MEAN-ABSOLUTE DEVIATION PORTFOLIO SELECTION MODEL WITH FUZZY RETURNS

Z. QIN, M. WEN AND C. GU

Abstract. In this paper, we consider portfolio selection problem in which security returns are regarded as fuzzy variables rather than random variables. We first introduce a concept of absolute deviation for fuzzy variables and prove some useful properties, which imply that absolute deviation may be used to measure risk well. Then we propose two mean-absolute deviation models by defining risk as absolute deviation to search for optimal portfolios. Furthermore, we design a hybrid intelligent algorithm by integrating genetic algorithm and fuzzy simulation to solve the proposed models. Finally, we illustrate this approach with two numerical examples.

1. Introduction

One of the most important directions in finance is portfolio selection theory which deals with how to select a combination of securities to achieve the investor's investment objective. As the founder of this theory, Markowitz [15] established a mean-variance model for portfolio selection problem by trading off the mean and variance of the investment. Since then, based on the assumption that security returns are random variables, a variety of enlarged and improved models have been developed in the framework of probability theory such as [1, 4, 6, 16, 18, 19]. Mean-semivariance model was introduced by Markowitz [16] to replace variance with semivariance in order to overcome the inefficiency of variance in asymmetrical cases. Mean-absolute deviation model was proposed by Konno and Yamazaki [6] by defining absolute deviation of random variables as a risk measure. The purpose of the model is to cope with very large-scale portfolio selection problem because it leads to a linear programming instead of a quadratic programming. In addition, the authors showed the fact that the mean-absolute deviation model gave essentially the same results as the mean-variance model if all the returns are normally distributed random variables.

If there are enough historical data in real life, it is reasonable to regard the returns as random variables and probability theory provides a powerful tool to handle portfolio selection problem. However, the investors often encounter the cases of lack of data. In order to deal with this situation, fuzzy set theory provides alternative approach by estimating security returns by some experts such as fund

Received: September 2010; Revised: October 2010; Accepted: December 2010

Key words and phrases: Uncertainty modelling, Fuzzy variable, Fuzzy portfolio selection, Credibility theory, Hybrid intelligent algorithm.

managers. In fact, several portfolio selection models have been established in fuzzy environment according to different methods, for instance, fuzzy set theory [2, 3, 22], possibility theory [20, 21, 27] or credibility theory [5, 9, 17]. In particular, credibility theory set up an axiomatic foundation for fuzzy set theory and provides a powerful approach to model fuzzy portfolio selection problems.

Up to now, there is no research on fuzzy portfolio selection taking absolute deviation as risk measure. Motivated by Konno and Yamazaki's work [6], this paper is to establish mean-absolute deviation models for portfolio selection problem with fuzzy returns of securities. In several particular cases, it can also be proved that mean-absolute deviation models may lead to linear programming in fuzzy environment.

The rest of the paper is organized as follows. In Section 2, some basic definitions and properties about fuzzy variables are reviewed for facilitating the understanding of the paper. In Section 3, the absolute deviation for fuzzy variable is defined and some important properties are discussed. The formulation of fuzzy mean-absolute deviation models is presented in Section 4 by defining risk as absolute deviation. In order to solve the proposed model, a hybrid intelligent algorithm is designed by integrating fuzzy simulation and genetic algorithm in Section 5. After that, two numerical examples are illustrated to show the effectiveness of the approach in Section 6. Finally, some conclusions are listed in Section 7.

2. **Preliminaries**

Fuzzy set theory was initiated by Zadeh [26] in 1965 and then was applied to many fields. In order to measure a fuzzy event, Liu and Liu [13] proposed a credibility measure in 2002 and then Li and Liu [7] gave a sufficient and necessary condition for it. Credibility theory was founded and refined by Liu [11] to characterize fuzzy phenomena. Next, some definitions and properties of credibility theory are recalled.

Let ξ be a fuzzy variable with membership function μ. Then for any $B \subset \Re$, the credibility measure of $\xi \in B$ was defined by Liu and Liu [13] as

$$\mathrm{Cr}\{\xi \in B\} = \frac{1}{2}\left(\sup_{x \in B} \mu(x) + 1 - \sup_{x \in B^c} \mu(x)\right), \tag{1}$$

which is also called the *credibility inversion theorem*. In order to rank fuzzy variables, the expected value of ξ was defined by Liu and Liu [13] as

$$E[\xi] = \int_0^{+\infty} \mathrm{Cr}\{\xi \geq r\}\mathrm{d}r - \int_{-\infty}^0 \mathrm{Cr}\{\xi \leq r\}\mathrm{d}r \tag{2}$$

provided that at least one of the two integrals is finite. If fuzzy variables ξ and η are independent, then we have

$$E[a\xi + b\eta] = aE[\xi] + bE[\eta] \tag{3}$$

for any $a, b \in \Re$. Since a fuzzy variable ξ and a constant are clearly independent, we have $E[a\xi + b] = aE[\xi] + b$.

Furthermore, if ξ is a fuzzy variable with finite expected value, then its variance was defined by Liu and Liu [13] as

$$V[\xi] = E\left[(\xi - E[\xi])^2\right]. \tag{4}$$

It is easy to prove that $V[\xi] = 0$ if and only if $\text{Cr}\{\xi = e\} = 1$. In addition, the semivariance of ξ was defined by Huang [5] as

$$S[\xi] = E\left[((\xi - E[\xi]) \wedge 0)^2\right] \tag{5}$$

in which $\wedge$ is the minimum operator.

Example 2.1. A triangular fuzzy variable ξ is fully determined by the triplet (a, b, c) of crisp numbers with $a < b < c$ and its membership function is given by

$$\mu(x) = \begin{cases} (x-a)/(b-a), & \text{if } a \le x \le b, \\ (x-c)/(b-c), & \text{if } b \le x \le c, \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

In what follows, we write $\xi = (a, b, c)$. It is easy to prove that $E[\xi] = (a + 2b + c)/4$ and

$$V[\xi] = \frac{33\alpha^3 + 21\alpha^2\beta + 11\alpha\beta^2 - \beta^3}{384\alpha}$$

where $\alpha = \max\{b - a, c - b\}$ and $\beta = \min\{b - a, c - b\}$. Especially, if $\alpha = \beta$, then we get $b - a = c - b$ and $E[\xi] = b$, $V[\xi] = (c - a)^2/24$ and $S[\xi] = V[\xi] = (c - a)^2/24$.

Example 2.2. If ξ is a normally distributed fuzzy variable with the following membership function

$$\mu(x) = 2\left(1 + \exp\left(\frac{\pi|x - e|}{\sqrt{6}\sigma}\right)\right)^{-1}, \ x \in \Re, \tag{7}$$

then Li and Liu [8] proved that $E[\xi] = e$ and $V[\xi] = \sigma^2$. Since the membership function is symmetric, we have $S[\xi] = V[\xi] = \sigma^2$.

3. **Absolute Deviation**

In this section, we first define the concept of absolute deviation of fuzzy variables and then prove some useful properties, which imply that it can be used as an appropriate definition of risk.

Definition 3.1. Let ξ be a fuzzy variable with finite expected value e. Then its absolute deviation is defined as

$$A[\xi] = E[|\xi - e|]. \tag{8}$$

Example 3.2. Assume that ξ is an equipossible fuzzy variable with membership function $\mu(x) = 1$, $a \le x \le b$. It follows from (2) that $E[\xi] = (a + b)/2$. If $0 < r \le (b - a)/2$, according to the credibility inversion theorem, we get

$$\text{Cr}\{|\xi - e| \ge r\} = \text{Cr}\left\{\left\{\xi \ge \frac{a+b}{2} + r\right\} \bigcup \left\{\xi \le \frac{a+b}{2} - r\right\}\right\} = \frac{1}{2}(1 + 1 - 1) = \frac{1}{2}.$$

If $r > (b-a)/2$, it is easy to prove that $\text{Cr}\{|\xi - e| \geq r\} = 0$. Hence, its absolute deviation is

$$A[\xi] = \int_0^{+\infty} \text{Cr}\{|\xi - e| \geq r\}\text{d}r = \int_0^{(b-a)/2} \frac{1}{2}\text{d}r = \frac{b-a}{4}.$$

Example 3.3. Let ξ be a triangular fuzzy variable (a, b, c). It follows from Definition 3.1 that

$$A[\xi] = \frac{(c-a)^2 + 12\alpha^2}{64\alpha}$$

where $\alpha = (b-a) \vee (c-b)$. Especially, if ξ is symmetric, we have $b - a = c - b$ and $A[\xi] = (c-a)/8$.

Theorem 3.4. *Let ξ be a fuzzy variable with finite expected value e. Then we have*

$$A[a\xi + b] = |a|A[\xi] \tag{9}$$

for any real numbers a and b.

Proof. For any $a, b \in \Re$, we have $E[a\xi + b] = aE[\xi] + b$. Furthermore, it follows from Definition 3.1 that

$$A[a\xi + b] = E\left[|a\xi + b - ae - b|\right] = |a|E[|\xi - e|] = |a|A[\xi].$$

The proof is complete. □

Theorem 3.5. *Let ξ be a fuzzy variable with finite expected value e. Then $A[\xi] = 0$ if and only if $\text{Cr}\{\xi = e\} = 1$.*

Proof. If $A[\xi] = 0$, then $E[|\xi - e|] = 0$. Note that

$$E[|\xi - e|] = \int_0^{+\infty} \text{Cr}\{|\xi - e| \geq r\}\text{d}r$$

which implies $\text{Cr}\{|\xi - e| \geq r\} = 0$ for any real number $r > 0$. Hence we have $\text{Cr}\{|\xi - e| = 0\} = 1$, i.e., $\text{Cr}\{\xi = e\} = 1$. Conversely, if $\text{Cr}\{\xi = e\} = 1$, then we have $\text{Cr}\{|\xi - e| = 0\} = 1$ and $\text{Cr}\{|\xi - e| \geq r\} = 0$ for any $r > 0$. Thus, we obtain

$$A[\xi] = \int_0^{+\infty} \text{Cr}\{|\xi - e| \geq r\}\text{d}r = 0.$$

The proof is complete. □

Theorem 3.6. *Let ξ be a fuzzy variable taking values in the closed interval $[a, b]$. If the expected value of ξ is e, then we have*

$$A[\xi] \leq \frac{2(b-e)(e-a)}{b-a}, \tag{10}$$

and the fuzzy variable with the maximum absolute deviation is

$$\xi = \begin{cases} a, & \text{with credibility } (b-e)/(b-a) \\ b, & \text{with credibility } (e-a)/(b-a). \end{cases} \tag{11}$$

Proof. Assume that ξ is defined on the credibility space $(\Theta, \mathcal{P}, \mathrm{Cr})$. For each $\theta \in \Theta$, we have $a \leq \xi(\theta) \leq b$ and

$$\xi(\theta) = \frac{b - \xi(\theta)}{b - a} a + \frac{\xi(\theta) - a}{b - a} b.$$

Define $f(x) = |x - e|$. It follows from the convexity of f that

$$|\xi(\theta) - e| \leq \frac{b - \xi(\theta)}{b - a} |a - e| + \frac{\xi(\theta) - a}{b - a} |b - e|.$$

Taking expected values on both sides, we obtain

$$A[\xi] \leq \frac{b - e}{b - a}(e - a) + \frac{e - a}{b - a}(b - e) = \frac{2(b - e)(e - a)}{b - a}.$$

The second conclusion is easy to be verified by calculating the absolute deviation. The proof is complete. □

Theorem 3.7. *Let ξ be a normally distributed fuzzy variable with expected value e and variance σ^2. Then we have*

$$A[\xi] = \frac{\sqrt{6} \ln 2}{\pi} \sqrt{V[\xi]}. \tag{12}$$

Proof. It is easy to prove that the membership function of $|\xi - e|$ is

$$\mu(x) = 2 \left(1 + \exp\left(\frac{\pi x}{\sqrt{6}\sigma}\right)\right)^{-1}, \ x \geq 0.$$

For any $x \geq 0$, it follows from the credibility inversion theorem that

$$\mathrm{Cr}\{|\xi - e| \geq x\} = \left(1 + \exp\left(\frac{\pi x}{\sqrt{6} m}\right)\right)^{-1}.$$

Please note that $V[\xi] = \sigma^2$ according to Example 2.2. It follows from equation (2) that

$$\begin{aligned}
A[\xi] &= \int_0^{+\infty} \mathrm{Cr}\{|\xi - e| \geq x\} \mathrm{d}x \\
&= \int_0^{+\infty} \left(1 + \exp\left(\frac{\pi x}{\sqrt{6}\sigma}\right)\right)^{-1} \mathrm{d}x \\
&= \frac{\sqrt{6} \ln 2}{\pi} \sigma \\
&= \frac{\sqrt{6} \ln 2}{\pi} \sqrt{V[\xi]}.
\end{aligned}$$

The proof is complete. □

Theorem 3.8. *Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent normally distributed fuzzy variables. Then we have*

$$A[x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n] = |x_1|A[\xi_1] + |x_2|A[\xi_2] + \cdots + |x_n|A[\xi_n] \tag{13}$$

for any $x_1, x_2, \cdots, x_n \in \Re$.

Proof. Since $\xi_1, \xi_2, \cdots, \xi_n$ are independent, $x_1\xi_1+x_2\xi_2+\cdots+x_n\xi_n$ is also a normally distributed fuzzy variable with expected value $x_1E[\xi_1] + x_2E[\xi_2] + \cdots + x_nE[\xi_n]$ and variance $\left(|x_1|\sqrt{V[\xi_1]} + |x_2|\sqrt{V[\xi_2]} + \cdots + |x_n|\sqrt{V[\xi_n]}\right)^2$ (See Li and Liu [8]). It follows from Theorem 3.7 that

$$\begin{aligned}
&A[x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n]\\
&= \frac{\sqrt{6}\ln 2}{\pi}\sqrt{V[x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n]}\\
&= \frac{\sqrt{6}\ln 2}{\pi}\left(|x_1|\sqrt{V[\xi_1]} + |x_2|\sqrt{V[\xi_2]} + \cdots + |x_n|\sqrt{V[\xi_n]}\right)\\
&= |x_1|A[\xi_1] + |x_2|A[\xi_2] + \cdots + |x_n|A[\xi_n].
\end{aligned}$$

The proof is complete. □

Theorem 3.9. *If ξ is a symmetric triangular fuzzy variable (a, b, c), then we have*

$$A[\xi] = \frac{\sqrt{6}}{4}\sqrt{V[\xi]}. \tag{14}$$

Proof. The variance of a symmetric triangular fuzzy variable (a, b, c) is $V[\xi] = (c-a)^2/24$, which has been proved by Liu [11]. Furthermore, it follows from Example 3.3 that $A[\xi] = (c-a)/8$. Hence, we can get equation (14). The proof is complete. □

Remark 3.10. By the similar argument to the proof of Theorem 3.8, if $\xi_1, \xi_2, \cdots, \xi_n$ are independent symmetric triangular fuzzy variables, then equation (13) still holds in this case.

Theorems 3.7 and 3.9 indicate that the absolute deviation is equivalent to the standard deviation in some special cases such as symmetric triangular fuzzy variables and normally distributed fuzzy variables.

4. Mean-Absolute Deviation Model

We consider a capital market with n risk securities offering fuzzy returns. An investor allocates his/her money among these n risky securities to achieve some objective. We use the notations: ξ_i is the return of the ith security and x_i is the proportion of total fund invested in security i, $i = 1, 2, \cdots, n$. In general, ξ_i is calculated as $(p'_i + d_i - p_i)/p_i$ in which p_i is the opening price of the ith security in one period, p'_i is the estimated closing price and d_i is the estimated dividends during the coming period, which are given by experienced experts. If the investor presets an expected return level r, and meanwhile, wants to minimize the risk, then the mean-absolute deviation model is formulated as follows,

$$\left\{\begin{array}{ll}
\text{minimize} & A[\xi_1x_1 + \xi_2x_2 + \cdots + \xi_nx_n]\\
\text{subject to:} & E[\xi_1x_1 + \xi_2x_2 + \cdots + \xi_nx_n] \ge r,\\
& x_1 + x_2 + \cdots + x_n = 1,\\
& x_i \ge 0, \quad i = 1, 2, \cdots, n,
\end{array}\right. \tag{15}$$

where E denotes the expected value operator and A the absolute deviation operator. The first constraint indicates that the expected return is no less than the target return r. The second constraint ensures that all the money will be invested to these n securities. In addition, $x_i \geq 0$ implies that the short-selling or borrowing of security i are not allowed.

As an alternative of model (15), the investor may choose to maximize the expected return of a portfolio while limiting the risk of its return. In this case, mean-absolute deviation model for fuzzy portfolio selection becomes

$$\begin{cases} \text{maximize} & E[\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n] \\ \text{subject to:} & A[\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n] \leq d, \\ & x_1 + x_2 + \cdots + x_n = 1, \\ & x_i \geq 0, \quad i = 1, 2, \cdots, n, \end{cases} \tag{16}$$

where d denotes the maximum risk level the investors can tolerate.

Remark 4.1. If the absolute deviation A is replaced by variance V or semivariance S, then models (15) and (16) become the corresponding mean-variance or mean-semivariance models [5] for fuzzy portfolio selection.

Theorem 4.2. *Suppose that ξ_i are independent normally distributed fuzzy variables with expected value e_i and variance σ_i^2 for $1 \leq i \leq n$. Models (15) and (16) respectively degenerate the following linear programings,*

$$\begin{cases} \textit{minimize} & x_1\sigma_1 + x_2\sigma_2 + \cdots + x_n\sigma_n \\ \textit{subject to:} & x_1 e_1 + x_2 e_2 + \cdots + x_n e_n \geq r, \\ & x_1 + x_2 + \cdots + x_n = 1, \\ & x_i \geq 0, \quad i = 1, 2, \cdots, n, \end{cases} \tag{17}$$

and

$$\begin{cases} \textit{maximize} & x_1 e_1 + x_2 e_2 + \cdots + x_n e_n \\ \textit{subject to:} & x_1\sigma_1 + x_2\sigma_2 + \cdots + x_n\sigma_n \leq \pi d/(\sqrt{6}\ln 2), \\ & x_1 + x_2 + \cdots + x_n = 1, \\ & x_i \geq 0, \quad i = 1, 2, \cdots, n. \end{cases} \tag{18}$$

Proof. It follows immediately from Theorem 3.8. □

Example 4.3. Suppose that we are to construct an optimal portfolio from two risky securities. Let ξ_1 and ξ_2 denote the fuzzy returns on the risky securities. Especially, we assume that ξ_i is a normally distributed fuzzy variable with expected value e_i and variance σ_i for $i = 1, 2$, and ξ_1 and ξ_2 are independent. Without loss of generality, we let $e_1 > e_2$. If $r \in [e_2, e_1]$, then model (17) becomes

$$\begin{cases} \text{minimize} & x_1\sigma_1 + x_2\sigma_2 \\ \text{subject to:} & x_1 e_1 + x_2 e_2 \geq r, \\ & x_1 + x_2 = 1, \\ & x_i \geq 0, \quad i = 1, 2. \end{cases}$$

The argument breaks down into two cases.

Case 1: $\sigma_1 > \sigma_2$. In this case, it is easy to calculate that the portion of the optimal portfolio that should be invested in securities 1 and 2 respectively are

$$x_1^* = \frac{r - e_2}{e_1 - e_2}, \quad x_2^* = \frac{e_1 - r}{e_1 - e_2}.$$

Please note that x_1^* is increasing about r, which implies that the higher the expected return level is, the larger the investment proportion on security 1 is.

Case 2: $\sigma_1 \leq \sigma_2$. In this case, the minimum risk portfolio is $x_1^* = 0$ and $x_2^* = 1$. Since security 2 has higher return and lower risk, the investor prefers security 2 to security 1. Thus, the result is clearly consistent with the actual.

If $r < e_2$, then the first constraint holds for all portfolios (x_1, x_2). Hence, it is clear that the minimum risk portfolio should be

$$x_1^* = \begin{cases} 1, & \sigma_1 \leq \sigma_2, \\ 0, & \text{otherwise,} \end{cases} \qquad x_2^* = \begin{cases} 1, & \sigma_1 \geq \sigma_2, \\ 0, & \text{otherwise.} \end{cases} \tag{19}$$

If $r > e_1$, then the first constraint does not hold for all portfolios (x_1, x_2) since $x_1 e_1 + x_2 e_2 < x_1 r + x_2 r = r$.

The similar result can be obtained when we use model (18) to determine the optimal portfolio.

Theorem 4.4. *Suppose that $\xi_i = (a_i, b_i, c_i)$ are independent and symmetric triangular fuzzy variables. Then models (15) and (16) respectively degenerate the following linear programings,*

$$\begin{cases} \textit{minimize} & x_1(c_1 - a_1) + x_2(c_2 - a_2) + \cdots + x_n(c_n - a_n) \\ \textit{subject to:} & x_1 b_1 + x_2 b_2 + \cdots + x_n b_n \geq r, \\ & x_1 + x_2 + \cdots + x_n = 1, \\ & x_i \geq 0, \quad i = 1, 2, \cdots, n, \end{cases} \tag{20}$$

and

$$\begin{cases} \textit{maximize} & x_1 b_1 + x_2 b_2 + \cdots + x_n b_n \\ \textit{subject to:} & x_1(c_1 - a_1) + x_2(c_2 - a_2) + \cdots + x_n(c_n - a_n) \leq 8d, \\ & x_1 + x_2 + \cdots + x_n = 1, \\ & x_i \geq 0, \quad i = 1, 2, \cdots, n. \end{cases} \tag{21}$$

Proof. It follows immediately from Remark 3.10. □

Remark 4.5. It can be seen that the mean-absolute deviation models are equivalent to linear programmings only in some very particular cases. For general cases such as the one with dependent returns, it is very difficult or impossible to obtain the corresponding deterministic optimization models. Thus, we next design a heuristics — hybrid intelligent algorithm to solve general models.

5. **Hybrid Intelligent Algorithm**

Hybrid intelligent algorithms are first initiated by Liu and Iwamura [12] and extended by Liu [10]. Since then, they have been employed to solve the optimization problems with uncertain parameters. In particular, hybrid intelligent algorithms have been successfully used to solve fuzzy portfolio selection problems [5, 9, 17] as well as other fields [23, 24, 25]. In this section, we design a hybrid intelligent algorithm to solve the proposed models. First, we introduce fuzzy simulation technique which is used to compute expected value and absolute deviation of fuzzy returns. Second, we briefly review and adapt a genetic algorithm (GA) to calculate the proposed models. Finally, we integrate fuzzy simulation and genetic algorithm to construct a hybrid intelligent algorithm.

Fuzzy Simulation.

It is more difficult to obtain the exact values of expected value and absolute deviation of a portfolio with general fuzzy returns. In order to handle the problem, we employ fuzzy simulation introduced by [12], which is a feasible approach to approximately calculate those quantities. Liu [14] has proved the convergence of fuzzy simulation. The detailed procedures on fuzzy simulation may be found in Liu [10]. Here, we adapt fuzzy simulation techniques to suit our models in Section 4.

Assume that fuzzy variable ξ_i has a membership function μ_i for $i = 1, 2, \cdots, n$. We first calculate the value of $\text{Cr}\{\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n \geq r\}$ where r is a nonnegative real number. The detailed algorithm is as follows,

Step 1: Set $m = 1$ and take M as a sufficiently large integer;

Step 2: Randomly generate n numbers w_{im} such that the membership degrees $\mu_i(w_{im}) \geq \varepsilon$ for $i = 1, 2, \cdots, n$, respectively, where ε is a sufficiently small number;

Step 3: Compute $x_1 w_{1m} + x_2 w_{2m} + \cdots + x_n w_{nm}$. If the value is no less than r, let $u_m = \min_{1\leq i\leq n} \mu_i(w_{im})$ and $v_m = 0$; otherwise, let $u_m = 0$ and $v_m = \min_{1\leq i\leq n} \mu_i(w_{im})$;

Step 4: If $m > M$, then go to next step; otherwise, set $m = m + 1$ and go to Step 2;

step 5: Return $(\max_{1\leq m\leq M} u_m + 1 - \max_{1\leq m\leq M} v_m)/2$ as the value of $\text{Cr}\{\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n \geq r\}$.

The following procedure is given to calculate $E[\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n]$.

Step 1: Set $e_1 = 0$, $e_2 = 0$ and $j = 0$, and let $h = M/N$ where M is a sufficiently large positive number and N is a sufficiently large integer;

Step 2: Partition the intervals $[0, M]$ and $[-M, 0]$ into N small intervals $[z_j, z_{j+1}]$ and $[z'_j, z'_{j+1}]$, respectively, in which $z_j = jh$ and $z'_j = -M + jh$ for $i = 0, 1, \cdots, n-1$.

Step 3: Set $e_1 \leftarrow e_1 + h(\text{Cr}\{\sum \xi_i x_i \geq z_j\} + \text{Cr}\{\sum \xi_i x_i \geq z_{j+1}\})/2$;

Step 4: Set $e_2 \leftarrow e_2 + h(\text{Cr}\{\sum \xi_i x_i \leq z'_j\} + \text{Cr}\{\sum \xi_i x_i \leq z'_{j+1}\})/2$;

Step 5: If $j = N$, stop and return $e_1 - e_2$; otherwise, go to Step 3.

Let ξ be a fuzzy variable with finite expected value. According to the definition of absolute deviation of fuzzy variable, we know that $A[\xi]$ is essentially the expected value of fuzzy variable $|\xi - E[\xi]|$. Replace $\sum_{i=1}^n \xi_i x_i$ with $|\sum_{i=1}^n \xi_i x_i - E[\sum_{i=1}^n \xi_i x_i]$ and repeat the above procedures, we can obtain the simulated value of $A[\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n]$.

Genetic Algorithm

Genetic algorithm was initiated by Holland in 1975 as an adaptive heuristic search algorithm premised on the evolutionary ideas of natural selection and genetic. Since then, genetic algorithm has been widely experimented and applied in many fields. For instance, Liu [10] has successfully applied genetic algorithm to solve many optimization problems with fuzzy parameters. In the next part, we briefly introduce the main parts of genetic algorithm. The interested readers may consult [10] or [5, 9, 17].

Representation Structure: This paper employs real encoding, i.e., a solution $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$ is encoded by a chromosome $\boldsymbol{c} = (c_1, c_2, \cdots, c_n)$ where the genes $c_1, c_2, \cdots, c_n$ are restricted as nonnegative numbers. The decoding process is determined by the link $x_i = c_i/(c_1 + c_2 + \cdots + c_n)$, which ensures that $x_1 + x_2 + \cdots + x_n = 1$ always holds. A chromosome is called feasible if it satisfies the corresponding constraint conditions. For instance, in model (15), if a chromosome $(c_1, \cdots, c_n)$ satisfies

$$E\left[\frac{\xi_1 c_1}{c_1 + \cdots + c_n} + \frac{\xi_2 c_2}{c_1 + \cdots + c_n} + \cdots + \frac{\xi_n c_n}{c_1 + \cdots + c_n}\right] \geq r,$$

then it is a feasible chromosome.

Initialization: Let *pop_size* be the population size. The purpose of the initialization is to obtain *pop_size* feasible chromosomes as the first population. The procedure of initialization is summarized as follows,

Step 1: Let i=1;
Step 2: Randomly generate n nonnegative numbers $c_1, c_2, \cdots, c_n$ such that $\boldsymbol{c}_i = (c_1, c_2, \cdots, c_n)$ is a feasible chromosome, in which fuzzy simulation is used to check its feasibility;
Step 3: If $i = pop\_size$, stop; otherwise, set $i = i + 1$ and go to Step 2.

Evaluation Function: This paper employs the more popular rank-based evaluation function to measure the likelihood of reproduction of each chromosome. Let $v \in (0, 1)$. The rank-based evaluation function is defined as: $Eval(\mathbf{c}_i) = v(1-v)^{i-1}, \quad i = 1, 2, \cdots, pop\_size$. It is easy to see that $i = 1$ indicates the best individual and $i = pop\_size$ indicates the worst one.

Selection Process: This paper uses spinning the roulette wheel method to select each chromosome. At each time, one chromosome is chosen for a new child population. The next population is obtained after repeating the process *pop_size* times. Define $p_0 = 0$ and $p_i = \sum_{j=1}^i Eval(\boldsymbol{c}_i), i = 1, 2, \cdots, pop\_size$. The algorithm is summarized as follows,

Step 1: Let j=1;
Step 2: Generate a number r between 0 and p_{pop_size};
Step 3: Choose the chromosome $\boldsymbol{c}_i$ if $r \in (p_{i-1}, p_i]$;
Step 4: If $j >$ *pop_size*, stop; otherwise, set $j = j + 1$ and go to Step 2.

Crossover Operation: Let P_c be the probability of crossover, and generate a number $r \in [0, 1]$. If $r < P_c$, randomly select two parent chromosomes denoted by $\mathbf{c}_1$ and $\mathbf{c}_2$, and then produce two offsprings through crossover operator

$$\boldsymbol{c}'_1 = r \cdot \mathbf{c}_1 + (1-r) \cdot \mathbf{c}_2, \quad \boldsymbol{c}'_2 = (1-r) \cdot \mathbf{c}_1 + r \cdot \mathbf{c}_2.$$

If both $\boldsymbol{c}'_1$ and $\boldsymbol{c}'_2$ are feasible, take them as children to replace their parents. If at least one of them is infeasible, then redo the crossover operation until two feasible children are obtained or a given number of iterations is finished. The crossover operation is finished by repeating the above process *pop_size* times.

Mutation Operation: Let P_m be the probability of mutation, and generate a real number $s \in (0, 1)$. If $s < P_m$, then choose a chromosome $\mathbf{c}$ as the parent one for mutation. Next, generate n numbers $d_1, d_2, \cdots, d_n$ which belong to the interval $[-D, D]$ where $D > 0$ is an appropriate number given by the investor. A new chromosome may be created by $\boldsymbol{c}' = \boldsymbol{c} + (d_1, d_2, \cdots, d_n)$. If $\boldsymbol{c}'$ is feasible, take it as the child. Otherwise, redo the mutation operation until one feasible child is obtained or a given number of iterations is finished. The mutation operation is finished by repeating the above process *pop_size* times.

Hybrid Intelligent Algorithm

The hybrid intelligent algorithm is essentially fuzzy simulation based genetic algorithm in which fuzzy simulation is used to approximately compute some objectives and/or constraints. The procedure of the algorithm is summarized as follows:

Step 1: Initialize *pop_size* feasible chromosomes whose feasibility is checked by fuzzy simulation;
Step 2: Update the chromosomes by crossover and mutation operations;
Step 3: Compute the objective value for each chromosome by using fuzzy simulation;
Step 4: Calculate the fitness of each chromosome by the rank-based-evaluation function according to their objective values;
Step 5: Select the chromosomes by using spinning the roulette wheel;
Step 6: Repeat Step 2 to Step 5 for a given number of generations;
Step 7: Return the best chromosome and convert it into the optimal solution.

6. Numerical Examples

In this section, numerical examples illustrate the modelling idea and the effectiveness of the proposed algorithm. The data on security returns are from Huang [5] in which there are 10 securities in the set of candidate securities. The returns of the first seven ones are triangular fuzzy variables, and that of the other three ones take the membership functions shown in Table 1 (from [5]).

Security i	Fuzzy return	Security i	Fuzzy return
1	$(-0.3, 1.8, 2.3)$	6	$(-0.8, 2.5, 3.0)$
2	$(-0.4, 2.0, 2.2)$	7	$(-0.6, 1.8, 3.0)$
3	$(-0.5, 1.9, 2.7)$	8	$1/(1+(r-1.6)^4)$
4	$(-0.6, 2.2, 2.8)$	9	$1/(1+(5r-7.4)^2))$
5	$(-0.7, 2.4, 2.7)$	10	$\exp(-(r-1.6)^2)$

TABLE 1. Fuzzy Returns of 10 Securities (Units per Stock)

Example 6.1. Suppose the minimum expected return level that the investor can accept is 1.5. If the investor accepts absolute deviation as risk measure, then he/she can use the following mean-absolute deviation model to construct an optimal portfolio,

$$\begin{cases} \text{minimize} & A[\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_{10} x_{10}] \\ \text{subject to:} & E[\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_{10} x_{10}] \geq 1.5 \\ & x_1 + x_2 + \cdots + x_{10} = 1 \\ & x_i \geq 0, \quad i = 1, 2, \cdots, 10. \end{cases} \tag{22}$$

A run of the hybrid intelligent algorithm (2500 cycles in simulation, 500 generations in GA) shows that the optimal portfolio is

$$\boldsymbol{x}^* = (0.121, 0.140, 0.114, 0.100, 0.116, 0.086, 0.102, 0.069, 0.081, 0.071)$$

which represents the allocation of his money to each security. The corresponding minimal risk is 0.827.

Next, we test the effectiveness of the algorithm by changing the parameters in the GA. In order to compare the results, we employ the relative error defined as

$$\frac{\text{Actual absolute derivation} - \text{Minimal absolte derivation}}{\text{Minimal absolute deviation}} \times 100\%.$$

Here, the minimal absolute derivation is the minimum of all the obtained results. As summarized in Table 2, the relative errors do not exceed 2%, which shows that the hybrid intelligent algorithm is robust to set parameters and effective for solving the proposed model.

Example 6.2. Suppose that the absolute deviation (the risk) cannot exceed the minimum risk level 1.1, then the investor can employ the following fuzzy mean-absolute deviation model,

$$\begin{cases} \text{maximize} & E[\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_{10} x_{10}] \\ \text{subject to:} & A[\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_{10} x_{10}] \leq 1.1, \\ & x_1 + x_2 + \cdots + x_{10} = 1, \\ & x_i \geq 0, \quad i = 1, 2, \cdots, 10. \end{cases} \tag{23}$$

The parameters of GA are chosen as $pop\_size = 100$, $P_c = 0.3$ and $P_m = 0.2$. A run of the hybrid intelligent algorithm (2000 cycles in simulation, 500 generations in GA) shows that the investor should allocate his money according to Table 3.

No.	*pop_size*	P_c	P_m	Simulation Times	Generation	Absolute Deviation	Relative Error (%)
1	50	0.3	0.8	1500	600	0.8353	1.02
2	90	0.4	0.7	1300	400	0.8331	0.75
3	75	0.3	0.5	1000	300	0.8330	0.74
4	120	0.6	0.4	1500	400	0.8336	0.81
5	150	0.7	0.6	2500	200	0.8270	0.01
6	300	0.5	0.7	2000	220	0.8377	1.31
7	200	0.2	0.1	2200	250	0.8412	1.72
8	350	0.6	0.5	1500	250	0.8290	0.25
9	100	0.8	0.3	2500	500	0.8269	0
10	400	0.4	0.3	2500	100	0.8375	1.28

TABLE 2. Computational Solutions of Example 6.1

The corresponding maximum expected return is 1.72. In addition, the evolution of optimal objective value at each generation is shown in Figure 1. It can be seen that the result keep stable after 300 generations.

Security i	1	2	3	4	5	6	7	8	9	10
Proportion	0.00	0.43	2.26	6.40	9.06	58.75	2.83	1.93	0.00	18.34

TABLE 3. Allocation of the Money to 10 Securities (%)

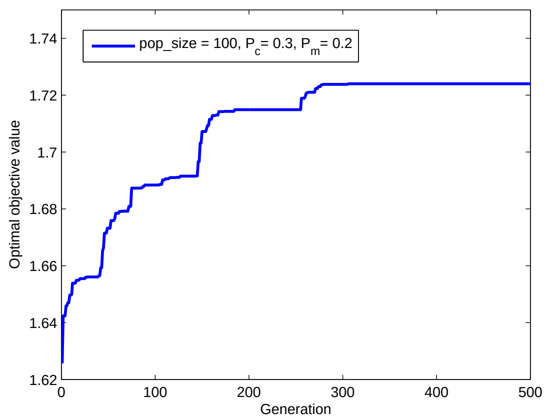

FIGURE 1. The Convergence of Objective Value in Example 7

In order to examine the sensitivity of the tolerable risk to total expected return, we experimented on this model by changing the tolerable risk. The computational results are summarized in Figure 2 which is almost consistent with the actual expectation. Generally speaking, hybrid intelligent algorithm only offers the approximate solution for the problem not the exact optimal solution. Therefore, there maybe exists larger computational errors in the results, which is the reason why the curve is not smooth.

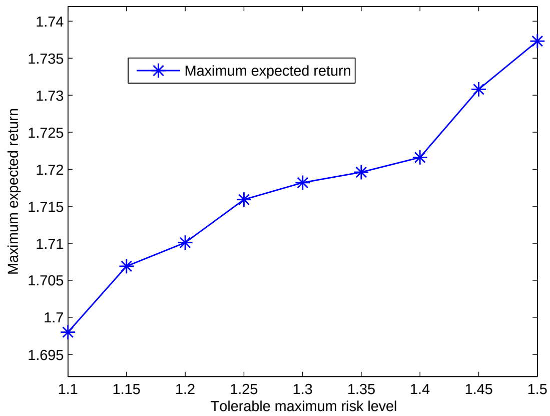

FIGURE 2. The Sensitivity to Changes of Tolerable Maximum Risk

7. Conclusions

The purpose of this paper was to investigate portfolio selection problem with fuzzy returns of securities by defining a new risk measure and proposing two mean-absolute deviation models. The main works of the paper included the following aspects: (i) a new definition of risk was presented in fuzzy environment and its some useful properties were proved; (ii) mean-absolute deviation models was proposed for fuzzy portfolio selection; (iii) a hybrid intelligent algorithm was designed and employed to solve the proposed models; (iv) two numerical examples were given to show the modelling idea and the effectiveness of the proposed algorithm.

Acknowledgements. This work was partially supported by National Natural Science Foundation of China Grant No. 71001003, Specialized Research Fund for the Doctoral Program of Higher Education of China Grant No. 20101102120022 and the Fundamental Research Funds for the Central Universities Grant No. YWF-10-02-028.

REFERENCES

[1] F. Abdelaziz, B. Aouni and R. Fayedh, *Multi-objective stochastic programming for portfolio selection*, Eur. J. Oper. Res., **177** (2007), 1811-1823.

[2] M. Arenas-Parra, A. Bilbao-Terol and M. Rodríguez-Uría, *A fuzzy goal programming approach to portfolio selection*, Eur. J. Oper. Res., **133** (2001), 287-297.

[3] A. Bilbao-Terol, B. Pérez-Gladish, M. Arenas-Parra and M. Rodrígez-Uría, *Fuzzy compromise programming for portfolio selection*, Appl. Math. Comput., **173** (2006), 251-264.

[4] M. Corazza and D. Favaretto, *On the existence of solutions to the quadratic mixed-integer mean-variance portfolio selection problem*, Eur. J. Oper. Res., **176** (2007), 1947-1980.

[5] X. Huang, *Mean-semivariance models for fuzzy protfolio selection*, J. Comput. Appl. Math., **217** (2008), 1-8.

[6] H. Konno and H. Yamazaki, *Mean-absolute deviation portfolio optimization model and its applications to Tokyo Stock Market*, Manage. Sci., **37** (1991), 519-531.

[7] X. Li and B. Liu, *A sufficient and necessary condition for credibility measures*, Int. J. Uncertain. Fuzz., **14** (2006), 527-535.

[8] X. Li and B. Liu, *Maximum entropy principle for fuzzy variables*, Int. J. Uncertain. Fuzz., **15** (2007), 43-53.

[9] X. Li, Z. Qin and S. Kar, *Mean-variance-skewness model for portfolio selection with fuzzy returns*, Eur. J. Oper. Res., **202** (2010), 239-247.

[10] B. Liu, *Theory and practice of uncertain programming*, Physica-Verlag, Heidelberg, 2002.

[11] B. Liu, *Uncertainty theory*, 2nd ed., Springer-Verlag, Berlin, 2007.

[12] B. Liu and K. Iwamura, *Chance constrained programming with fuzzy parameters*, Fuzzy Sets and Systems, **94** (1998), 227-237.

[13] B. Liu and Y. Liu, *Expected value of fuzzy variable and fuzzy expected value models*, IEEE T. Fuzzy Syst., **10** (2002), 445-450.

[14] Y. Liu, *Convergent results about the use of fuzzy simulation in fuzzy optimization problems*, IEEE T. Fuzzy Syst., **14** (2006), 295-304.

[15] H. Markowitz, *Porfolio selection*, J. Finance, **7** (1952), 77-91.

[16] H. Markowitz, *Portfolio selection: efficient diversification of investments*, Wiley, New York, 1959.

[17] Z. Qin, X. Li and X. Ji, *Portfolio selection based on fuzzy cross-entropy*, J. Comput. Appl. Math., **228** (2009), 139–149.

[18] Y. Simaan, *Estimation risk in portfolio selection: the mean vairance model versus the mean absolute deviation model*, Manage. Sci., **43** (1997), 1437-1446.

[19] M. G. Speranza, *Linear programming model for portfolio optimization*, Finance, **14** (1993), 107-123.

[20] H. Tanaka and P. Guo, *Portfolio selection based on upper and lower exponential possibility distributions*, Eur. J. Oper. Res., **114** (1999), 115-126.

[21] H. Tanaka, P. Guo and I. Türksen, *Portfolio selection based on fuzzy probabilities and possibility distributions*, Fuzzy Sets and Systems, **111** (2000), 387-397.

[22] E. Vercher, J. Bermúdez and J. Segura, *Fuzzy portfolio optimization under downside risk measures*, Fuzzy Sets and Systems, **158** (2007), 769-782.

[23] L. Yang, K. Li and Z. Gao, *Train timetable problem on a single-line railway with fuzzy passenger demand*, IEEE T. Fuzzy Syst., **17** (2009), 617-629.

[24] L. Yang and L. Liu, *Fuzzy fixed charge solid transportation problem and algorithm*, Appl. Soft Comput., **7** (2007), 879-889.

[25] L. Yang, Z. Gao and K. Li, *Railway freight transportation planning with mixed uncertainty of randomness and fuzziness*, Appl. Soft Comput., **11** (2011), 778-792.

[26] L. A. Zadeh, *Fuzzy sets*, Information and Control, **8** (1965), 338-353.

[27] W. Zhang and Z. Nie, *On admissible efficient portfolio selection problem*, Appl. Math. Comput., **159** (2004), 357-371.

Zhongfeng Qin, School of Economics and Management, Beihang University, Beijing 100191, China

E-mail address: qin@buaa.edu.cn, qzf05@mails.thu.edu.cn

Meilin Wen*, School of Reliability and Systems Engineering, Beihang University, Beijing 100191, China

E-mail address: wenmeilin@buaa.edu.cn

Changchao Gu, Sinopec Management Institute, Beijing 100012, China

E-mail address: guchangchao@gamil.com

*Corresponding author