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DIMENSION OF FUZZY HYPERVECTOR SPACES

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Abstract. In this paper we investigate the algebraic properties of dimension of fuzzy hypervector spaces. Also, we prove that two isomorphic fuzzy hypervector spaces have the same dimension.

1. **Introduction**

The notion of a hypergroup was introduced by F. Marty in 1934 [25]. Since then many researchers have worked on hyperalgebraic structures and developed this theory (for more see [9], [10], [16], [41]). In 1988, M. Scafati Tallini introduced the notion of hypervector spaces and studied basic properties of them (for more see [26-35]).

The concept of a fuzzy subset of a non-empty set was introduced by Zadeh in 1965 [42] as a function from a non-empty set X into the unit real interval $I = [0, 1]$. Rosenfeld [29] applied this to the theory of groups and then many researchers developed it in all the fields of algebra. The concepts of a fuzzy field and a fuzzy linear space over a fuzzy field were introduced and discussed by Nanda [27]. In 1977, Katsaras and Liu [21] formulated and studied the notion of fuzzy vector subspaces over the field of real or complex numbers.

Recently fuzzy set theory has been well developed in the context of hyperalgebraic structure theory. (for example see [1], [2], [6], [7], [8], [11], [12], [13], [14], [15], [17], [18], [19], [23], [30], [43]). In [1] the first author introduced and studied the notion of fuzzy hypervector space over valued fields. In [5] fuzzy basis of fuzzy hypervector spaces was studied. In this paper we follow [1], [4] and [5] and study more properties of dimension of fuzzy hypervector spaces. In this context we find some results of the algebraic properties of dimension of such hyperspaces under certain conditions. In particular two standard results from crisp theory are proved for case, namely $\dim(\tilde{V}_1 + \tilde{V}_2) = \dim(\tilde{V}_1) + \dim(\tilde{V}_2) - \dim(\tilde{V}_1 \cap \tilde{V}_2)$ and $\dim\left(\widetilde{\ker T}\right) + \dim\left(\widetilde{Im\ T}\right) = \dim\left(\tilde{V}\right)$. Also, we consider isomorphism of fuzzy hypervector spaces and prove that two isomorphic fuzzy hypervector spaces have the same dimension. Finally by an example we show that two fuzzy hypervector spaces with the same dimension need not to be isomorphic(note that if we consider, R-hypmod, the category of hypermodules over a fixed classical ring R, then it is

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easy to see that the free object in this category exists, for instance the free objects in R-mod, the category of R-modules, are free objects as objects in R-hypmod. Also, most important class of free objects in R-hypmod are hypervector spaces, of course in other classes of hypermodules it does not guarantee that the free objects always exists, for instance see [26]).

2. **Preliminaries**

In this section we present some definitions and simple properties of hypervector spaces and fuzzy hypervector spaces, that we shall use later. for more information see [3] and [5].

A map $\circ : H \times H \longrightarrow P^*(H)$ is called a hyperoperation or join operation, where $P^*(H)$ is the set of all non-empty subsets of H. The join operation is extended to subsets of H in a natural way, so that $A \circ B$ is given by

$$A \circ B = \bigcup\{a \circ b : a \in A \text{ and } b \in B \ \}.$$

The notations $a \circ A$ and $A \circ a$ are used for $\{a\} \circ A$ and $A \circ \{a\}$ respectively. Generally, the singleton $\{a\}$ is identified by its element a.

Definition 2.1. [31] Let K be a field and $(V, +)$ be an Abelian group. We define a hypervector space over K to be the quadruple $(V, +, \circ, K)$, where "$\circ$" is a mapping

$$\circ : K \times V \quad \longrightarrow \quad P^*(V),$$

such that for all $a, b \in K$ and $x, y \in V$ the following conditions hold:

(H_1) $a \circ (x + y) \subseteq a \circ x + a \circ y$,
(H_2) $(a + b) \circ x \subseteq a \circ x + b \circ x$,
(H_3) $a \circ (b \circ x) = (ab) \circ x$,
(H_4) $a \circ (-x) = (-a) \circ x = -(a \circ x)$,
(H_5) $x \in 1 \circ x$.

Remark 2.2. (i) In the right hand side of the right distributive law (H_1) the sum is meant in the sense of Frobenius, that is we consider the set of all sums of an element of $a \circ x$ with an element of $a \circ y$. Similarly it is in the left distributive law (H_2).

(ii) We say that $(V, +, \circ, K)$ is anti-left distributive if

$$\forall a, b \in K,\ \forall x \in V,\ (a + b) \circ x \supseteq a \circ x + b \circ x,$$

and strongly left distributive, if

$$\forall a, b \in K,\ \forall x \in V,\ (a + b) \circ x = a \circ x + b \circ x,$$

In a similar way we define the anti-right distributive and strongly right distributive hypervector spaces, respectively. The hypervector space $(V, +, \circ, K)$ is called strongly distributive if it is both strongly left and strongly right distributive.

(iii) The left hand side of the associative law (H_3) means the set-theoretical union of all the sets $a \circ y$, where y runs over the set $b \circ x$, i.e.

$$a \circ (b \circ x) = \bigcup_{y \in b \circ x} a \circ y.$$

(iv) Let $\Omega_V = 0 \circ \underline{0}$, where $\underline{0}$ is the zero of $(V,+)$. In [31] it is shown that if V is either strongly right or strongly left distributive, then Ω_V is a subgroup of $(V,+)$.

Example 2.3. In the classical vector space $(\mathbb{R}^3, +, ., \mathbb{R})$ we define:

$$\begin{cases} \circ : & \mathbb{R} \times \mathbb{R}^3 \longrightarrow P^*(\mathbb{R}^3) \\ & a \circ (x_0, y_0, z_0) = L, \end{cases}$$

where L is a line with the parametric equations:

$$L : \begin{cases} x = ax_0 \\ y = ay_0 \\ z = t \end{cases}$$

Then $V = (\mathbb{R}^3, +, \circ, \mathbb{R})$ is a strongly distributive hypervector space.

In the sequel of this note, unless otherwise specified, we assume that V is a hypervector space over the field K.

Definition 2.4. A non-empty subset W of V is a subhypervector space if W is itself a hypervector space with the hyperoperation on V.

Proposition 2.5. [2] *The intersection of a family of subhypervector spaces is a subhypervector space.*

Definition 2.6. If S is a non-empty subset of V, then the linear span of S is the smallest subhypervector space of V containing S, i.e.

$$\langle S \rangle = \bigcap_{S \subseteq W \leqslant V} W.$$

Lemma 2.7. [2] *If S is a non-empty subset of V, then*

$$\langle S \rangle = \left\{ t \in \sum_{i=1}^{n} a_i \circ s_i, a_i \in K, s_i \in S \ , n \in \ \mathbb{N} \right\}.$$

Definition 2.8. A subset S of V is called linearly independent if for any vectors $v_1, v_2, \ldots, v_n$ in S, and $c_1, \ldots, c_n \in K$, $\underline{0} \in c_1 \circ v_1 + \cdots + c_n \circ v_n$, implies that $c_1 = c_2 = \cdots = c_n = 0$. A subset S of V is called linearly dependent if it is not linearly independent.

Definition 2.9. A basis for V is a linearly independent subset β of V such that linearly spans V, i.e., $\langle \beta \rangle = V$. We say that V is finite dimensional if it has a finite basis.

Definition 2.10. A hypervector space V over K is said to be K-invertible or shortly invertible if and only if $u \in a \circ v$ implies that $v \in a^{-1} \circ u$, for $u, v \in V$, $a \in K \setminus \{0\}$.

Definition 2.11. Let V and W be hypervector spaces over K. A mapping $T : V \longrightarrow W$ is called
(i) weak linear transformation iff

$$T(x + y) = T(x) + T(y) \text{ and } T(a \circ x) \cap a \circ T(x) \neq \varnothing,$$

(ii) (inclusion) linear transformation iff

$$T(x+y) = T(x) + T(y) \text{ and } T(a \circ x) \subseteq a \circ T(x),$$

(iii) good transformation iff

$$T(x+y) = T(x) + T(y) \text{ and } T(a \circ x) = a \circ T(x).$$

Definition 2.12. Let $T : V \longrightarrow W$ be a linear transformation. Then the kernel of T is denoted by $\ker T$ and is defined by

$$\ker T = \{x \in V \ : T(x) \in \Omega_W\}.$$

Definition 2.13. (i) For a fuzzy subset μ of X, $\mu \in FS(X)$, the level subset μ_t is defined by

$$\mu_t = \{x \in X : \mu(x) \geq t\}, \quad t \in [0,1].$$

(ii) The image of μ is denoted by $Im(\mu)$ and is defined by

$$Im(\mu) = \mu(X) = \{\mu(x) : x \in X\}.$$

(iii) If $\mu \in FS(X)$ and $A \subseteq X$, then by $\underline{\mu}(A)$ and $\bar{\mu}(A)$ we mean

$$\underline{\mu}(A) = \bigwedge_{a \in A} \mu(a) \qquad \text{and} \qquad \bar{\mu}(A) = \bigvee_{a \in A} \mu(a).$$

(iv) **(Extension Principle)** Let $f : X \longrightarrow Y$ be a mapping and $\mu \in FS(X)$ and $\nu \in FS(Y)$. Then we define $f(\mu) \in FS(Y)$ and $f^{-1}(\nu) \in FS(X)$ respectively as follows:

$$f(\mu)(y) = \begin{cases} \bigvee\limits_{x \in f^{-1}(y)} \mu(x) & \text{if } f^{-1}(y) \neq \varnothing, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$f^{-1}(\nu)(x) = \nu(f(x)), \quad \forall x \in X.$$

Definition 2.14. Let K be a field and $\nu \in FS(K)$. Suppose the following conditions hold:

(i) $\nu(a+b) \geq \nu(a) \wedge \nu(b), \quad \forall a, b \in K,$
(ii) $\nu(-a) \geq \nu(a), \quad \forall a \in K,$
(iii) $\nu(ab) \geq \nu(a) \wedge \nu(b), \quad \forall a, b \in K,$
(iv) $\nu(a^{-1}) \geq \nu(a), \quad \forall a \in K \setminus \{0\},$
(v) $\nu(1) = \nu(0) = 1.$

Then we call ν a fuzzy field in K and denote it by ν_K.

Obviously, Definition 2.14 is a generalization of the notion of classical field.

Definition 2.15. Let μ be a fuzzy subset of V and ν be a fuzzy field of K. Then the pair $\tilde{V} = (V, \mu)$ is said to be a fuzzy hypervector space of V over the fuzzy field ν_K, if for all $x, y \in V$ and all $a \in K$, the following conditions are satisfied:

(i) $\mu(x+y) \geq \mu(x) \wedge \mu(y)$,
(ii) $\mu(-x) \geq \mu(x)$,
(iii) $\bigwedge_{y \in a \circ x} \mu(y) \geq \nu(a) \wedge \mu(x)$, $(\underline{\mu}(a \circ x) \geq \nu(a) \wedge \mu(x))$.

If we consider $\nu = \chi_K$, the characteristic function of K, then $\tilde{V} = (V, \mu)$ is called a fuzzy subhyperspace of V.

Example 2.16. In Example 2.3, set $\mu((0,0,0)) = \frac{1}{2}$, $\mu(\mathbb{R} \times \{0\} \times \{0\} \setminus (0,0,0)) = \frac{1}{3}$, $\mu\left(\mathbb{R}^3 \setminus \mathbb{R} \times \{0\} \times \{0\}\right) = \frac{1}{5}$. Then $\tilde{V} = (V, \mu)$ is a fuzzy subhyperspace of V.

Proposition 2.17. [1] *Let $\mu \in FS(V)$ and $\nu \in FS(K)$. Then μ is a fuzzy subhyperspace if and only if μ_α is a subhypervector space of V, for all $\alpha \in Im(\mu)$. μ_α is called a level subhyperspace of V.*

Definition 2.18. [5] Let $\tilde{V} = (V, \mu)$ be a fuzzy subhyperspace of V. We say that a finite set of vectors $\{x_1, x_2, \ldots, x_n\}$ is fuzzy linearly independent in $\tilde{V}$ if and only if $\{x_1, x_2, \ldots, x_n\}$ is linearly independent in V and for all $a_1, a_2, \ldots, a_n \in K$,

$$\underline{\mu}\left(\sum_{i=1}^{n} a_i \circ x_i\right) = \bigwedge_{i=1}^{n} \underline{\mu}\left(a_i \circ x_i\right).$$

A set of vectors is fuzzy linearly independent in $\tilde{V}$ if all its finite subsets are fuzzy linearly independent in $\tilde{V}$.

Definition 2.19. A fuzzy basis for a fuzzy subhyperspace $\tilde{V} = (V, \mu)$ is a fuzzy linearly independent basis for V.

Example 2.20. For the classical vector space $(\mathbb{R}^3, +, ., \mathbb{R})$ with basis $\{i, j, k\}$, the set $\{i,j\}$ is a fuzzy basis of the fuzzy subhyperspace μ as it is defined in Example 2.16.

Definition 2.21. A set B is said to be upper well ordered if for all non-empty subsets $C \subseteq B$, $\sup C \in C$.

Lemma 2.22. [5] *If $\tilde{V} = (V, \mu)$ is a fuzzy subhyperspace such that $\mu(V)$ is upper well ordered, and W is a proper subhypervector space of V, then there exists $v \in V \setminus W$ such that*

$$\forall w \in W, \quad \mu(v+w) = \mu(v) \wedge \mu(w).$$

Lemma 2.23. [5] *If $\tilde{V} = (V, \mu)$ is a fuzzy subhyperspace such that $\mu(V)$ is upper well ordered, and if $\acute{\beta}$ is a fuzzy basis for $\tilde{W} = (W, \mu\,|_W)$, where W is a proper subhypervector space of V, then there exists $v \in V \setminus W$ such that $\overset{*}{\beta} = \acute{\beta} \cup \{v\}$ is a fuzzy basis for $\tilde{U} = \left(U = \left\langle \overset{*}{\beta} \right\rangle, \mu\,|_U\right)$, where $\left\langle \overset{*}{\beta} \right\rangle$ is the hypervector space spanned by $\overset{*}{\beta}$.*

Theorem 2.24. [5] *If V is finite dimensional, then $\tilde{V} = (V, \mu)$ has a fuzzy basis.*

3. Dimension of Fuzzy Hypervector Spaces

Definition 3.1. Let $\tilde{V}_1 = (V, \mu_1)$ and $\tilde{V}_2 = (V, \mu_2)$ be two fuzzy subhyperspaces of V. Define the intersection of $\tilde{V}_1$ and $\tilde{V}_2$ to be

$$\tilde{V}_1 \cap \tilde{V}_2 = (V,\ \mu_1 \wedge \mu_2).$$

Define the sum of $\tilde{V}_1$ and $\tilde{V}_2$ to be

$$\tilde{V}_1 + \tilde{V}_2 = (V,\ \mu_1 + \mu_2),$$

where $\mu_1 + \mu_2$ is

$$\begin{aligned}(\mu_1 + \mu_2)(x) &= \bigvee_{\substack{x = x_1 + x_2 \\ x_1, x_2 \in V}} (\mu_1(x_1) \wedge \mu_2(x_2)) \\ &= \bigvee_{x_1 \in V} (\mu_1(x_1) \wedge \mu_2(x - x_1)).\end{aligned}$$

Proposition 3.2. [5] *Let $\tilde{V}_1 = (V, \mu_1)$ and $\tilde{V}_2 = (V, \mu_2)$ be two fuzzy subhyperspaces of V. Then the following conditions hold:*

(i) $\tilde{V}_1 \cap \tilde{V}_2$ *is a fuzzy subhyperspace of* V,

(ii) $\tilde{V}_1 + \tilde{V}_2$ *is a fuzzy subhyperspace of* V,

(iii) *If $\mu_1(V)$ and $\mu_2(V)$ are upper well ordered, then $\tilde{V}_1 \cap \tilde{V}_2$ and $\tilde{V}_1 + \tilde{V}_2$ have fuzzy basis.*

Definition 3.3. We define the dimension of a fuzzy hypervector space $\tilde{V} = (V, \mu)$ to be

$$\dim \tilde{V} = \bigvee \left\{ \sum_{v \in \beta} \mu(v) : \beta \text{ is a basis for } V \right\}.$$

Clearly dim is a function from the class of all fuzzy hypervector spaces to $[0, \infty]$. A fuzzy hypervector space $\tilde{V}$ is finite dimensional if and only if $\dim \tilde{V} < \infty$.

Theorem 3.4. [5] *If $\tilde{V} = (V, \mu)$ is finite dimensional, then*

$$\dim(\tilde{V}) = \sum_{v \in \beta^*} \mu(v),$$

where β^* is any fuzzy basis for $\tilde{V}$.

We now investigate deeper the properties of dimension of fuzzy hypervector spaces. An important result from crisp theory that we would like to have in fuzzy setting is: If $\tilde{V}_1 = (V, \mu_1)$ and $\tilde{V}_2 = (V, \mu_2)$ are two fuzzy hypervector spaces then

$$\dim(\tilde{V}_1 + \tilde{V}_2) = \dim(\tilde{V}_1) + \dim(\tilde{V}_2) - \dim(\tilde{V}_1 \cap \tilde{V}_2) \tag{1}$$

Unfortunately this is not always true.

Example 3.5. For the classical vector space $(\mathbb{R}^3, +, ., \mathbb{R})$ with basis $\{i, j, k\}$and its subspace $\left(\mathbb{R}^2 \times \{0\}, +, ., \mathbb{R}\right)$ with basis $\{i, j\}$, let

$$\begin{cases} \circ: & \mathbb{R} \times \mathbb{R}^3 \longrightarrow P^*(\mathbb{R}^3) \\ & a \circ (x, y, z) = (ax, ay, az) + \left(\mathbb{R}^2 \times \{0\}\right). \end{cases}$$

Then $(\mathbb{R}^3, +, \circ, \mathbb{R})$ is a strongly distributive hypervector space with basis $\{k\}$. Now consider $V = (\mathbb{R}^3, +, \circ, \mathbb{R})$, $\mu_1 \equiv \frac{1}{3}$, $\mu_2 \equiv \frac{1}{5}$, then $\mu_1 + \mu_2 \equiv \frac{1}{5}$. Clearly $\dim(\tilde{V}_1) = \frac{1}{3}$, $\dim(\tilde{V}_2) = \frac{1}{5}$, $\dim(\tilde{V}_1 + \tilde{V}_2) = \frac{1}{5}$ and $\dim(\tilde{V}_1 \cap \tilde{V}_2) = \frac{1}{5}$. Thus $\dim(\tilde{V}_1 + \tilde{V}_2) \neq \dim(\tilde{V}_1) + \dim(\tilde{V}_2) - \dim(\tilde{V}_1 \cap \tilde{V}_2)$. But if $\mu_1 \equiv \frac{1}{3}$, $\mu_2\left(\mathbb{R}^3 \backslash (0,0,0)\right) = \frac{1}{5}$ and $\mu_2\left((0,0,0)\right) = \frac{1}{3}$, then $\dim(\tilde{V}_1) = \frac{1}{3}$, $\dim(\tilde{V}_2) = \frac{1}{5}$, $\dim(\tilde{V}_1 + \tilde{V}_2) = \frac{1}{3}$, $\dim(\tilde{V}_1 \cap \tilde{V}_2) = \frac{1}{5}$ and (1) holds.

We now shall prove that under certain conditions (1) holds.

Lemma 3.6. *Let V be finite dimensional, $\tilde{V}_1 = (V, \mu_1)$ and $\tilde{V}_2 = (V, \mu_2)$ be two fuzzy subhyperspaces such that $\mu_1(\underline{0}) \geq \bar{\mu}_2(V \backslash \{\underline{0}\})$ and $\mu_2(\underline{0}) \geq \bar{\mu}_1(V \backslash \{\underline{0}\})$. Then there exists a basis β^* for V which is also a fuzzy basis for $\tilde{V}_1$, $\tilde{V}_2$, $\tilde{V}_1 \cap \tilde{V}_2$ and $\tilde{V}_1 + \tilde{V}_2$. In addition, if $A_1 = \{x \in V : \mu_1(x) < \mu_2(x)\}$ and $A_2 = V \backslash A_1$, then for all $v \in \beta^* \cap A_1$,*

$$(\mu_1 \wedge \mu_2)(v) = \mu_1(v) \qquad \textit{and} \qquad (\mu_1 + \mu_2)(v) = \mu_2(v),$$

and for all $v \in \beta^* \cap A_2$,

$$(\mu_1 \wedge \mu_2)(v) = \mu_2(v) \qquad \textit{and} \qquad (\mu_1 + \mu_2)(v) = \mu_1(v).$$

Proof. (By induction on $\dim V$). In the case $\dim V = 1$ the statement is clearly true. Now suppose that the theorem is true for all fuzzy hypervector spaces with dimension of the underlying hypervector space equal to n.

Let $\dim V = n + 1 > 1$ and let $\tilde{\beta}_1 = \{v_1, v_2, \ldots, v_n, v_{n+1}\}$ be any fuzzy basis for $\tilde{V}_1$. Without loss of generality, assume that $\mu_1(v_1) \leq \mu_1(v_i)$, $i = 2, \ldots, n+1$. Let $W = \langle v_2, \ldots, v_n, v_{n+1} \rangle$. Since $n + 1 > 1$, $W \neq \{\underline{0}\}$ and $\dim W = n$. Define the following two fuzzy hypervector spaces:

$$\tilde{W}_1 = (W, \mu_1|_W) \quad \text{and} \quad \tilde{W}_2 = (W, \mu_2|_W).$$

By the inductive hypothesis there exists a basis β for W, which is also a fuzzy basis for $\tilde{W}_1$, $\tilde{W}_2$, $\tilde{W}_1 \cap \tilde{W}_2$ and $\tilde{W}_1 + \tilde{W}_2$. Also for all $v \in \beta \cap A_1$,

$$(\mu_1|_W \wedge \mu_2|_W)(v) = \mu_1|_W(v) \qquad \text{and} \qquad (\mu_1|_W + \mu_2|_W)(v) = \mu_2|_W(v),$$

and for all $v \in \beta \cap A_2$,

$$(\mu_1|_W \wedge \mu_2|_W)(v) = \mu_2|_W(v) \qquad \text{and} \qquad (\mu_1|_W + \mu_2|_W)(v) = \mu_1|_W(v).$$

Now we shall show that β can be extended to β^*, such that β^* is a fuzzy basis for $\tilde{V}_1$, $\tilde{V}_2$, $\tilde{V}_1 \cap \tilde{V}_2$ and $\tilde{V}_1 + \tilde{V}_2$. Furthermore, for all $v \in \beta^* \cap A_1$,

$$(\mu_1 \wedge \mu_2)(v) = \mu_1(v) \qquad \text{and} \qquad (\mu_1 + \mu_2)(v) = \mu_2(v),$$

and for all $v \in \beta^* \cap A_2$,

$$(\mu_1 \wedge \mu_2)(v) = \mu_2(v) \qquad \text{and} \qquad (\mu_1 + \mu_2)(v) = \mu_1(v).$$

First we have to show that for all $x \in W$,

$$(\mu_1 + \mu_2)|_W (x) = (\mu_1|_W + \mu_2|_W)(x) \tag{2}$$

Let $x \in W \setminus \{\underline{0}\}$. Then

$$\begin{aligned} (\mu_1 + \mu_2)|_W (x) &= \bigvee_{x_1 \in V} (\mu_1(x_1) \wedge \mu_2(x - x_1)) \\ &= \left(\bigvee_{x_1 \in W} (\mu_1(x_1) \wedge \mu_2(x - x_1)) \right) \\ &\quad \vee \left(\bigvee_{x_2 \in V \setminus W} (\mu_1(x_2) \wedge \mu_2(x - x_2)) \right). \end{aligned}$$

Since $x \in W \setminus \{\underline{0}\}$, we have

$$\mu_1(x) \wedge \mu_2(x - x) = \mu_1(x) \wedge \mu_2(\underline{0}) \leq \bigvee_{x_1 \in W} (\mu_1(x_1) \wedge \mu_2(x - x_1)),$$

$$\mu_1(\underline{0}) \wedge \mu_2(x - \underline{0}) = \mu_1(\underline{0}) \wedge \mu_2(x) \leq \bigvee_{x_1 \in W} (\mu_1(x_1) \wedge \mu_2(x - x_1)).$$

Since $\mu_1(\underline{0}) \geq \bar{\mu}_2(W \setminus \{\underline{0}\})$ and $\mu_2(\underline{0}) \geq \bar{\mu}_1(W \setminus \{\underline{0}\})$, so

$$\mu_1(x) \wedge \mu_2(\underline{0}) = \mu_1(x) \quad \textit{and} \quad \mu_1(\underline{0}) \wedge \mu_2(x) = \mu_2(x),$$

and this leads to the following inequality:

$$\mu_1(x) \vee \mu_2(x) \leq \bigvee_{x_1 \in W} (\mu_1(x_1) \wedge \mu_2(x - x_1)) \tag{3}$$

Suppose that

$$\bigvee_{x_1 \in W} (\mu_1(x_1) \wedge \mu_2(x - x_1)) < \bigvee_{x_2 \in V \setminus W} (\mu_1(x_2) \wedge \mu_2(x - x_2)) \tag{4}$$

This means that there exists $\tilde{x} \in V \setminus W$, such that

$$\bigvee_{x_1 \in W} (\mu_1(x_1) \wedge \mu_2(x - x_1)) < \mu_1(\tilde{x}) \wedge \mu_2(x - \tilde{x}).$$

In view of (3) we must have

$$\mu_1(x) \vee \mu_2(x) < \mu_1(\tilde{x}) \wedge \mu_2(x - \tilde{x}) \tag{5}$$

Since $\tilde{x} \in V \setminus W$ and $\bar{\mu}_1(V \setminus W) = \mu_1(v_1) \leq \mu_1(v_i)$, for all $i = 2, \ldots, n+1$, we must have $\mu_1(x) \geq \mu_1(\tilde{x})$. Thus (5) becomes $\mu_1(x) \vee \mu_2(x) < \mu_1(x) \wedge \mu_2(x - \tilde{x})$. It is easy to check that the last inequality never holds (use the properties of $\wedge$, $\vee$ and $<$). This means that our assumption (4) is false. Therefore, we must have

$$\bigvee_{x_1 \in W} (\mu_1(x_1) \wedge \mu_2(x - x_1)) \geq \bigvee_{x_2 \in V \backslash W} (\mu_1(x_2) \wedge \mu_2(x - x_2)) .$$

Clearly, it is also true if $x = \underline{0}$. Thus for all $x \in W$:

$$\begin{aligned} (\mu_1 + \mu_2)\,|_W\,(x) &= \bigvee_{x_1 \in V} (\mu_1(x_1) \wedge \mu_2(x - x_1)) \\ &= \left(\bigvee_{x_1 \in W} (\mu_1(x_1) \wedge \mu_2(x - x_1)) \right) \\ &\quad \vee \left(\bigvee_{x_2 \in V \backslash W} (\mu_1(x_2) \wedge \mu_2(x - x_2)) \right) . \end{aligned}$$

But since

$$\bigvee_{x_1 \in W} (\mu_1(x_1) \wedge \mu_2(x - x_1)) \geq \bigvee_{x_2 \in V \backslash W} (\mu_1(x_2) \wedge \mu_2(x - x_2)) ,$$

we get

$$\begin{aligned} (\mu_1 + \mu_2)\,|_W\,(x) &= \bigvee_{x_1 \in W} (\mu_1(x_1) \wedge \mu_2(x - x_1)) \\ &= \bigvee_{x_1 \in W} (\mu_1\,|_W\,(x_1) \wedge \mu_2\,|_W\,(x - x_1)) \\ &= (\mu_1\,|_W + \mu_2\,|_W\,)\,(x). \end{aligned}$$

This establishes (2). Clearly (2) implies that β is fuzzy linearly independent in $\tilde{V}_1 + \tilde{V}_2$.

Let $v^* \in V \backslash W$, such that $\mu_2\,(v^*) = \bar{\mu}_2\,(V \backslash W)$. Clearly such v^* exists, since μ_2 takes on a finite number of values. By Lemma 2.22, and Lemma 2.23, $\beta^* = \beta \cup \{v^*\}$ is an extension of fuzzy basis β of $\tilde{W}_2$ to a fuzzy basis of $\tilde{V}_2$. Since $\bar{\mu}_1\,(V \backslash W) = \mu_1\,(v_1)$, then β^* is also an extension of fuzzy basis β of $\tilde{W}_1$ to a fuzzy basis of $\tilde{V}_1$.

Now we must show that β^* is an extension of the fuzzy basis β of $\tilde{W}_1 \cap \tilde{W}_2$ to a fuzzy basis of $\tilde{V}_1 \cap \tilde{V}_2$. If $v^* \in A_1$, then since μ_1 is constant on $V \backslash W$, so

$$\overline{(\mu_1 \wedge \mu_2)}(A_1 \cap V \backslash W) = \mu_1\,(v^*) ,$$

and for all $z \in A_2 \cap V \backslash W$,

$$(\mu_1 \wedge \mu_2)\,(z) \leq \mu_1\,(v^*) .$$

It follows that for $v^* \in A_1$, we have

$$(\mu_1 \wedge \mu_2)\,(v^*) = \overline{(\mu_1 \wedge \mu_2)}\,(V \backslash W) .$$

If $v^* \in A_2$, then $\mu_2(v^*) \leq \mu_1(v^*)$. Since $\mu_2(v^*) = \bar{\mu}_2\,(V \backslash W)$ and μ_1 is constant on $V \backslash W$, we get $A_1 \cap V \backslash W = \emptyset$. Therefore, for $v^* \in A_2$, we have

$$(\mu_1 \wedge \mu_2)\,(v^*) = \overline{(\mu_1 \wedge \mu_2)}\,(V \backslash W) .$$

By Lemma 2.23, we may now conclude that β^* extends the fuzzy basis β for $\tilde{W}_1 \cap \tilde{W}_2$ to a fuzzy basis for $\tilde{V}_1 \cap \tilde{V}_2$.

Now we prove that β^* is also an extension of the fuzzy basis β of $\tilde{W}_1 + \tilde{W}_2$ to a fuzzy basis for $\tilde{V}_1 + \tilde{V}_2$. Suppose that there exists $z \in V \backslash W$, such that $(\mu_1 + \mu_2)(v^*) < (\mu_1 + \mu_2)(z)$. Clearly, the vector z can be written in the form $z \in a \circ (v^* + w)$, where $a \neq 0$ and $w \in W$. Therefore, we have

$$\begin{aligned}(\mu_1 + \mu_2)(v^*) &< (\mu_1 + \mu_2)(z) \\ &\leq \underline{(\mu_1 + \mu_2)}(a \circ (v^* + w)) \\ &= (\mu_1 + \mu_2)(v^* + w).\end{aligned}$$

This means that there exists $x_1 \in V$, such that for all $\tilde{x}$,

$$\mu_1(\tilde{x}) \wedge \mu_2(v^* - \tilde{x}) < \mu_1(x_1) \wedge \mu_2(v^* + w - x_1) \tag{6}$$

In particular (6) is true if $\tilde{x} = \underline{0}$, i.e.

$$\mu_1(\underline{0}) \wedge \mu_2(v^*) < \mu_1(x_1) \wedge \mu_2(v^* + w - x_1).$$

But since $\mu_1(\underline{0}) \geq \bar{\mu}_2(V \backslash \{\underline{0}\})$ we have

$$\mu_2(v^*) < \mu_1(x_1) \wedge \mu_2(v^* + w - x_1) \tag{7}$$

If $x_1 \in W$, then since $w \in W$ we must have $w - x_1 \in W$. Thus by Lemma 2.22, $\mu_2(v^* + w - x_1) = \mu_2(v^*) \wedge \mu_2(w - x_1)$ and so (7) becomes $\mu_2(v^*) < \mu_1(x_1) \wedge \mu_2(v^*) \wedge \mu_2(w - x_1)$, which is impossible. Thus $x_1 \in V \backslash W$. Let $\tilde{x} = v^*$ in (6). Since $\mu_2(\underline{0}) \geq \bar{\mu}_1(V \backslash \{\underline{0}\})$ we have

$$\mu_1(v^*) < \mu_1(x_1) \wedge \mu_2(v^* + w - x_1) \tag{8}$$

Recall that $\mu_1(V \backslash W) = \mu_1(v_1)$ and thus $\mu_1(v^*) = \mu_1(x_1)$. This again means that the inequality (8) is false. That is, for all $z \in V \backslash W$, $(\mu_1 + \mu_2)(v^*) \geq \underline{(\mu_1 + \mu_2)}(z)$. Therefore, by Lemma 2.23, β^* is an extension of the fuzzy basis β of $\tilde{W}_1 + \tilde{W}_2$ to a fuzzy basis of $\tilde{V}_1 + \tilde{V}_2$.

Now we prove that if $v^* \in A_1$, then $(\mu_1 + \mu_2)(v^*) = \mu_2(v^*)$, and if $v^* \in A_2$, then $(\mu_1 + \mu_2)(v^*) = \mu_1(v^*)$. From the definition we have:

$$(\mu_1 + \mu_2)(v^*) = \bigvee_{x_1 \in V} (\mu_1(x_1) \wedge \mu_2(v^* - x_1)).$$

Let $\tilde{x}$ be such that

$$\bigvee_{x_1 \in V} (\mu_1(x_1) \wedge \mu_2(v^* - x_1)) = \mu_1(\tilde{x}) \wedge \mu_2(v^* - \tilde{x}).$$

By substituting $x_1 = \underline{0}$ and then $x_1 = \tilde{x}$ and recalling that $\mu_1(\underline{0}) \geq \bar{\mu}_2(V \backslash \{\underline{0}\})$ and $\mu_2(\underline{0}) \geq \bar{\mu}_1(V \backslash \{\underline{0}\})$, we obtain

$$\mu_1(v^*) \vee \mu_2(v^*) \leq \mu_1(\tilde{x}) \wedge \mu_2(v^* - \tilde{x}).$$

Suppose that

$$\mu_1(v^*) \vee \mu_2(v^*) < \mu_1(\tilde{x}) \wedge \mu_2(v^* - \tilde{x}). \tag{9}$$

If $\tilde{x} \in W$, then by Lemma 2.22, (as $\beta = \beta^* \cup \{v^*\}$ is a fuzzy basis for $\tilde{V_2}$), (9) becomes

$$\mu_1(v^*) \vee \mu_2(v^*) < \mu_1(\tilde{x}) \wedge \mu_2(v^*) \wedge \mu_2(\tilde{x}).$$

This is never true, and thus $\tilde{x} \in V \backslash W$. But since $\mu_1(v^*) = \mu_1(\tilde{x})$, inequality (9) never holds, and so

$$\mu_1(v^*) \vee \mu_2(v^*) = \mu_1(\tilde{x}) \wedge \mu_2(v^* - \tilde{x}) = (\mu_1 + \mu_2)(v^*) \tag{10}$$

equation (10) clearly leads to the required result.

This completes the proof. □

Lemma 3.6, is a very valuable tool in manipulating fuzzy hypervector spaces. The next result immediately follows from Lemma 3.6.

Theorem 3.7. *Let V be finite dimensional, $\tilde{V_1} = (V, \mu_1)$ and $\tilde{V_2} = (V, \mu_2)$ be two fuzzy hypervector spaces such that $\mu_1(\underline{0}) \geq \bar{\mu}_2(V \backslash \{\underline{0}\})$ and $\mu_2(\underline{0}) \geq \bar{\mu}_1(V \backslash \{\underline{0}\})$. Then*

$$\dim(\tilde{V_1} + \tilde{V_2}) = \dim(\tilde{V_1}) + \dim(\tilde{V_2}) - \dim(\tilde{V_1} \cap \tilde{V_2}).$$

Proof. Let β be the fuzzy basis in Lemma 3.6. Then

$$\begin{aligned}
\dim(\tilde{V_1} + \tilde{V_2}) &= \sum_{v \in \beta} (\mu_1 + \mu_2)(v) \\
&= \sum_{v \in A_1 \cap \beta} \mu_2(v) + \sum_{v \in A_2 \cap \beta} \mu_1(v) \\
&= \sum_{v \in A_1 \cap \beta} \mu_2(v) + \sum_{v \in A_2 \cap \beta} \mu_1(v) + \sum_{v \in A_2 \cap \beta} \mu_2(v) \\
&\quad + \sum_{v \in A_1 \cap \beta} \mu_1(v) - \sum_{v \in A_2 \cap \beta} \mu_2(v) - \sum_{v \in A_1 \cap \beta} \mu_1(v) \\
&= \sum_{v \in \beta} \mu_1(v) + \sum_{v \in \beta} \mu_2(v) - \sum_{v \in A_2 \cap \beta} \mu_2(v) - \sum_{v \in A_1 \cap \beta} \mu_1(v) \\
&= \dim(\tilde{V_1}) + \dim(\tilde{V_2}) - \left(\sum_{v \in \beta} (\mu_1 \wedge \mu_2)(v) \right) \\
&= \dim(\tilde{V_1}) + \dim(\tilde{V_2}) - \dim(\tilde{V_1} \cap \tilde{V_2}).
\end{aligned}$$

□

Definition 3.8. Let $\tilde{V} = (V, \mu)$ be a fuzzy hypervector space and let $T : V \longrightarrow W$ be a linear transformation. Then $\widetilde{\ker} T = (\ker T, \mu|_{\ker T})$ and $\widetilde{Im}\, T = (Im\, T, T(\mu)|_{ImT})$.

Theorem 3.9. *Let V be strongly left distributive and finite dimensional. Let $\tilde{V} = (V, \mu)$ be a fuzzy hypervector space and $T : V \longrightarrow W$ be an injective good transformation. Then*

$$\dim\left(\widetilde{\ker} T\right) + \dim\left(\widetilde{Im}\, T\right) = \dim\left(\tilde{V}\right).$$

Proof. Let $\beta_{\ker} = \{t_1, \ldots, t_m\}$ be a fuzzy basis for $\widetilde{\ker T}$. We extend $\beta_{\ker}$ to a fuzzy basis $\beta = \{t_1, \ldots, t_m, x_1, \ldots, x_n\}$ for $\tilde{V}$. (It is possible by repeated application of Lemma 2.23). Setting $\beta_{ex} = \{x_1, \ldots, x_n\}$.

First we show that $T(\beta_{ex}) = \beta_{Im}$ is a fuzzy basis for $\widetilde{Im}\ T$.

Step 1: β_{Im} generates ImT:

If $T(x) \in ImT$, then $x \in V$, and so $x \in \sum_{i=1}^{m} a_i \circ t_i + \sum_{j=1}^{n} b_j \circ x_j,\ a_i, b_j \in K,\ i = 1, \ldots, m,\ j = 1, \ldots, n$. Hence

$$\begin{aligned} T(x) &\in \sum_{i=1}^{m} a_i \circ T(t_i) + \sum_{j=1}^{n} b_j \circ T(x_j) \\ &\subseteq \sum_{i=1}^{m} a_i \circ \Omega_W + \sum_{j=1}^{n} b_j \circ T(x_j) \\ &= \Omega_W + \sum_{j=1}^{n} b_j \circ T(x_j) \\ &= \sum_{j=1}^{n} b_j \circ T(x_j) \\ &\subseteq \langle \beta_{Im} \rangle . \end{aligned}$$

Thus β_{Im} generates ImT.

Step 2: β_{Im} is linearly independent:

If $\underline{0} \in \sum_{j=1}^{n} b_j \circ T(x_j)$, then $\underline{0} = T(x)$, such that $x \in \sum_{j=1}^{n} b_j \circ x_j$ and $x \in \ker T$. Thus $x \in \sum_{i=1}^{m} a_i \circ t_i$, and so

$$\begin{aligned} \underline{0} &\in \sum_{i=1}^{m} a_i \circ t_i - \sum_{j=1}^{n} b_j \circ x_j \\ &= \sum_{i=1}^{m} a_i \circ t_i + \sum_{j=1}^{n} (-b_j) \circ x_j . \end{aligned}$$

Hence $a_i = 0,\ i = 1, \ldots, m$, and $b_j = 0,\ j = 1, \ldots, n$.

Step 3: β_{Im} is fuzzy linearly independent:

β_{Im} is linearly independent by step 2. Now let $a_1, \ldots, a_n \in K$, where not all are zero. Then we have

$$\begin{aligned} \underline{T(\mu)}\left(\sum_{i=1}^{n} a_i \circ T(x_i)\right) &= \bigwedge_{w \in \sum_{i=1}^{n} a_i \circ T(x_i)} T(\mu)(w) \\ &= \bigwedge_{\substack{w = w_1 + \cdots + w_n, \\ w_i \in a_i \circ T(x_i)}} T(\mu)(w) \\ &= \begin{cases} \bigwedge_{\substack{w = w_1 + \cdots + w_n, \\ w_i \in a_i \circ T(x_i)}} \left(\bigvee_{v \in T^{-1}(w)} \mu(v) \right) & \text{if } T^{-1}(w) \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Since $w_1 + \cdots + w_n \in Im\ T$, thus

$$\underline{T(\mu)}\left(\sum_{i=1}^{n} a_i \circ T(x_i)\right) = \bigwedge_{w_i \in a_i \circ T(x_i)} \left(\bigvee_{v \in T^{-1}(w_1+\cdots+w_n)} \mu(v)\right).$$

Also

$$\left.\begin{array}{r} v \in T^{-1}(w_1 + \cdots + w_n) \\ w_i \in a_i \circ T(x_i),\ i = 1, \ldots, n \end{array}\right\} \begin{array}{ll} \Longleftrightarrow & T(v) = w_1 + \cdots + w_n \\ & \quad \in a_1 \circ T(x_1) + \cdots + a_n \circ T(x_n) \\ \Longleftrightarrow & T(v) \in T(a_1 \circ x_1 + \cdots + a_n \circ x_n) \\ \Longleftrightarrow & v \in a_1 \circ x_1 + \cdots + a_n \circ x_n \\ \Longleftrightarrow & v = v_1 + \cdots + v_n,\ v_i \in a_i \circ x_i \\ \Longleftrightarrow & v - v_1 - \cdots - v_n = \underline{0} \\ \Longleftrightarrow & T(v - v_1 - \cdots - v_n) = T(\underline{0}) = \underline{0} \in 0 \circ \underline{0}_W \\ \Longleftrightarrow & v - v_1 - \cdots - v_n \in \ker T \\ \Longleftrightarrow & v \in \ker T + (v_1 + \cdots + v_n), \quad v_i \in a_i \circ x_i. \end{array}$$

Hence

$$\underline{T(\mu)}\left(\sum_{i=1}^{n} a_i \circ T(x_i)\right) = \bigwedge_{w_i \in a_i \circ T(x_i)} \left(\bigvee_{v \in \ker T + (v_1+\cdots+v_n)} \mu(v)\right).$$

If $x \in \ker T$, then $x = \underline{0}$ or $x \in \sum_{i=1}^{m} b_i \circ t_i$, where $t_i \in \beta_{\ker}$ and not all b_i are zero. So if $v \in \ker T + (v_1 + \cdots + v_n)$, then

$$\begin{array}{lllll} & \text{either} & v \in \underline{0} + (v_1 + \cdots + v_n) & \text{or} & v \in \sum_{i=1}^{m} b_i \circ t_i + (v_1 + \cdots + v_n) \\ \Longrightarrow & \text{either} & v \in \sum_{i=1}^{n} a_i \circ x_i & \text{or} & v \in \sum_{i=1}^{m} b_i \circ t_i + \sum_{i=1}^{n} a_i \circ x_i \\ \Longrightarrow & \text{either} & \mu(v) \geq \underline{\mu}\left(\sum_{i=1}^{n} a_i \circ x_i\right) & \text{or} & \mu(v) \geq \underline{\mu}\left(\sum_{i=1}^{m} b_i \circ t_i + \sum_{i=1}^{n} a_i \circ x_i\right) \end{array}$$

Hence by Definition 2.18,

$$\text{either} \quad \mu(v) \geq \bigwedge_{i=1}^{n} \underline{\mu}\,(a_i \circ x_i) \quad \text{or} \quad \mu(v) \geq \left(\bigwedge_{i=1}^{m} \underline{\mu}\,(b_i \circ t_i)\right) \bigwedge \left(\bigwedge_{i=1}^{n} \underline{\mu}\,(a_i \circ x_i)\right).$$

Clearly,

$$\bigwedge_{i=1}^{n} \underline{\mu}\,(a_i \circ x_i) \geq \left(\bigwedge_{i=1}^{m} \underline{\mu}\,(b_i \circ t_i)\right) \bigwedge \left(\bigwedge_{i=1}^{n} \underline{\mu}\,(a_i \circ x_i)\right).$$

Thus

$$\underline{T(\mu)}\left(\sum_{i=1}^{n} a_i \circ T(x_i)\right) = \bigwedge_{i=1}^{n} \underline{\mu}\,(a_i \circ x_i).$$

By the same argument

$$\underline{T(\mu)}\,(a_i \circ T(x_i)) = \underline{\mu}\,(a_i \circ x_i).$$

Hence

$$\underline{T(\mu)}\left(\sum_{i=1}^{n} a_i \circ T(x_i)\right) = \bigwedge_{i=1}^{n} \underline{T(\mu)}\left(a_i \circ T(x_i)\right).$$

Therefore β_{Im} is a fuzzy basis for ImT.

Now by definition of dimension of a fuzzy subhyperspace we have:

$$\begin{aligned} \dim\left(\tilde{V}\right) &= \sum_{v \in \beta_{\ker} \cup \beta_{ex}} \mu(v) \\ &= \sum_{v \in \beta_{\ker}} \mu(v) + \sum_{v \in \beta_{ex}} \mu(v). \end{aligned}$$

But by the above if $x \in \langle \beta_{ex} \rangle$, then it follows that $T(\mu)\left(T(x)\right) = \mu\left(x\right)$. Consequently,

$$\begin{aligned} \dim\left(\tilde{V}\right) &= \sum_{v \in \beta_{\ker}} \mu(v) + \sum_{v \in \beta_{ex}} T(\mu)\left(T(v)\right) \\ &= \sum_{v \in \beta_{\ker}} \mu(v) + \sum_{w \in \beta_{Im}} T(\mu)\left(w\right) \\ &= \dim\left(\widetilde{\ker T}\right) + \dim\left(\widetilde{Im\ T}\right). \end{aligned}$$

□

4. **Isomorphism of Fuzzy Hypervector Spaces**

Definition 4.1. Let $\tilde{V} = (V, \mu)$ and $\tilde{W} = (W, \eta)$ be two fuzzy hypervector spaces. Then μ is said to be isomorphic to η, written as $\mu \cong \eta$, if there exists a good isomorphism $T : V \longrightarrow W$, such that $T(\mu) = \eta$. In other words, the following diagram is commutative:

Clearly, the relation of isomorphism in the set of all fuzzy hypervector spaces is an equivalence relation.

Lemma 4.2. [28] *Let μ ba a fuzzy subset of a set X and let $x \in X$. Then $\mu(x) = \sup\left\{t : x \in \mu_t\right\}$.*

Theorem 4.3. *Let $\tilde{V} = (V, \mu)$ and $\tilde{W} = (W, \eta)$ be two fuzzy subhyperspaces. Then $\mu \cong \eta$, if and only if $Im\ \mu = Im\ \eta$ and $T(\mu_t) = \eta_t$, $\forall t \in [0, 1]$, where $T : V \longrightarrow W$ is a good isomorphism.*

Proof. Since the map T is one-to-one and onto, it follows that $\left(T(\mu)\right)_t = T(\mu_t)$, $\forall t \in [0, 1]$. Now for any $w \in W$, by Lemma 4.2 we have

$$\begin{aligned} \left(T(\mu)\right)(w) &= \sup\left\{t : w \in \left(T(\mu)\right)_t\right\} \\ &= \sup\left\{t : w \in T(\mu_t)\right\} \\ &= \sup\left\{t : w \in \eta_t\right\} \\ &= \eta(w). \end{aligned}$$

Hence $T(\mu) = \eta$ and so $\mu \cong \eta$. □

Theorem 4.4. *Let* $(V,+,\circ,K)$ *and* $(W,\dot{+},\star,K)$ *be finite dimensional hypervector spaces and let* $\tilde{V}=(V,\mu)$ *and* $\tilde{W}=(W,\eta)$ *be two fuzzy subhyperspaces. If* $\mu\cong\eta$*, then* $\dim\mu=\dim\eta$.

Proof. Let $T:V\longrightarrow W$ be a good isomorphism, such that $T(\mu)=\eta$. Let $\beta=\{x_1,\dots,x_n\}$ be any fuzzy basis of μ. At first we show that $T(\beta)=\{T(x_1),\dots,T(x_n)\}$ is a basis of $T(V)=W$. If $\underline{0}_W\in\sum_{i=1}^{n}a_i\star T(x_i)$, then $\underline{0}_W=s_1\dot{+}\cdots\dot{+}s_n$, such that $s_i\in a_i\star T(x_i)=T(a_i\circ x_i),\ i=1,\dots,n$, and so $s_i=T(t_i),\ t_i\in a_i\circ x_i,\ i=1,\dots,n$. Thus

$$\begin{aligned}T(\underline{0}_V)&=\underline{0}_W=T(t_1)\dot{+}\cdots\dot{+}T(t_n)\\&=T(t_1+\cdots+t_n),\end{aligned}$$

and hence $\underline{0}_V=t_1+\cdots+t_n\in\sum_{i=1}^{n}a_i\circ x_i$. Then $a_1=a_2=\cdots=a_n=0$, and so $T(\beta)$ is linearly independent. It is clear that $T(\beta)$ generates W. Also for all $a_1,\dots,a_n\in K$, we have

$$\begin{aligned}\underline{T(\mu)}\left(\sum_{i=1}^{n}a_i\star T(x_i)\right)&=\underline{T(\mu)}\left(T\left(\sum_{i=1}^{n}a_i\circ x_i\right)\right)\\&=\bigwedge_{s\in T\left(\sum_{i=1}^{n}a_i\circ x_i\right)}T(\mu)\,(s)\\&=\bigwedge_{t\in\sum_{i=1}^{n}a_i\circ x_i}T(\mu)\,(T\,(t))\\&=\bigwedge_{t\in\sum_{i=1}^{n}a_i\circ x_i}\mu\,(t)\\&=\underline{\mu}\left(\sum_{i=1}^{n}a_i\circ x_i\right)\\&=\bigwedge_{i=1}^{n}\underline{\mu}\,(a_i\circ x_i)\\&=\bigwedge_{i=1}^{n}\underline{T(\mu)}\,(T\,(a_i\circ x_i))\\&=\bigwedge_{i=1}^{n}\underline{T(\mu)}\,(a_i\circ T(x_i))\,.\end{aligned}$$

Thus $T(\beta)$ is a fuzzy basis of $T(\mu)$. Therefore,

$$\begin{aligned}\dim\eta&=\dim T(\mu)\\&=\sum_{T(x)\in T(\beta)}(T(\mu))\,(T(x))\\&=\sum_{x\in T^{-1}(T(\beta))}\mu(x)\\&=\sum_{x\in\beta}\mu(x)\\&=\dim\mu.\end{aligned}$$

□

The converse of the above theorem turns out to be false. This is shown in the following example.

Example 4.5. Consider strongly left distributive hypervector space $(\mathbb{R}^3, +, \circ, \mathbb{R})$ in Example 2.3, with basis $\{i, j\}$. Let $V = W = (\mathbb{R}^3, +, \circ, \mathbb{R})$, and $A = \mathbb{R} \times \{0\} \times \{0\}$ and $B = \{0\} \times \mathbb{R} \times \{0\}$. Consider the fuzzy subsets μ and η of V defined by

$$\mu(x) = \begin{cases} 1 & x \in A, \\ 0.3 & \text{otherwise}, \end{cases} \qquad \text{and} \qquad \eta(x) = \begin{cases} 0.9 & x \in B, \\ 0.4 & \text{otherwise}. \end{cases}$$

It follows from Proposition 2.17, that μ and η are fuzzy subhyperspaces of V. Furthermore, for any $a, b \in \mathbb{R}$, we have

$$\underline{\mu}(a \circ i + b \circ j) = \mu(a, b, 0) = \begin{cases} 1 & b = 0, \\ 0.3 & b \neq 0, \end{cases}$$

$$\underline{\eta}(a \circ i + b \circ j) = \eta(a, b, 0) = \begin{cases} 0.9 & a = 0, \\ 0.4 & a \neq 0, \end{cases}$$

$$\min\left\{\underline{\mu}(a \circ i), \underline{\mu}(b \circ j)\right\} = \min\{\mu(a,0,0), \mu(0,b,0)\} = \begin{cases} 1 & b = 0, \\ 0.3 & b \neq 0, \end{cases}$$

and

$$\min\left\{\underline{\eta}(a \circ i), \underline{\eta}(b \circ j)\right\} = \min\{\eta(a,0,0), \eta(0,b,0)\} = \begin{cases} 0.9 & a = 0, \\ 0.4 & a \neq 0. \end{cases}$$

These observations yield

$$\underline{\mu}(a \circ i + b \circ j) = \min\left\{\underline{\mu}(a \circ i), \underline{\mu}(b \circ j)\right\} \quad \text{and} \quad \underline{\eta}(a \circ i + b \circ j) = \min\left\{\underline{\eta}(a \circ i), \underline{\eta}(b \circ j)\right\}.$$

Thus the set $\{i, j\}$ is a fuzzy basis for both μ and η. Therefore,

$$\dim \mu = \mu(i) + \mu(j) = 1.3 \qquad \text{and} \qquad \dim \eta = \eta(i) + \eta(j) = 1.3.$$

Hence, $\dim \mu = \dim \eta$. However, since $Im\mu \neq Im\eta$, it follows from Theorem 4.3 that μ is not isomorphic to η.

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