

STRUCTURAL PROPERTIES OF FUZZY GRAPHS

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ABSTRACT. Matroids are important combinatorial structures and connect closely with graphs. Matroids and graphs were all generalized to fuzzy setting respectively. This paper tries to study connections between fuzzy matroids and fuzzy graphs. For a given fuzzy graph, we first induce a sequence of matroids from a sequence of crisp graph, i.e., cuts of the fuzzy graph. A fuzzy matroid, named graph fuzzy matroid, is then constructed by using the sequence of matroids. An equivalent description of graphic fuzzy matroids is given and their properties of fuzzy bases and fuzzy circuits are studied.

1. Introduction

Fuzzy graph theory generalized Euler's crisp graph theory and was introduced by Rosenfeld [19] in 1975. Fuzzy analogs of some basic concepts in graph theory like paths, cycles, trees and connectedness, etc. were also defined by Rosenfeld. Many authors subsequently contributed to the development of fuzzy graphs. For example, Bhattacharya [1] showed that a fuzzy group can be induced by a fuzzy graph and introduced the notions of eccentricity and center of fuzzy graphs; Morde-son [17] defined different operations on fuzzy graphs and studied properties of these operations; Sunitha and Vijayakumar [21] characterized fuzzy trees; Bhutani and Rosenfeld [2, 3, 4] defined strong arcs, fuzzy end nodes and M-strong fuzzy graphs, and properties of these definitions were studied respectively; Blue et al. [5] presented a taxonomy of fuzzy graphs that treats fuzziness in vertex existence, edge existence, edge connectivity, and edge weight. And meanwhile, fuzzy graph theory also has numerous applications in various fields such as information theory, network analysis, clustering analysis, etc. For a detailed exposition of theoretic and applied aspects of fuzzy graphs, we refer the reader to [16].

The concept of matroid was introduced by Whitney [22] in 1935. One of two fundamental classes of matroids comes from graph theory (the second such class consists of matroids derived from linear algebra). Matroids have a close connection with graphs and many important matroid problems are motivated by corresponding problems for graphs. One of the more obvious indications of the pervasive influence of graphs throughout matroid theory is in the terminology of the subject, which borrows heavily from graph theory.

Goetschel and Voxman [6] first introduced fuzzy matroids and many results in matroid theory were generalized to fuzzy matroids in subsequent papers [7, 8, 9, 10,

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14, 13]. Shi [20] introduced a new approach to fuzzification of matroids; Yao and Shi [23] constructed base axioms and circuit axioms for fuzzifying matroids; Lu et al. [15] studied relations among matroids, fuzzy matroids and fuzzifying matroids from a categorical viewpoint. As mentioned above, matroids and graphs connect closely with each other. It is natural to study the connection of fuzzification of matroids and fuzzy graphs. Huang [11] makes a try in this aspect and constructs a fuzzifying matroid from a fuzzy graph. Moreover, [11] investigates the relation of graphic fuzzifying matroids and representable fuzzifying matroids.

Fuzzy matroids introduced by Goetschel and Voxman and fuzzifying matroids defined by Shi are two different kinds of fuzzification of matroids (for more kinds of fuzzy matroids and their relations, see [12]). Except for Huang's work on the connection of fuzzifying matroids and fuzzy graphs, few results concerning related work can be found. This paper tries to study the connection of fuzzy matroids and fuzzy graphs. We construct a fuzzy matroid from a given fuzzy graph and call it a graphic fuzzy matroid. Moreover, properties of graphic fuzzy matroids are discussed. Some problems worthy of consideration in the future are also presented.

The arrangement of this paper is as follows. Section 2 contains some fundamental definitions and results for fuzzy sets, fuzzy matroids and fuzzy graphs which will be needed in this paper. In Section 3 we present an approach to getting a sequence of matroids from a given fuzzy graph. Based on the sequence of matroids defined in Section 2, a fuzzy matroid (named a graphic fuzzy matroid) is constructed and an equivalent description of graphic fuzzy matroids is given in Section 4. Section 5 discusses the properties of fuzzy bases and fuzzy circuits for graphic fuzzy matroids and a concluding remark is given in the last section.

2. Preliminaries

This section presents some fundamental notions and results on fuzzy sets, fuzzy matroids and fuzzy graphs.

Let X be a finite set. A fuzzy set μ on X is a mapping $\mu : X \rightarrow [0, 1]$. We use $[a]$ to denote the fuzzy set taking constant value a on X ($a \in [0, 1]$) and χ_Y the characteristic function of Y ($Y \subseteq X$). The fuzzy set $[a] \wedge \chi_{\{x\}}$ ($a > 0$) (usually denoted by x_a) is called a fuzzy point. We denote the family of all fuzzy sets on X by $[0, 1]^X$, and denote the set of all fuzzy points on X by $J([0, 1]^X)$. For any $\mu, \nu \in [0, 1]^X$, we write $\mu \leq \nu$ if $\mu(x) \leq \nu(x)$ for each $x \in X$, and $\mu < \nu$ if $\mu \leq \nu$ and $\mu \neq \nu$. $\mu \vee \nu$ and $\mu \wedge \nu$ are fuzzy sets which satisfy $(\mu \vee \nu)(x) = \max\{\mu(x), \nu(x)\}$ and $(\mu \wedge \nu)(x) = \min\{\mu(x), \nu(x)\}$ ($\forall x \in X$). For each $\mu \in [0, 1]^X$, the non-negative real number $\sum_{x \in X} \mu(x)$ (written as $|\mu|$) is called the cardinality of μ ; we write $\text{supp } \mu = \{x \in X \mid \mu(x) > 0\}$, $m(\mu) = \min\{\mu(x) \mid x \in \text{supp } \mu\}$, $M(\mu) = \max\{\mu(x) \mid x \in \text{supp } \mu\}$, $\mu_{[a]} = \{x \in X \mid \mu(x) \geq a\}$ (where $0 \leq a \leq 1$), and $R^+(\mu) = \{\mu(x) \mid \mu(x) > 0, x \in X\}$.

Definition 2.1. (Oxley [18]). A matroid M is a pair (X, \mathcal{I}) consisting of a finite set X and a nonempty family \mathcal{I} of subsets of X which satisfies the following two conditions:

- (I1) If $Y \in \mathcal{I}$ and $Z \subseteq Y$, then $Z \in \mathcal{I}$ (hereditary property).

(I2) If $Y, Z \in \mathcal{I}$ and $|Y| < |Z|$ (where $|Y|$ denotes the cardinality of Y), then there exists $W \in \mathcal{I}$ such that $Y \subset W \subseteq Y \cup Z$ (independence augmentation property).

Let $M=(X, \mathcal{I})$ be a matroid. The members of \mathcal{I} are called the independent sets of M . A set in \mathcal{I} that is maximal in the sense of inclusion is called a base of the matroid M . A subset Y of X is called dependent if $Y \notin \mathcal{I}$. A minimal, in the sense of inclusion, dependent subset of X is called a circuit of the matroid M . We use the symbol $\mathcal{B}(M)$ (resp. $\mathcal{C}(M)$) to denote the family of all bases (resp. all circuits) of the matroid M . Matroids have many equivalent descriptions. Besides independent sets and bases, matroids can also be determined in terms of closed sets, rank functions, closure operators, circuits, etc. For a more detailed discussion of axiom systems of matroids, see [18].

Goetschel and Voxman in 1988 generalized Definition 1.1 to fuzzy setting and introduced the concept of fuzzy matroids as follows.

Definition 2.2. (Goetschel and Voxman [6]). Let X be a finite set, and $\Psi \subseteq [0, 1]^X$ is a nonempty family of fuzzy sets satisfying:

(FI1) If $\mu \in \Psi$, $\nu \in [0, 1]^X$ and $\nu < \mu$, then $\nu \in \Psi$.

(FI2) If $\mu, \nu \in \Psi$ and $0 < |\text{supp } \mu| < |\text{supp } \nu|$, then there exists $\omega \in \Psi$ such that

(i) $\mu < \omega \leq \mu \vee \nu$.

(ii) $m(\omega) \geq \min\{m(\mu), m(\nu)\}$.

Then we call $\mathcal{M} = (X, \Psi)$ a fuzzy matroid, and call Ψ the family of independent fuzzy sets of \mathcal{M} . $\mu \in \Psi$ is called a fuzzy base if whenever $\nu \in \Psi$ and $\mu \leq \nu$ then $\mu = \nu$. $\omega \in [0, 1]^X$ is a fuzzy circuit of \mathcal{M} if $\omega \notin \Psi$ and $\omega \setminus \setminus a \in \Psi$ for each $a \in \text{supp } \omega$, where $\omega \setminus \setminus a$ is defined by

$$\omega \setminus \setminus a(x) = \begin{cases} \mu(x), & x \neq a, \\ 0, & x = a. \end{cases}$$

Properties of fuzzy bases and fuzzy circuits are studied in [7] and [8] respectively.

Example 2.3. Let $X = \{a, b, c, d\}$ and $\Phi = \{\mu \in [0, 1]^X \mid m(\mu) = M(\mu) = \frac{1}{2} \text{ and } |\text{supp } \mu| = 2\}$. Again, let $\Psi_1 = \{\nu \in [0, 1]^X \mid \text{there exists } \mu \in \Phi \text{ such that } \nu \leq \mu\}$ and $\Psi_2 = \{\nu \in [0, 1]^X \mid \nu < [\frac{1}{2}] \wedge \chi_a\}$. Then we can check that (X, Ψ_1) is a fuzzy matroid and Φ is the family of all fuzzy bases of (X, Ψ_1) . Further, $\omega \in [0, 1]^X$ is a fuzzy circuit of (X, Ψ_1) if and only if $|\text{supp } \omega| = 3$ and $M(\omega) \leq \frac{1}{2}$. While it is easy to verify that (X, Ψ_2) is a fuzzy matroid with no fuzzy bases and fuzzy circuits. This example will be mentioned again in Section 4.

From Example 2.3 we can see that a fuzzy matroid does not necessarily have fuzzy bases. In Section 4, a fuzzy matroid (i.e. graphic fuzzy matroid) shall be induced from a fuzzy graph, and an approach to finding fuzzy bases of a graphic fuzzy matroid will be presented in Section 5. Now we introduce some preliminaries on fuzzy graphs.

Definition 2.4. A fuzzy graph (V, μ, ρ) is a nonempty set V together with a pair of functions $\mu : V \rightarrow [0, 1]$ and $\rho : V \times V \rightarrow [0, 1]$ such that for all x, y in V , $\rho(x, y) \leq \mu(x) \wedge \mu(y)$. We call μ the fuzzy vertex set and ρ the fuzzy edge

set of (V, μ, ρ) respectively. Note we will assume that, unless otherwise specified, the underlying set is V and that is finite. Therefore, for the sake of notational convenience, we omit V in the sequel and use the notation (μ, ρ) .

The fuzzy graph (ν, τ) is called a partial fuzzy subgraph of (μ, ρ) if $\nu \leq \mu$ and $\tau \leq \rho$, denoted by $(\nu, \tau) \subseteq (\mu, \rho)$. Similarly, the fuzzy graph (P, ν, τ) is called a fuzzy subgraph of (V, μ, ρ) if $P \subseteq V, \nu(x) = \mu(x)$ for all $x \in P$ and $\tau(x, y) = \rho(x, y)$ for all $x, y \in P$. A fuzzy subgraph is obviously a special case of a partial fuzzy subgraph. A path in a fuzzy graph (μ, ρ) is a sequence of distinct vertices x_0, x_1, \dots, x_n (except possibly x_0 and x_n) such that $\rho(x_{i-1}, x_i) > 0$ and $\bigwedge \rho(x_{i-1}, x_i) (1 \leq i \leq n)$ is the strength of the path; a cycle is a path with $x_0 = x_n (n \geq 2)$. We call a fuzzy graph a forest if the graph consisting of its nonzero edges is a forest, and a tree if this graph is also connected. The definitions of fuzzy circle and fuzzy tree seem a little more complicated and we list these two concepts as follows.

Definition 2.5. (μ, ρ) is a fuzzy circle if and only if (μ, ρ) is a cycle and there does not exist a unique $(x, y) \in \text{supp } \rho$ such that $\rho(x, y) = \bigwedge \{\rho(u, v) \mid (u, v) \in \text{supp } \rho\}$. A fuzzy graph (μ, ρ) is a fuzzy forest if it has a partial fuzzy spanning subgraph (ν, τ) which is a forest, where for all $(x, y) \in \text{supp } \rho - \text{supp } \tau$, there is a path in (ν, τ) between x and y whose strength is greater than $\mu(x, y)$. A connected fuzzy forest is a fuzzy tree.

To illustrate the two concepts above, see Fig. 4. The triangle in Fig. 4(a) is a fuzzy cycle (thus Fig. 4(a) is not a fuzzy tree), while the cycle in Fig. 4(b) is not a fuzzy cycle (so Fig. 4(b) is a fuzzy tree). For undefined definitions involving fuzzy graphs, see [16].

3. A Sequence of Matroids from Cuts of Fuzzy Graphs

In this section, we show an approach to getting a sequence of crisp matroids from a fuzzy graph (μ, ρ) . At first, we introduce one of two fundamental examples of matroids induced from graphs, another examples comes from linear algebra.

Example 3.1. Let $G = (V, E)$ be a crisp graph and let

$$\mathcal{I} = \{X \subseteq E \mid X \text{ does not contain a cycle of } G^1\}.$$

Then we can check that \mathcal{I} is the collection of independent sets of a matroid on E , called the cycle matroid of G , denoted by $M(G)$. Consider the crisp graph G in Figure 1(a), and let $M(G) = (E, \mathcal{I})$, where $E = \{e_1, e_2, e_3, e_4, e_5\}$, then

$$\mathcal{I} = \{\{e_1, e_2, e_3\}, \{e_1, e_2, e_4\}, \{e_1, e_3, e_4\}, \{e_1, e_2\}, \{e_1, e_3\}, \{e_1, e_4\}, \{e_2, e_3\}, \{e_2, e_4\}, \{e_3, e_4\}, \{e_1\}, \{e_2\}, \{e_3\}, \{e_4\}, \emptyset\},$$

$$\mathcal{B}(M(G)) = \text{Max } \mathcal{I} = \{\{e_1, e_2, e_3\}, \{e_1, e_2, e_4\}, \{e_1, e_3, e_4\}\},$$

$$\mathcal{C}(M(G)) = \text{Min } (2^E - \mathcal{I}) = \{\{e_2, e_3, e_4\}\}.$$

¹To be precise, X does not contain the edge set of a cycle, or equivalently, $G[X]$, the subgraph induced by X , is a forest. Since the ground set of $M(G)$ is the edge set of G , we shall refer to certain subgraphs of G when we mean just their edge sets. This commonplace practice should not cause confusion.

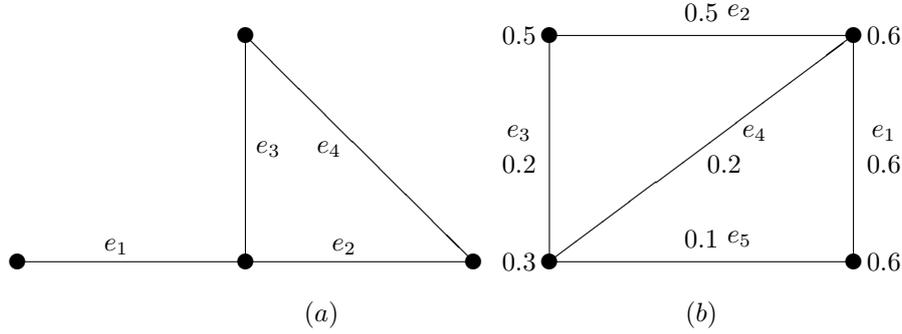


FIGURE 1. A Graph and a Fuzzy Graph

Let (μ, ρ) be a fuzzy graph. For any $r \in [0, 1]$, we know that $(\mu_{[r]}, \rho_{[r]})$ is a crisp graph, and then denote the cycle matroid of $(\mu_{[r]}, \rho_{[r]})$ by M_r . The collection of independent sets of M_r is denoted by $\mathcal{I}(M_r)$, and usually by \mathcal{I}_r for short. If $a, b \in [0, 1]$ and $a < b$, obviously $(\mu_{[b]}, \rho_{[b]})$ is a subgraph of $(\mu_{[a]}, \rho_{[a]})$. Moreover, we can easily verify that \mathcal{I}_b is a subset of \mathcal{I}_a . That is, we have the following result.

Proposition 3.2. *Let (μ, ρ) be a fuzzy graph. For every pair of $a, b \in [0, 1]$ and $a < b$, two matroids M_a and M_b can be induced by the fuzzy graph (μ, ρ) following the preceding definition. Moreover, $\mathcal{I}_b \subseteq \mathcal{I}_a$ holds.*

Note that the underlying set V of fuzzy graph (μ, ρ) considered in this paper is finite, thus matroids induced by a fuzzy graph are finite. By Proposition 3.2, an interesting remark can be

Remark 3.3. Let (V, μ, ρ) be a fuzzy graph. For every $a \in [0, 1]$, let M_a be the cycle matroid induced by the crisp graph $(\mu_{[a]}, \rho_{[a]})$. Let $E_a = \rho_{[a]}$ and denote $\text{supp } \rho$ by E throughout this paper unless otherwise specified. Since E is a finite set, there is at most a finite number of matroids that can be defined on E . By proposition 3.2, there is a finite sequence $0 = a_0 < a_1 < a_2 < \dots < a_n \leq 1$ satisfying

- (i) If $a_i < a < b < a_{i+1}$ ($0 \leq i \leq n - 1$), then $\mathcal{I}_a = \mathcal{I}_b$.
- (ii) If $a_i < a < a_{i+1} < b < a_{i+2}$ ($0 \leq i \leq n - 2$), then $\mathcal{I}_a \supset \mathcal{I}_b$.
- (iii) If $0 < a < b \leq 1$, then $\mathcal{I}_a \supseteq \mathcal{I}_b$.

The above sequence a_1, \dots, a_n is called the fundamental sequence of (V, μ, ρ) , and denoted by $\langle a_1, a_2, \dots, a_n \rangle$. $(E, \mathcal{I}_{a_1}), (E, \mathcal{I}_{a_2}), \dots, (E, \mathcal{I}_{a_n})$ is a sequence of matroids induced by the fuzzy graph (V, μ, ρ) such that $\mathcal{I}_{a_{i+1}} \subset \mathcal{I}_{a_i}$ ($i = 1, 2, \dots, n - 1$)².

Example 3.4. Consider the fuzzy graph (μ, ρ) defined in Figure 1(b). Crisp graphs $(\mu_{[a]}, \rho_{[a]})$ are shown in Figure 2. We can check that the fundamental sequence of

² \mathcal{I}_{a_i} is induced by the crisp $(\mu_{[a_i]}, \rho_{[a_i]})$ ($i = 1, 2, \dots, n$). That is to say, (V, μ, ρ) in fact induces a sequence of matroids $(E_{a_1}, \mathcal{I}_{a_1}), (E_{a_2}, \mathcal{I}_{a_2}), \dots, (E_{a_n}, \mathcal{I}_{a_n})$. Since $E_{a_i} \subseteq E$, this sequence of matroids certainly can be considered as matroids defined on E ($i = 1, 2, \dots, n$). To do this, just because a fuzzy matroid shall be defined on E from the induced matroidal sequence.

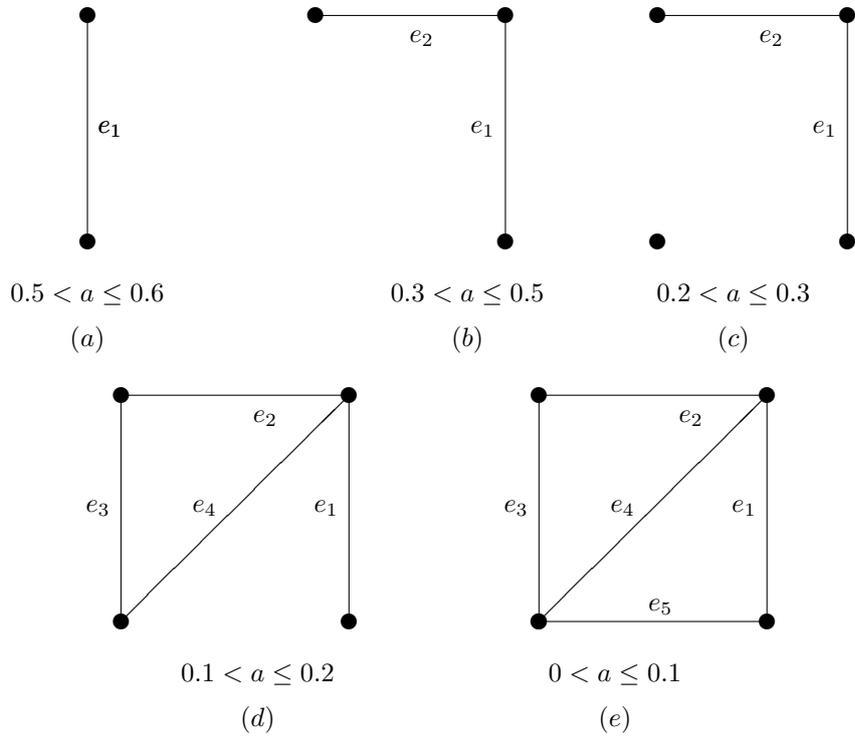


FIGURE 2. Cuts of a Fuzzy Graph

(μ, ρ) is $\langle 0.1, 0.2, 0.5, 0.6 \rangle$, and $(E, \mathcal{I}_{0.1}), (E, \mathcal{I}_{0.2}), (E, \mathcal{I}_{0.5}), (E, \mathcal{I}_{0.6})$ is a sequence of matroids induced by the fuzzy graph (μ, ρ) . In fact, we have

$$\mathcal{I}_a = \begin{cases} \{\emptyset\}, & a \in (0.6, 1], \\ \{\{e_1\}, \emptyset\}, & a \in (0.5, 0.6], \\ 2^{\{e_1, e_2\}}, & a \in (0.2, 0.5], \\ 2^{\{e_1, e_2, e_3\}} \cup 2^{\{e_1, e_2, e_4\}} \cup 2^{\{e_1, e_3, e_4\}}, & a \in (0.1, 0.2], \\ \bigcup \{2^X \mid X \in \wp_3(E)\} - \{\{e_1, e_4, e_5\}, \{e_2, e_3, e_4\}\}, & a \in (0, 0.1], \end{cases}$$

where $\wp_3(E)$ denotes the set of all three-element subsets of $E = \{e_1, e_2, e_3, e_4, e_5\}$.

In Example 3.4, the fundamental sequence of (μ, ρ) is just the range of ρ . This result also holds for the general case. The proof is trivial and we omit it.

Proposition 3.5. *Let (V, μ, ρ) be a fuzzy graph. Then the fundamental sequence of (V, μ, ρ) is $\{\rho(x, y) > 0 \mid x, y \in V\}$.*

The following example shows different fuzzy graphs may induce the same sequence of matroids.

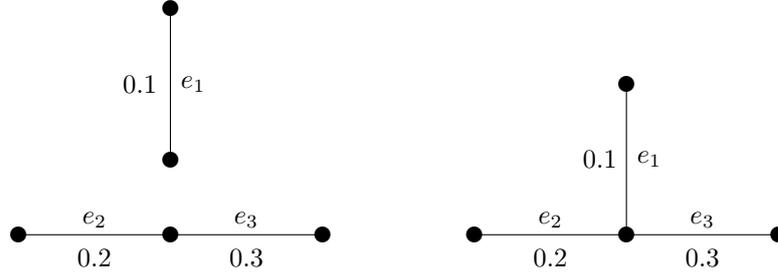


FIGURE 3. Two Fuzzy Graphs in Example 3.6

Example 3.6. Consider the two fuzzy graphs in Figure 3, we can easily check that they have the same fundamental sequence $\langle 0.1, 0.2, 0.3 \rangle$ and sequence of matroids (V, \mathcal{I}_a) , where

$$\mathcal{I}_a = \begin{cases} \{\emptyset\}, & a \in (0.3, 1], \\ 2^{\{e_3\}}, & a \in (0.2, 0.3], \\ 2^{\{e_2, e_3\}}, & a \in (0, 0.2], \\ 2^{\{e_1, e_2, e_3\}}, & a \in (0, 0.1]. \end{cases}$$

By Example 3.6, we have

Remark 3.7. Let \mathbf{FG} be the set of all fuzzy graph and \mathbf{MS} be the set of all sequences of matroids. Given a fuzzy graph (μ, ρ) , there is a unique sequence of matroids defined above. Thus the map $f : \mathbf{FG} \rightarrow \mathbf{MS}$ constructed in this way is well-defined. Example 3.6 shows that f is not injective, and we shall point out that f is not surjective in the next section.

4. Fuzzy Matroids Induced by Fuzzy Graphs

By decomposition theorem of fuzzy sets, we can get a fuzzy set if we know its all α -level sets. In Example 3.4, all levels families of independent sets \mathcal{I}_a are obtained, thus “fuzzy independent sets” can be constructed by decomposition theorem of fuzzy sets. We continue to consider Example 3.4 in the following.

Let Ψ be a family of fuzzy sets defined on $E = \{e_1, e_2, e_3, e_4, e_5\}$ and $\omega \in \Psi$ if and only if

$$\omega_{[a]} \text{ satisfies } \begin{cases} \omega_{[a]} \in \{\emptyset\}, & a \in (0.6, 1], \\ \omega_{[a]} \in \{\{e_1\}, \emptyset\}, & a \in (0.5, 0.6], \\ \omega_{[a]} \in 2^{\{e_1, e_2\}}, & a \in (0.2, 0.5], \\ \omega_{[a]} \in 2^{\{e_1, e_2, e_3\}} \cup 2^{\{e_1, e_2, e_4\}} \cup 2^{\{e_1, e_3, e_4\}}, & a \in (0.1, 0.2], \\ \omega_{[a]} \in \bigcup \{2^X \mid X \in \wp_3(E)\} - \{C_1, C_2\}, & a \in (0, 0.1], \end{cases}$$

where $C_1 = \{e_1, e_4, e_5\}$ and $C_2 = \{e_2, e_3, e_4\}$. Then we can check that Ψ is a fuzzy matroid on V . In fact, we can construct a fuzzy matroid from every fuzzy graph in this way.

Proposition 4.1. *Let (V, μ, ρ) be a fuzzy graph. For every $a \in [0, 1]$, let M_a be the cycle matroid induced by the crisp graph $(\mu_{[a]}, \rho_{[a]})$ and \mathcal{I}_a be the family of all independent sets of M_a . Suppose that $\Psi = \{\omega \in [0, 1]^E \mid \omega_{[a]} \in \mathcal{I}_a \text{ for each } a \in (0, 1]\}$, then Ψ is a fuzzy matroid on E .*

Proof. We show that Ψ satisfies (FI1) and (FI2).

Suppose $\omega_1 \in \Psi, \omega_2 \in [0, 1]^E$ and $\omega_2 < \omega_1$. By the definition of Ψ , we have $\omega_{1[a]} \in \mathcal{I}_a (\forall a \in (0, 1])$. Note that $\omega_2 < \omega_1$ implies $\omega_{2[a]} \subseteq \omega_{1[a]}$ and (E, \mathcal{I}_a) is a fuzzy matroid, then (I2) of Definition 2.1 implies $\omega_{2[a]} \in \mathcal{I}_a (\forall a \in (0, 1])$. Therefore, $\omega_2 \in \Psi$ holds from the definition of Ψ and Ψ satisfies (FI1).

Suppose $\omega_1, \omega_2 \in \Psi$ and $|\text{supp } \omega_1| < |\text{supp } \omega_2|$. That is $\text{supp } \omega_1 = \omega_{1[m(\omega_1)]} \subset \omega_{2[m(\omega_2)]}$. Let $b = \min\{m(\omega_1), m(\omega_2)\}$, then $\text{supp } \omega_1 = \omega_{1[m(\omega_1)]} = \omega_{1[b]} \in \mathcal{I}_b$. Similarly, $\text{supp } \omega_2 \in \mathcal{I}_b$. Because (E, \mathcal{I}_b) is a cycle matroid, there exists an independent set A satisfying

$$\text{supp } \omega_1 \subset A \subseteq \text{supp } \omega_1 \cup \text{supp } \omega_2.$$

Define $\omega \in [0, 1]^E$ as follows:

$$\omega(x) = \begin{cases} \omega_1(x), & x \in \text{supp } \omega_1, \\ b, & x \in A - \text{supp } \omega_1, \\ 0, & \text{otherwise.} \end{cases}$$

Then we can easily check $\omega_a \in \mathcal{I}_a (\forall a \in (0, 1])$, thus $\omega \in \Psi$. Obviously, $\omega_1 < \omega \leq \omega_1 \vee \omega_2$ and $m(\omega) = \min\{m(\omega_1), m(\omega_2)\}$. Therefore, Ψ has the property (FI2) and then (V, Ψ) is a fuzzy matroid. \square

Fuzzy matroids in Proposition 4.1 are called graphic fuzzy matroids induced by fuzzy graphs. Given a fuzzy graph (μ, ρ) , we have introduced an approach to constructing a graphic fuzzy matroid. We now give another explanation of graphic fuzzy matroids.

Given a graph (V, E) , we define a family of edge sets \mathcal{I} in Example 3.1, then \mathcal{I} determines a matroid on E . For a fuzzy graph (V, μ, ρ) , we naturally consider the fuzzy analog of \mathcal{I} defined as follows:

$$\Psi_\rho = \{\omega \in [0, 1]^{V \times V} \mid \omega \triangleleft \rho \text{ and } \omega \text{ does not contain a cycle of } (V, \mu, \rho)\},$$

where $\omega \triangleleft \rho$ means $\omega(x, y) = \rho(x, y)$ or $\omega(x, y) = 0$ for any $(x, y) \in V \times V$ ³. Unfortunately, Ψ_ρ usually is not a fuzzy matroid on $V \times V$. For example, we consider the fuzzy graph (V, μ, ρ) in Fig. 1(b) again. Define

$$\omega_1(x) = \begin{cases} 0.2, & x = e_1, \\ 0, & \text{otherwise.} \end{cases} \quad \omega_2(x) = \begin{cases} 0.6, & x = e_1, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that $\omega_2 \in \Psi_\rho, \omega_1 < \omega_2$ and $\omega_1 \notin \Psi_\rho$, so Ψ_ρ does not satisfy (FI1).

For every $a \in (0, 1]$, let

$$\rho_a = \rho \wedge [a],$$

³i.e., ω is the edge set of a fuzzy subgraph of (V, μ, ρ)

then (V, μ, ρ_a) is a partial fuzzy subgraph of (μ, ρ) and $\omega_1 \in \Psi_{\rho_{0.2}}$ holds clearly. This result also holds for the general case, because we have

Proposition 4.2. *Let (μ, ρ) be a fuzzy graph and Ψ be the graphic fuzzy matroid induced by (μ, ρ) . Then*

$$\Psi = \bigcup_{(\nu, \tau) \subseteq (\mu, \rho)} \Psi_\tau.$$

Proof. Suppose that $\omega \in \Psi_\tau$ for some partial fuzzy subgraph (ν, τ) of (μ, ρ) . To show $\omega \in \Psi$, by the definition of Ψ , we need only to prove that $\omega_{[b]} \in \mathcal{I}_b$ ($\forall b \in (0, 1]$), where \mathcal{I}_b is the family of independent sets of cycle matroid of $(\mu_{[b]}, \rho_{[b]})$. If $b > \max\{\omega(x) \mid x \in \text{supp } \omega\}$, then $\omega_{[b]} = \emptyset \in \mathcal{I}_b$. We now consider the case $b \leq \max\{\omega(x) \mid x \in \text{supp } \omega\}$. By the definition of Ψ_τ , $\omega_{[b]}$ is an edge set of $(\nu_{[b]}, \tau_{[b]})$ and does not contain cycles. Since (ν, τ) is a partial fuzzy subgraph of (μ, ρ) , $(\nu_{[b]}, \tau_{[b]})$ is a subgraph of $(\mu_{[b]}, \rho_{[b]})$. Therefore, $\omega_{[b]}$ consists of edges of $(\mu_{[b]}, \rho_{[b]})$ and contains no cycle. It follows from the definition of graphic fuzzy matroid that $\omega_{[b]} \in \mathcal{I}_b$.

Conversely, suppose $\omega \in \Psi$, we prove $\omega \in \Psi_\tau$ for some partial fuzzy subgraph (ν, τ) of (μ, ρ) . At first, we assert that $\omega \leq \rho$ holds. If $\omega \not\leq \rho$, then there exists $\mathbf{x} \in V \times V$ such that $\omega(\mathbf{x}) > \rho(\mathbf{x})$. That is, $\mathbf{x} \in \omega_{[c]} - \rho_{[c]}$ for some $c \in (0, 1]$. Note that $\omega \in \Psi$ implies $\omega_{[b]} \in \mathcal{I}_b$ and \mathcal{I}_b is a family of subsets of $\rho_{[b]}$, so $\omega_{[b]} \subseteq \rho_{[b]}$ ($\forall b \in (0, 1]$), which contradicts $\mathbf{x} \in \omega_{[c]} - \rho_{[c]}$. Thus (μ, ω) is a partial fuzzy subgraph of (μ, ρ) , and we now show $\omega \in \Psi_\omega$ in the following. Assume that ω contains a cycle of (μ, ω) , then $\omega_{[d]}$ contains a cycle of $(\mu_{[d]}, \omega_{[d]})$ for some $d \in (0, 1]$. Because $(\mu_{[d]}, \omega_{[d]})$ is a subgraph of $(\mu_{[d]}, \rho_{[d]})$, $\omega_{[d]}$ contains a cycle of $(\mu_{[d]}, \rho_{[d]})$, which contradicts the fact $\omega_{[d]} \in \mathcal{I}_d$. Thus $\omega \in \Psi_\omega$ and this concludes the proof. \square

Remark 4.3. Let (μ, ρ) be a fuzzy graph and then a graphic fuzzy matroid can be induced by (μ, ρ) . Conversely, for a given fuzzy matroid, there need not exist a fuzzy graph which can induce the fuzzy matroid (see Proposition 5.7), which shows that f in Remark 3.7 is not surjective.

5. Fuzzy Bases and Fuzzy Circuits

In this section, we study properties of fuzzy bases and fuzzy circuits of graphic fuzzy matroids. As pointed out in Example 2.3, a fuzzy matroid (X, Ψ) does not necessarily have fuzzy bases, then closeness was proposed to ensure existence of fuzzy bases (see [5, Theorem 1.10]). By the definition of graphic fuzzy matroids, it is easy to see that every graphic fuzzy matroid is closed and thus has fuzzy bases. We now construct a fuzzy base from a graphic fuzzy matroid in the following.

Suppose that (μ, ρ) is a fuzzy graph and Ψ is the graphic fuzzy matroid induced by (μ, ρ) . Let

$$\{a_1, a_2, \dots, a_n\} = \{\rho(\mathbf{x}) \mid \mathbf{x} \in \text{supp } \rho\},$$

where $0 < a_1 \leq a_2 \leq \dots \leq a_n \leq 1$. Then we can get a fuzzy base by the following three steps.

1. Choose a spanning forest A_1 of $(\mu_{[a_1]}, \rho_{[a_1]})$;
2. Choose a maximal spanning forest A_i of $(\mu_{[a_i]}, \rho_{[a_i]})$ such that $A_i \subseteq A_{i-1}$ (i.e., $A_i = \text{Max}\{A \mid A \subseteq \rho_{[a_i]} \cap A_{i-1} \text{ and } A \text{ does not contain cycles of } (\mu_{[a_i]}, \rho_{[a_i]})\}$, $\forall i = 2, 3, \dots, n$);

3. Let

$$\omega = \bigvee_{i \in \{1, 2, \dots, n\}} \chi_{A_i} \wedge [a_i],$$

then ω is a fuzzy base of Ψ . The proof is trivial and we omit it.

Bases of the cycle matroid in Example 3.1 correspond to spanning trees (or forests). Every spanning tree of a connected graph with n vertexes has $n - 1$ edges. The corresponding result in matroid theory is that every base of a matroid has the same cardinality. For graphic fuzzy matroids, the fuzzy analog of this result need not be true.

Example 5.1. Consider the graphic fuzzy matroid (E, Ψ) induced by the fuzzy graph in Figure 1(b). We now construct two fuzzy bases following the above steps. Obviously, $\{a_1, a_2, \dots, a_n\} = \{\rho(\mathbf{x}) \mid \mathbf{x} \in \text{supp } \rho\} = \{0.1, 0.2, 0.5, 0.6\}$. The graphs $(\mu_{[a_i]}, \rho_{[a_i]})$ are presented in Figure 2 ($i = 1, 2, 3, 4$). First, we choose $A_1 = \{e_1, e_2, e_3\}$, and then we choose $A_2 = \{e_1, e_2, e_3\}$, $A_3 = \{e_1, e_2\}$ and $A_4 = \{e_1\}$ respectively. Let $\omega_1 = (\chi_{A_1} \wedge [0.1]) \vee (\chi_{A_2} \wedge [0.2]) \vee (\chi_{A_3} \wedge [0.5]) \vee (\chi_{A_4} \wedge [0.6])$,

then ω_1 is a fuzzy base. Similarly, the following ω_2 is another fuzzy base:

$$\omega_2 = (\chi_{A_1} \wedge [0.1]) \vee (\chi_{A_2} \wedge [0.2]) \vee (\chi_{A_3} \wedge [0.5]) \vee (\chi_{A_4} \wedge [0.6]),$$

where $A_1 = \{e_3, e_4, e_5\}$, $A_2 = \{e_3, e_4\}$, $A_3 = \emptyset$ and $A_4 = \emptyset$. In fact,

$$\omega_1(x) = \begin{cases} 0.6, & x = e_1, \\ 0.5, & x = e_2, \\ 0.2, & x = e_3, \\ 0, & \text{otherwise.} \end{cases} \quad \omega_2(x) = \begin{cases} 0.2, & x = e_3, \\ 0.2, & x = e_4, \\ 0.1, & x = e_5, \\ 0, & \text{otherwise.} \end{cases}$$

We can also verify that ω_1 and ω_2 are fuzzy bases directly. Since $\omega_{i[a]}$ is an element of \mathcal{I}_a ($\forall a \in (0, 1]$ and $i = 1, 2$), then ω_1 and ω_2 are fuzzy independent sets of the matroid (E, Ψ) . Obviously, ω_1 and ω_2 are maximal elements in Ψ . Therefore, ω_1 and ω_2 are fuzzy bases of (E, Ψ) , while we have $|\omega_1| = 1.3 \neq 0.5 = |\omega_2|$.

For cycle matroids, bases correspond to spanning trees of graphs (cf. Example 3.1). For graphic fuzzy matroids, we have

Remark 5.2. (1) Suppose that Ψ is the graphic fuzzy matroid induced by a fuzzy graph (μ, ρ) . Then it is easy to prove that τ is a fuzzy base of Ψ if and only if (μ, τ) is a spanning tree of (μ, ρ) .

(2) A fuzzy tree (as a partial fuzzy subgraph of (μ, ρ)) need not be a fuzzy independent set of Ψ . Moreover, a fuzzy spanning tree is not necessarily a fuzzy base of Ψ (see Example 5.4)

Note that the fuzzy graph discussed above has cycles. If a fuzzy graph with no cycle, we have

Proposition 5.3. *Let (μ, ρ) be a fuzzy graph without cycles and Ψ be the graphic fuzzy matroid induced by (μ, ρ) . Then Ψ has only one fuzzy base.*

Proof. If (μ, ρ) has no cycle, then we only have one choice in every step of the above three steps, which concludes the proof. \square

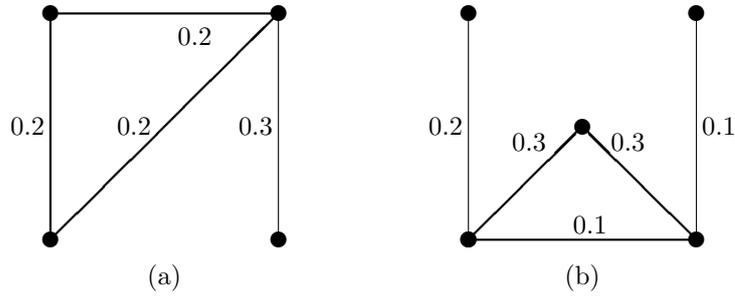


FIGURE 4. Two Fuzzy Graphs in Example 5.4

The following example shows that different cases about cardinalities of fuzzy bases may occur for fuzzy graph with cycles.

Example 5.4. Consider the two fuzzy graphs in Figure 4. We can check that all the three fuzzy bases of graphic fuzzy matroids induced by Figure 4(a) have the same cardinality, while there exist fuzzy bases of graphic fuzzy matroids induced by Figure 4(b) which have different cardinalities. Note that the fuzzy graph (μ, ρ) in Figure 4(b) is a fuzzy tree, but $\rho_{[0.1]} \notin \mathcal{I}_{0.1}$. By Proposition 4.1, ρ is not an independent set of the graphic fuzzy matroid induced by (μ, ρ) .

It is easy to check that two cycles (presented in thick lines) in Figure 4 correspond to fuzzy circuits of graphic fuzzy matroids induced by the two fuzzy graphs respectively. Moreover, every partial fuzzy subgraph of the two circles (with three edges) is also a fuzzy circuit of the induced graphic fuzzy matroid. That is to say, the two cycles correspond to maximal fuzzy circuits of graphic fuzzy matroids induced by Figure 4 respectively. In fact, this result also holds in general case.

Proposition 5.5. *Let (μ, ρ) be a fuzzy graph and (E, Ψ) be the graphic matroid induced by (μ, ρ) . Then the fuzzy subgraph (ν, τ) of (μ, ρ) is a cycle if and only if τ is a maximal fuzzy circuit of (E, Ψ) .*

Proof. If (ν, τ) is a fuzzy subgraph of (μ, ρ) and is a cycle, then $\text{supp } \tau$ is a cycle of the crisp graph $(\text{supp } \nu, \text{supp } \tau)$. For any $x \in \text{supp } \tau$, $\text{supp } \tau \setminus x$ is a chain in $(\text{supp } \nu, \text{supp } \tau)$, thus $\text{supp } \tau \setminus x$ is an independent set of the cycle matroid $(\tau_{[m(\tau)]}, \mathcal{I}_{m(\tau)})$. Obviously, for every $a \in (0, 1]$, $(\tau \setminus x)_{[a]} \subseteq \text{supp } \tau \setminus x$, so $(\tau \setminus x)_{[a]} \in \mathcal{I}_a$. By Proposition 4.1, $\tau \setminus x$ is fuzzy independent set of (E, Ψ) . Moreover, $\tau_{[m(\tau)]}$ is a cycle of $(\text{supp } \nu, \text{supp } \tau)$, it follows from Proposition 4.1 that $\tau \notin \Psi$. Therefore, τ is a fuzzy circuit. Suppose that τ' is a fuzzy circuit and $\tau < \tau'$ and we derive a contradiction. Then (μ, τ') is a partial fuzzy subgraph of (μ, ρ) . Note that (ν, τ) is a fuzzy subgraph of (μ, ρ) and $\tau < \tau'$, thus $\text{supp } \tau \subsetneq \text{supp } \tau'$. Since $\text{supp } \tau$ is a cycle, then there exist $y \in \text{supp } \tau' - \text{supp } \tau$ such that $\text{supp } \tau' - \{y\}$ contains a cycle. That is, $(\tau' \setminus y)_{[m(\tau')]}$ contains a cycle, thus $\tau' \setminus y \notin \Psi$ holds from Proposition 4.1, which contradicts the fact that τ' is a fuzzy circuit.

Conversely, to prove (ν, τ) is a cycle, we need only to show that $\text{supp } \tau$ is a cycle in the crisp graph $(\text{supp } \nu, \text{supp } \tau)$. if $\text{supp } \tau$ is not a cycle, then two cases may occur.

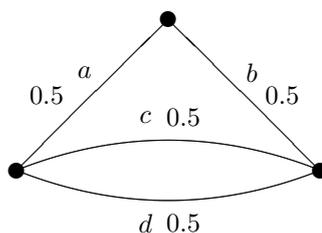


FIGURE 5. A Fuzzy Graph in the Proof of Proposition 5.7

Case 1: τ contains a cycle as a proper subset, then a similar argument as in the above proof can be given to contradict the fact that τ is a fuzzy circuit.

Case 2: τ does not contain a cycle, then $\text{supp } \tau$ is an independent set of the cycle matroid $(\tau_{[m(\tau)]}, \mathcal{I}_{m(\tau)})$. It is easy to verify that in this case τ is a fuzzy independent set of (E, Ψ) , contrary to hypothesis.

Thus $\text{supp } \tau$ is a cycle and this concludes the proof. \square

Remark 5.6. As pointed out in Remark 5.2, a fuzzy spanning tree does not correspond to a fuzzy base of the graphic fuzzy matroid. The similar situation also occurs for fuzzy cycles. Note Figure 4(b) is not a fuzzy cycle, but is a fuzzy circuit of the related graphic fuzzy matroid.

At last, as an application of Proposition 5.5, we show that the fuzzy matroid (X, Ψ_1) in Example 2.3 can not be induced by some fuzzy graph.

Proposition 5.7. *Let (X, Ψ_1) be the fuzzy matroid in Example 2.3. Then there does not exist a fuzzy graph which can induce (X, Ψ_1) by the approach in Proposition 4.1.*

Proof. Suppose that there exists a fuzzy graph (μ, ρ) which can induce the fuzzy matroid (X, Ψ_1) and derive a contradiction. Since $\omega \in [0, 1]^X$ is a fuzzy circuit of (X, Ψ_1) if and only if $|\text{supp } \omega| = 3$ and $M(\omega) \leq \frac{1}{2}$, then $a_{\frac{1}{2}} \vee b_{\frac{1}{2}} \vee c_{\frac{1}{2}}$ is a maximal fuzzy circuit of (X, Ψ_1) . By Proposition 5.5, $a_{\frac{1}{2}} \vee b_{\frac{1}{2}} \vee c_{\frac{1}{2}}$ is a cycle of (μ, ρ) . Similarly, $a_{\frac{1}{2}} \vee b_{\frac{1}{2}} \vee d_{\frac{1}{2}}$ is a cycle of (μ, ρ) (see Figure 5). Thus $d_{\frac{1}{2}} \vee c_{\frac{1}{2}}$ is a cycle. It holds from Proposition 5.5 that $d_{\frac{1}{2}} \vee c_{\frac{1}{2}}$ is a fuzzy circuit of (X, Ψ_1) . But $|\text{supp } d_{\frac{1}{2}} \vee c_{\frac{1}{2}}| = 2$, it is a contradiction. \square

6. Conclusions

In this paper, we construct a fuzzy matroid from a given fuzzy graph. Properties on fuzzy bases and fuzzy circuits of the induced fuzzy matroid are obtained. Based on our discussion in this paper, we list our future work as follows.

1. Two new fuzzy matroids can be introduced. First, when we generalize cycle matroids to fuzzy setting in Section 4, the corresponding family of fuzzy sets need not be a fuzzy matroid. To remedy this deficiency, a new fuzzy matroid should be proposed if a minor revision of (FI1) in Definition 2.2 is given as follows.

(FI1*) If $\mu \in \Psi$, then for every $A \subseteq \text{supp } \mu$, $\mu \wedge \chi_A \in \Psi$.

Second, a direct generalization of cycle matroid in Example 3.1 is to define the family of fuzzy independent sets $\Psi \subseteq [0, 1]^E$ as $\{\tau \in \Psi \mid \tau \text{ does not contain a fuzzy cycle of } (\mu, \rho)^4\}$.

2. It is well-known that cycles and spanning trees of graphs correspond to circuits and bases of matroids. While fuzzy cycles and fuzzy spanning trees do not correspond to fuzzy circuits and fuzzy bases (see Remark 5.2 and Remark ??). How to revise the definitions of fuzzy cycles and fuzzy trees so that the correspondence also holds in fuzzy setting?

3. What is the relation between graphic fuzzy matroid in this paper and Huang's graphic fuzzifying matroids in [11]?

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⁴I.e., there does not exist a fuzzy subgraph (ν, σ) of (μ, ρ) which is a fuzzy cycle such that $\sigma \leq \tau$.

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