) \xrightarrow{F} \beta_t(y)$, as $y_n \xrightarrow{F} y$. We may suppose that for some $\eta < \delta/2$ we have

$$F_{\beta_t(s)}(\delta) \geq F_{\beta_t(s)}(\delta/2) > 1 - \delta/2 > 1 - \delta,$$

for each $s \in G$, provided that $F_s(\eta) > 1 - \eta$. Now consider any $u \in B(t, r)$, where $r = \min\{\eta, \delta/2\}$. Write $u = st$ with $F_s(r) > 1 - r > 1 - \delta/2$. Put $y = s^2$. Then

$$F_y(\delta) \geq F_y(r) = F_{s^2}(r) \geq \tau_M(F_s, F_s)(r) > 1 - r > 1 - \delta.$$

Since $stst = (stst^{-1}s^{-2})s^2t^2$, we have $u^2 = \beta_t(s)s^2t^2$, and

$$F_{\beta_t(s)y}(\delta) \geq \tau_M(F_{\beta_t(s)}, F_y)(\delta) > 1 - \delta.$$

Therefore, $\beta_t(s)yp \in B(p, \delta)$. Now $u^2 = \beta_t(s)ypz$. Since f is midconvex,

$$f(u) \leq \frac{1}{2}(f(\beta_t(s)yp) + f(z)) \leq \frac{1}{2}(M + f(z)).$$

Because $t \in G$ was arbitrary it shows that f is locally bounded above at each point. $\square$

By Theorem 3.2 and Theorem 3.3 we have the following corollary.

Corollary 3.4. *Let (G, F, τ_M) be a probabilistic normed group such that F is an abelian probabilistic group-norm and let f be a real-valued midconvex function on G. If f is locally bounded above at some point $p \in G$ then f is locally bounded at all points.*

Theorem 3.5. *Suppose that (G, F, τ_M) is a probabilistic normed group such that F is an abelian probabilistic group-norm and f is a midconvex function on G. If f is locally bounded above at p, then f is continuous at p.*

Proof. Let (G, F, τ_M) be a probabilistic normed group. Suppose that f is locally bounded above at $p \in G$. Therefore by Corollary 3.4, f is also locally bounded at each point of G. For $r > 0$ and $x \in G$, we define $\psi_x(r)$ as

$$\psi_x(r) = \inf_{B(x,r)} f.$$

Since $B(x, r_1) \subseteq B(x, r_2)$ as $r_1 < r_2$ we get

$$\psi_x(r_1) = \inf_{B(x,r_1)} f \geq \inf_{B(x,r_2)} f = \psi_x(r_2).$$

This implies that the function ψ_x is decreasing. Define m_f on G by

$$m_f(x) = \lim_{r \to 0^+} \psi_x(r) = \lim_{r \to 0^+} \inf_{B(x,r)} f,$$

for all $x \in G$. Similarly, for $r > 0$ and $x \in G$, define $\mu_x(r)$ as

$$\mu_x(r) = \sup_{B(x,r)} f.$$

It is easily seen that each function μ_x is an increasing function. Again, define M_f on G by

$$M_f(x) = \lim_{r \to 0^+} \mu_x(r) = \lim_{r \to 0^+} \sup_{B(x,r)} f.$$

Note that since f is locally bounded above at $p \in G$, by Theorem 3.4, both functions m_f, M_f are real-valued. We have

$$m_f(p) = \lim_{r \to 0^+} \inf_{B(p,r)} f \leq f(p) \leq \lim_{r \to 0^+} \sup_{B(p,r)} f = M_f(p).$$

For each $n \in \mathbb{N}$ there exists $y_n \in B(p, \frac{1}{n})$ such that

$$\inf_{B(p,\frac{1}{n})} f(y_n) \leq f(y_n) \leq \inf_{B(p,\frac{1}{n})} f(y_n) + \frac{1}{n}.$$

This implies that $y_n \xrightarrow{F} p$ and

$$\lim_{n \to \infty} f(y_n) = m_f(p).$$

Similarly, there exists $z_n \in G$ such that $z_n \xrightarrow{F} p$ and

$$\lim_{n \to \infty} f(z_n) = M_f(p).$$

For each $n \in \mathbb{N}$, let $u_n = y_n^{-1} z_n^2$. Let $\varepsilon > 0$ be given. There exists N such that for all $n > N$,

$$\begin{aligned} F_{u_n p^{-1}}(\varepsilon) = F_{y_n^{-1} z_n^2 p^{-1}}(\varepsilon) &\geq \tau_M(F_{y_n^{-1} z_n}, F_{z_n p^{-1}})(\varepsilon) \\ &= \tau_M(F_{y_n^{-1} p p^{-1} z_n}, F_{z_n p^{-1}})(\varepsilon) \\ &\geq \tau_M(\tau_M(F_{y_n^{-1} p}, F_{p^{-1} z_n}), F_{z_n p^{-1}})(\varepsilon) \\ &> 1 - \varepsilon. \end{aligned}$$

Thus $u_n \xrightarrow{F} p$ and we get

$$\liminf_{n \to +\infty} f(u_n) \leq M_f(p).$$

By midconvexity of f we have

$$2f(z_n) \leq f(y_n) + f(u_n),$$

i.e.,

$$f(u_n) \geq 2f(z_n) - f(y_n).$$

Hence, by letting $n \to +\infty$, we obtain

$$\liminf_{n \to +\infty} f(u_n) \geq 2M_f(p) - m_f(p).$$

Hence $M_f(p) \leq m_f(p)$. Since $m_f(p) \leq f(p) \leq M_f(p)$, therefore $m_f(p) = f(p) = M_f(p)$ and so f is continuous at p. □

4. **Conclusion**

In this paper the theory of probabilistic normed groups has been introduced with the objective of generalizing normed groups and obtaining a probabilistic version of some known results in convex analysis. As we have seen in Example 2.2, each normed group can be considered as a probabilistic normed group. We have introduced the probabilistic normed groups as probabilistic counterparts of normed groups. In particular, we have given results on probabilistic group-norms and their relations with invariant probabilistic metrics. We have also investigated the continuity in probabilistic group-norm of inner automorphisms of a group and the continuity of left and right shifts. The midconvex functions have been studied on probabilistic normed groups and the given results are generalization of some known results. In addition, the locally boundedness of midconvex functions have been investigated in probabilistic group-norm. Our results extend some known results in convex analysis.

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