A FUZZY MINIMUM RISK MODEL FOR THE RAILWAY TRANSPORTATION PLANNING PROBLEM

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Abstract. The railway transportation planning under the fuzzy environment is investigated in this paper. As a main result, a new modeling method, called minimum risk chance-constrained model, is presented based on the credibility measure. For the convenience of solving the mathematical model, the crisp equivalents of chance functions are analyzed under the condition that the involved fuzzy parameters are trapezoidal fuzzy variables. An approximate model is also constructed for the problem based on an improved discretization method for fuzzy variables and the relevant convergence theorems. To obtain an approximate solution, a tabu search algorithm is designed for the presented model. Finally, some numerical experiments are performed to show the applications of the model and the algorithm.

1. Introduction

Railway freight transportation planning is an important issue in the railway transportation. An efficient transportation plan can not only decrease the transportation cost, but also improve the transportation efficiency and productivity. In recent decades, this problem has been investigated by many researchers and a variety of models and algorithms have been designed. Gorman [2] investigated a joint train-scheduling and demand-flow problem for a US freight railroad and proposed a "tabu-enhanced" genetic search algorithm to search for an optimal solution. In addition, an operating-plan model was proposed by Gorman [3] for minimizing the schedule-related costs of service with the consideration of the rail-operating capabilities, and a heuristic algorithm, which is a combination of genetic and tabu search, was designed to search for optimal solutions. Newton et al. [12] investigated the railroad blocking problem and treated it as a network design problem. A branch and bound algorithm was also developed to solve the problem. Crainic et al.[1] presented a modeling and algorithmic framework for the tactical planning of freight transportation. The relevant approach had been applied to planning problems of freight transportation by rail and by truck. Keaton[4] explored the railroad operating plan problem and modeled it as a mixed-integer programming, and then designed Lagrangian relaxation and heuristic approaches to solve the problem. Yang et al.[18] investigated modeling methods for railway freight transportation problem with random and fuzzy parameters.

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In the literature, most researchers investigated the transportation plan design problem under the condition that the involved parameters were fixed quantities. Nevertheless, since the transportation plan is generally made in advance and it will be employed for at least a given service period, some parameters will probably be different from the predetermined values due to uncertainty of the transportation system. For instance, the transportation cost will fluctuate in different times due to the existing competition among the different transportation modes. Thus, it is not suitable to treat these parameters as fixed quantities in the process of making a decision. To obtain a reasonable transportation plan, it is necessary to investigate the uncertainties in the transportation system. Generally, if the sample data are enough and the statistical methods are feasible, then the uncertainties can be treated as random parameters by statistical methods. However, if there are not enough sample data, a suitable way is to treat the uncertain parameters as fuzzy variables after analyzing their characteristics. Here, we investigate how to model this problem under the fuzzy environment. For some modeling methods under the fuzzy environment, see Liu [6], Yang et al. [16, 17, 19], Qin et al.[15, 13, 14].

The rest of the paper is organized as follows. In Section 2, we construct a mathematical model for the problem as a minimum risk chance-constrained model on the basis of the credibility measure. Noting that there exist chance functions in the model, we analyze their properties in Section 3, where the involved fuzzy variables are supposed to be trapezoidal fuzzy variables. When the objective function and chance functions can not be calculated by analytical methods, a common way is to use fuzzy simulation to simulate their approximate values. However, an example shows that fuzzy simulation is not always effective in simulating the credibility of fuzzy events. To make up for this deficiency, in Section 4, we present an improved discretization method for fuzzy variables. Some convergence theorems about the discrete fuzzy variables are also discussed. Based on the discretization method and the convergence theorems, we construct an approximate model for the problem. To obtain an approximate optimal solution of this problem, in Section 5 a tabu search algorithm is designed for the mathematical model, where the objective function and the feasibility of solutions can be analyzed by analytical methods or by the approximate model. In Section 6, some numerical experiments are performed to show the applications of the model and the algorithm.

2. **Description of the Problem**

Here, we give a model for the railway transportation planing problem under fuzzy environment. For this purpose, we firstly introduce some basic concepts in credibility theory, which will be employed to construct the mathematical model of the problem.

Fuzziness is a common phenomenon in real decision-making systems. To deal with this uncertainty in a mathematical way, the notion of fuzzy set was presented by Zadeh [20] in 1965, which is described by a membership function from a universal set Θ to the interval $[0, 1]$. The use of fuzzy variable is another way to describe a fuzzy phenomenon, which is defined as a general function from the possibility space $(\Theta, \mathscr{P}(\Theta), \text{Pos})$ to the set of real numbers (see Nahmias [11]), just like the definition of a random variable. In fact, fuzzy variables and fuzzy sets coincide with each other, since any fuzzy set can be represented by a fuzzy variable, and conversely, for any fuzzy variable, its membership function can be deduced by the possibility measure. For a given fuzzy variable ξ, its membership function can be deduced by the following way:

$$\mu_\xi(x) = \text{Pos}\{\theta \in \Theta | \xi(\theta) = x\}. \tag{1}$$

By using the membership function, the possibility measure of fuzzy event $\{\xi \in B\}$ can be computed by $\text{Pos}\{\xi \in B\} = \sup_{x \in B} \mu_\xi(x)$. The necessity measure of $\{\xi \in B\}$ is the duality of its possibility measure, which is defined by $\text{Nec}\{\xi \in B\} = 1 - \sup_{x \in B^c} \mu_\xi(x)$. The credibility measure, presented by Liu and Liu[5], is the average of the possibility measure and the necessity measure, i.e., $\text{Cr}\{\xi \in B\} = (\text{Pos}\{\xi \in B\} + \text{Nec}\{\xi \in B\})/2$.

Next, we give a detailed description of the railway transportation planing problem discussed in this paper. For this problem, the transportation activities are carried out on a railway transportation network rather than on a single rail line. For simplicity, the railway network is abbreviated by a graph (G, E), in which $i, j \in G$ represent the nodes (stations) and $(i, j) \in E$ represents the arc (rail section) from node i to node j. In the railway network, there exists more than one pair of origins and destinations. An origin station and its destination station will be denoted by "OD" for short, between which there is a transportation demand for the transportation activities. Because we consider the problem in a railway network, generally there exist more than one potential paths between each origin station and its destination station. The aim of the problem is thus to assign the transportation demand of each OD to their potential paths so that the total transportation cost can be minimized and meanwhile the capacity constraints of each station and each rail line are not violated.

To construct the mathematical model of this problem, we need to determine the decision variables, objective function and system constraints, respectively.

• **Decision Variables**

The main purpose of this problem is to determine the transportation amount on each potential route and the corresponding service frequency. Thus, the decision variables include:

x_{kt}: the amount of commodity flow on path t for OD k, which is a decision variable.

z_{kt}: the frequency of train services on path t for OD k, which is a decision variable.

• **Objective Function**

Objective function includes two parts, i.e., total transportation cost and the total fixed charges. To formulate the objective function, we first introduce some relevant parameters as follows:

$k = 1, 2, \cdots, m$: the notations of different ODs in the railway network.

T_k: the total number of potential paths for OD k.

$\tilde{e}_{ij}^k$: the unit transportation cost of commodity flow on arc (i, j) for OD k, which will be treated as a fuzzy variable.

$\tilde{h}_k$: the fixed charge of a train service for OD k, which will be treated as a fuzzy variable.

$y_{kt} = 1$, if path t is selected to serve for OD k; $y_{kt} = 0$, otherwise.

$u_{ij}^{kt} = 1$, if arc (i, j) is a segment of path t for OD k; $u_{ij}^{kt} = 0$, otherwise.

By using these notations, the total relevant cost of the railway transportation system, denoted by $F(\boldsymbol{x}, \boldsymbol{z})$, consists of the total transportation cost and the total fixed charge, i.e.,

$$F(\boldsymbol{x}, \boldsymbol{z}) = \sum_{k=1}^{m} \sum_{t=1}^{T_k} \sum_{(i,j)\in E} \tilde{e}_{ij}^k x_{kt} u_{ij}^{kt} y_{kt} + \sum_{k=1}^{m} \sum_{t=1}^{T_k} \tilde{h}_k z_{kt},$$

where the first part of $F(\boldsymbol{x}, \boldsymbol{z})$ is the total transportation cost and the second part is the total fixed charge of train services.

Note that for any feasible solution $(\boldsymbol{x}, \boldsymbol{z})$, the total relevant cost is actually a fuzzy variable due to the existence of fuzzy parameters. However, it is meaningless to minimize the total relevant cost directly when solving the problem. For this reason, we use the minimum risk criterion to reconstruct the objective function in the following way. We first give a maximized acceptable target T for the decision-maker. The risk criterion refers to the credibility that the total relevant cost exceeds the predetermined target T. Then in seeking an optimal solution, we minimize this risk criterion rather than the total relevant cost directly. We thus have the following objective function

$$f(\boldsymbol{x}, \boldsymbol{z}) = \mathrm{Cr}\left\{ \sum_{k=1}^{m} \sum_{t=1}^{T_k} \sum_{(i,j)\in E} \tilde{e}_{ij}^k x_{kt} u_{ij}^{kt} y_{kt} + \sum_{k=1}^{m} \sum_{t=1}^{T_k} \tilde{h}_k z_{kt} \geq T \right\}.$$

• **System Constraints**

System constraints include line capacity constraint, station capacity constraint, demand constraint, etc. The parameters are listed as follows.

a_k: the transportation demand for OD k.

$\tilde{p}_i$: the turnover capacity at node i, which will be treated as a fuzzy variable.

q_k: the capacity of each train for OD k.

$\tilde{c}_{ij}$: the passing capacity on arc (i, j), which will be treated as a fuzzy variable.

Z^+: the set of non-negative integers.

$r_i^{kt} = 1$, if node i lies on path t for OD k; $r_i^{kt} = 0$, otherwise.

As for the line capacity constraint, it is required that the total transported commodities on each arc not exceed its passing capacity. Note that the passing capacity $\tilde{c}_{ij}$ is represented by a fuzzy variable. To make the constraint meaningful, we shall employ a chance-constrained method to construct this constraint. That is, the capacity constraint is allowed to be violated, but the credibility that this inequality holds should not be less than a predetermined credibility confidence level

α_{ij}. That is, we have the following credibility constraint:

$$\text{Cr}\left\{\sum_{k=1}^{m}\sum_{t=1}^{T_k} x_{kt}u_{ij}^{kt}y_{kt} \le \tilde{c}_{ij}\right\} \ge \alpha_{ij}, \quad (i,j)\in E.$$

Since the turnover capacity of each station is also considered in the transportation system, we need to ensure that the total amount of commodities passing through each station should not exceed its capacity. We also use the chance-constrained method to deal with this inequality. That is, we first give a credibility confidence level β_i, and then suppose the credibility that the total commodities passing through each station do not exceed its turnover capacity should exceed β_i. We then have the following chance constraint:

$$\text{Cr}\left\{\sum_{k=1}^{m}\sum_{t=1}^{T_k} x_{kt}r_{i}^{kt}y_{kt} \le \tilde{p}_{i}\right\} \ge \beta_i, i\in G.$$

In addition, note that the demand of each OD should be satisfied. It follows that

$$\sum_{t=1}^{T_k} x_{kt} = a_k, k=1,2,\cdots,m.$$

Furthermore, the total commodities transported on path t for OD k should not be larger than the transportation capacity of the involved train services. Then,

$$x_{kt} \le z_{kt}q_k, k=1,2,\cdots,m, t=1,2,\cdots,T_k.$$

We now present a mathematical model for the railway transportation planning problem as the following minimum risk chance-constrained model:

$$\left\{\begin{array}{l}
\min \text{Cr}\left\{\displaystyle\sum_{k=1}^{m}\sum_{t=1}^{T_k}\sum_{(i,j)\in E} \tilde{e}_{ij}^{k}x_{kt}u_{ij}^{kt}y_{kt} + \sum_{k=1}^{m}\sum_{t=1}^{T_k}\tilde{h}_k z_{kt} \ge T\right\} \\
\text{s.t.} \\
\quad \text{Cr}\left\{\displaystyle\sum_{k=1}^{m}\sum_{t=1}^{T_k} x_{kt}u_{ij}^{kt}y_{kt} \le \tilde{c}_{ij}\right\} \ge \alpha_{ij}, \quad (i,j)\in E \\
\quad \text{Cr}\left\{\displaystyle\sum_{k=1}^{m}\sum_{t=1}^{T_k} x_{kt}r_{i}^{kt}y_{kt} \le \tilde{p}_{i}\right\} \ge \beta_i, i\in G \\
\quad \displaystyle\sum_{t=1}^{T_k} x_{kt} = a_k, k=1,2,\cdots,m \\
\quad x_{kt} \le z_{kt}q_k, k=1,2,\cdots,m, t=1,2,\cdots,T_k \\
\quad x_{kt} \ge 0, k=1,2,\cdots,m, t=1,2,\cdots,T_k \\
\quad y_{kt} \in \{0,1\}, k=1,2,\cdots,m, t=1,2,\cdots,T_k \\
\quad z_{kt} \in Z^{+}, k=1,2,\cdots,m, t=1,2,\cdots,T_k.
\end{array}\right. \tag{2}$$

3. Theoretical Analysis of the Model

It is clear that there are chance functions in the objective function and constraints. For solving the model conveniently, some mathematical properties of chance functions will be analyzed here.

For any feasible solution $(\boldsymbol{x}, \boldsymbol{z})$, we need to compute the following objective value:

$$f(\boldsymbol{x}, \boldsymbol{z}) = \text{Cr}\left\{\sum_{k=1}^{m}\sum_{t=1}^{T_k}\sum_{(i,j)\in E}\tilde{e}_{ij}^{k}x_{kt}u_{ij}^{kt}y_{kt} + \sum_{k=1}^{m}\sum_{t=1}^{T_k}\tilde{h}_k z_{kt} \geq T\right\}.$$

Actually, if the involved fuzzy variables $\tilde{e}_{ij}$ and $\tilde{h}_k$, $(i,j) \in E$, $k = 1, 2, \cdots, m$, are special fuzzy variables, for instance, interval fuzzy variables, triangular fuzzy variables or trapezoidal fuzzy variables, we can then compute the objective by analytical methods. Note that an interval fuzzy variable and a triangular fuzzy variable can be treated as special cases of trapezoidal fuzzy variables. In the following discussion, the involved fuzzy variables are supposed to be trapezoidal fuzzy variables.

We first provide the following results. For trapezoidal fuzzy variables $\tilde{a} = (a_1, a_2, a_3, a_4)$ and $\tilde{b} = (b_1, b_2, b_3, b_4)$, we have

$$\begin{aligned} &\tilde{a} + \tilde{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4), \\ &k\tilde{a} = (ka_1, ka_2, ka_3, ka_4), \quad k \geq 0. \end{aligned} \tag{3}$$

Theorem 3.1. *Suppose that $\tilde{e}_{ij}^{k} = (\tilde{e}_{ij}^{k1}, \tilde{e}_{ij}^{k2}, \tilde{e}_{ij}^{k3}, \tilde{e}_{ij}^{k4})$ and $\tilde{h}_k = (\tilde{h}_k^1, \tilde{h}_k^2, \tilde{h}_k^3, \tilde{h}_k^4)$, $(i,j) \in E$, $k = 1, 2, \cdots, m$, are trapezoidal fuzzy variables. Then, the objective function can be computed according to the following formula:*

$$f(\boldsymbol{x}, \boldsymbol{z}) = \begin{cases} 1, & T \leq f_1(\boldsymbol{x}, \boldsymbol{z}) \\ 1 - \dfrac{T - f_1(\boldsymbol{x}, \boldsymbol{z})}{2(f_2(\boldsymbol{x}, \boldsymbol{z}) - f_1(\boldsymbol{x}, \boldsymbol{z}))}, & f_1(\boldsymbol{x}, \boldsymbol{z}) \leq T \leq f_2(\boldsymbol{x}, \boldsymbol{z}) \\ 0.5, & f_2(\boldsymbol{x}, \boldsymbol{z}) \leq T \leq f_3(\boldsymbol{x}, \boldsymbol{z}) \\ \dfrac{T - f_4(\boldsymbol{x}, \boldsymbol{z})}{2(f_3(\boldsymbol{x}, \boldsymbol{z}) - f_4(\boldsymbol{x}, \boldsymbol{z}))}, & f_3(\boldsymbol{x}, \boldsymbol{z}) \leq T \leq f_4(\boldsymbol{x}, \boldsymbol{z}) \\ 0, & T \geq f_4(\boldsymbol{x}, \boldsymbol{z}), \end{cases} \tag{4}$$

where,

$$f_p(\boldsymbol{x}, \boldsymbol{z}) = \sum_{k=1}^{m}\sum_{t=1}^{T_k}\sum_{(i,j)\in E}\tilde{e}_{ij}^{kp}x_{kt}u_{ij}^{kt}y_{kt} + \sum_{k=1}^{m}\sum_{t=1}^{T_k}\tilde{h}_k^{p} z_{kt}, \ p = 1, \ 2, \ 3, \ 4.$$

Proof. Since $\tilde{e}_{ij}^{k}$ and $\tilde{h}_k$ are trapezoidal fuzzy variables, by using the equation (3), we have

$$\sum_{k=1}^{m}\sum_{t=1}^{T_k}\sum_{(i,j)\in E}\tilde{e}_{ij}^{k}x_{kt}u_{ij}^{kt}y_{kt} + \sum_{k=1}^{m}\sum_{t=1}^{T_k}\tilde{h}_k z_{kt} = (f_1(\boldsymbol{x}, \boldsymbol{z}), f_2(\boldsymbol{x}, \boldsymbol{z}), f_3(\boldsymbol{x}, \boldsymbol{z}), f_4(\boldsymbol{x}, \boldsymbol{z})).$$

By using the definition of credibility measure, we can easily complete the proof. □

For any given solution $(\boldsymbol{x}, \boldsymbol{z})$, we need to check whether it satisfies the following inequalities or not, to determine its feasibility:

$$g_{ij}(\boldsymbol{x}, \boldsymbol{z}) = \mathrm{Cr}\left\{\sum_{k=1}^{m}\sum_{t=1}^{T_k} x_{kt}u_{ij}^{kt}y_{kt} \le \tilde{c}_{ij}\right\} \ge \alpha_{ij},\ (i,j) \in E,$$

$$h_i(\boldsymbol{x}, \boldsymbol{z}) = \mathrm{Cr}\left\{\sum_{k=1}^{m}\sum_{t=1}^{T_k} x_{kt}r_i^{kt}y_{kt} \le \tilde{p}_i\right\} \ge \beta_i,\ i \in G.$$

Actually, we can transform the aforementioned inequalities to their crisp equivalents for some special $\tilde{c}_{ij}$ and $\tilde{p}_i$. We first give the following results.

Lemma 3.2. [16] *Suppose that ξ is a fuzzy variable with the continuous membership function $\mu_\xi(x)$, and $r_0 = \sup\{r|\mu_\xi(r) = 1\}$ such that $\mu_\xi(x)$ is non-decreasing on $(-\infty, r_0]$ and non-increasing on $[r_0, +\infty]$. Then, we have $\mathrm{Cr}\{h(x) \le \xi\} \ge \alpha$ if and only if $h(x) \le F_\alpha$, where,*

$$F_\alpha = \begin{cases} \sup\{F|F = \mu_\xi^{-1}(2\alpha)\}, & \text{if } \alpha \le 0.5 \\ \sup\{F|F = \mu_\xi^{-1}(2(1-\alpha)), F < r_0\}, & \text{if } \alpha > 0.5. \end{cases} \tag{5}$$

Lemma 3.3. [16] *Let ξ be a trapezoidal fuzzy variable having the form (a, b, c, d). Then, the crisp equivalent of $\mathrm{Cr}\{h(x) \le \xi\} \ge \alpha$ is $h(x) \le F_\alpha$, where,*

$$F_\alpha = \begin{cases} 2\alpha c + (1-2\alpha)d, & \text{if } \alpha \le 0.5 \\ (2\alpha - 1)a + 2(1-\alpha)b, & \text{if } \alpha > 0.5. \end{cases}$$

Theorem 3.4. *Let $\tilde{c}_{ij} = (\tilde{c}_{ij}^1, \tilde{c}_{ij}^2, \tilde{c}_{ij}^3, \tilde{c}_{ij}^4)$ be a trapezoidal fuzzy variable. Then, the first constraint in model (2) is equivalent to:*

$$\sum_{k=1}^{m}\sum_{t=1}^{T_k} x_{kt}u_{ij}^{kt}y_{kt} \le F_{\alpha_{ij}},$$

where,

$$F_{\alpha_{ij}} = \begin{cases} 2\alpha_{ij}\tilde{c}_{ij}^3 + (1-2\alpha_{ij})\tilde{c}_{ij}^4, & \text{if } \alpha_{ij} \le 0.5 \\ (2\alpha_{ij} - 1)\tilde{c}_{ij}^1 + 2(1-\alpha_{ij})\tilde{c}_{ij}^2, & \text{if } \alpha_{ij} > 0.5. \end{cases}$$

Proof. The proof is obvious by using Lemma 3.3. □

Theorem 3.5. *Let $\tilde{p}_i = (\tilde{p}_i^1, \tilde{p}_i^2, \tilde{p}_i^3, \tilde{p}_i^4)$ be a trapezoidal fuzzy variable. Then, the second constraint in model (2) is equivalent to:*

$$\sum_{k=1}^{m}\sum_{t=1}^{T_k} x_{kt}r_i^{kt}y_{kt} \le F_{\beta_i},$$

where,

$$F_{\beta_i} = \begin{cases} 2\beta_i\tilde{p}_i^3 + (1-2\beta_i)\tilde{p}_i^4, & \text{if } \beta_i \le 0.5 \\ (2\beta_i - 1)\tilde{p}_i^1 + 2(1-\beta_i)\tilde{p}_i^2, & \text{if } \beta_i > 0.5. \end{cases}$$

Proof. The proof is obvious by using Lemma 3.3. □

4. **Approximation of the Model**

As discussed in Section 3, if the involved fuzzy variables in the minimum risk chance-constrained model are trapezoidal fuzzy variables, we can use analytical methods to compute the objective and check the feasibility of solutions. However, if the fuzzy variables are more complex and the analytical methods are inappropriate, we need to turn to fuzzy simulation to compute the objective function and check feasibility. Liu [6, 7] designed the following simulation algorithm to simulate the credibility of the fuzzy event $\mathrm{Cr}\{\xi \le r\}$.

Algorithm 4.1. Fuzzy Simulation Algorithm

Step 1: Randomly generate a sequence $\{\theta_k\}_{k=1}^N$ from the set Θ.
Step 2: Write $\nu_k = \mathrm{Pos}\{\theta_k\}$.
Step 3: Return the following value:

$$L = \frac{1}{2}\left(\max_{1\le k\le N}\left\{\nu_k \middle| \xi(\theta_k) \le r\right\} + \min_{1\le k\le N}\left\{1-\nu_k \middle| \xi(\theta_k) > r\right\}\right).$$

Although this procedure has been frequently employed in solving fuzzy programming models, we may wonder whether the fuzzy simulation algorithm is always effective. To get a straightforward answer, we give the following two examples.

Example 4.2. Let $\xi = (3,4,5)$ be a triangular fuzzy variable. Then, by theoretical analysis, we have $\mathrm{Cr}\{\xi \le 4.5\} = 0.75$. If we use the above fuzzy simulation algorithm, we can obtain the approximate value $\mathrm{Cr}\{\xi \le 4.5\} \approx 0.7497$ after sampling 500 samples from the support $[3,5]$ of ξ. For this example, the fuzzy simulation algorithm is effective.

Example 4.3. Let ξ be a fuzzy variable with the following membership function:

$$\mu_\xi(x) = \begin{cases} \frac{1}{2}(x-3), & \text{if } 3 \le x < 4 \\ 1, & \text{if } x = 4 \\ \frac{1}{2}(5-x), & \text{if } 4 < x \le 5 \\ 0, & \text{otherwise.} \end{cases}$$

By theoretical analysis, we have $\mathrm{Cr}\{\xi \le 4.5\} = 0.875$. Using the above simulation algorithm, we obtain an approximate value $\mathrm{Cr}\{\xi \le 4.5\} \approx 0.6249$ after sampling 500 samples from the support $[3,5]$ of ξ. If we increase the sample data, the simulated values fluctuate around 0.625. Using the following formula to calculate the relative error,

$$\text{relative error} = \frac{\text{actual value} - \text{simulated value}}{\text{simulated value}} \times 100\%,$$

we obtain relative error $= 40\%$. This fact shows that the fuzzy simulated algorithm is not effective in this case.

Remark 4.4. We can see from the above two examples that the simulation algorithm in Example 4.2 is convergent, but it is not convergent in Example 4.3. The main reason that the simulation algorithm is invalid is that in the process of sampling data from the support, it is impossible to sample the discontinuity points of the membership function automatically in a computer. Thus, in the simulation, the information at the discontinuity points of the membership function has been lost, causing the large relative error. So, we need to design a more suitable discretization method for fuzzy variables.

4.1. **Discretization Method of Fuzzy Variables.** Discretization methods are always employed in the theoretical analysis of uncertainty theory. For instance, in random simulation, the process of sampling data is actually a discretion method of random variables. Additionally, Liu [8] presented a discretization method to investigate the convergence of the fuzzy simulation for continuous fuzzy variables. To solve the two-stage fuzzy random programming problems, Liu [9, 10] also presented a discretization method for fuzzy random variables. In the following, we shall propose an improved discretization method of fuzzy variables based on the methods by Liu [8] to give more intuitions, being much similar to the fuzzy simulation procedure. The designed method is not only effective for continuous variables but also for piecewise continuous fuzzy variables. Also, it is much easier for us to determine the membership functions for generated discrete fuzzy variables.

Let ξ be a fuzzy variable with the bounded support denoted by $[a, b]$ with $|a| < \infty, |b| < \infty$, where the support of fuzzy variable ξ is the closure of the set $\{x|\mu_\xi(x) > 0\}$. By the following discretization method, we can generate a sequence of discrete fuzzy variables $\{\varsigma_i\}$ via the fuzzy variable ξ.

Algorithm 4.5. Discretization Method of Fuzzy Variables

Step 1: Randomly generate an element $\theta \in \Theta$ with $\mathrm{Pos}\{\theta\} = 1$, and denote $x_0 = \xi(\theta)$.

Step 2: For any fixed integer i, compute the followings:

$$
\begin{aligned}
l_i^+ &= \inf\left\{\frac{k}{i} | k \in Z, s.t.\ \frac{k}{i} \geq x_0\right\}\\
l_i^- &= \sup\left\{\frac{k}{i} | k \in Z, s.t.\ \frac{k}{i} \leq x_0\right\}.
\end{aligned} \tag{6}
$$

If $l_i^- = l_i^+$, then let $l_i^- \leftarrow x_0 - \frac{1}{i}, l_i^+ \leftarrow x_0 + \frac{1}{i}$.

Step 3: Compute the following discrete fuzzy variable ς_i:

$$
\varsigma_i(\theta) = \begin{cases} \inf\left\{\frac{k}{i} | k \in Z, s.t.\ \frac{k}{i} \geq \xi(\theta)\right\}, & \text{if } \xi(\theta) \in [a, l_i^-] \\ x_0, & \text{if } \xi(\theta) \in (l_i^-, l_i^+) \\ \sup\left\{\frac{k}{i} | k \in Z, s.t.\ \frac{k}{i} \leq \xi(\theta)\right\}, & \text{if } \xi(\theta) \in [l_i^+, b]. \end{cases} \tag{7}
$$

In the above method, if the support of ξ is bounded, then for any fixed i, the support of ς_i is a finite set. For the convenience of description, we denote the support of ς_i by $\{a_1, a_2, \cdots, a_{N_i}\}$ such that $a_1 < a_2 < \cdots < a_{N_i}$ and $a_t = x_0$ for some $1 \leq t \leq N_i$. Now, we deduce the possibility distribution of discrete fuzzy variable ς_i using (1). That is,

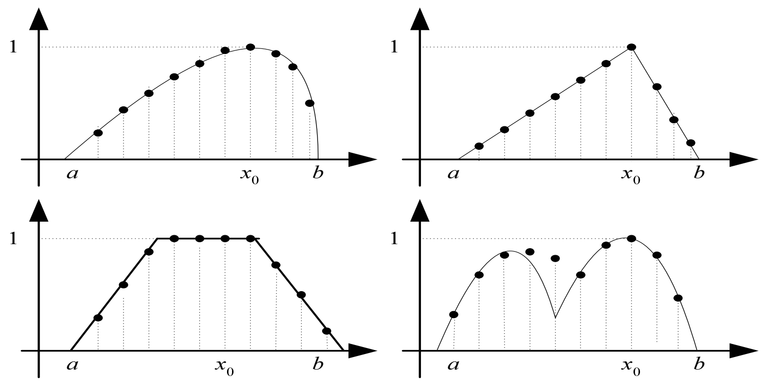

FIGURE 1. Discretization Process of Some Fuzzy Variables

$$\mathrm{Pos}\{\varsigma_i = a_k\} = \mathrm{Pos}\{\theta|\varsigma_i(\theta) = a_k\} = \begin{cases} \mathrm{Pos}\left\{\theta|a \le \xi(\theta) \le a_1\right\}, & \text{if } a_k = a_1, \\ \mathrm{Pos}\left\{\theta|a_{k-1} < \xi(\theta) \le a_k\right\}, & \text{if } a_1 < a_k < x_0, \\ 1, & \text{if } a_k = x_0, \\ \mathrm{Pos}\left\{\theta|a_k \le \xi(\theta) < a_{k+1}\right\}, & \text{if } x_0 < a_k < a_{N_i}, \\ \mathrm{Pos}\left\{\theta|a_{N_i} \le \xi(\theta) \le b\right\}, & \text{if } a_k = a_{N_i}. \end{cases}$$

We give Figure 1 to depict the discretization method of fuzzy variables, where the black dots represent the possibility distributions of the corresponding discrete fuzzy variables.

Definition 4.6. [21] Let $(\Theta, \mathscr{P}(\Theta), \mathrm{Pos})$ be a possibility space. Then, the set

$$\Theta^+ = \{\theta \in \Theta | \mathrm{Pos}\{\theta\} > 0\}$$

is called the kernel of the possibility space $(\Theta, \mathscr{P}(\Theta), \mathrm{Pos})$.

Theorem 4.7. Let $\{\varsigma_i\}$ be a sequence of discrete fuzzy variables generated by discretization method *(7)*. Then, $\{\varsigma_i\}$ converges to ξ uniformly on the Kernel Θ^+ of Θ.

Proof. It is easy to prove that for any $\theta \in \Theta^+$, we have $\xi(\theta) \in [a, b]$. It follows from equation (7) that for any $\theta \in \Theta^+$, we have

$$|\varsigma_i(\theta) - \xi(\theta)| \le \frac{1}{i}.$$

Letting $i \to \infty$, we have that the sequence $\{\varsigma_i\}$ converges to ξ uniformly on Θ^+. □

4.2. **Some Convergence Theorems.** In the following discussion, without specific statements, we suppose that the considered fuzzy variables have bounded supports with the form $[a, b]$. However, if the support of a fuzzy variable is not bounded, we can then construct a bounded fuzzy variable to approximate the fuzzy variable. For this purpose, we first give the definition of truncated fuzzy variables.

Definition 4.8. Let ξ be a fuzzy variable with the membership function $\mu_\xi(x)$

such that $\mu_\xi(x_0) = 1$. For any $a < x_0$ and $b > x_0$, a fuzzy variable $\bar{\xi}$ with the membership function,

$$\mu_{\bar{\xi}}(x) = \begin{cases} \mu_\xi(x), & \text{if } a \le x \le b \\ 0, & \text{otherwise,} \end{cases}$$

is called a truncated fuzzy variable of ξ.

Theorem 4.9. Let ξ be a fuzzy variable such that $\lim_{x\to\infty} \mu_\xi(x) = 0$ and $\lim_{x\to-\infty} \mu_\xi(x) = 0$. Then, for any $\varepsilon > 0$, there exists a truncated fuzzy variable $\bar{\xi}$ of ξ such that

$$|\text{Cr}\{\xi \le r\} - \text{Cr}\{\bar{\xi} \le r\}| \le \varepsilon, \quad |\text{Cr}\{\xi \ge r\} - \text{Cr}\{\bar{\xi} \ge r\}| \le \varepsilon. \tag{8}$$

Proof. Without loss of generality, suppose $0 < \varepsilon < 0.5$. Since $\lim_{x\to-\infty} \mu_\xi(x) = 0$, there exists a constant a such that $\mu_\xi(x) < 2\varepsilon$, for any $x \le a$. We thus have

$$\text{Cr}\{\xi \le r\} = \frac{1}{2}\text{Pos}\{\xi \le r\} \le \varepsilon,$$

for any $r \le a$. On the other hand, since $\lim_{x\to\infty} \mu_\xi(x) = 0$, there exists a constant b such the $\mu_\xi(x) < 2\varepsilon$, for any $x \ge b$. We thus have

$$\text{Cr}\{\xi \ge r\} = \frac{1}{2}\text{Pos}\{\xi \ge r\} \le \varepsilon,$$

for any $r \ge b$. We define the membership function of truncated fuzzy variable $\bar{\xi}$ as follows:

$$\mu_{\bar{\xi}}(x) = \begin{cases} \mu_\xi(x), & \text{if } a \le x \le b, \\ 0, & \text{otherwise.} \end{cases}$$

In the following, we only prove $|\text{Cr}\{\xi \le r\} - \text{Cr}\{\bar{\xi} \le r\}| \le \varepsilon$, and the other result can be proved in a similar way. We divide the proof into the following four parts:

(i) If $r < a$, then we have $\text{Cr}\{\bar{\xi} \le r\} = 0$, which implies $|\text{Cr}\{\xi \le r\} - \text{Cr}\{\bar{\xi} \le r\}| \le \varepsilon$.

(ii) If $b > r \ge a$ and $\text{Cr}\{\xi \le r\} \le \varepsilon$, then we have

$$\varepsilon \ge \text{Cr}\{\xi \le r\} = \frac{1}{2}\text{Pos}\{\xi \le r\} = \frac{1}{2}\sup_{x\le r}\mu_\xi(x) \ge \frac{1}{2}\sup_{a\le x\le r}\mu_\xi(x) = \frac{1}{2}\sup_{a\le x\le r}\mu_{\bar{\xi}}(x) = \text{Cr}\{\bar{\xi} \le r\} \ge 0,$$

which implies $|\text{Cr}\{\xi \le r\} - \text{Cr}\{\bar{\xi} \le r\}| \le \varepsilon$.

(iii) If $b > r \ge a$ and $0.5 \ge \text{Cr}\{\xi \le r\} > \varepsilon$, then we have

$$\begin{aligned} \text{Cr}\{\xi \le r\} = \frac{1}{2}\sup_{x\le r}\mu_\xi(x) &= \frac{1}{2}\sup_{a\le x\le r}\mu_\xi(x) \vee \frac{1}{2}\sup_{x<a}\mu_\xi(x) \\ &= \frac{1}{2}\sup_{a\le x\le r}\mu_\xi(x) = \frac{1}{2}\sup_{a\le x\le r}\mu_{\bar{\xi}}(x) = \text{Cr}\{\bar{\xi} \le r\}, \end{aligned}$$

which implies $|\text{Cr}\{\xi \le r\} - \text{Cr}\{\bar{\xi} \le r\}| = 0 \le \varepsilon$.

(iv) $b > r \geq a$ and $1 \geq \mathrm{Cr}\{\xi \leq r\} > 0.5$. For this case, it is easy to prove $\mathrm{Cr}\{\bar{\xi} \leq r\} \geq 0.5$. Thus, we have

$$|\mathrm{Cr}\{\xi \leq r\} - \mathrm{Cr}\{\bar{\xi} \leq r\}| = \frac{1}{2}|\mathrm{Pos}\{\xi > r\} - \mathrm{Pos}\{\bar{\xi} > r\}|,$$

which implies that it is sufficient to prove $|\mathrm{Pos}\{\xi > r\} - \mathrm{Pos}\{\bar{\xi} > r\}| \leq 2\varepsilon$. If $\mathrm{Pos}\{\xi > r\} > 2\varepsilon$, then we have

$$\begin{aligned}\mathrm{Pos}\{\xi > r\} = \sup_{x>r} \mu_\xi(x) &= \sup_{r<x\leq b} \mu_\xi(x) \vee \sup_{x>b} \mu_\xi(x) \\ &= \sup_{r<x\leq b} \mu_\xi(x) = \sup_{r<x\leq b} \mu_{\bar{\xi}}(x) = \mathrm{Pos}\{\bar{\xi} > r\},\end{aligned}$$

which implies $|\mathrm{Cr}\{\xi \leq r\} - \mathrm{Cr}\{\bar{\xi} \leq r\}| = 0 \leq \varepsilon$. On the other hand, if $\mathrm{Pos}\{\xi > r\} \leq 2\varepsilon$, then we have

$$\begin{aligned}2\varepsilon \geq \mathrm{Pos}\{\xi > r\} = \sup_{x>r} \mu_\xi(x) &= \sup_{r<x\leq b} \mu_\xi(x) \vee \sup_{x>b} \mu_\xi(x) \\ &\geq \sup_{r<x\leq b} \mu_\xi(x) = \sup_{r<x\leq b} \mu_{\bar{\xi}}(x) = \mathrm{Pos}\{\bar{\xi} > r\} \geq 0,\end{aligned}$$

which implies $|\mathrm{Cr}\{\xi \leq r\} - \mathrm{Cr}\{\bar{\xi} \leq r\}| \leq \varepsilon$.

(v) If $r \geq b$, then we have $\mathrm{Cr}\{\bar{\xi} \leq r\} = 1$. Additionally, we have

$$1 \geq \mathrm{Cr}\{\xi \leq r\} = \frac{1}{2}(1 + 1 - \mathrm{Pos}\{\xi > r\}) = \frac{1}{2}(1 + 1 - \sup_{x>r} \mu_\xi(x)\}) \geq 1 - \varepsilon.$$

It follows that $|\mathrm{Cr}\{\xi \leq r\} - \mathrm{Cr}\{\bar{\xi} \leq r\}| \leq \varepsilon$. The proof is now complete. □

Remark 4.10. Theorem 4.9 states that if the support of fuzzy variable ξ is not bounded, then we can construct an approximate fuzzy variable $\bar{\xi}$, which has a bounded support.

Remark 4.11. In Theorem 4.9, if ε is a sufficiently small constant, then we have the following relationships:

$$\mathrm{Cr}\{\xi \leq r\} \approx \mathrm{Cr}\{\bar{\xi} \leq r\}, \quad \mathrm{Cr}\{\xi \geq r\} \approx \mathrm{Cr}\{\bar{\xi} \geq r\}.$$

That is, to calculate $\mathrm{Cr}\{\xi \leq r\}$ or $\mathrm{Cr}\{\xi \geq r\}$, it is sufficient to consider the bounded fuzzy variable $\bar{\xi}$ instead of ξ.

In the following, we consider convergence theorems for the credibility of fuzzy events, which can also be treated as the mathematical foundation of fuzzy simulation.

Theorem 4.12. Suppose that $\{\varsigma_i\}$ is a sequence of discrete fuzzy variables generated by the discretization method (7). Then, for any continuity point r^* of $\mathrm{Cr}\{\xi \leq r\}$ ($\mathrm{Cr}\{\xi \geq r\}$), we have

$$\lim_{i\to\infty} \mathrm{Cr}\{\varsigma_i \leq r^*\} = \mathrm{Cr}\{\xi \leq r^*\}.$$

$$\left(\lim_{i\to\infty} \mathrm{Cr}\{\varsigma_i \geq r^*\} = \mathrm{Cr}\{\xi \geq r^*\}\right).$$

Proof. We only prove the first part. The second part can be proved similarly. It is sufficient to prove that for any $\varepsilon > 0$, there exists $N > 0$ such that $|\mathrm{Cr}\{\varsigma_i \le r^*\} - \mathrm{Cr}\{\xi \le r^*\}| < \varepsilon$, for any $i > N$. Since r^* is a continuity point of the function $\mathrm{Cr}\{\xi \le r\}$, for the above $\varepsilon > 0$, there exists a constant $i_0 > 0$ such that for any $|r - r^*| \le \frac{1}{i_0}$,

$$|\mathrm{Cr}\{\xi \le r\} - \mathrm{Cr}\{\xi \le r^*\}| < \varepsilon,$$

which implies $\mathrm{Cr}\left\{\xi + \frac{1}{i_0} \le r^*\right\} > \mathrm{Cr}\{\xi \le r^*\} - \varepsilon$ and $\mathrm{Cr}\left\{\xi - \frac{1}{i_0} \le r^*\right\} < \mathrm{Cr}\{\xi \le r^*\} + \varepsilon$. It follows from Theorem 4.7, that

$$|\varsigma_{i_0}(\theta) - \xi(\theta)| \le \frac{1}{i_0}$$

for any $\theta \in \Theta^+$. Then, we have

$$\left\{\theta \in \Theta^+ | \xi(\theta) - \frac{1}{i_0} \le r^*\right\} \supset \{\theta \in \Theta^+ | \varsigma_{i_0}(\theta) \le r^*\} \supset \left\{\theta \in \Theta^+ | \xi(\theta) + \frac{1}{i_0} \le r^*\right\},$$

which implies:

$$\mathrm{Cr}\left\{\xi - \frac{1}{i_0} \le r^*\right\} \ge \mathrm{Cr}\{\varsigma_{i_0} \le r^*\} \ge \mathrm{Cr}\left\{\xi + \frac{1}{i_0} \le r^*\right\}.$$

By using the continuity at r^*, we have

$$\mathrm{Cr}\{\xi \le r^*\} + \varepsilon > \mathrm{Cr}\left\{\xi - \frac{1}{i_0} \le r^*\right\} \ge \mathrm{Cr}\{\varsigma_{i_0} \le r^*\} \ge \mathrm{Cr}\left\{\xi + \frac{1}{i_0} \le r^*\right\} > \mathrm{Cr}\{\xi \le r^*\} - \varepsilon.$$

That is, $|\mathrm{Cr}\{\varsigma_{i_0} \le r^*\} - \mathrm{Cr}\{\xi \le r^*\}| < \varepsilon$. Note that this inequality holds for any ς_i with $i \ge i_0$, and the theorem is thus proved. □

Corollary 4.13. Suppose that $\{\varsigma_i\}$ is a sequence of discrete fuzzy variables generated by the discretization method (7). Then, for almost all $r \in \Re$, we have

$$\lim_{i\to\infty} \mathrm{Cr}\{\varsigma_i \le r\} = \mathrm{Cr}\{\xi \le r\},$$
$$\lim_{i\to\infty} \mathrm{Cr}\{\varsigma_i \ge r\} = \mathrm{Cr}\{\xi \ge r\}.$$

Proof. Since $\mathrm{Cr}\{\xi \le r\}$ and $\mathrm{Cr}\{\xi \ge r\}$ are monotone functions with respect to r, it follows that they are continuous almost everywhere with respect to r. By using Theorem 4.12, the results are obvious. □

Corollary 4.14. Suppose that $\{\varsigma_i\}$ is a sequence of discrete fuzzy variables generated by the discretization method (7). If ξ is a continuous fuzzy variable, then for any $r \in \Re$, we have

$$\lim_{i\to\infty} \mathrm{Cr}\{\varsigma_i \le r\} = \mathrm{Cr}\{\xi \le r\},$$
$$\lim_{i\to\infty} \mathrm{Cr}\{\varsigma_i \ge r\} = \mathrm{Cr}\{\xi \ge r\}.$$

Example 4.15. Let us also consider the credibility of the fuzzy event in Example 4.3. By using the discretization method, we can obtain a discrete fuzzy variable ς_{50}. Then, we get

$$\mathrm{Cr}\{\varsigma_{50} \le 4.5\} = 0.88 \approx 0.875 = \mathrm{Cr}\{\xi \le 4.5\}.$$

Thus, when computing $\text{Cr}\{\xi \le 4.5\}$, we can replace ξ with ς_{50} to obtain an approximate value, which can be calculated by analytical methods. This case shows that this approximation method is effective in calculating the credibility of fuzzy events.

For $\boldsymbol{\xi}$ a fuzzy vector having the form $\boldsymbol{\xi} = (\xi_1, \xi_2, \cdots, \xi_n)$ defined on the product space $\Theta_1 \times \Theta_2 \times \cdots \times \Theta_n$, consider the fuzzy function $f(\boldsymbol{x}, \boldsymbol{\xi})$. For the fuzzy vector $\boldsymbol{\xi} = (\xi_1, \xi_2, \cdots, \xi_n)$, suppose that the support of $\boldsymbol{\xi}$ is $S = \prod_{k=1}^{n}[a_k, b_k]$. We generate the discrete fuzzy variables ς_{ji}, for each ξ_j, $j = 1, 2, \cdots, n$ by using the discretization method (7). Denote $\boldsymbol{\varsigma}_i = (\varsigma_{1i}, \varsigma_{2i}, \cdots, \varsigma_{ni})$. Then, we have

$$\|\boldsymbol{\varsigma}_i(\boldsymbol{\theta}) - \xi(\boldsymbol{\theta})\| = \sqrt{\sum_{j=1}^{n}(\varsigma_{ji}(\boldsymbol{\theta}) - \xi_j(\boldsymbol{\theta}))^2} \le \frac{\sqrt{n}}{i},$$

for any $\boldsymbol{\theta} \in \Theta_1^+ \times \Theta_2^+ \times \cdots \times \Theta_n^+$, which implies that $\{\boldsymbol{\varsigma}_i\}$ converges to $\boldsymbol{\xi}$ uniformly on $\Theta_1^+ \times \Theta_2^+ \times \cdots \times \Theta_n^+$.

Theorem 4.16. Suppose that for any fixed $\boldsymbol{x}$, the function $f(\boldsymbol{x}, \boldsymbol{\xi})$ is continuous on the support of $\boldsymbol{\xi}$ and the support of $\boldsymbol{\xi}$ is a compact set. Then, for almost all $r \in \Re$, we have

$$\lim_{i\to\infty} \text{Cr}\{f(\boldsymbol{x}, \boldsymbol{\varsigma}_i) \le r\} = \text{Cr}\{f(\boldsymbol{x}, \boldsymbol{\xi}) \le r\}$$

$$\lim_{i\to\infty} \text{Cr}\{f(\boldsymbol{x}, \boldsymbol{\varsigma}_i) \ge r\} = \text{Cr}\{f(\boldsymbol{x}, \boldsymbol{\xi}) \ge r\}.$$

Proof. Note that the support of $\boldsymbol{\xi}$ is compact and $f(\boldsymbol{x}, \boldsymbol{\xi})$ is continuous on the support of $\boldsymbol{\xi}$. Then, $f(\boldsymbol{x}, \boldsymbol{\xi})$ is uniformly continuous on the support of $\boldsymbol{\xi}$ for any fixed $\boldsymbol{x}$. Using this result, we can easily prove that $f(\boldsymbol{x}, \boldsymbol{\varsigma}_i)$ converges to $f(\boldsymbol{x}, \boldsymbol{\xi})$ uniformly on $\Theta_1^+ \times \Theta_2^+ \times \cdots \times \Theta_n^+$. It follows from Corollary 4.13 that

$$\lim_{i\to\infty} \text{Cr}\{f(\boldsymbol{x}, \boldsymbol{\varsigma}_i) \le r\} = \text{Cr}\{f(\boldsymbol{x}, \boldsymbol{\xi}) \le r\}$$

$$\lim_{i\to\infty} \text{Cr}\{f(\boldsymbol{x}, \boldsymbol{\varsigma}_i) \ge r\} = \text{Cr}\{f(\boldsymbol{x}, \boldsymbol{\xi}) \ge r\},$$

for almost all $r \in \Re$. The proof is thus complete. □

4.3. **Approximate Model of the Problem.** We shall investigate the approximate model of the problem by using the discretization method. When solving model (2), we need to compute the following objective function for each feasible solution $(\boldsymbol{x}, \boldsymbol{z})$:

$$f(\boldsymbol{x}, \boldsymbol{z}) : (\boldsymbol{x}, \boldsymbol{z}) \to \text{Cr}\left\{\sum_{k=1}^{m}\sum_{t=1}^{T_k}\sum_{(i,j)\in E} \tilde{e}_{ij}^{k} x_{kt} u_{ij}^{kt} y_{kt} + \sum_{k=1}^{m}\sum_{t=1}^{T_k} \tilde{h}_k z_{kt} \ge T\right\}.$$

Actually, if the involved fuzzy variables $\tilde{e}_{ij}^{k}, \tilde{h}_k$, $(i,j) \in E$, $k = 1, 2, \cdots, m$, are complex, we generally can not calculate this objective function by analytical means. In order to obtain an approximate objective, we can replace fuzzy variables $\tilde{e}_{ij}^{k}, \tilde{h}_k$ by their discrete fuzzy variables using the discretization method (7). Denote the discrete fuzzy variables of $\tilde{e}_{ij}^{k}$ and $\tilde{h}_k$ by $\tilde{e}_{ij}^{kn}$ and $\tilde{h}_{kn}$, respectively. Then, the approximate objective function is:

$$f_n(\boldsymbol{x},\boldsymbol{z}) = \mathrm{Cr}\left\{\sum_{k=1}^{m}\sum_{t=1}^{T_k}\sum_{(i,j)\in E}\tilde{e}_{ij}^{kn}x_{kt}u_{ij}^{kt}y_{kt} + \sum_{k=1}^{m}\sum_{t=1}^{T_k}\tilde{h}_{kn}z_{kt} \geq T\right\}.$$

By using Theorem 4.16, we have $f_n(\boldsymbol{x},\boldsymbol{z}) \to f(\boldsymbol{x},\boldsymbol{z})$ as $n\to\infty$. Moreover, for any solution $(\boldsymbol{x},\boldsymbol{z})$, we can calculate $f_n(\boldsymbol{x},\boldsymbol{z})$ by analytical methods.

In addition, if it is difficult for us to check the feasibility of solutions in model (2) analytically, then we can replace the chance functions with their approximations by using convergence theorems. Denote the discrete fuzzy variables of $\tilde{c}_{ij}$ and $\tilde{p}_i$ by $\tilde{c}_{ij}^{n}$ and $\tilde{p}_i^{n}$, respectively. Then, the approximate chance functions can be written as:

$$g_{ij}^{n}(\boldsymbol{x},\boldsymbol{z}) = \mathrm{Cr}\left\{\sum_{k=1}^{m}\sum_{t=1}^{T_k}x_{kt}u_{ij}^{kt}y_{kt} \leq \tilde{c}_{ij}^{n}\right\},$$

$$h_{i}^{n}(\boldsymbol{x},\boldsymbol{z}) = \mathrm{Cr}\left\{\sum_{k=1}^{m}\sum_{t=1}^{T_k}x_{kt}r_{i}^{kt}y_{kt} \leq \tilde{p}_{i}^{n}\right\}.$$

By using Corollary 4.13, we have $g_{ij}^{n}(\boldsymbol{x},\boldsymbol{z}) \to g_{ij}(\boldsymbol{x},\boldsymbol{z})$ and $h_i^n(\boldsymbol{x},\boldsymbol{z}) \to h_i(\boldsymbol{x},\boldsymbol{z})$, as $n\to\infty$. Thus, if n is large enough, then we can ensure that $g_{ij}^{n}(\boldsymbol{x},\boldsymbol{z})$ and $h_i^n(\boldsymbol{x},\boldsymbol{z})$ are as close to $g_{ij}(\boldsymbol{x},\boldsymbol{z})$ and $h_i(\boldsymbol{x},\boldsymbol{z})$ as possible, respectively, where $g_{ij}^{n}(\boldsymbol{x},\boldsymbol{z})$ and $h_i^n(\boldsymbol{x},\boldsymbol{z})$ can be calculated analytically.

5. **Tabu Search Algorithm**

In the following, we shall design tabu search algorithm to obtain an approximate optimal solution of the problem. The followings are technical details of the algorithm.

1. Solution Representation

We can see from the model that the aim is to minimize the credibility that the total relevant cost exceeds the predetermined target T, where the fixed charge is one of the most important parts in the total cost. Thus, in the process of optimizing the objective, it is better to make the total fixed charge minimized. For this purpose, the frequency of train service needs to be minimized so as to decrease the fixed charge. Accordingly, we need to ensure that the service trains are fully loaded, if possible. For this purpose, we set $x_{kt} = n\cdot q_k$ $(n\in Z, n\geq 0)$ or $x_{kt} < q_k$ in the initial solution. Thus, when x_{kt} is determined, the service frequency on path t for OD k can also be obtained. In the following, we use the following array to denote a solution:

$$X = (\overbrace{x_{11},x_{12},\cdots,x_{1T_1}}^{T_1};\overbrace{x_{21},x_{22},\cdots,x_{2T_2}}^{T_2};\cdots;\overbrace{x_{m1},x_{m2},\cdots,x_{mT_m}}^{T_m}).$$

In the above representation, we require that $x_{kt} \geq 0$. If $x_{kt} > 0$, then the amount of commodities transported on path t for OD k is x_{kt}; otherwise, there is no transportation assignment on path t for OD k.

2. Neighborhood Structure

Tabu search circulates the searching process from one solution to another, by which we can obtain a sequence of seed solutions, denoted by $\{X_l\}$. If a solution in this sequence satisfies the optimal conditions, then it will be treated as an approximate optimal solution. Thus, in tabu search, an important work is to design a criterion to determine the next solution X_{l+1} from the current considered solution X_l. Generally, solution X_{l+1} should be chosen in the neighborhood of solution X_l. In the following, we determine the structure of neighborhood for each solution.

We first define a neighbor X' of solution X as follows:

$$X = (\overbrace{x_{11}, x_{12}, \cdots, x_{1T_1}}^{T_1}; \cdots; \overbrace{x_{k1}, \cdots, x_{kt}, \cdots, x_{kt'}, \cdots, x_{kT_k}}^{T_k}; \cdots; \overbrace{x_{m1}, x_{m2}, \cdots, x_{mT_m}}^{T_m})$$

$$X' = (\overbrace{x_{11}, x_{12}, \cdots, x_{1T_1}}^{T_1}; \cdots; \overbrace{x_{k1}, \cdots, \bar{x}_{kt}, \cdots, \bar{x}_{kt'}, \cdots, x_{kT_k}}^{T_k}; \cdots; \overbrace{x_{m1}, x_{m2}, \cdots, x_{mT_m}}^{T_m}),$$

where the solution X' is the same as X except for the elements $\bar{x}_{kt}$ and $\bar{x}_{kt'}$. In solution X, the element x_{kt} is supposed to be a non-zero element, and the elements $\bar{x}_{kt}$ and $\bar{x}_{kt'}$ in solution X' are generated by the operations on x_{kt} and $x_{kt'}$. That is, we first decrease the commodity flow by u ($u \leq x_{kt}$) units on path t for OD k, and then add the u units commodity flow to path t' for OD k, i.e., we let

$$\bar{x}_{kt} = x_{kt} - u, \quad \bar{x}_{kt'} = x_{kt'} + u.$$

In the real conditions, if the constant u is generated randomly, then the total frequency of train services will increase, which will cause the increase of the total fixed charge. To make up for this deficiency, we stipulate that the constant u should be the capacity of the train service multiplied by some non-negative integer, that is,

$$u \leq x_{kt}, \quad u = z \cdot q_k, \quad z \in \{0, 1, 2, \cdots\}.$$

If X' is a feasible solution, it is called a feasible neighbor of solution X. We stipulate that the neighborhood of solution X consists of all of such feasible solutions. Actually, if we can not obtain the neighborhood in an acceptable time, we can randomly generate N feasible neighbors and put them into the neighborhood as an approximation.

3. Tabu Moves

In tabu search, if we obtain a solution X_{l+1} from the neighborhood of solution X_l, then at the next iteration, we need to select an optimal solution in the neighborhood of X_{l+1}, in which X_l is also a feasible neighbor. To avoid selecting X_l at the next iteration (if X_l is selected as X_{l+2}, then a local cycle will probably appear), when constructing the neighborhood of X_{l+1}, we suppose that the neighbor X_l of X_{l+1} is forbidden to be produced. That is, if X_{l+1} is obtained by the following operations on X_l,

$$\bar{x}_{kt} \leftarrow x_{kt} - u, \quad \bar{x}_{kt'} \leftarrow x_{kt'} + u,$$

then in constructing the neighbors of X_{l+1} (or for the next w iterations), the following operations will be forbidden

$$x_{kt} \leftarrow \bar{x}_{kt} + u, \quad x_{kt'} \leftarrow \bar{x}_{kt'} - u.$$

The tabu move can be denoted by a triplet $(x_{kt}, x_{kt'}, u)$.

4. Aspiration Criterion

The aim of tabu move is to avoid the local recycling in the search process. However, in some circumstances, we need to design an aspiration criterion for tabu search in order to obtain the optimal solution as fast as possible. Suppose that we obtain a feasible neighbor of X_l at iteration l, denoted by $\bar{X}$, which is generated by a tabu move. If the objective value of $\bar{X}$ is better than that of the best solution encountered so far, then the relevant tabu move will be canceled, which implies that the solution $\bar{X}$ can be selected as X_{l+1}.

5. The Procedure of Tabu Search Algorithm

Using the above technical details, we list the steps of tabu search as follows.
X_l**:** The solution at iteration l.
X^***:** The best solution encountered up to iteration l.
l^***:** The index of iteration at which X^* is obtained.
$N(X_l)$: The neighborhood of X_l.
Max_iter: A given parameter for iteration.

- Search for the potential feasible routes of each OD k ($k = 1, 2, \cdots, m$).
- Determine the representation of a solution based on the information of routes. Then, randomly generate a feasible solution X_1.
- Let $X^* = X_1, l^* = 1, l = 1$, and initialize the tabu list.

while $(l - l^* \leq Max\_iter)$
{ • For solution X_l, produce the neighborhood $N(X_l)$ firstly.
• By approximation or analytically, calculate objective values of elements in $N(X_l)$.
• According to the objective values, select the best solution X' in $N(X_l)$ (X' is non-tabu or satisfies an aspiration criterion).
• If solution X' is better than X^*, then let $X_{l+1} = X'$, $X^* = X'$, $l^* = l$; otherwise, let $X_{l+1} = X'$.
• Update the tabu list, and let $l++$. }

- Output X^*.

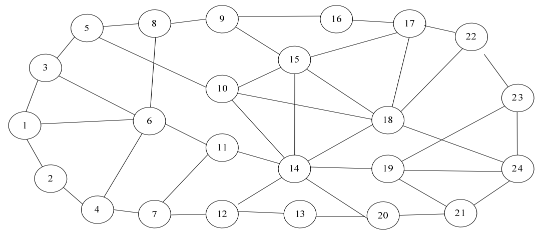

Figure 2. The Rail Transportation Network in Numerical Experiments

6. **Numerical Experiments**

Here, some numerical experiments are given to show the applications of the model and the algorithm. In the experiments, the railway transportation network consists of 24 nodes and 42 arcs (see Figure 2). To construct the mathematical model of the problem, the needed parameters are listed in Tables 1 and 2, respectively. Additionally, five ODs will be considered in the transportation network. For each OD, the relevant parameters are listed in Table 3.

(i,j)	d_{ij}	$\tilde{c}_{ij}$	$\tilde{e}_{ij}^k$	(i,j)	d_{ij}	$\tilde{c}_{ij}$	$\tilde{e}_{ij}^k$
(1, 2)	34	$\mathcal{EXP}(90)$	(60, 62, 67)	(12, 13)	43	$\mathcal{EXP}(84)$	(33, 37, 42)
(1, 3)	35	$\mathcal{EXP}(95)$	(35, 37, 40)	(12, 14)	65	$\mathcal{EXP}(88)$	(44, 45, 50)
(1, 6)	67	$\mathcal{EXP}(86)$	(54, 62, 64)	(13, 20)	67	$\mathcal{EXP}(67)$	(42, 44, 46)
(2, 4)	87	$\mathcal{EXP}(89)$	(15, 16, 20)	(14, 15)	87	$\mathcal{EXP}(90)$	(35, 38, 40)
(3, 5)	67	$\mathcal{EXP}(86)$	(87, 92, 93)	(14, 18)	45	$\mathcal{EXP}(99)$	(60, 61, 65)
(3, 6)	53	$\mathcal{EXP}(89)$	(35, 42, 44)	(14, 19)	61	$\mathcal{EXP}(85)$	(46, 48, 50)
(4, 6)	37	$\mathcal{EXP}(84)$	(41, 43, 46)	(14, 20)	79	$\mathcal{EXP}(94)$	(35, 38, 40)
(4, 7)	32	$\mathcal{EXP}(95)$	(20, 26, 30)	(15, 17)	55	$\mathcal{EXP}(97)$	(76, 81, 82)
(5, 8)	44	$\mathcal{EXP}(97)$	(53, 60, 61)	(15, 18)	75	$\mathcal{EXP}(91)$	(70, 85, 87)
(5, 10)	66	$\mathcal{EXP}(85)$	(36, 38, 46)	(16, 17)	71	$\mathcal{EXP}(95)$	(37, 42, 45)
(6, 8)	68	$\mathcal{EXP}(86)$	(80, 83, 95)	(17, 18)	85	$\mathcal{EXP}(84)$	(49, 51, 67)
(6, 11)	46	$\mathcal{EXP}(95)$	(83, 89, 90)	(17, 22)	68	$\mathcal{EXP}(79)$	(80, 84, 85)
(7, 11)	36	$\mathcal{EXP}(84)$	(51, 52, 60)	(18, 22)	75	$\mathcal{EXP}(89)$	(51, 56, 58)
(7, 12)	78	$\mathcal{EXP}(96)$	(61, 63, 70)	(18, 24)	44	$\mathcal{EXP}(84)$	(40, 41, 50)
(8, 9)	67	$\mathcal{EXP}(89)$	(78, 79, 85)	(19, 21)	78	$\mathcal{EXP}(92)$	(60, 68, 70)
(9, 15)	66	$\mathcal{EXP}(86)$	(83, 87, 90)	(19, 23)	34	$\mathcal{EXP}(88)$	(33, 36, 38)
(9, 16)	87	$\mathcal{EXP}(98)$	(76, 82, 84)	(19, 24)	65	$\mathcal{EXP}(99)$	(54, 61, 63)
(10, 14)	95	$\mathcal{EXP}(83)$	(64, 74, 80)	(20, 21)	34	$\mathcal{EXP}(98)$	(85, 89, 92)
(10, 15)	67	$\mathcal{EXP}(93)$	(57, 61, 63)	(21, 24)	33	$\mathcal{EXP}(90)$	(61, 62, 70)
(10, 18)	65	$\mathcal{EXP}(95)$	(45, 47, 52)	(22, 23)	68	$\mathcal{EXP}(95)$	(70, 74, 75)
(11, 14)	64	$\mathcal{EXP}(82)$	(25, 27, 32)	(23, 24)	77	$\mathcal{EXP}(93)$	(40, 52, 54)

Table 1. The Parameters in Numerical Experiments

Station (i)	Turnover Capacity ($\tilde{p}_i$)	Station (i)	Turnover Capacity ($\tilde{p}_i$)
1	$\mathcal{EXP}(20)\cdot 10$	13	$\mathcal{EXP}(36)\cdot 11$
2	$\mathcal{EXP}(22)\cdot 11$	14	$\mathcal{EXP}(26)\cdot 20$
3	$\mathcal{EXP}(25)\cdot 15$	15	$\mathcal{EXP}(29)\cdot 19$
4	$\mathcal{EXP}(30)\cdot 20$	16	$\mathcal{EXP}(42)\cdot 14$
5	$\mathcal{EXP}(45)\cdot 23$	17	$\mathcal{EXP}(21)\cdot 22$
6	$\mathcal{EXP}(33)\cdot 10$	18	$\mathcal{EXP}(41)\cdot 13$
7	$\mathcal{EXP}(23)\cdot 16$	19	$\mathcal{EXP}(53)\cdot 15$
8	$\mathcal{EXP}(44)\cdot 11$	20	$\mathcal{EXP}(22)\cdot 20$
9	$\mathcal{EXP}(45)\cdot 12$	21	$\mathcal{EXP}(35)\cdot 10$
10	$\mathcal{EXP}(26)\cdot 11$	22	$\mathcal{EXP}(26)\cdot 12$
11	$\mathcal{EXP}(32)\cdot 12$	23	$\mathcal{EXP}(22)\cdot 11$
12	$\mathcal{EXP}(31)\cdot 11$	24	$\mathcal{EXP}(27)\cdot 10$

Table 2. Turnover Capacity ($\tilde{p}_i$) of Each Station

Index (k)	O	D	UB	a_k	q_k	$\tilde{h}_k$
1	1	13	300	80	10	$(1150, 1200, 1300)$
2	4	17	295	90	10	$(1400, 1500, 1550)$
3	12	9	290	65	10	$(1750, 1800, 1850)$
4	21	5	285	30	10	$(1950, 2000, 2060)$
5	24	8	295	46	10	$(3300, 3500, 3560)$

Table 3. The Parameters for Different ODs

There are two classes of fuzzy variables in Table 1, Table 2 and Table 3. The notation with the form "(a, b, c)" represents the triangular fuzzy variable, and the notation with the form "$\mathcal{EXP}(a)$" represents the fuzzy variable with the membership function $\mu(x) = e^{-(x-a)^2}$. In the experimental model, the predetermined target is set to be $T = 131000$, and the chance-constrained confidence levels, respectively, are $\alpha_{ij} = 0.9, (i, j) \in E$, $\beta_i = 0.9$, $i = 1, 2, \cdots, 24$.

In the numerical experiments, we first need to determine the potential routes for each OD so that we can determine the representation of the solution in tabu search. The potential paths for each OD consist of paths whose lengths are not larger than the upper bound (UB) listed in Table 3. In the railway transportation network, there are six potential paths for OD 1, five potential paths for OD 2, three potential paths for OD 3, four potential paths for OD 4 and four potential paths for OD 5. Thus, the total number of the potential paths in the problem is 22.

In this example, the fuzzy parameters in the objective function are all triangular fuzzy variables. By using the result in Section 3, we can compute the objective function by analytical methods for any solution. As for the system constraints, we adopt the discrete fuzzy variable for each fuzzy parameter. Note that fuzzy variables with the form $\mathcal{EXP}(a)$ have not bounded supports, and thus we generate

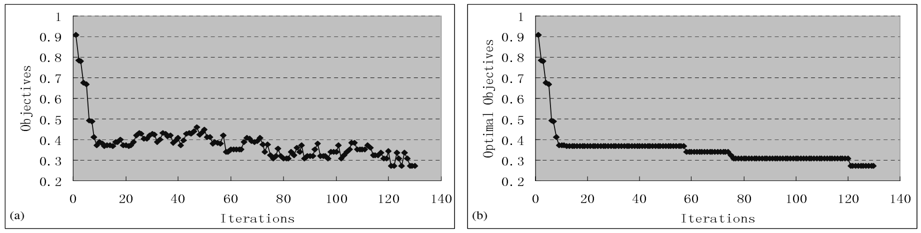

FIGURE 3. Variations of the Seed Solutions and the Best Solutions

truncated fuzzy variables for such fuzzy variables by Theorem 4.9, where we set $\varepsilon = e^{-4}/2$. To keep the accuracy of the approximate model, we set $n = 100$. Then, the distance between a fuzzy parameter and its discrete fuzzy variable ς_{100} is lower that 0.01, i.e.,

$$|\varsigma_{100} - \xi| \leq \frac{1}{100} = 0.01.$$

The discrete fuzzy variable of a fuzzy parameter is generated by the discretization method (7) given in Section 4.

We implemented the tabu search using Microsoft Visual C++ to seek an approximate optimal solution of this problem. To show the performance of the algorithm, we adopted different parameters in tabu search, and the computational results are given in Table 4. We can see that although the parameters are different, we always can get the optimal solutions with the objective value 0.2728. This fact shows the robustness of the implemented algorithm. We obtained an approximate optimal transportation plan given in Table 5.

In order to investigate the performance of the tabu search, we analyze the computational results when Max_iter=500 and Tabu Tenure=7. For convenience, the seed solution (referred by X_l in the procedure of tabu search in Section 5) at each iteration and the best solution encountered so far at each iteration are shown in Figure 3 (a) and Figure 3 (b), respectively. We can see from Figure 3 (a) that the tendency of the sequence $\{X_l\}$ decreases roughly as the iteration increases. However, on the micro-level, this variation tendency is broken at some iterations. For instance, at iteration 47, the value of the seed solution is larger than those of other seed solutions from iteration 8 to iteration 60. Also, a similar variation takes place at iteration 106. In this search process, some local optimal solutions are also obtained, for instance, at iterations 41, 59, 102, respectively. When the iteration number is 121, the minimized objective is found. As for Figure 3 (b), the best objectives varies according to a ladder-like diagram with the iteration number increase. For the first 8 iterations, the best objective values encountered so far decrease quickly. But, when the iteration is larger than 8, the variation tendencies of best solutions slow down. At the iteration 121, the optimal objective value is obtained.

Max_iter	Tabu Tenure	l^*	Optimal Obj. Val.
50	3	249	0.2728
50	5	270	0.2728
100	4	391	0.2728
100	6	380	0.2728
100	7	121	0.2728
200	4	391	0.2728
200	6	380	0.2728
400	5	270	0.2728
400	7	121	0.2728
500	7	121	0.2728

Table 4. The Optimal Solutions by Using Different Parameters

OD	Transportation Amount	Frequency	Optimal Paths
1	80	8	$1 \to 2 \to 4 \to 7 \to 12 \to 13$
2	10	1	$4 \to 6 \to 8 \to 9 \to 15 \to 17$
2	10	1	$4 \to 6 \to 11 \to 14 \to 15 \to 17$
2	60	6	$4 \to 6 \to 11 \to 14 \to 18 \to 17$
2	10	1	$4 \to 7 \to 11 \to 14 \to 15 \to 17$
3	65	7	$12 \to 14 \to 15 \to 9$
4	30	3	$21 \to 24 \to 18 \to 10 \to 5$
5	46	5	$24 \to 18 \to 10 \to 5 \to 8$

Table 5. The Optimal Solution

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References

[1] T. G. Crainic and J. Roy, *O. R. tools for tactical freight transportation planning*, European Journal of Operational Research, **33(3)** (1988), 290-297.

[2] M. F. Gorman, *An application of genetic and tabu searches to the freight railroad operating plan problem*, Annals of Operations Research, **78** (1998), 51-69.

[3] M. F. Gorman, *Santa fe railway uses an operating-plan model to improve its service design*, Interfaces, **28(4)** (1998), 1-12.

[4] M. H. Keaton, *Designing optimal railroad operating plans: lagrangian relaxation and heuristic approaches*, Transportation Research B, **23(6)** (1989), 415-431.

[5] B. Liu and Y. K. Liu, *Expected value of fuzzy variable and fuzzy expected value model*, IEEE Transactions on Fuzzy Systems, **10(4)** (2002), 445-450.

[6] B. Liu, *Theory and practice of uncertain programming*, Physica-Verlag, Heidelberg, 2002.

[7] B. Liu, *Uncertainty theory: an introduction to its axiomatic foundations*, Springer-Verlag, Berlin, 2004.
[8] Y. K. Liu, *Convergence results about the use of fuzzy simulation in fuzzy optimization problems*, IEEE Transactions on Fuzzy Systems, **14** (2006), 295-304.
[9] Y. K. Liu, *The approximate method for two-stagefuzzy random programming with recourse*, IEEE Transactions on Fuzzy Systems, **15** (2007), 1197-1208.
[10] Y. K. Liu, *The convergence results about approximating fuzzy random minimum risk problems*, Applied Mathematics and Computation, **205(2)** (2008), 608-621.
[11] S. Nahmias, *Fuzzy variable*, Fuzzy Sets and Systems, **1** (1978), 97-110.
[12] H. N. Newton, C. Barnhart and P. H. Vance, *Constructing railroad blocking plans to minimize handling costs*, Transportation Science, **32(4)** (1998), 330-345.
[13] Z. Qin and X. Gao, *Fractional Liu process with application to finance*, Mathematical and Computer Modeling, **50** (2009), 1538-1543.
[14] Z. Qin, X. Li and X. Ji, *Portfolio selection based on fuzzy cross-entropy*, Journal of Computational and Applied Mathematics, **228** (2009), 139-149.
[15] Z. Qin and X. Ji, *Logistics network design for product recovery in fuzzy environment*, European Journal of Operational Research, **202** (2010), 279-290.
[16] L. Yang and L. Liu, *Fuzzy fixed charge solid transportation problem and algorithm*, Applied Soft Computing, **7** (2007), 879-889.
[17] L. Yang, K. Li and Z. Gao, *Train timetable problem on a single-line railway with fuzzy passenger demand*, IEEE Transactions on Fuzzy Systems, **17(3)** (2009), 617-629.
[18] L. Yang, Z. Gao and K. Li, *Railway freight transportation planning with mixed uncertainty of randomness and fuzziness*, Applied Soft Computing, **11(1)** (2011), 778-792 .
[19] L. Yang, X. Ji, Z. Gao and K. Li, *Logistics distribution centers location problem and algorithm under fuzzy environment*, Journal of Computational and Applied Mathematics, **208(2)** (2007), 303-315.
[20] L. A. Zadeh, *Fuzzy sets*, Information and Control, **8** (1965), 338-353.
[21] J. Zhou and B. Liu, *Convergence concept of bifuzzy sequence*, Asian Information-Science-Life, **2(3)** (2004), 297-310.

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