

## THE $p$ -CENTER PROBLEM ON FUZZY NETWORKS AND REDUCTION OF COST

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ABSTRACT. Here we consider the  $p$ -center problem on different types of fuzzy networks. In particular, we are interested in the networks with interval and triangular fuzzy arc lengths and vertex-weights. A methodology to obtain the best satisfaction level of the decision maker who wishes to reduce the cost within the tolerance limits is proposed. Illustrative examples are provided.

### 1. Introduction

The  $p$ -center problem is a well-known facility location problem. It arises in the following way. Suppose some demand points are given by the vertices of a network, and the weight of each vertex represents the demand at that point, and there is a path between every pair of vertices in order to transport products from one vertex to another. The decision maker is to locate  $p$  locations within the network where the facilities are to be located such that the maximum cost to transport products to each demand point from the nearest facility is minimized. It is assumed that these costs are directly proportional to the distances to be covered and the quantities of product to be transported. If the facilities are considered to be located at the vertices only, the  $p$ -center problem can be defined as follows.

Let  $G = (V, E)$  be a connected undirected network, where  $V = \{v_1, v_2, \dots, v_n\}$  is the set of vertices of  $G$  and  $E$  is the set of edges. As  $G$  is connected, there exists a path between every pair of vertices. The distance  $d(v_i, v_j)$  between two vertices  $v_i$  and  $v_j$  is denoted by  $d_{ij}$ , the length of the shortest path joining the vertices  $v_i$  and  $v_j$ . The weight  $w_i$  is associated with the vertex  $v_i$  for all  $i = 1, 2, \dots, n$ . Let us consider all the subsets of  $\{1, 2, \dots, n\}$  with  $p$  number of elements. Let those subsets be  $A_1, A_2, \dots, A_m$ . So for a given set covering  $A_i$ , the distance to be covered in order to serve the vertex  $v_j$  is given by  $\delta_{ij} = \min_{l \in A_i} d_{lj}$ . Also for each  $A_i$ , we define  $\rho_i = \max_{v_j \in V} \delta_{ij} \cdot w_j$ . Thus the  $p$ -center problem is to find the set  $A_k$  for which the corresponding  $\rho$  is minimum among all  $\rho_i$ 's and that minimum is called the  $p$ -radius of the network. Hence, if  $\rho^* = \min_i \rho_i$  be the  $p$ -radius, then the  $p$ -center is given by  $A_k$  such that  $\rho_k = \rho^*$ . Clearly, the  $p$ -center of a network is not unique.

Kariv and Hakimi [17] showed that the problem is NP-complete. They described an algorithm of time complexity  $O(|E|^p |V|^{2p-1} \log |V| / (p-1)!)$  (or  $O(|E|^p |V|^{2p-1} / (p-1)!)$ ) for finding the  $p$ -center of a vertex-weighted (or vertex-unweighted) network.

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Tamir [32] improved the complexity bounds by using dynamic data structure, and showed that the weighted and unweighted  $p$ -center on network can be found in  $O(|E|^p|V|^p \log^2 |V|)$  and  $O(|E|^p|V|^p \log^3 |V|)$  time.  $p$ -center problems on different subclasses of graph such as trees, circular-arc graphs etc. have been investigated by some authors [2, 10]. Also exact algorithms to solve the vertex  $p$ -center problem and capacitated vertex  $p$ -center problems are found in [1, 28]. The location problems correspond to large-scale optimization models that can not be solved in polynomial time, and so efficient heuristics are required [15, 19, 20]. Recently Nayeem and Pal [24] have proposed a genetic algorithmic approach to solve the  $p$ -center problem where the computational times are reported to be reasonably lower.

The classical location problems deal with the networks for which all the information, i.e., the set of vertices, the set of edges, the weights of vertices and edges are known and certain. But the data and the environments which are available in reality are vague and imprecise in nature in most cases. Thus the fuzzy considerations arise. Let us consider a network whose vertices represent some cities/towns in a country and the edges represent the roads linking the cities/towns. Now the property of being a town has no crisp boundary. A locality with 50,000 inhabitants may not be considered as a town by someone, while a locality with 25,000 inhabitants may be considered as a town by some other else. These can be managed easily by taking both the localities as towns with different membership values. Hence the vertex set becomes a fuzzy set in this case.

Similarly depending upon characteristics like goodness, roughness, crowd, etc. of the roads or links, the edge set becomes fuzzy. Moreover, the demands and the distances or transportation cost per unit item between a demand point and a facility are almost impossible to obtain as crisp values. Hence the cases of fuzzy weights take place.

The concept of fuzzy graphs was first proposed by Rosenfeld [29], though it originates from the pioneering work of Zadeh [33]. A lot of works have been already done on this topic in [5, 6, 7, 12, 14]. An extensive review can be found in Mordeson and Nair [22]. Some network problems, namely, shortest path problem [25, 27], PERT problem [9, 26] etc. with fuzzy arc-lengths are also found in the literature. Fuzzy location problems are addressed in [23]. The fuzzy  $p$ -median problem and a global analysis of the solutions are provided by Canós et al. [3, 4]. They give an algorithm to minimize the transportation cost with an acceptable reduction of the covered demand to cut down the cost by a significant amount.

Different fuzzy networks and location models are given in Section 2. In this paper, we consider networks with the edge-weights or vertex-weights as interval numbers and triangular fuzzy numbers and give a straight-forward method to find the  $p$ -center. This method largely depends on the ranking methods of the imprecise numbers under consideration. So we develop the arithmetic and comparison rules of interval numbers and triangular fuzzy numbers in Section 3, and using those rules we obtain the method to find the  $p$ -center in Section 4. The most important part of this paper is Section 5, as we propose there a method to obtain the grade of satisfaction of the decision maker who wishes to fix the value of the  $p$ -radius by a crisp number within the tolerance limits.

## 2. Fuzzy Network and Different location Models

Fuzzyness of a network may arise in many ways. Rosenfeld [29] has given the definition of a fuzzy graph in the following way.

**Definition 2.1.** A fuzzy graph is a structure  $G = (V, E, \sigma, \mu)$ , where  $V = \{v_1, v_2, \dots, v_n\}$  is the vertex set,  $\sigma : V \rightarrow [0, 1]$  is the membership function of the vertex set,  $E = \{e_{ij} : v_i, v_j \in V, i \neq j\}$  is the edge set, provided that

$$\max \mu(e_{ij}) \leq \min\{\sigma(v_i), \sigma(v_j)\} \text{ for all } v_i, v_j \in V,$$

i.e., the membership  $\mu$  of every edge between two vertices  $v_i$  and  $v_j$  must be less than or equal to that of each of  $v_i$  and  $v_j$ .

We denote a fuzzy graph  $G$  as  $\tilde{G} = (\tilde{V}, \tilde{E})$ .

If the membership value of each vertex of  $\tilde{G}$  is 1, then the vertex set  $\tilde{V}$  becomes a crisp set. Then we write  $\tilde{G} = (V, \tilde{E})$ .

If  $\tilde{E}$  is a crisp set, i.e., each edge of  $\tilde{G}$  has the membership value 1 in  $\tilde{E}$ , then from Definition 2.1 it follows that  $\tilde{V}$  is also a crisp set. Thus in this case,  $\tilde{G}$  becomes the crisp graph  $G = (V, E)$ . Although in reality in a few cases fuzzy graphs with fuzzy vertex set and crisp edge set may be found. As for example, during some military actions, roads through some odd locations in hill regions are constructed instantly and those are destroyed just after reaching of the whole company to the next location. This is done in order to reuse some equipments for construction and to avoid the chance of taking advantage of the roads by the enemies. But, to follow Rosenfelds definition, we ignore this kind of exceptional fuzzy graphs. Albeit these can be treated in the same way as that of the others which are considered in this paper.

With reference to a fuzzy graph  $\tilde{G} = (\tilde{V}, \tilde{E})$ , we can have a fuzzy network as follows.

Let  $w(\cdot)$  and  $l(\cdot)$  be two functions defined for every members of  $\tilde{V}$  and  $\tilde{E}$  called the weight and length respectively. Now if  $w(\cdot)$  assumes fuzzy values for all or some of the members of  $\tilde{V}$ , then we denote the fuzzy valued weight function as  $\tilde{w}(\cdot)$  and similarly the fuzzy valued length function is denoted by  $\tilde{l}(\cdot)$ . The structure  $\tilde{N} = (\tilde{V}, \tilde{E}, \tilde{w}, \tilde{l})$  is a called fuzzy network.

A fuzzy network can be classified into the following classes. We may consider fuzzy location models of the following types.

- (1) (a) The most general fuzzy network  $\tilde{N} = (\tilde{V}, \tilde{E}, \tilde{w}, \tilde{l})$ , where the vertex set and edge set both are fuzzy sets and the weight function and length function are both fuzzy-valued.
- (b) The vertex set and the edge set are fuzzy sets and either the weight function or the length function or both are crisp-valued, i.e.,  $\tilde{N} = (\tilde{V}, \tilde{E}, \tilde{w}, l)$  or,  $\tilde{N} = (\tilde{V}, \tilde{E}, w, \tilde{l})$  or,  $\tilde{N} = (\tilde{V}, \tilde{E}, w, l)$ .
- (2) (a) Vertex set  $V$  is crisp, the edge set  $\tilde{E}$  is fuzzy and the weight and length functions are both fuzzy-valued, i.e.,  $\tilde{N} = (V, \tilde{E}, \tilde{w}, \tilde{l})$ .

- (b)  $V$  is a crisp set,  $\tilde{E}$  is a fuzzy set and either the weight function or the length function or the both are crisp-valued, i.e.,  $\tilde{N} = (V, \tilde{E}, w, \tilde{l})$  or,  $\tilde{N} = (V, \tilde{E}, \tilde{w}, l)$  or,  $\tilde{N} = (V, \tilde{E}, w, l)$ .
- (3) (a) Vertex set and edge set both are crisp sets and both the weight function and the length function are fuzzy-valued, i.e.,  $\tilde{N} = (V, E, \tilde{w}, \tilde{l})$ .
- (b)  $V$  and  $E$  are both crisp sets and either the weight function or the length function is fuzzy-valued, but not both. If both the weight function and the length function are crisp, then the network becomes a crisp network. i.e.,  $\tilde{N} = (V, E, w, \tilde{l})$  or,  $\tilde{N} = (V, E, \tilde{w}, l)$ .

The fuzzy  $p$ -center problem, or more generally the fuzzy location problems can be modelled in either of the above classes of fuzzy networks. Using the  $\alpha$ -cut techniques, the fuzzy location models of Type (1) or Type (2) can be reduced to solving a finite series of models of Type (3) [23].

**Definition 2.2.** The  $\alpha$ -cut,  $\alpha \in [0, 1]$ , of a fuzzy graph  $G = (V, E, \sigma, \mu)$  is the classical graph  $G^\alpha = (V^\alpha, E^\alpha)$  with  $V^\alpha = \{v \in V : \sigma(v) \geq \alpha\}$  and  $E^\alpha = \{e \in E : \mu(e) \geq \alpha\}$ .

In this paper, we solve the  $p$ -center problem on a fuzzy network of Type (3)(b), i.e., on a network with crisp sets of vertices and edges and (i) imprecise edge-weights, or (ii) imprecise vertex-weights. They both on combination give rise to the solution of a model of Type (3)(a). In particular, we consider the imprecise numbers as interval numbers or triangular fuzzy numbers.

### 3. Arithmetic of Imprecise Numbers

The arithmetic of imprecise numbers, especially, the comparison rules of those play an important role in solving this problem. In this section, interval numbers and triangular fuzzy numbers are introduced. Addition and comparison techniques between two such numbers are also discussed.

**3.1. Interval Number and Interval Arithmetic.** As said in the previous section, the distance ‘about 5 KM’ may be considered as  $[4.5, 5.5]$  KM. Clearly, to someone it may appear to lie within, say,  $[4.6, 5.3]$  KM, etc.

In general, an interval number is defined as

$$A = [a_L, a_R] = \{a : a_L \leq a \leq a_R\} \quad (1)$$

where,  $a_L$  and  $a_R$  are the real numbers called the left end point and the right end point of the interval  $A$ .

Another way to represent an interval number in terms of midpoint and width is

$$A = \langle m(A), w(A) \rangle, \text{ where, } m(A) = \text{midpoint of } A = \frac{a_R + a_L}{2}$$

$$\text{and } w(A) = \text{half width of } A = \frac{a_R - a_L}{2}. \quad (2)$$

A crisp real number  $k$  may be considered as a degenerate interval  $[k, k] = \langle k, 0 \rangle$ . The addition of two interval numbers  $A = [a_L, a_R]$  and  $B = [b_L, b_R]$  is given by

$$A \oplus B = [a_L + b_L, a_R + b_R]. \quad (3)$$

Alternately, in mean-width notations, if  $A = \langle m_1, w_1 \rangle$  and  $B = \langle m_2, w_2 \rangle$  then,

$$A \oplus B = \langle m_1 + m_2, w_1 + w_2 \rangle. \quad (4)$$

The product of two interval numbers  $A = [a_L, a_R]$  and  $B = [b_L, b_R]$  is given by

$$A \odot B = [\min\{a_L \cdot b_L, a_R \cdot b_L, a_L \cdot b_R, a_R \cdot b_R\}, \max\{a_L \cdot b_L, a_R \cdot b_L, a_L \cdot b_R, a_R \cdot b_R\}]. \quad (5)$$

If  $A$  and  $B$  are both positive, then (5) becomes  $A \odot B = [a_L \cdot b_L, a_R \cdot b_R]$ . The negation of an interval number  $A = [a_L, a_R]$  given by  $-A = [-a_R, -a_L]$ .

The subtraction of two interval numbers  $A = [a_L, a_R]$  and  $B = [b_L, b_R]$  is given by

$$A \ominus B = [a_L - b_R, a_R - b_L]. \quad (6)$$

Alternately, in mean-width notations, if  $A = \langle m_1, w_1 \rangle$  and  $B = \langle m_2, w_2 \rangle$  then,

$$A \ominus B = \langle m_1 - m_2, w_1 + w_2 \rangle. \quad (7)$$

The quotient of two interval numbers  $A = [a_L, a_R]$  and  $B = [b_L, b_R]$  is given by

$$A \oslash B = [\min\{a_L/b_L, a_R/b_L, a_L/b_R, a_R/b_R\}, \max\{a_L/b_L, a_R/b_L, a_L/b_R, a_R/b_R\}]. \quad (8)$$

If  $A$  and  $B$  are both positive, then (8) becomes  $A \oslash B = [a_L/b_L, a_R/b_R]$ .

Comparison between two interval numbers is very important in interval arithmetic. This is discussed in the following.

**3.1.1. Comparison of Interval Numbers.** As in [21],  $A$  is strictly less than  $B$  if and only if  $a_R < b_L$  and this is denoted by  $A < B$ .

And  $A$  is contained in  $B$  if and only if  $a_L \geq b_L$  and  $a_R \leq b_R$  and this is denoted by  $A \subseteq B$ .

Two more order relations  $\leq_{LR}$  and  $\leq_{MW}$  are introduced in [13] as,

$$A \leq_{LR} B \text{ iff } a_L \leq b_L \text{ and } a_R \leq b_R \quad (9)$$

$$\text{and } A \leq_{MW} B \text{ iff } m_1 \leq m_2 \text{ and } w_1 \geq w_2 \quad (10)$$

along with the strict order relations  $<_{LR}$  and  $<_{MW}$  as,

$$A <_{LR} B \text{ iff } A \leq_{LR} B \text{ and } A \neq B \quad (11)$$

$$\text{and } A <_{MW} B \text{ iff } A \leq_{MW} B \text{ and } A \neq B. \quad (12)$$

Some probabilistic views to order relation are incorporated in [31]. But the order relation which seems to us most significant and which is used in our algorithm is given by Sengupta et al. [30]. They have introduced an acceptability index ( $\mathcal{A}$ -index) to the proposition ‘ $A$  is inferior to  $B$ ’ as

$$\mathcal{A}(A \prec B) = \frac{m_2 - m_1}{w_1 + w_2}. \quad (13)$$

In connection with this ‘acceptability index’, we define ‘total dominance’ and ‘partial dominance’ of two interval numbers  $A = \langle m_1, w_1 \rangle$  and  $B = \langle m_2, w_2 \rangle$  one over another as follows:

**Definition 3.1.** If  $\mathcal{A}(A \prec B) \geq 1$  then,  $A$  is said to be ‘totally dominating’ over  $B$  in the sense of minimization and  $B$  is said to be ‘totally dominating’ over  $A$  in the sense of maximization. We denote this by  $A \prec B$ .

**Definition 3.2.** If  $0 < \mathcal{A}(A \prec B) < 1$  then  $A$  is said to be ‘partially dominating’ over  $B$  in the sense of minimization and  $B$  is said to be ‘partially dominating’ over  $A$  in the sense of maximization. This is denoted by  $A \prec_P B$ .

But, when  $\mathcal{A}(A \prec B) = 0$ , i.e.,  $m_1 = m_2$  then we may not get an order relation from the above definitions. Then we may emphasize on the widths of the interval numbers  $A$  and  $B$ .

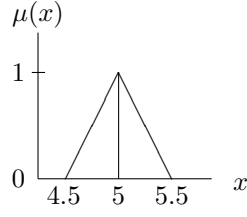
If  $w_1 > w_2$  then the left end point of  $A$  is less than that of  $B$  and on finding a minimum distance, there is a chance that the distance may lie on  $A$ . But at the same time, since the right end point of  $A$  is greater than that of  $B$ , if one prefers  $A$  to  $B$  in minimization then in worst case, he may be looser than one who prefers  $B$  to  $A$ . Thus in such a situation an optimistic decision-maker would prefer  $A$  to  $B$  whereas a pessimistic decision-maker would do the converse.

**3.1.2. Numerical Examples. Example 1:** Let  $A = [160, 170] = \langle 165, 5 \rangle$  and  $B = [180, 186] = \langle 183, 3 \rangle$ . Then  $\mathcal{A}(A \prec B) = \frac{183-165}{5+3} = 2.25 > 1$ . So in minimization,  $A$  is totally dominating over  $B$ .

**Example 2:** Let  $A = [160, 170] = \langle 165, 5 \rangle$  and  $C = [166, 180] = \langle 173, 7 \rangle$ . Then  $\mathcal{A}(A \prec B) = \frac{173-165}{5+7} = \frac{2}{3} = 0.67$ . So in minimization,  $A$  is partially dominating over  $B$  with level of satisfaction 0.67.

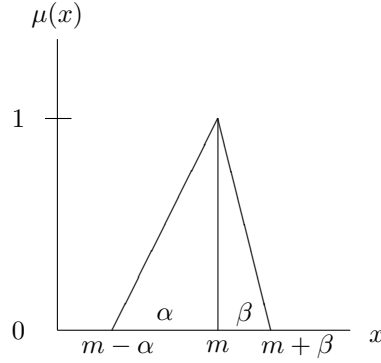
**3.2. Triangular Fuzzy Number and its Arithmetic.** In the previous section, we have considered the distance ‘about 5 KM’ as an interval  $[4.5, 5.5]$ . Similarly, we can fuzzify it as a triangular fuzzy number with the distance 5 KM with membership degree 1 and the distances lying between 4.5 - 5 KM and 5 - 5.5 KM with membership values within 0 and 1 as follows.

$$\mu(x) = \begin{cases} \frac{x-4.5}{0.5} & \text{for } 4.5 < x \leq 5 \\ \frac{5.5-x}{0.5} & \text{for } 5 \leq x < 5.5. \end{cases} \quad (14)$$

FIGURE 1. Membership function of  $\langle 5, 0.5, 0.5 \rangle$ .

In general, a triangular fuzzy number is represented by a triplet  $\tilde{A} = \langle m, \alpha, \beta \rangle$  with the membership function

$$\mu(x) = \begin{cases} 0 & \text{for } x \leq m - \alpha \\ 1 - \frac{m-x}{\alpha} & \text{for } m - \alpha < x < m \\ 1 & \text{for } x = m \\ 1 - \frac{x-m}{\beta} & \text{for } m < x < m + \beta \\ 0 & \text{for } x \geq m + \beta. \end{cases} \quad (15)$$

FIGURE 2. Membership function of  $\langle m, \alpha, \beta \rangle$ .

i.e.,  $m$  is the point whose membership value is 1 and  $\alpha$  and  $\beta$  are the left spread and right spread respectively.

Another way of representation of a triangular fuzzy number is given by the triplet  $(\underline{a}, a, \bar{a})$ . Here,  $a$  has the membership value 1 and left spread and right spread are  $a - \underline{a}$  and  $\bar{a} - a$  respectively. Like interval numbers, a crisp real number  $k$  may be represented as a triangular fuzzy number as  $(k, k, k) = \langle k, 0, 0 \rangle$ .

Let  $\tilde{M} = \langle m, \alpha, \beta \rangle$  and  $\tilde{N} = \langle n, \gamma, \delta \rangle$  be two triangular fuzzy numbers. Then the fuzzy sum of these two numbers is given by

$$\tilde{M} \oplus \tilde{N} = \langle m + n, \alpha + \beta, \gamma + \delta \rangle. \quad (16)$$

and similarly the other binary operations may be defined as follows.

$$\tilde{M} \ominus \tilde{N} = \langle m - n, \alpha + \delta, \beta + \gamma \rangle \quad (17)$$

$$\widetilde{M} \odot \widetilde{N} = \begin{cases} \langle mn, m\gamma + n\alpha, m\delta + n\beta \rangle, & \text{when } m \geq 0, n \geq 0. \\ \langle mn, n\alpha - m\delta, n\beta - m\gamma \rangle, & \text{when } m \leq 0, n \geq 0. \\ \langle mn, -n\beta - m\delta, -n\alpha - m\gamma \rangle, & \text{when } m \leq 0, n \leq 0. \end{cases} \quad (18)$$

and

$$\widetilde{M} \oslash \widetilde{N} = \left\langle \frac{m}{n}, \frac{m\delta + n\alpha}{n^2}, \frac{m\gamma + n\beta}{n^2} \right\rangle. \quad (19)$$

In particular, if  $\widetilde{N}$  be a crisp number  $k$ , then  $\widetilde{M} \odot \widetilde{N} = \widetilde{M}.k = \langle m.k, \alpha.k, \beta.k \rangle$  and  $\widetilde{M} \oslash \widetilde{N} = \widetilde{M}/k = \langle m/k, \alpha/k, \beta/k \rangle$ .

Clearly, the product and quotient of a triangular fuzzy number and a crisp number is a triangular fuzzy number.

**3.2.1. Comparison of Triangular Fuzzy Numbers.** Like interval numbers, various order relations for triangular fuzzy numbers are available in the literature [8, 18]. A generalized order relation for flat fuzzy numbers is given by Okada and Soper [27] and as a special case, order relation for triangular fuzzy numbers can be developed. They have used three notations  $\prec, \preceq, \preceq_h$ . For triangular fuzzy numbers  $\widetilde{a} = \langle a, \alpha, \beta \rangle$  and  $\widetilde{b} = \langle b, \gamma, \delta \rangle$  those are accomplished as

(i)  $\widetilde{a} \preceq \widetilde{b}$  iff  $a \leq b$ ,  $a - \alpha \leq b - \gamma$  and  $a + \beta \leq b + \delta$ ,

(ii)  $\widetilde{a} \prec \widetilde{b}$  iff  $\widetilde{a} \preceq \widetilde{b}$  and  $\widetilde{a} \neq \widetilde{b}$ , i.e.,  $a = b$ ,  $\alpha = \gamma$  and  $\beta = \delta$  all do not hold simultaneously,

(iii)  $\widetilde{a} \preceq_h \widetilde{b}$  iff  $a \leq b$ ,  $a - \alpha(1-h) \leq b - \gamma(1-h)$  and  $a - \beta(1-h) \leq b - \delta(1-h)$ .

As they have asserted  $\widetilde{a} \preceq_h \widetilde{b}$  will hold for two flat fuzzy numbers  $\widetilde{a} = (\underline{a}, \bar{a}, \alpha, \beta)$  with membership function

$$\mu_{\widetilde{a}}(x) = \begin{cases} L\left(\frac{\underline{a}-x}{\alpha}\right) & \text{for } x < \underline{a} \\ 1 & \text{for } \underline{a} \leq x \leq \bar{a} \\ R\left(\frac{x-\bar{a}}{\beta}\right) & \text{for } x > \bar{a} \end{cases} \quad (20)$$

and  $\widetilde{b} = (\underline{b}, \bar{b}, \gamma, \delta)$  with membership function

$$\mu_{\widetilde{b}}(x) = \begin{cases} L\left(\frac{\underline{b}-x}{\gamma}\right) & \text{for } x < \underline{b} \\ 1 & \text{for } \underline{b} \leq x \leq \bar{b} \\ R\left(\frac{x-\bar{b}}{\delta}\right) & \text{for } x > \bar{b} \end{cases} \quad (21)$$

iff  $\underline{a} \leq \underline{b}$ ,  $\bar{a} \leq \bar{b}$ ,  $\underline{a} - \alpha L^{-1}(h) \leq \underline{b} - \gamma L^{-1}(h)$  and  $\bar{a} + \beta R^{-1}(h) \leq \bar{b} + \delta R^{-1}(h)$ .

Now for triangular fuzzy numbers,  $L(x) = R(x) = \begin{cases} 1-x & \text{for } 0 \leq x \leq 1 \\ 0 & \text{for } x > 1 \end{cases}$  and the above results follow.

Clearly,  $\widetilde{a} \preceq_h \widetilde{b}$  holds with  $h = 1$  iff  $a \leq b$ . Thus, the decision-maker is to prefer  $\widetilde{a}$  to  $\widetilde{b}$  with satisfaction level 1 in case of minimization only if  $a \leq b$ .

That is,  $\preceq_h$  becomes insignificant in case of triangular fuzzy numbers. But if we introduce the  $\mathcal{A}$ -index idea also here, then the result will hold with different satisfaction grades for different widths of the numbers. These can be incorporated as in the following.



**Definition 3.3.** The acceptability index ( $\mathcal{A}$ -index) of the proposition ‘ $\tilde{a} = \langle a, \alpha, \beta \rangle$  is preferred to  $\tilde{b} = \langle b, \gamma, \delta \rangle$ ’ is given by

$$\mathcal{A}(\tilde{a} \prec \tilde{b}) = \frac{b - a}{\beta + \gamma}. \quad (22)$$

Using this  $\mathcal{A}$ -index we may define the following ranking orders.

**Definition 3.4.** If  $\mathcal{A}(\tilde{a} \prec \tilde{b}) \geq 1$  then  $\tilde{a}$  is said to be totally dominating over  $\tilde{b}$  in case of minimization and the case is converse in case of maximization and this is denoted by  $\tilde{a} \prec \tilde{b}$ .

**Definition 3.5.** If  $0 < \mathcal{A}(\tilde{a} \prec \tilde{b}) < 1$  then  $\tilde{a}$  is said to be ‘partially dominating’ over  $\tilde{b}$  in the sense of minimization and  $\tilde{b}$  is said to be ‘partially dominating’ over  $\tilde{a}$  in the sense of maximization. This is denoted by  $\tilde{a} \prec_P \tilde{b}$ .

Now to consider the preference of the decision maker when the points attaining the membership values 1 for two triangular fuzzy numbers are the same, we have to emphasize on the left spreads and right spreads of the numbers as in the case of the interval numbers. Let  $\tilde{a} = \langle a, \alpha, \beta \rangle$  and  $\tilde{b} = \langle a, \gamma, \delta \rangle$ . Now if  $\beta = \delta$  also, then the number with larger left spread will be preferred in minimization and the number with smaller left spread will be preferred in maximization. Similarly, if  $\alpha = \gamma$ , then the number with smaller right spread will be preferred in minimization and the number with larger right spread will be preferred in maximization. In this connection, we define two more order relations as below.

**Definition 3.6.** Let  $\tilde{a} = \langle a, \alpha, \beta \rangle$  and  $\tilde{b} = \langle a, \gamma, \delta \rangle$ . If  $\beta = \delta$  and  $\alpha > \gamma$  then  $\tilde{a}$  is said to be right dominating over  $\tilde{b}$  in case of minimization and the case is converse in case of maximization and this is denoted by  $\tilde{a} \prec_R \tilde{b}$ .

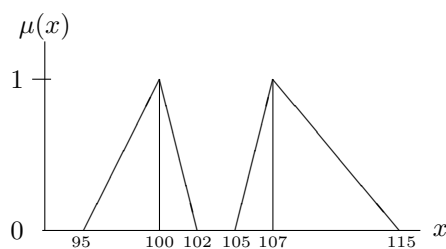
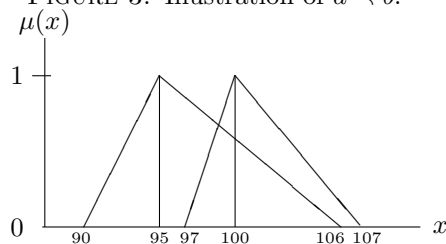
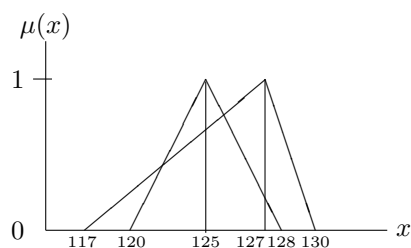
**Definition 3.7.** Let  $\tilde{a} = \langle a, \alpha, \beta \rangle$  and  $\tilde{b} = \langle a, \gamma, \delta \rangle$ . If  $\alpha = \gamma$  and  $\beta < \delta$  then  $\tilde{a}$  is said to be left dominating over  $\tilde{b}$  in case of minimization and the case is converse in case of maximization and this is denoted by  $\tilde{a} \prec_L \tilde{b}$ .

But if both  $\alpha \neq \gamma$  and  $\beta \neq \delta$ , then it is important whether the decision maker is optimistic or pessimistic. An optimistic decision maker will be indifferent about the equality of  $\alpha$  and  $\gamma$  in maximization and that of  $\beta$  and  $\delta$  in case of minimization. Whereas, a pessimistic decision maker will be indifferent about the equality of  $\alpha$  and  $\gamma$  in minimization and that of  $\beta$  and  $\delta$  in case of maximization.

**3.2.2. Numerical Examples. Example 3:** Let  $\tilde{a} = (95, 100, 102)$  and  $\tilde{b} = (105, 107, 115)$  be two triangular fuzzy numbers. Then  $\mathcal{A}(\tilde{a} \prec \tilde{b}) = \frac{107-100}{2+2} = 1.75 > 1$ . Thus  $\tilde{a} \prec \tilde{b}$  which is also compatible with our intuition.

**Example 4:** Let  $\tilde{a} = (90, 95, 106)$  and  $\tilde{b} = (97, 100, 107)$  be two triangular fuzzy numbers. Then  $\mathcal{A}(\tilde{a} \prec \tilde{b}) = \frac{100-95}{11+3} = \frac{5}{14} < 1$ . Thus  $\tilde{a} \prec_P \tilde{b}$  with degree of satisfaction 0.36.

**Example 5:** Let  $\tilde{a} = (120, 125, 128)$  and  $\tilde{b} = (117, 127, 130)$  be two triangular fuzzy numbers. Then  $\mathcal{A}(\tilde{a} \prec \tilde{b}) = \frac{127-125}{3+10} = \frac{2}{13} < 1$ . Thus  $\tilde{a} \prec_P \tilde{b}$  with degree of satisfaction 0.15 only.

FIGURE 3. Illustration of  $\tilde{a} \prec \tilde{b}$ .FIGURE 4. Illustration of  $\tilde{a} \prec_P \tilde{b}$ .FIGURE 5. Illustration of  $\tilde{a} \prec_P \tilde{b}$ .

### 3.3. A Unified Algorithm Involving the Dominance of Interval Numbers and Triangular Fuzzy Numbers.

Two interval numbers  $A = \langle m_A, w_A \rangle$  and  $B = \langle m_B, w_B \rangle$  are said to be *non-dominating* if (i)  $m_A = m_B$  and (ii)  $w_A \neq w_B$ .

Likewise, two triangular fuzzy numbers  $\tilde{A} = \langle a, \alpha, \beta \rangle$  and  $\tilde{B} = \langle b, \gamma, \delta \rangle$  are said to be *non-dominating* if (i)  $a = b$  and (ii)  $\alpha \neq \gamma$  and  $\beta \neq \delta$ .

The following function computes the minimum between two interval numbers (or triangular fuzzy numbers). This reduces to finding the crisp minimum when the numbers are degenerate interval numbers (or triangular fuzzy numbers).

FUNCTION  $\widetilde{\min}(A, B)$

if  $A = \langle m_A, w_A \rangle$  (or  $\langle a, \alpha, \beta \rangle$ ) and  $B = \langle m_B, w_B \rangle$  (or  $\langle b, \gamma, \delta \rangle$ ) are not

*non-dominating* then

if  $((A \prec B)$  or  $(A \prec_P B)$  or  $(A \prec_R B)$  or  $(A \prec_L B)$ ) then  
     minimum =  $A$ ;

else

```

    minimum = B;
endif;
else
  if ( $w_A < w_B$ ) (or  $\alpha < \gamma$ ) then
    if the decision maker is optimistic, then minimum=B;
    if the decision maker is pessimistic, then minimum=A;
  endif;
endif;
return(minimum);
END min.

```

Similarly, in the following we have given another function  $\widetilde{\max}$  which determines the maximum between two interval numbers (or triangular fuzzy numbers). This also gives the crisp maximum when the numbers are degenerate interval numbers (or triangular fuzzy numbers).

```

FUNCTION  $\widetilde{\max}(A, B)$ 
  if  $A = \langle m_A, w_A \rangle$  (or  $\langle a, \alpha, \beta \rangle$ ) and  $B = \langle m_B, w_B \rangle$  (or  $\langle b, \gamma, \delta \rangle$ )
  are not
  non-dominating then
    if ( $(A < B)$  or  $(A <_P B)$  or  $(A <_R B)$  or  $(A <_L B)$ ) then
      maximum = B;
    else
      maximum = A;
    endif;
  else
    if ( $w_A > w_B$ ) (or  $\beta > \delta$ ) then
      if the decision maker is optimistic, then maximum=A;
      if the decision maker is pessimistic, then maximum=B;
    endif;
  endif;
  return(maximum);
END  $\widetilde{\max}$ .

```

#### 4. Solution Methodology of the Fuzzy $p$ -center Problem

In this section, we discuss the solution methodology of the fuzzy  $p$ -center problem on a network (i)  $\widetilde{N} = (V, E, w, \widetilde{l})$ , or (ii)  $\widetilde{N} = (V, E, \widetilde{w}, l)$ . In reality these cases arise frequently, since, it is more appropriate to measure a length or distance by a fuzzy number, instead of a crisp number.

Let  $V$  be the set of vertices;  $W$  be the set of points at which the facilities can be located, which we have also considered as  $V$  in this paper;  $d_{ij}$  be the distance

between the vertices  $v_i$  and  $v_j$ ; and  $w_i$  be the weight associated with the vertex  $v_i$ . An integer programming (IP) formulation of the  $p$ -center problem can be given as follows [11].

$$\begin{aligned}
& \text{Minimize } \rho \\
& \text{subject to } \sum_{v_j \in W} x_{ij} = 1 \text{ for all } v_i \in V \\
& \quad x_{ij} \leq y_j \quad \forall v_i \in V, v_j \in W \\
& \quad \sum_{j \in W} y_j \leq p \\
& \quad \sum_{j \in W} d_{ij} x_{ij} w_i \leq \rho \quad \forall v_i \in V \\
& \quad x_{ij}, y_j \in \{0, 1\} \quad \forall v_i \in V, v_j \in W.
\end{aligned} \tag{23}$$

where, the variable  $x_{ij}$  assumes the value 1 if the facility site  $j$  is assigned to cover the demand  $w_i$  at the vertex  $v_i$  and the value 0 otherwise. The binary variable  $y_j$  assumes the value 1 if the facility site  $j$  is chosen for locating a facility and the value 0 otherwise, and the variable  $\rho$  on minimization gives the  $p$ -radius of the network.

When the edge-weights or the vertex-weights are imprecise in nature, the above IP problem becomes fuzzy. Different types of fuzzy IP problems are described by Herrera and Verdegay [16]. It is well known that solving to optimality the  $p$ -center problem using the IP formulation is very much time consuming even for small instances. Solving of the fuzzy version of the IP formulation of the problem is more time consuming and complicated. The following enumeration approach is comparatively better when the number of vertices is not too large.

We start by enumerating all possible subsets of the vertex set  $V$  with cardinality  $p$  and by numbering them as  $C_1, C_2, \dots, C_m$ ;  $m = {}^n C_p$ . Now we construct the matrix  $\Delta = (\delta_{ij})_{n \times m}$  as follows.

$$\delta_{ij} = d(v_i, C_j) = \widetilde{\min}_{u \in C_j} d(v_i, u).$$

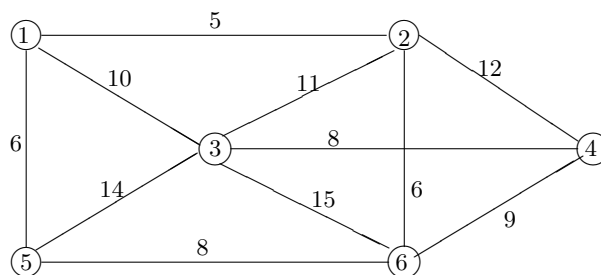
Denoting the column maxima  $\widetilde{\max}_{1 \leq i \leq n} \delta_{ij} w_i$  by  $M_j$  for all  $j = 1, 2, \dots, m$ , the  $p$ -radius is obtained by finding the minimum among all  $M_j, j = 1, 2, \dots, m$  and the corresponding  $C_j$  is the required  $p$ -center. This can be written in the form of a procedure as follows.

PROCEDURE FUZZYPCENTER( $V, D, p$ )

**Input:** The vertex set  $V$  and the distance matrix  $D = (d_{ij})_{n \times n}$ ,  $n = |V|$  being the number of vertices of the graph, and the number of facilities ( $p$ ) to be located.

**Output:** The  $p$ -center and the  $p$ -radius of the graph.

**Step 1:** Construct all the subsets of  $V$  having cardinality  $p$  and enumerate them as  $C_1, C_2, \dots, C_m$ ;  $m = {}^n C_p$ .

FIGURE 6. A network with  $n = 6$ .

	1	2	3	4	5	6
1	0	5	10	17	6	11
2	5	0	11	12	11	6
3	10	11	0	8	14	15
4	17	12	8	0	17	9
5	6	11	14	17	0	8
6	11	6	15	9	8	0

TABLE 1. Distance matrix of the network in Figure 6.

**Step 2:** Construct the matrix  $\Delta = (\delta_{ij})_{n \times m}$  as

$$\delta_{ij} = d(v_i, C_j) = \widetilde{\min}_{u \in C_j} \{d(v_i, u)\}.$$

**Step 3:** for ( $j = 1$  to  $m$ ) do

$$\text{find } M_j = \widetilde{\max}_{1 \leq i \leq n} \{\delta_{ij} \cdot w_i\}$$

Endfor;

**Step 4:** Find  $M_k = \widetilde{\min}_{1 \leq j \leq m} \{M_j\}$ .  $M_k$  is the  $p$ -radius of the graph.

**Step 5:** Find  $C_{j^*}$  such that  $M_k = \widetilde{\max}_{1 \leq i \leq n} \{\delta_{ij^*} \cdot w_i\}$ .

$C_{j^*}$  is the  $p$ -center of the graph.

END FUZZYPCENTER

Although, different approximate algorithms are available in the literature [24] to find the  $p$ -center of a network, but none are found to deal with imprecise data.

**4.1. Illustrative Examples.** We illustrate the procedure stated in the earlier section with the help of a network with  $n = 6$  and  $p = 2$ . The network is given in the Figure 6.

**4.1.1. Network with Imprecise Vertex-weights.** In the following, we consider that the vertex weights of the network of Figure 6 are imprecise and the edge weights are taken as crisp numbers. The distance matrix is given in the Table 1.

At first, we enumerate all the 2-member subsets of the vertex set  $V = \{1, 2, \dots, 6\}$  as follows.

$$\begin{aligned} C_1 &= \{1, 2\}, C_2 = \{1, 3\}, C_3 = \{1, 4\}, C_4 = \{1, 5\}, C_5 = \{1, 6\}, \\ C_6 &= \{2, 3\}, C_7 = \{2, 4\}, C_8 = \{2, 5\}, C_9 = \{2, 6\}, \\ C_{10} &= \{3, 4\}, C_{11} = \{3, 5\}, C_{12} = \{3, 6\}, \\ C_{13} &= \{4, 5\}, C_{14} = \{4, 6\}, \\ C_{15} &= \{5, 6\}. \end{aligned}$$

Then the matrix  $\Delta = (\delta_{ij})_{6 \times 15}$  is obtained as below.

	$C_1$	$C_2$	$C_3$	$C_4$	$C_5$	$C_6$	$C_7$	$C_8$	$C_9$	$C_{10}$	$C_{11}$	$C_{12}$	$C_{13}$	$C_{14}$	$C_{15}$
1	0	0	0	0	0	5	5	5	5	10	6	10	6	11	6
2	0	5	5	5	5	0	0	0	0	11	11	6	11	6	6
3	10	0	8	10	10	0	8	11	11	0	0	0	8	8	14
4	12	8	0	17	9	8	0	12	9	0	8	8	0	0	9
5	6	6	6	0	6	11	11	0	8	14	0	8	0	8	0
6	6	11	9	8	0	6	6	6	0	9	8	0	8	0	0

TABLE 2. The  $\Delta$ -matrix of the network in Figure 6.

(a) *Vertex weights are interval numbers:* The vertex weights are taken as  $w_1 = \langle 11, 1 \rangle$ ,  $w_2 = \langle 2.5, 0.5 \rangle$ ,  $w_3 = \langle 10.5, 1.5 \rangle$ ,  $w_4 = \langle 5.5, 0.5 \rangle$ ,  $w_5 = \langle 3, 1 \rangle$ ,  $w_6 = \langle 4.5, 0.5 \rangle$ . Thus we get the Table 3.

Finally, we get the 2-radius of the network as  $\widetilde{\min}_{1 \leq j \leq 15} M_j = \langle 49.5, 5.5 \rangle$  and the 2-center as  $C_2 = \{1, 3\}$ .

(b) *Vertex weights are triangular fuzzy numbers:* Now we consider the vertex weights of the network as triangular fuzzy numbers. Those are taken as  $\widetilde{w}_1 = \langle 11.5, 1, 0.5 \rangle$ ,  $\widetilde{w}_2 = \langle 2.5, 0.5, 0.5 \rangle$ ,  $\widetilde{w}_3 = \langle 10.5, 1.5, 1 \rangle$ ,  $\widetilde{w}_4 = \langle 5.5, 0.5, 0.5 \rangle$ ,  $\widetilde{w}_5 = \langle 3, 1, 0.5 \rangle$ ,  $\widetilde{w}_6 = \langle 4.5, 0.5, 1 \rangle$ .

Then we get the Table 4.

Finally, we get the 2-radius of the network as  $\widetilde{\min}_{1 \leq j \leq 15} \widetilde{M}_j = \langle 49.5, 5.5, 11 \rangle$  and the 2-center as  $C_2 = \{1, 3\}$ .

**4.1.2. Network with Imprecise Edge-weights.** Here, we consider the same network with imprecise edge weights and the vertex weights are considered as crisp numbers.

(a) *Edge weights are interval numbers:* Let the edge weights are given as in Figure 7.

In mean-width form the distance matrix is given by Table 6.

Using this distance matrix, the matrix  $\Delta = (\delta_{ij})_{6 \times 15}$  is to be found. Here,  $d(2, C_{11}) = \widetilde{\min}\{d_{23}, d_{25}\} = \widetilde{\min}\{\langle 11.5, 0.5 \rangle, \langle 11.5, 1.5 \rangle\} = \langle 11.5, 1.5 \rangle$  if the decision

$v_i$	$\delta_{i1}.w_i$	$\delta_{i2}.w_i$	$\delta_{i3}.w_i$	$\delta_{i4}.w_i$	$\delta_{i5}.w_i$	$w_i$
1	$\langle 0, 0 \rangle$	$\langle 0, 0 \rangle$	$\langle 0, 0 \rangle$	$\langle 0, 0 \rangle$	$\langle 0, 0 \rangle$	$\langle 11, 1 \rangle$
2	$\langle 0, 0 \rangle$	$\langle 12.5, 2.5 \rangle$	$\langle 12.5, 2.5 \rangle$	$\langle 12.5, 2.5 \rangle$	$\langle 12.5, 2.5 \rangle$	$\langle 2.5, 0.5 \rangle$
3	$\langle 105, 15 \rangle$	$\langle 0, 0 \rangle$	$\langle 84, 12 \rangle$	$\langle 105, 15 \rangle$	$\langle 105, 15 \rangle$	$\langle 10.5, 1.5 \rangle$
4	$\langle 66, 6 \rangle$	$\langle 44, 4 \rangle$	$\langle 0, 0 \rangle$	$\langle 93.5, 8.5 \rangle$	$\langle 49.5, 4.5 \rangle$	$\langle 5.5, 0.5 \rangle$
5	$\langle 18, 6 \rangle$	$\langle 18, 6 \rangle$	$\langle 18, 6 \rangle$	$\langle 0, 0 \rangle$	$\langle 18, 6 \rangle$	$\langle 3, 1 \rangle$
6	$\langle 27, 3 \rangle$	$\langle 49.5, 5.5 \rangle$	$\langle 40.5, 4.5 \rangle$	$\langle 36, 4 \rangle$	$\langle 0, 0 \rangle$	$\langle 4.5, 0.5 \rangle$
$M_j$	$\langle 105, 15 \rangle$	$\langle 49.5, 5.5 \rangle$	$\langle 84, 12 \rangle$	$\langle 105, 15 \rangle$	$\langle 105, 15 \rangle$	

---

$v_i$	$\delta_{i6}.w_i$	$\delta_{i7}.w_i$	$\delta_{i8}.w_i$	$\delta_{i9}.w_i$	$\delta_{i10}.w_i$	$w_i$
1	$\langle 55, 5 \rangle$	$\langle 55, 5 \rangle$	$\langle 55, 5 \rangle$	$\langle 55, 5 \rangle$	$\langle 110, 10 \rangle$	$\langle 11, 1 \rangle$
2	$\langle 0, 0 \rangle$	$\langle 0, 0 \rangle$	$\langle 0, 0 \rangle$	$\langle 0, 0 \rangle$	$\langle 27.5, 5.5 \rangle$	$\langle 2.5, 0.5 \rangle$
3	$\langle 0, 0 \rangle$	$\langle 84, 12 \rangle$	$\langle 115.5, 16.5 \rangle$	$\langle 115.5, 16.5 \rangle$	$\langle 0, 0 \rangle$	$\langle 10.5, 1.5 \rangle$
4	$\langle 44, 4 \rangle$	$\langle 0, 0 \rangle$	$\langle 66, 6 \rangle$	$\langle 49.5, 4.5 \rangle$	$\langle 0, 0 \rangle$	$\langle 5.5, 0.5 \rangle$
5	$\langle 33, 11 \rangle$	$\langle 33, 11 \rangle$	$\langle 0, 0 \rangle$	$\langle 24, 8 \rangle$	$\langle 42, 14 \rangle$	$\langle 3, 1 \rangle$
6	$\langle 27, 3 \rangle$	$\langle 27, 3 \rangle$	$\langle 27, 3 \rangle$	$\langle 0, 0 \rangle$	$\langle 40.5, 4.5 \rangle$	$\langle 4.5, 0.5 \rangle$
$M_j$	$\langle 55, 5 \rangle$	$\langle 84, 12 \rangle$	$\langle 115.5, 16.5 \rangle$	$\langle 115.5, 16.5 \rangle$	$\langle 110, 10 \rangle$	

---

$v_i$	$\delta_{i11}.w_i$	$\delta_{i12}.w_i$	$\delta_{i13}.w_i$	$\delta_{i14}.w_i$	$\delta_{i15}.w_i$	$w_i$
1	$\langle 66, 6 \rangle$	$\langle 110, 10 \rangle$	$\langle 66, 6 \rangle$	$\langle 121, 11 \rangle$	$\langle 66, 6 \rangle$	$\langle 11, 1 \rangle$
2	$\langle 27.5, 5.5 \rangle$	$\langle 15, 3 \rangle$	$\langle 27.5, 5.5 \rangle$	$\langle 15, 3 \rangle$	$\langle 15, 3 \rangle$	$\langle 2.5, 0.5 \rangle$
3	$\langle 0, 0 \rangle$	$\langle 0, 0 \rangle$	$\langle 84, 12 \rangle$	$\langle 84, 12 \rangle$	$\langle 147, 21 \rangle$	$\langle 10.5, 1.5 \rangle$
4	$\langle 44, 4 \rangle$	$\langle 44, 4 \rangle$	$\langle 0, 0 \rangle$	$\langle 0, 0 \rangle$	$\langle 49.5, 4.5 \rangle$	$\langle 5.5, 0.5 \rangle$
5	$\langle 0, 0 \rangle$	$\langle 24, 8 \rangle$	$\langle 0, 0 \rangle$	$\langle 24, 8 \rangle$	$\langle 0, 0 \rangle$	$\langle 3, 1 \rangle$
6	$\langle 36, 4 \rangle$	$\langle 0, 0 \rangle$	$\langle 36, 4 \rangle$	$\langle 0, 0 \rangle$	$\langle 0, 0 \rangle$	$\langle 4.5, 0.5 \rangle$
$M_j$	$\langle 66, 6 \rangle$	$\langle 110, 10 \rangle$	$\langle 84, 12 \rangle$	$\langle 121, 11 \rangle$	$\langle 147, 21 \rangle$	

TABLE 3. The  $\Delta$ -matrix being multiplied with the interval number weights of the vertices.

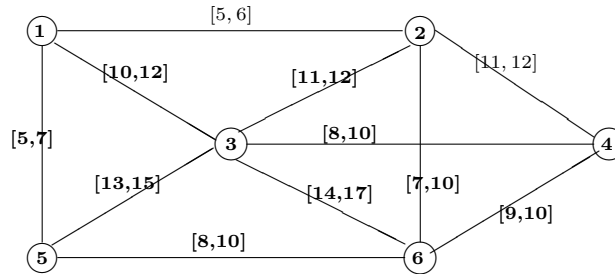


FIGURE 7. A network with interval edge-weights.

maker is optimistic, and  $\langle 11.5, 0.5 \rangle$  if the decision maker is pessimistic. Thus according to the different views of the decision maker, we get the following matrices shown in Table 7 and Table 9.

CASE - I: Optimistic View

Multiplying with the corresponding weights, we get the Table 8.

$v_i$	$\delta_{i1}.\widetilde{w}_i$	$\delta_{i2}.\widetilde{w}_i$	$\delta_{i3}.\widetilde{w}_i$	$\delta_{i4}.\widetilde{w}_i$	$\delta_{i5}.\widetilde{w}_i$	$\widetilde{w}_i$
1	$\langle 0, 0, 0 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 11.5, 1, 0.5 \rangle$
2	$\langle 0, 0, 0 \rangle$	$\langle 12.5, 2.5, 2.5 \rangle$	$\langle 12.5, 2.5, 2.5 \rangle$	$\langle 12.5, 2.5, 2.5 \rangle$	$\langle 12.5, 2.5, 2.5 \rangle$	$\langle 2.5, 0.5, 0.5 \rangle$
3	$\langle 105, 15, 10 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 84, 12, 8 \rangle$	$\langle 105, 15, 10 \rangle$	$\langle 105, 15, 10 \rangle$	$\langle 10.5, 1.5, 1 \rangle$
4	$\langle 66, 6, 6 \rangle$	$\langle 44, 4, 4 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 93.5, 8.5, 8.5 \rangle$	$\langle 49.5, 4.5, 4.5 \rangle$	$\langle 5.5, 0.5, 0.5 \rangle$
5	$\langle 18, 6, 3 \rangle$	$\langle 18, 6, 3 \rangle$	$\langle 18, 6, 3 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 18, 6, 3 \rangle$	$\langle 3, 1, 0.5 \rangle$
6	$\langle 27, 3, 6 \rangle$	$\langle 49.5, 5.5, 11 \rangle$	$\langle 40.5, 4.5, 9 \rangle$	$\langle 36, 4, 8 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 4.5, 0.5, 1 \rangle$
$\widetilde{M}_j$	$\langle 105, 15, 10 \rangle$	$\langle 49.5, 5.5, 11 \rangle$	$\langle 84, 12, 8 \rangle$	$\langle 105, 15, 10 \rangle$	$\langle 105, 15, 10 \rangle$	
$v_i$	$\delta_{i6}.\widetilde{w}_i$	$\delta_{i7}.\widetilde{w}_i$	$\delta_{i8}.\widetilde{w}_i$	$\delta_{i9}.\widetilde{w}_i$	$\delta_{i10}.\widetilde{w}_i$	$\widetilde{w}_i$
1	$\langle 57.5, 5, 2.5 \rangle$	$\langle 57.5, 5, 2.5 \rangle$	$\langle 57.5, 5, 2.5 \rangle$	$\langle 57.5, 5, 2.5 \rangle$	$\langle 115, 10, 5 \rangle$	$\langle 11.5, 1, 0.5 \rangle$
2	$\langle 0, 0, 0 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 27.5, 5.5, 5.5 \rangle$	$\langle 2.5, 0.5, 0.5 \rangle$
3	$\langle 0, 0, 0 \rangle$	$\langle 84, 12, 8 \rangle$	$\langle 115.5, 16.5, 11 \rangle$	$\langle 115.5, 16.5, 11 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 10.5, 1.5, 1 \rangle$
4	$\langle 44, 4, 4 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 66, 6, 6 \rangle$	$\langle 49.5, 4.5, 4.5 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 5.5, 0.5, 0.5 \rangle$
5	$\langle 33, 11, 5.5 \rangle$	$\langle 33, 11, 5.5 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 24, 8, 4 \rangle$	$\langle 42, 14, 7 \rangle$	$\langle 3, 1, 0.5 \rangle$
6	$\langle 27, 3, 6 \rangle$	$\langle 27, 3, 6 \rangle$	$\langle 27, 3, 6 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 40.5, 4.5, 9 \rangle$	$\langle 4.5, 0.5, 1 \rangle$
$\widetilde{M}_j$	$\langle 57.5, 5, 2.5 \rangle$	$\langle 84, 12, 8 \rangle$	$\langle 115.5, 16.5, 11 \rangle$	$\langle 115.5, 16.5, 11 \rangle$	$\langle 115, 10, 5 \rangle$	
$v_i$	$\delta_{i11}.\widetilde{w}_i$	$\delta_{i12}.\widetilde{w}_i$	$\delta_{i13}.\widetilde{w}_i$	$\delta_{i14}.\widetilde{w}_i$	$\delta_{i15}.\widetilde{w}_i$	$\widetilde{w}_i$
1	$\langle 69, 6, 3 \rangle$	$\langle 115, 10, 5 \rangle$	$\langle 69, 6, 3 \rangle$	$\langle 126.5, 11, 5.5 \rangle$	$\langle 69, 6, 3 \rangle$	$\langle 11.5, 1, 0.5 \rangle$
2	$\langle 27.5, 5.5, 5.5 \rangle$	$\langle 15, 3, 3 \rangle$	$\langle 27.5, 5.5, 5.5 \rangle$	$\langle 15, 3, 3 \rangle$	$\langle 15, 3, 3 \rangle$	$\langle 2.5, 0.5, 0.5 \rangle$
3	$\langle 0, 0, 0 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 84, 12, 8 \rangle$	$\langle 84, 12, 8 \rangle$	$\langle 147, 21, 14 \rangle$	$\langle 10.5, 1.5, 1 \rangle$
4	$\langle 44, 4, 4 \rangle$	$\langle 44, 4, 4 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 49.5, 4.5, 4.5 \rangle$	$\langle 5.5, 0.5, 0.5 \rangle$
5	$\langle 0, 0, 0 \rangle$	$\langle 24, 8, 4 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 24, 8, 4 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 3, 1, 0.5 \rangle$
6	$\langle 36, 4, 8 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 36, 4, 8 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 4.5, 0.5, 1 \rangle$
$\widetilde{M}_j$	$\langle 69, 6, 3 \rangle$	$\langle 115, 10, 5 \rangle$	$\langle 84, 12, 8 \rangle$	$\langle 126.5, 11, 5.5 \rangle$	$\langle 147, 21, 14 \rangle$	

TABLE 4. The  $\Delta$ -matrix being multiplied with the triangular fuzzy number weights of the vertices.

	1	2	3	4	5	6	Weights
1	[0, 0]	[5, 6]	[10, 12]	[16, 18]	[5, 7]	[12, 16]	11
2	[5, 6]	[0, 0]	[11, 12]	[11, 12]	[10, 13]	[7, 10]	3
3	[10, 12]	[11, 12]	[0, 0]	[8, 10]	[13, 15]	[14, 17]	10
4	[16, 18]	[11, 12]	[8, 10]	[0, 0]	[17, 20]	[9, 10]	5
5	[5, 7]	[10, 13]	[13, 15]	[17, 20]	[0, 0]	[8, 10]	3
6	[12, 16]	[7, 10]	[14, 17]	[9, 10]	[8, 10]	[0, 0]	4

TABLE 5. Distance matrix of the network of Figure 7.

Thus we get the 2-radius of the network as  $\widetilde{\min}_{1 \leq j \leq 15} M_j = \langle 56, 8 \rangle$  and the 2-center as  $C_2 = \{1, 3\}$ .

#### CASE - II: Pessimistic View

Multiplying with the corresponding weights, we get the Table 10.

Thus in this case also, we get the 2-radius of the network as  $\widetilde{\min}_{1 \leq j \leq 15} M_j = \langle 56, 8 \rangle$  and the 2-center as  $C_2 = \{1, 3\}$ .



	1	2	3	4	5	6	Weights
1	$\langle 0, 0 \rangle$	$\langle 5.5, 0.5 \rangle$	$\langle 11, 1 \rangle$	$\langle 17, 1 \rangle$	$\langle 6, 1 \rangle$	$\langle 14, 2 \rangle$	11
2	$\langle 5.5, 0.5 \rangle$	$\langle 0, 0 \rangle$	$\langle 11.5, 0.5 \rangle$	$\langle 11.5, 0.5 \rangle$	$\langle 11.5, 1.5 \rangle$	$\langle 8.5, 1.5 \rangle$	3
3	$\langle 11, 1 \rangle$	$\langle 11.5, 0.5 \rangle$	$\langle 0, 0 \rangle$	$\langle 9, 1 \rangle$	$\langle 14, 1 \rangle$	$\langle 15.5, 1.5 \rangle$	10
4	$\langle 17, 1 \rangle$	$\langle 11.5, 0.5 \rangle$	$\langle 9, 1 \rangle$	$\langle 0, 0 \rangle$	$\langle 18.5, 1.5 \rangle$	$\langle 9.5, 0.5 \rangle$	5
5	$\langle 6, 1 \rangle$	$\langle 11.5, 1.5 \rangle$	$\langle 14, 1 \rangle$	$\langle 18.5, 1.5 \rangle$	$\langle 0, 0 \rangle$	$\langle 9, 1 \rangle$	3
6	$\langle 14, 2 \rangle$	$\langle 8.5, 1.5 \rangle$	$\langle 15.5, 1.5 \rangle$	$\langle 9.5, 0.5 \rangle$	$\langle 9, 1 \rangle$	$\langle 0, 0 \rangle$	4

TABLE 6. Distance matrix of the network of Figure 7 in mean-width form.

	$C_1$	$C_2$	$C_3$	$C_4$	$C_5$	Weights
1	$\langle 0, 0 \rangle$	$\langle 0, 0 \rangle$	$\langle 0, 0 \rangle$	$\langle 0, 0 \rangle$	$\langle 0, 0 \rangle$	11
2	$\langle 0, 0 \rangle$	$\langle 5.5, 0.5 \rangle$	$\langle 5.5, 0.5 \rangle$	$\langle 5.5, 0.5 \rangle$	$\langle 5.5, 0.5 \rangle$	3
3	$\langle 11, 1 \rangle$	$\langle 0, 0 \rangle$	$\langle 9, 1 \rangle$	$\langle 11, 1 \rangle$	$\langle 11, 1 \rangle$	10
4	$\langle 11.5, 0.5 \rangle$	$\langle 9, 1 \rangle$	$\langle 0, 0 \rangle$	$\langle 17, 1 \rangle$	$\langle 9.5, 0.5 \rangle$	5
5	$\langle 6, 1 \rangle$	$\langle 6, 1 \rangle$	$\langle 6, 1 \rangle$	$\langle 0, 0 \rangle$	$\langle 6, 1 \rangle$	3
6	$\langle 8.5, 1.5 \rangle$	$\langle 14, 2 \rangle$	$\langle 9.5, 0.5 \rangle$	$\langle 9, 1 \rangle$	$\langle 0, 0 \rangle$	4

	$C_6$	$C_7$	$C_8$	$C_9$	$C_{10}$	Weights
1	$\langle 5.5, 0.5 \rangle$	$\langle 5.5, 0.5 \rangle$	$\langle 5.5, 0.5 \rangle$	$\langle 5.5, 0.5 \rangle$	$\langle 11, 1 \rangle$	11
2	$\langle 0, 0 \rangle$	$\langle 0, 0 \rangle$	$\langle 0, 0 \rangle$	$\langle 0, 0 \rangle$	$\langle 11.5, 0.5 \rangle$	3
3	$\langle 0, 0 \rangle$	$\langle 9, 1 \rangle$	$\langle 11.5, 0.5 \rangle$	$\langle 11.5, 0.5 \rangle$	$\langle 0, 0 \rangle$	10
4	$\langle 9, 1 \rangle$	$\langle 0, 0 \rangle$	$\langle 11.5, 0.5 \rangle$	$\langle 9.5, 0.5 \rangle$	$\langle 0, 0 \rangle$	5
5	$\langle 11.5, 1.5 \rangle$	$\langle 11.5, 1.5 \rangle$	$\langle 0, 0 \rangle$	$\langle 9, 1 \rangle$	$\langle 14, 1 \rangle$	3
6	$\langle 8.5, 1.5 \rangle$	$\langle 8.5, 1.5 \rangle$	$\langle 8.5, 1.5 \rangle$	$\langle 0, 0 \rangle$	$\langle 9.5, 0.5 \rangle$	4

	$C_{11}$	$C_{12}$	$C_{13}$	$C_{14}$	$C_{15}$	Weights
1	$\langle 6, 1 \rangle$	$\langle 11, 1 \rangle$	$\langle 6, 1 \rangle$	$\langle 14, 2 \rangle$	$\langle 6, 1 \rangle$	11
2	$\langle 11.5, 1.5 \rangle$	$\langle 8.5, 1.5 \rangle$	$\langle 11.5, 1.5 \rangle$	$\langle 8.5, 1.5 \rangle$	$\langle 8.5, 1.5 \rangle$	3
3	$\langle 0, 0 \rangle$	$\langle 0, 0 \rangle$	$\langle 9, 1 \rangle$	$\langle 9, 1 \rangle$	$\langle 14, 1 \rangle$	10
4	$\langle 9, 1 \rangle$	$\langle 9, 1 \rangle$	$\langle 0, 0 \rangle$	$\langle 0, 0 \rangle$	$\langle 9.5, 0.5 \rangle$	5
5	$\langle 0, 0 \rangle$	$\langle 9, 1 \rangle$	$\langle 0, 0 \rangle$	$\langle 9, 1 \rangle$	$\langle 0, 0 \rangle$	3
6	$\langle 9, 1 \rangle$	$\langle 0, 0 \rangle$	$\langle 9, 1 \rangle$	$\langle 0, 0 \rangle$	$\langle 0, 0 \rangle$	4

TABLE 7. The  $\Delta$ -matrix of the network in Figure 7 from optimistic view.

(b) *Edge weights are triangular fuzzy numbers*: Let us consider the edge weights as triangular fuzzy numbers as indicated in the Figure 8. The distance matrix of the network is given in Table 11. From this matrix the  $\Delta$ -matrix can be obtained as given in Table 12.

Multiplying with the corresponding weights, we get the Table 13.

Hence we get the 2-radius of the network as  $\min_{1 \leq j \leq 15} \widetilde{M}_j = \langle 54, 6, 10 \rangle$  and the 2-center as  $C_2 = \{1, 3\}$ .

$v_i$	$\delta_{i1}.w_i$	$\delta_{i2}.w_i$	$\delta_{i3}.w_i$	$\delta_{i4}.w_i$	$\delta_{i5}.w_i$	<i>Weights</i>
1	$\langle 0, 0 \rangle$	$\langle 0, 0 \rangle$	$\langle 0, 0 \rangle$	$\langle 0, 0 \rangle$	$\langle 0, 0 \rangle$	11
2	$\langle 0, 0 \rangle$	$\langle 16.5, 1.5 \rangle$	$\langle 16.5, 1.5 \rangle$	$\langle 16.5, 1.5 \rangle$	$\langle 16.5, 1.5 \rangle$	3
3	$\langle 110, 10 \rangle$	$\langle 0, 0 \rangle$	$\langle 90, 10 \rangle$	$\langle 110, 10 \rangle$	$\langle 110, 10 \rangle$	10
4	$\langle 57.5, 2.5 \rangle$	$\langle 45, 5 \rangle$	$\langle 0, 0 \rangle$	$\langle 85, 5 \rangle$	$\langle 47.5, 2.5 \rangle$	5
5	$\langle 18, 3 \rangle$	$\langle 18, 3 \rangle$	$\langle 18, 3 \rangle$	$\langle 0, 0 \rangle$	$\langle 18, 3 \rangle$	3
6	$\langle 34, 6 \rangle$	$\langle 56, 8 \rangle$	$\langle 38, 2 \rangle$	$\langle 36, 4 \rangle$	$\langle 0, 0 \rangle$	4
$M_j$	$\langle 110, 10 \rangle$	$\langle 56, 8 \rangle$	$\langle 90, 10 \rangle$	$\langle 110, 10 \rangle$	$\langle 110, 10 \rangle$	

$v_i$	$\delta_{i6}.w_i$	$\delta_{i7}.w_i$	$\delta_{i8}.w_i$	$\delta_{i9}.w_i$	$\delta_{i10}.w_i$	<i>Weights</i>
1	$\langle 60.5, 5.5 \rangle$	$\langle 60.5, 5.5 \rangle$	$\langle 60.5, 5.5 \rangle$	$\langle 60.5, 5.5 \rangle$	$\langle 121, 11 \rangle$	11
2	$\langle 0, 0 \rangle$	$\langle 0, 0 \rangle$	$\langle 0, 0 \rangle$	$\langle 0, 0 \rangle$	$\langle 34.5, 1.5 \rangle$	3
3	$\langle 0, 0 \rangle$	$\langle 90, 10 \rangle$	$\langle 115, 5 \rangle$	$\langle 115, 5 \rangle$	$\langle 0, 0 \rangle$	10
4	$\langle 45, 5 \rangle$	$\langle 0, 0 \rangle$	$\langle 57.5, 2.5 \rangle$	$\langle 47.5, 2.5 \rangle$	$\langle 0, 0 \rangle$	5
5	$\langle 34.5, 4.5 \rangle$	$\langle 34.5, 4.5 \rangle$	$\langle 0, 0 \rangle$	$\langle 27, 3 \rangle$	$\langle 42, 3 \rangle$	3
6	$\langle 34, 6 \rangle$	$\langle 34, 6 \rangle$	$\langle 34, 6 \rangle$	$\langle 0, 0 \rangle$	$\langle 38, 2 \rangle$	4
$M_j$	$\langle 60.5, 5.5 \rangle$	$\langle 90, 10 \rangle$	$\langle 115, 5 \rangle$	$\langle 115, 5 \rangle$	$\langle 121, 11 \rangle$	

$v_i$	$\delta_{i11}.w_i$	$\delta_{i12}.w_i$	$\delta_{i13}.w_i$	$\delta_{i14}.w_i$	$\delta_{i15}.w_i$	<i>Weights</i>
1	$\langle 66, 11 \rangle$	$\langle 121, 11 \rangle$	$\langle 66, 11 \rangle$	$\langle 154, 22 \rangle$	$\langle 66, 11 \rangle$	11
2	$\langle 34.5, 4.5 \rangle$	$\langle 25.5, 4.5 \rangle$	$\langle 34.5, 4.5 \rangle$	$\langle 25.5, 4.5 \rangle$	$\langle 25.5, 4.5 \rangle$	3
3	$\langle 0, 0 \rangle$	$\langle 0, 0 \rangle$	$\langle 90, 10 \rangle$	$\langle 90, 10 \rangle$	$\langle 140, 10 \rangle$	10
4	$\langle 45, 5 \rangle$	$\langle 45, 5 \rangle$	$\langle 0, 0 \rangle$	$\langle 0, 0 \rangle$	$\langle 47.5, 2.5 \rangle$	5
5	$\langle 0, 0 \rangle$	$\langle 27, 3 \rangle$	$\langle 0, 0 \rangle$	$\langle 27, 3 \rangle$	$\langle 0, 0 \rangle$	3
6	$\langle 36, 4 \rangle$	$\langle 0, 0 \rangle$	$\langle 36, 4 \rangle$	$\langle 0, 0 \rangle$	$\langle 0, 0 \rangle$	4
$M_j$	$\langle 66, 11 \rangle$	$\langle 121, 11 \rangle$	$\langle 90, 10 \rangle$	$\langle 154, 22 \rangle$	$\langle 140, 10 \rangle$	

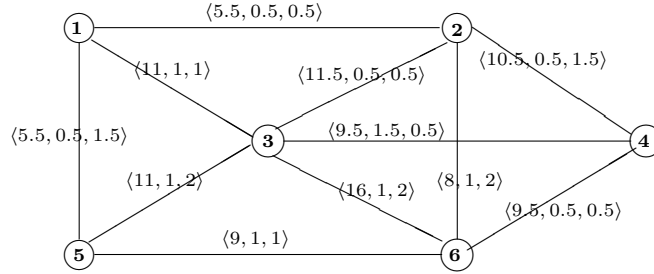
TABLE 8. The  $\Delta$ -matrix being multiplied with the weights of the vertices.

FIGURE 8. Network with triangular fuzzy edge-weights.

## 5. Reduction of Cost

In Section 4, we have seen that if the vertex weights or the edge weights are imprecise, then the  $p$ -radius of the network is also an imprecise number of same

	$C_1$	$C_2$	$C_3$	$C_4$	$C_5$	<i>Weights</i>
1	$\langle 0, 0 \rangle$	$\langle 0, 0 \rangle$	$\langle 0, 0 \rangle$	$\langle 0, 0 \rangle$	$\langle 0, 0 \rangle$	11
2	$\langle 0, 0 \rangle$	$\langle 5.5, 0.5 \rangle$	$\langle 5.5, 0.5 \rangle$	$\langle 5.5, 0.5 \rangle$	$\langle 5.5, 0.5 \rangle$	3
3	$\langle 11, 1 \rangle$	$\langle 0, 0 \rangle$	$\langle 9, 1 \rangle$	$\langle 11, 1 \rangle$	$\langle 11, 1 \rangle$	10
4	$\langle 11.5, 0.5 \rangle$	$\langle 9, 1 \rangle$	$\langle 0, 0 \rangle$	$\langle 17, 1 \rangle$	$\langle 9.5, 0.5 \rangle$	5
5	$\langle 6, 1 \rangle$	$\langle 6, 1 \rangle$	$\langle 6, 1 \rangle$	$\langle 0, 0 \rangle$	$\langle 6, 1 \rangle$	3
6	$\langle 8.5, 1.5 \rangle$	$\langle 14, 2 \rangle$	$\langle 9.5, 0.5 \rangle$	$\langle 9, 1 \rangle$	$\langle 0, 0 \rangle$	4
	$C_6$	$C_7$	$C_8$	$C_9$	$C_{10}$	<i>Weights</i>
1	$\langle 5.5, 0.5 \rangle$	$\langle 5.5, 0.5 \rangle$	$\langle 5.5, 0.5 \rangle$	$\langle 5.5, 0.5 \rangle$	$\langle 11, 1 \rangle$	11
2	$\langle 0, 0 \rangle$	$\langle 0, 0 \rangle$	$\langle 0, 0 \rangle$	$\langle 0, 0 \rangle$	$\langle 11.5, 0.5 \rangle$	3
3	$\langle 0, 0 \rangle$	$\langle 9, 1 \rangle$	$\langle 11.5, 0.5 \rangle$	$\langle 11.5, 0.5 \rangle$	$\langle 0, 0 \rangle$	10
4	$\langle 9, 1 \rangle$	$\langle 0, 0 \rangle$	$\langle 11.5, 0.5 \rangle$	$\langle 9.5, 0.5 \rangle$	$\langle 0, 0 \rangle$	5
5	$\langle 11.5, 1.5 \rangle$	$\langle 11.5, 1.5 \rangle$	$\langle 0, 0 \rangle$	$\langle 9, 1 \rangle$	$\langle 14, 1 \rangle$	3
6	$\langle 8.5, 1.5 \rangle$	$\langle 8.5, 1.5 \rangle$	$\langle 8.5, 1.5 \rangle$	$\langle 0, 0 \rangle$	$\langle 9.5, 0.5 \rangle$	4
	$C_{11}$	$C_{12}$	$C_{13}$	$C_{14}$	$C_{15}$	<i>Weights</i>
1	$\langle 6, 1 \rangle$	$\langle 11, 1 \rangle$	$\langle 6, 1 \rangle$	$\langle 14, 2 \rangle$	$\langle 6, 1 \rangle$	11
2	$\langle 11.5, 0.5 \rangle$	$\langle 8.5, 1.5 \rangle$	$\langle 11.5, 0.5 \rangle$	$\langle 8.5, 1.5 \rangle$	$\langle 8.5, 1.5 \rangle$	3
3	$\langle 0, 0 \rangle$	$\langle 0, 0 \rangle$	$\langle 9, 1 \rangle$	$\langle 9, 1 \rangle$	$\langle 14, 1 \rangle$	10
4	$\langle 9, 1 \rangle$	$\langle 9, 1 \rangle$	$\langle 0, 0 \rangle$	$\langle 0, 0 \rangle$	$\langle 9.5, 0.5 \rangle$	5
5	$\langle 0, 0 \rangle$	$\langle 9, 1 \rangle$	$\langle 0, 0 \rangle$	$\langle 9, 1 \rangle$	$\langle 0, 0 \rangle$	3
6	$\langle 9, 1 \rangle$	$\langle 0, 0 \rangle$	$\langle 9, 1 \rangle$	$\langle 0, 0 \rangle$	$\langle 0, 0 \rangle$	4

TABLE 9. The  $\Delta$ -matrix of the network in Figure 7 from pessimistic view.

kind. The  $p$ -radius value may be considered as the cost to cover the furthest vertex from the  $p$ -center set.

Suppose, in a proposed local area network (LAN), there are five servers and hundred clients which are to be served by the servers. Also suppose that a client can be connected to a server or a client by a single cable with no joining allowed. Then the length of the longest cable required to cover all the clients deserves much importance than the sum of the length of all the cables required. This can be modelled easily as a  $p$ -center problem with  $n = 100$  and  $p = 5$ .

Again the length of a cable may vary depending upon the situation of a client within a building or a room. So the length of each cable may be considered as interval numbers. Consequently the model consists of a network with interval edge weights. Suppose the  $p$ -radius value for the model is  $\langle m, w \rangle$  and we have no cable of length  $\geq m$ . Then we have to check whether a  $p$ -center set can be found with the cost ( $p$ -radius value) as the length of the longest cable available, i.e., we have to find a  $p$ -center set with some reduced cost. Also, it is clear that a  $p$ -center set with reduced cost can not cover all the vertices with the weight assigned to it. In the following, we define the ‘degree of attainment’ of a crisp number within an interval (or a triangular fuzzy number), i.e., we describe a measure by which the crisp number is less than the mean of the interval (or the point with membership

$v_i$	$\delta_{i1}.w_i$	$\delta_{i2}.w_i$	$\delta_{i3}.w_i$	$\delta_{i4}.w_i$	$\delta_{i5}.w_i$	<i>Weights</i>
1	$\langle 0, 0 \rangle$	$\langle 0, 0 \rangle$	$\langle 0, 0 \rangle$	$\langle 0, 0 \rangle$	$\langle 0, 0 \rangle$	11
2	$\langle 0, 0 \rangle$	$\langle 16.5, 1.5 \rangle$	$\langle 16.5, 1.5 \rangle$	$\langle 16.5, 1.5 \rangle$	$\langle 16.5, 1.5 \rangle$	3
3	$\langle 110, 10 \rangle$	$\langle 0, 0 \rangle$	$\langle 90, 10 \rangle$	$\langle 110, 10 \rangle$	$\langle 110, 10 \rangle$	10
4	$\langle 57.5, 2.5 \rangle$	$\langle 45, 5 \rangle$	$\langle 0, 0 \rangle$	$\langle 85, 5 \rangle$	$\langle 47.5, 2.5 \rangle$	5
5	$\langle 18, 3 \rangle$	$\langle 18, 3 \rangle$	$\langle 18, 3 \rangle$	$\langle 0, 0 \rangle$	$\langle 18, 3 \rangle$	3
6	$\langle 34, 6 \rangle$	$\langle 56, 8 \rangle$	$\langle 38, 2 \rangle$	$\langle 36, 4 \rangle$	$\langle 0, 0 \rangle$	4
$M_j$	$\langle 110, 10 \rangle$	$\langle 56, 8 \rangle$	$\langle 90, 10 \rangle$	$\langle 110, 10 \rangle$	$\langle 110, 10 \rangle$	
$v_i$	$\delta_{i6}.w_i$	$\delta_{i7}.w_i$	$\delta_{i8}.w_i$	$\delta_{i9}.w_i$	$\delta_{i10}.w_i$	<i>Weights</i>
1	$\langle 60.5, 5.5 \rangle$	$\langle 60.5, 5.5 \rangle$	$\langle 60.5, 5.5 \rangle$	$\langle 60.5, 5.5 \rangle$	$\langle 121, 11 \rangle$	11
2	$\langle 0, 0 \rangle$	$\langle 0, 0 \rangle$	$\langle 0, 0 \rangle$	$\langle 0, 0 \rangle$	$\langle 34.5, 1.5 \rangle$	3
3	$\langle 0, 0 \rangle$	$\langle 90, 10 \rangle$	$\langle 115, 5 \rangle$	$\langle 115, 5 \rangle$	$\langle 0, 0 \rangle$	10
4	$\langle 45, 5 \rangle$	$\langle 0, 0 \rangle$	$\langle 57.5, 2.5 \rangle$	$\langle 47.5, 2.5 \rangle$	$\langle 0, 0 \rangle$	5
5	$\langle 34.5, 4.5 \rangle$	$\langle 34.5, 4.5 \rangle$	$\langle 0, 0 \rangle$	$\langle 27, 3 \rangle$	$\langle 42, 3 \rangle$	3
6	$\langle 34, 6 \rangle$	$\langle 34, 6 \rangle$	$\langle 34, 6 \rangle$	$\langle 0, 0 \rangle$	$\langle 38, 2 \rangle$	4
$M_j$	$\langle 60.5, 5.5 \rangle$	$\langle 90, 10 \rangle$	$\langle 115, 5 \rangle$	$\langle 115, 5 \rangle$	$\langle 121, 11 \rangle$	
$v_i$	$\delta_{i11}.w_i$	$\delta_{i12}.w_i$	$\delta_{i13}.w_i$	$\delta_{i14}.w_i$	$\delta_{i15}.w_i$	<i>Weights</i>
1	$\langle 66, 11 \rangle$	$\langle 121, 11 \rangle$	$\langle 66, 11 \rangle$	$\langle 154, 22 \rangle$	$\langle 66, 11 \rangle$	11
2	$\langle 34.5, 1.5 \rangle$	$\langle 25.5, 4.5 \rangle$	$\langle 34.5, 1.5 \rangle$	$\langle 25.5, 4.5 \rangle$	$\langle 25.5, 4.5 \rangle$	3
3	$\langle 0, 0 \rangle$	$\langle 0, 0 \rangle$	$\langle 90, 10 \rangle$	$\langle 90, 10 \rangle$	$\langle 140, 10 \rangle$	10
4	$\langle 45, 5 \rangle$	$\langle 45, 5 \rangle$	$\langle 0, 0 \rangle$	$\langle 0, 0 \rangle$	$\langle 47.5, 2.5 \rangle$	5
5	$\langle 0, 0 \rangle$	$\langle 27, 3 \rangle$	$\langle 0, 0 \rangle$	$\langle 27, 3 \rangle$	$\langle 0, 0 \rangle$	3
6	$\langle 36, 4 \rangle$	$\langle 0, 0 \rangle$	$\langle 36, 4 \rangle$	$\langle 0, 0 \rangle$	$\langle 0, 0 \rangle$	4
$M_j$	$\langle 66, 11 \rangle$	$\langle 121, 11 \rangle$	$\langle 90, 10 \rangle$	$\langle 154, 22 \rangle$	$\langle 140, 10 \rangle$	

TABLE 10. The  $\Delta$ -matrix of the network in Figure 7 being multiplied with the vertex weights.

value 1). The minimum degree of attainment by which the weights of the vertices can be covered is taken as the ‘satisfaction grade’ of the corresponding  $p$ -center.

**Definition 5.1.** The ‘degree of attainment’ of a crisp number  $\alpha$  within an interval  $A = [a_L, a_U]$  is defined as

$$\mu(A, \alpha) = \begin{cases} 1 & \text{if } \alpha \geq a_U \\ \frac{\alpha - a_L}{a_U - a_L} & \text{if } a_L \leq \alpha \leq a_U \\ 0 & \text{if } \alpha \leq a_L \end{cases} \quad (24)$$

**Definition 5.2.** The ‘degree of attainment’ of a crisp number  $\alpha$  within a triangular fuzzy number  $\tilde{A} = (a, a, \bar{a})$  is defined as

$$\mu(\tilde{A}, \alpha) = \begin{cases} 1 & \text{if } \alpha \geq a \\ \frac{\alpha - a}{a - \bar{a}} & \text{if } a \leq \alpha \leq \bar{a} \\ 0 & \text{if } \alpha \leq a \end{cases} \quad (25)$$

	1	2	3	<i>Weights</i>
1	$\langle 0, 0, 0 \rangle$	$\langle 5.5, 0.5, 0.5 \rangle$	$\langle 11, 1, 1 \rangle$	11
2	$\langle 5.5, 0.5, 0.5 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 11.5, 0.5, 0.5 \rangle$	3
3	$\langle 11, 1, 1 \rangle$	$\langle 11.5, 0.5, 0.5 \rangle$	$\langle 0, 0, 0 \rangle$	10
4	$\langle 16, 1, 2 \rangle$	$\langle 10.5, 0.5, 1.5 \rangle$	$\langle 9.5, 1.5, 0.5 \rangle$	5
5	$\langle 5.5, 0.5, 1.5 \rangle$	$\langle 11, 1, 2 \rangle$	$\langle 11, 1, 2 \rangle$	3
6	$\langle 13.5, 1.5, 2.5 \rangle$	$\langle 8, 1, 2 \rangle$	$\langle 16, 1, 2 \rangle$	4
	4	5	6	<i>Weights</i>
1	$\langle 16, 1, 2 \rangle$	$\langle 5.5, 0.5, 1.5 \rangle$	$\langle 13.5, 1.5, 2.5 \rangle$	11
2	$\langle 10.5, 0.5, 1.5 \rangle$	$\langle 11, 1, 2 \rangle$	$\langle 8, 1, 2 \rangle$	3
3	$\langle 9.5, 1.5, 0.5 \rangle$	$\langle 11, 1, 2 \rangle$	$\langle 16, 1, 2 \rangle$	10
4	$\langle 0, 0, 0 \rangle$	$\langle 18.5, 1.5, 1.5 \rangle$	$\langle 9.5, 0.5, 0.5 \rangle$	5
5	$\langle 18.5, 1.5, 1.5 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 9, 1, 1 \rangle$	3
6	$\langle 9.5, 0.5, 0.5 \rangle$	$\langle 9, 1, 1 \rangle$	$\langle 0, 0, 0 \rangle$	4

TABLE 11. The distance matrix of the network of Figure 8.

	$C_1$	$C_2$	$C_3$	$C_4$	$C_5$	<i>Weights</i>
1	$\langle 0, 0, 0 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 0, 0, 0 \rangle$	11
2	$\langle 0, 0, 0 \rangle$	$\langle 5.5, 0.5, 0.5 \rangle$	$\langle 5.5, 0.5, 0.5 \rangle$	$\langle 5.5, 0.5, 0.5 \rangle$	$\langle 5.5, 0.5, 0.5 \rangle$	3
3	$\langle 11, 1, 1 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 9.5, 1.5, 0.5 \rangle$	$\langle 11, 1, 2 \rangle$	$\langle 11, 1, 1 \rangle$	10
4	$\langle 10.5, 0.5, 1.5 \rangle$	$\langle 9.5, 1.5, 0.5 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 16, 1, 2 \rangle$	$\langle 9.5, 0.5, 0.5 \rangle$	5
5	$\langle 5.5, 0.5, 1.5 \rangle$	$\langle 5.5, 0.5, 1.5 \rangle$	$\langle 5.5, 0.5, 1.5 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 5.5, 0.5, 1.5 \rangle$	3
6	$\langle 8, 1, 2 \rangle$	$\langle 13.5, 1.5, 2.5 \rangle$	$\langle 9.5, 0.5, 0.5 \rangle$	$\langle 9, 1, 1 \rangle$	$\langle 0, 0, 0 \rangle$	4
	$C_6$	$C_7$	$C_8$	$C_9$	$C_{10}$	<i>Weights</i>
1	$\langle 5.5, 0.5, 0.5 \rangle$	$\langle 5.5, 0.5, 0.5 \rangle$	$\langle 5.5, 0.5, 1.5 \rangle$	$\langle 5.5, 0.5, 0.5 \rangle$	$\langle 11, 1, 1 \rangle$	11
2	$\langle 0, 0, 0 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 10.5, 0.5, 1.5 \rangle$	3
3	$\langle 0, 0, 0 \rangle$	$\langle 9.5, 1.5, 0.5 \rangle$	$\langle 11, 1, 2 \rangle$	$\langle 11.5, 0.5, 0.5 \rangle$	$\langle 0, 0, 0 \rangle$	10
4	$\langle 9.5, 1.5, 0.5 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 10.5, 0.5, 1.5 \rangle$	$\langle 9.5, 0.5, 0.5 \rangle$	$\langle 0, 0, 0 \rangle$	5
5	$\langle 11, 1, 2 \rangle$	$\langle 11, 1, 2 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 9, 1, 1 \rangle$	$\langle 11, 1, 2 \rangle$	3
6	$\langle 8, 1, 2 \rangle$	$\langle 8, 1, 2 \rangle$	$\langle 8, 1, 2 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 9.5, 0.5, 0.5 \rangle$	4
	$C_{11}$	$C_{12}$	$C_{13}$	$C_{14}$	$C_{15}$	<i>Weights</i>
1	$\langle 5.5, 0.5, 1.5 \rangle$	$\langle 11, 1, 1 \rangle$	$\langle 5.5, 0.5, 1.5 \rangle$	$\langle 13.5, 1.5, 2.5 \rangle$	$\langle 5.5, 0.5, 1.5 \rangle$	11
2	$\langle 11, 1, 2 \rangle$	$\langle 8, 1, 2 \rangle$	$\langle 10.5, 0.5, 1.5 \rangle$	$\langle 8, 1, 2 \rangle$	$\langle 8, 1, 2 \rangle$	3
3	$\langle 0, 0, 0 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 9.5, 1.5, 0.5 \rangle$	$\langle 9.5, 1.5, 0.5 \rangle$	$\langle 11, 1, 2 \rangle$	10
4	$\langle 9.5, 1.5, 0.5 \rangle$	$\langle 9.5, 1.5, 0.5 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 9.5, 0.5, 0.5 \rangle$	5
5	$\langle 0, 0, 0 \rangle$	$\langle 9, 1, 1 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 9, 1, 1 \rangle$	$\langle 0, 0, 0 \rangle$	3
6	$\langle 9, 1, 1 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 9, 1, 1 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 0, 0, 0 \rangle$	4

TABLE 12. The  $\Delta$ -matrix of the network of Figure 8.

Next we state the bounds within which the reduced cost may be fixed.

Suppose the weight of the vertex  $v_i$  is  $w_i = [a_i, b_i]$  for all  $i = 1, 2, \dots, n$ . Then we solve the problem twice, once taking the weights as  $a_i$ , for which the result obtained is, say,  $\rho_L$  and next taking the weights as  $b_i$ , for which the obtained  $p$ -radius is,

$v_i$	$\widetilde{\delta}_{i1}.w_i$	$\widetilde{\delta}_{i2}.w_i$	$\widetilde{\delta}_{i3}.w_i$	$\widetilde{\delta}_{i4}.w_i$	$\widetilde{\delta}_{i5}.w_i$	Weights
1	$\langle 0, 0, 0 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 0, 0, 0 \rangle$	11
2	$\langle 0, 0, 0 \rangle$	$\langle 16.5, 1.5, 1.5 \rangle$	$\langle 16.5, 1.5, 1.5 \rangle$	$\langle 16.5, 1.5, 1.5 \rangle$	$\langle 16.5, 1.5, 1.5 \rangle$	3
3	$\langle 110, 10, 10 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 95, 15, 5 \rangle$	$\langle 110, 10, 20 \rangle$	$\langle 110, 10, 10 \rangle$	10
4	$\langle 52.5, 2.5, 7.5 \rangle$	$\langle 47.5, 7.5, 2.5 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 80, 5, 10 \rangle$	$\langle 47.5, 2.5, 2.5 \rangle$	5
5	$\langle 16.5, 1.5, 4.5 \rangle$	$\langle 16.5, 1.5, 4.5 \rangle$	$\langle 16.5, 1.5, 4.5 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 16.5, 1.5, 4.5 \rangle$	3
6	$\langle 32, 4, 8 \rangle$	$\langle 54, 6, 10 \rangle$	$\langle 38, 2, 2 \rangle$	$\langle 36, 4, 4 \rangle$	$\langle 0, 0, 0 \rangle$	4
$\widetilde{M}_j$	$\langle 110, 10, 10 \rangle$	$\langle 54, 6, 10 \rangle$	$\langle 95, 15, 5 \rangle$	$\langle 110, 10, 10 \rangle$	$\langle 110, 10, 10 \rangle$	
$v_i$	$\widetilde{\delta}_{i6}.w_i$	$\widetilde{\delta}_{i7}.w_i$	$\widetilde{\delta}_{i8}.w_i$	$\widetilde{\delta}_{i9}.w_i$	$\widetilde{\delta}_{i10}.w_i$	Weights
1	$\langle 60.5, 5.5, 5.5 \rangle$	$\langle 60.5, 5.5, 5.5 \rangle$	$\langle 60.5, 5.5, 16.5 \rangle$	$\langle 60.5, 5.5, 5.5 \rangle$	$\langle 121, 11, 11 \rangle$	11
2	$\langle 0, 0, 0 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 31.5, 1.5, 4.5 \rangle$	3
3	$\langle 0, 0, 0 \rangle$	$\langle 95, 15, 5 \rangle$	$\langle 110, 10, 20 \rangle$	$\langle 115, 5, 5 \rangle$	$\langle 0, 0, 0 \rangle$	10
4	$\langle 47.5, 7.5, 2.5 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 52.5, 2.5, 7.5 \rangle$	$\langle 47.5, 2.5, 2.5 \rangle$	$\langle 0, 0, 0 \rangle$	5
5	$\langle 33, 3, 6 \rangle$	$\langle 33, 3, 6 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 27, 3, 3 \rangle$	$\langle 33, 3, 6 \rangle$	3
6	$\langle 32, 4, 8 \rangle$	$\langle 32, 4, 8 \rangle$	$\langle 32, 4, 8 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 38, 2, 2 \rangle$	4
$\widetilde{M}_j$	$\langle 60.5, 5.5, 5.5 \rangle$	$\langle 95, 15, 5 \rangle$	$\langle 110, 10, 20 \rangle$	$\langle 115, 5, 5 \rangle$	$\langle 121, 11, 11 \rangle$	
$v_i$	$\widetilde{\delta}_{i11}.w_i$	$\widetilde{\delta}_{i12}.w_i$	$\widetilde{\delta}_{i13}.w_i$	$\widetilde{\delta}_{i14}.w_i$	$\widetilde{\delta}_{i15}.w_i$	Weights
1	$\langle 60.5, 5.5, 16.5 \rangle$	$\langle 121, 11, 11 \rangle$	$\langle 60.5, 5.5, 16.5 \rangle$	$\langle 148.5, 16.5, 27.5 \rangle$	$\langle 60.5, 5.5, 16.5 \rangle$	11
2	$\langle 33, 3, 6 \rangle$	$\langle 24, 3, 6 \rangle$	$\langle 31.5, 1.5, 4.5 \rangle$	$\langle 24, 3, 6 \rangle$	$\langle 24, 3, 6 \rangle$	3
3	$\langle 0, 0, 0 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 95, 15, 5 \rangle$	$\langle 95, 15, 5 \rangle$	$\langle 110, 10, 20 \rangle$	10
4	$\langle 47.5, 7.5, 2.5 \rangle$	$\langle 47.5, 7.5, 2.5 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 47.5, 2.5, 2.5 \rangle$	5
5	$\langle 0, 0, 0 \rangle$	$\langle 27, 3, 3 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 27, 3, 3 \rangle$	$\langle 0, 0, 0 \rangle$	3
6	$\langle 36, 4, 4 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 36, 4, 4 \rangle$	$\langle 0, 0, 0 \rangle$	$\langle 0, 0, 0 \rangle$	4
$\widetilde{M}_j$	$\langle 60.5, 5.5, 16.5 \rangle$	$\langle 121, 11, 11 \rangle$	$\langle 95, 15, 5 \rangle$	$\langle 148.5, 16.5, 27.5 \rangle$	$\langle 110, 10, 20 \rangle$	

TABLE 13. The  $\Delta$ -matrix of the network of Figure 8 being multiplied with the vertex weights.

say,  $\rho_U$ . Now we can state the following theorem involving the bounds  $\rho_L$  and  $\rho_U$  of the  $p$ -radius.

**Theorem 5.3.** *Let  $G = (V, E)$  be a network of which each vertex  $v_i \in V$  assumes an interval weight  $w_i = [a_i, b_i]$  for all  $i = 1, 2, \dots, n$ . The crisp  $p$ -radius problem with corresponding weights  $a_i$  has the solution  $\rho_L$  and the same with the weights  $b_i$  has the solution  $\rho_U$ . For a fixed value within  $\rho_L$  and  $\rho_U$ , there exists at least one  $p$ -center with non-zero satisfaction grade.*

Similarly, when the weights of the vertices are triangular fuzzy number, we have the following theorem.

**Theorem 5.4.** *Let  $G = (V, E)$  be a network of which each vertex  $v_i \in V$  assumes a triangular fuzzy number weight  $w_i = (\alpha_i, \beta_i, \gamma_i)$  for all  $i = 1, 2, \dots, n$ . The crisp  $p$ -radius problem with corresponding weights  $\alpha_i$  has the solution  $\rho_L$  and the same with the weights  $\gamma_i$  has the solution  $\rho_U$ . For a fixed value within  $\rho_L$  and  $\rho_U$ , there exists at least one  $p$ -center with non-zero satisfaction grade.*

If  $\rho$  be a crisp number fixed by the decision maker as the  $p$ -radius, then we denote the degree of attainment of  $\frac{\rho}{\delta_{ij}}$  within  $w_i$  by  $\mu_{ij}$ , i.e.,  $\mu_{ij} = \mu\left(w_i, \frac{\rho}{\delta_{ij}}\right)$ . The

minimum of all  $\mu_{ij}, i = 1, 2, \dots, n$  is denoted by  $\lambda_j$ . Our aim is to find the best possible degree of satisfaction  $\lambda^* = \max_{1 \leq j \leq m} \{\lambda_j\}$ . Now we can give the algorithm.

**ALGORITHM BEST- $p$ -CENTER**

**Input:** The  $\Delta$ -matrix of the network and the value  $\rho$  to which the  $p$ -radius value is fixed.

**Output:** The best possible  $p$ -center  $C^*$ .

- Step 1:** Order all vertices  $v_{i_1}, v_{i_2}, \dots, v_{i_n}$  according to decreasing values of  $\delta_{ij}$ . Clearly, if  $n' = n - p$ , then  $\delta_{i_k j} = 0$  for all  $k > n'$ , since  $v_{i_k}, k > n'$  are those vertices where the facilities are located.
- Step 2:** Compute  $\mu_{i_k j}$  for all  $k = 1, 2, \dots, n'$ , if all are found to have non-zero values. But if  $\mu_{i_{k'} j} = 0$ , then leave the calculations for all  $k > k'$ .
- Step 3:** For a column  $j$  of which each  $\mu_{i_k j}$  is non-zero, find  $\lambda_j$ . Clearly  $\lambda_j = 0$  for all the rest columns.
- Step 4.** Finally, obtain the best possible degree of satisfaction as  $\lambda^* = \max_{1 \leq j \leq m} \{\lambda_j\}$ . The set  $C^*$  corresponding to  $\lambda^*$  is the best possible  $p$ -center.

END BEST- $p$ -CENTER.

**5.1. Illustrative Examples.** Now we illustrate the algorithm with the help of the network given in Figure 7. The edge weights and vertex weights are taken as same as in Section 4.

(a) *Vertex weights are interval numbers:* At first, we find the bounds within which the  $p$ -radius can be fixed. The vertex weights are given by  $w_1 = \langle 11, 1 \rangle = [10, 12]$ ,  $w_2 = \langle 2.5, 0.5 \rangle = [2, 3]$ ,  $w_3 = \langle 10.5, 1.5 \rangle = [9, 12]$ ,  $w_4 = \langle 5.5, 0.5 \rangle = [5, 6]$ ,  $w_5 = \langle 3, 1 \rangle = [2, 4]$ ,  $w_6 = \langle 4.5, 0.5 \rangle = [4, 5]$ . Thus we have to solve the crisp problem once taking the weights  $w_1 = 10, w_2 = 2, w_3 = 9, w_4 = 5, w_5 = 2, w_6 = 4$  and next taking the weights  $w_1 = 12, w_2 = 3, w_3 = 12, w_4 = 6, w_5 = 4, w_6 = 5$ . Solving those, we get  $\rho_L = 44$  and  $\rho_U = 55$ . Those corresponds to the  $p$ -center  $C_6 = \{2, 3\}$  in both the cases. Let the decision maker wishes to fix the maximum cost i.e., the  $p$ -radius value by 54. We have to find the  $p$ -center with best possible satisfaction grade.

Arranging the  $\delta_{ij}$ 's in decreasing order, we construct the following table. The subscript of each entry indicates the corresponding vertex.

Now we calculate  $\frac{\rho}{\delta_{i_k j}}$  and  $\mu_{i_k j}$  in the Table 15.  $\mu_{i_k j}$ s are given in the subscripts. Once we find that the  $\mu$ -value of an entry is zero, the successive entries in that column are left blank.

So we obtain the 2-center as  $C_2 = \{1, 3\}$  in this case also, but the grade of satisfaction is 0.91.

(b) *Vertex weights are triangular fuzzy numbers:* Now we consider the vertex weights of the network as triangular fuzzy numbers. Those are given by  $\widetilde{w}_1 = \langle 11.5, 1, 0.5 \rangle = (10.5, 11.5, 12)$ ,  $\widetilde{w}_2 = \langle 2.5, 0.5, 0.5 \rangle = (2, 2.5, 3)$ ,  $\widetilde{w}_3 = \langle 10.5, 1.5, 1 \rangle =$

$C_1$	$C_2$	$C_3$	$C_4$	$C_5$	$C_6$	$C_7$	$C_8$	$C_9$	$C_{10}$	$C_{11}$	$C_{12}$	$C_{13}$	$C_{14}$	$C_{15}$
12 <sub>4</sub>	11 <sub>6</sub>	9 <sub>6</sub>	17 <sub>4</sub>	10 <sub>3</sub>	11 <sub>5</sub>	11 <sub>5</sub>	12 <sub>4</sub>	11 <sub>3</sub>	14 <sub>5</sub>	11 <sub>2</sub>	10 <sub>1</sub>	11 <sub>2</sub>	11 <sub>1</sub>	14 <sub>3</sub>
10 <sub>3</sub>	8 <sub>4</sub>	8 <sub>3</sub>	10 <sub>3</sub>	9 <sub>4</sub>	8 <sub>4</sub>	8 <sub>3</sub>	11 <sub>3</sub>	9 <sub>4</sub>	11 <sub>2</sub>	8 <sub>4</sub>	8 <sub>4</sub>	8 <sub>3</sub>	8 <sub>3</sub>	9 <sub>4</sub>
6 <sub>5</sub>	6 <sub>5</sub>	6 <sub>5</sub>	8 <sub>6</sub>	6 <sub>5</sub>	6 <sub>6</sub>	6 <sub>6</sub>	6 <sub>6</sub>	8 <sub>5</sub>	10 <sub>1</sub>	8 <sub>6</sub>	8 <sub>5</sub>	8 <sub>6</sub>	8 <sub>5</sub>	6 <sub>1</sub>
6 <sub>6</sub>	5 <sub>2</sub>	5 <sub>2</sub>	5 <sub>2</sub>	5 <sub>2</sub>	5 <sub>1</sub>	5 <sub>1</sub>	5 <sub>1</sub>	5 <sub>1</sub>	9 <sub>6</sub>	6 <sub>1</sub>	6 <sub>2</sub>	6 <sub>1</sub>	6 <sub>2</sub>	6 <sub>2</sub>
0 <sub>1</sub>	0 <sub>1</sub>	0 <sub>1</sub>	0 <sub>1</sub>	0 <sub>1</sub>	0 <sub>2</sub>	0 <sub>2</sub>	0 <sub>2</sub>	0 <sub>2</sub>	0 <sub>3</sub>	0 <sub>3</sub>	0 <sub>3</sub>	0 <sub>4</sub>	0 <sub>4</sub>	0 <sub>5</sub>
0 <sub>2</sub>	0 <sub>3</sub>	0 <sub>4</sub>	0 <sub>5</sub>	0 <sub>6</sub>	0 <sub>3</sub>	0 <sub>4</sub>	0 <sub>5</sub>	0 <sub>6</sub>	0 <sub>4</sub>	0 <sub>5</sub>	0 <sub>6</sub>	0 <sub>5</sub>	0 <sub>6</sub>	0 <sub>6</sub>

TABLE 14. Arrangement of the  $\delta_{ij}$ 's in decreasing order.

	$C_1$	$C_2$	$C_3$	$C_4$	$C_5$	$C_6$	$C_7$	$C_8$
	4.5 <sub>0</sub>	4.91 <sub>0,91</sub>	6 <sub>1</sub>	3.18 <sub>0</sub>	5.4 <sub>0</sub>	4.91 <sub>1</sub>	4.91 <sub>1</sub>	4.5 <sub>0</sub>
	–	6.75 <sub>1</sub>	6.75 <sub>0</sub>	–	–	6.75 <sub>1</sub>	6.75 <sub>0</sub>	–
	–	9 <sub>1</sub>	–	–	–	9 <sub>1</sub>	–	–
	–	10.8 <sub>1</sub>	–	–	–	10.8 <sub>0,4</sub>	–	–
$\lambda_j$	0	0.91	0	0	0	0.4	0	0
	$C_9$	$C_{10}$	$C_{11}$	$C_{12}$	$C_{13}$	$C_{14}$	$C_{15}$	
	4.91 <sub>0</sub>	3.86 <sub>0,93</sub>	4.91 <sub>1</sub>	5.4 <sub>0</sub>	4.91 <sub>1</sub>	4.91 <sub>0</sub>	3.86 <sub>0</sub>	
	–	4.91 <sub>1</sub>	6.75 <sub>1</sub>	–	6.75 <sub>0</sub>	–	–	
	–	5.4 <sub>0</sub>	6.75 <sub>1</sub>	–	–	–	–	
	–	–	9 <sub>0</sub>	–	–	–	–	
$\lambda_j$	0	0	0	0	0	0	0	

TABLE 15. The values of  $\frac{\rho}{\delta_{i.kj}}$ 's and  $\mu_{i.kj}$ 's (in the subscripts) for interval vertex weights.

$(9, 10.5, 11.5), \widetilde{w}_4 = \langle 5.5, 0.5, 0.5 \rangle = (5, 5.5, 6), \widetilde{w}_5 = \langle 3, 1, 0.5 \rangle = (2, 3, 3.5), \widetilde{w}_6 = \langle 4.5, 0.5, 1 \rangle = (4, 4.5, 5.5)$ . Thus in this case, we have to solve the crisp problem once taking the weights  $w_1 = 10.5, w_2 = 2, w_3 = 9, w_4 = 5, w_5 = 2, w_6 = 4$  and next taking the weights  $w_1 = 12, w_2 = 3, w_3 = 11.5, w_4 = 6, w_5 = 3.5, w_6 = 5.5$ . Solving those, we get  $\rho_L = 44$  which corresponds to  $C_2 = \{1, 3\}$  and  $\rho_U = 60$  which corresponds to the  $p$ -center  $C_6 = \{2, 3\}$ . Let the decision maker wishes to fix the maximum cost i.e., the  $p$ -radius value by 50. We have to find the  $p$ -center with best possible satisfaction grade. As earlier, we get the final table as Table 16.

So we obtain the 2-center as  $C_2 = \{1, 3\}$  in this case, and the grade of satisfaction is 1.

## 6. Conclusion

In this paper, we have considered and solved the  $p$ -center problems on different kinds of fuzzy graphs and provided several illustrative examples. Also we have considered a fuzzy aspect of the problem by fixing the maximum cost (i.e.,  $p$ -radius value) within some bounds which can also be determined. We have given



	$C_1$	$C_2$	$C_3$	$C_4$	$C_5$	$C_6$	$C_7$	$C_8$
	4.17 <sub>0</sub>	4.54 <sub>1</sub>	5.56 <sub>1</sub>	2.94 <sub>0</sub>	5 <sub>0</sub>	4.54 <sub>1</sub>	4.54 <sub>1</sub>	4.17 <sub>0</sub>
	–	6.25 <sub>1</sub>	6.25 <sub>0</sub>	–	–	6.25 <sub>1</sub>	6.25 <sub>0</sub>	–
	–	8.33 <sub>1</sub>	–	–	–	8.33 <sub>1</sub>	–	–
	–	10 <sub>1</sub>	–	–	–	10 <sub>0</sub>	–	–
$\lambda_j$	0	1	0	0	0	0	0	0
	$C_9$	$C_{10}$	$C_{11}$	$C_{12}$	$C_{13}$	$C_{14}$	$C_{15}$	
	4.54 <sub>0</sub>	3.57 <sub>1</sub>	4.54 <sub>1</sub>	5 <sub>0</sub>	4.54 <sub>1</sub>	4.54 <sub>0</sub>	3.57 <sub>0</sub>	
	–	4.54 <sub>1</sub>	6.25 <sub>1</sub>	–	6.25 <sub>0</sub>	–	–	
	–	5 <sub>0</sub>	6.25 <sub>1</sub>	–	–	–	–	
	–	–	8.33 <sub>0</sub>	–	–	–	–	
$\lambda_j$	0	0	0	0	0	0	0	

TABLE 16. The values of  $\frac{\rho}{\delta_{i_k j}}$ 's and  $\mu_{i_k j}$ 's (in the subscripts) for triangular fuzzy weights.

an algorithm to find a  $p$ -center with specified cost and to obtain the satisfaction degree of that  $p$ -center.

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