A COMMON FIXED POINT THEOREM FOR ψ-WEAKLY COMMUTING MAPS IN L-FUZZY METRIC SPACES

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Abstract. In this paper, a common fixed point theorem for ψ-weakly commuting maps in L-fuzzy metric spaces is proved.

1. Introduction and Preliminaries

The notion of fuzzy sets was introduced by Zadeh [26] and various concepts of fuzzy metric spaces were considered in [7, 8, 14, 15]. Many authors have studied fixed point theory in fuzzy metric spaces. The most interesting references are [3, 4, 10, 11, 16, 18, 25].

In the sequel, we shall adopt the usual terminology, notation and conventions of L-fuzzy metric spaces introduced by Saadati et al. [21, 22] and [1].

Definition 1.1. [10] Let $L = (L, \leq_L)$ be a complete lattice, and $U$ a non-empty set called universe. An L-fuzzy set $A$ on $U$ is defined as a mapping $A : U \rightarrow L$.

For each $u$ in $U$, $A(u)$ represents the degree (in $L$) to which $u$ satisfies $A$.

Lemma 1.2. [6] Consider the set $L^*$ and operation $\leq_{L^*}$ defined by:

$L^* = \{(x_1, x_2) : (x_1, x_2) \in [0, 1]^2 \text{ and } x_1 + x_2 \leq 1\},$

$(x_1, x_2) \leq_{L^*} (y_1, y_2) \iff x_1 \leq y_1 \text{ and } x_2 \geq y_2, \text{ for every } (x_1, x_2), (y_1, y_2) \in L^*.$

Then $(L^*, \leq_{L^*})$ is a complete lattice.

Definition 1.3. [2] An intuitionistic fuzzy set $A_{\zeta, \eta}$ on a universe $U$ is an object $A_{\zeta, \eta} = \{(\zeta_u, \eta_u) : u \in U\}$, where, for all $u \in U$, $\zeta_u \in [0, 1]$ and $\eta_u \in [0, 1]$ are called the membership degree and the non-membership degree, respectively, of $u$ in $A_{\zeta, \eta}$, and furthermore satisfy $\zeta_u + \eta_u \leq 1$.

Classically, a triangular norm $T$ on $([0, 1], \leq)$ is defined as an increasing, commutative, associative mapping $T : [0, 1]^2 \rightarrow [0, 1]$ satisfying $T(1, x) = x$, for all $x \in [0, 1]$. These definitions can be straightforwardly extended to any lattice $\mathcal{L} = (L, \leq_L)$. Define first $0_{\mathcal{L}} = \inf L$ and $1_{\mathcal{L}} = \sup L$.

Definition 1.4. A triangular norm (t-norm) on $\mathcal{L}$ is a mapping $T : L^2 \rightarrow L$ satisfying the following conditions:

(i) $T(x, 1_{\mathcal{L}}) = x$; (boundary condition)

(ii) $T(x, y) \in T(y, x)$; (commutativity)

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(iii) \((\forall(x, y, z) \in L^3)(T(x, T(y, z)) = T(T(x, y), z)); \) (associativity)
(iv) \((\forall(x, x', y, y') \in L^4)(x \leq_L x' \text{ and } y \leq_L y' \Rightarrow T(x, y) \leq_L T(x', y')). \)

(For involutive negation.

A t-norm can also be defined recursively as an \((n+1)\)-ary operation \((n \in \mathbb{N}\setminus\{0\})\) by \(T^1 = T\) and
\[
T^n(x_{(1)}, \cdots, x_{(n+1)}) = T(T^{n-1}(x_{(1)}, \cdots, x_{(n)}), x_{(n+1)})
\]
for \(n \geq 2\) and \(x_{(i)} \in L\).

**Definition 1.5.** [5] A t-norm \(T\) on \(L^*\) is called \(t\)-representable if and only if there exist a t-norm \(T\) and a t-conorm \(S\) on \([0, 1]\) such that, for all \(x = (x_1, x_2), y = (y_1, y_2) \in L^*\),
\[
T(x, y) = (T(x_1, y_1), S(x_2, y_2)).
\]

**Definition 1.6.** A negation on \(L\) is an decreasing mapping \(N : L \to L\) satisfying \(N(0_L) = 1_L\) and \(N(1_L) = 0_L\). If \(N(N(x)) = x\), for all \(x \in L\), then \(N\) is called an involutive negation.

If, for all \(x \in [0, 1]\), \(N_s(x) = 1 - x\), we say that \(N_s\) is the standard negation on \([0, 1], \leq\).

**Definition 1.7.** The 3-tuple \((X, \mathcal{M}, T)\) is said to be an \(L\)-fuzzy metric space if \(X\) is an arbitrary (non-empty) set, \(T\) is a continuous \(t\)-norm on \(L\) and \(\mathcal{M}\) is an \(L\)-fuzzy set on \(X \times [0, +\infty]\) satisfying the following conditions for every \(x, y, z\) in \(X\) and \(t, s\) in \([0, +\infty]\):

(a) \(\mathcal{M}(x, y, t) \geq_L 0_L\);
(b) \(\mathcal{M}(x, y, t) = 1_L\) for all \(t > 0\) if and only if \(x = y\);
(c) \(\mathcal{M}(x, y, t) = \mathcal{M}(y, x, t)\);
(d) \(T(\mathcal{M}(x, y, t), \mathcal{M}(y, z, s)) \leq_L \mathcal{M}(x, z, t + s)\);
(e) \(\mathcal{M}(x, y, \cdot) : [0, +\infty] \to L\) is continuous.

In this case \(\mathcal{M}\) is called an \(L\)-fuzzy metric. If \(\mathcal{M} = \mathcal{M}_{M,N}\) is an intuitionistic fuzzy set (see Definition 1.3) then the 3-tuple \((X, \mathcal{M}_{M,N}, T)\) is said to be an intuitionistic fuzzy metric space.

**Example 1.8.** [24] Let \((X, d)\) be a metric space. Set \(T(a, b) = (a_1b_1, \min(a_2 + b_2, 1))\) for all \(a = (a_1, a_2)\) and \(b = (b_1, b_2)\) in \(L^*\) and let \(M\) and \(N\) be fuzzy sets on \(X \times [0, \infty]\) defined as follows:
\[
\mathcal{M}_{M,N}(x, y, t) = (M(x, y, t), N(x, y, t)) = \left(\frac{t}{t + md(x, y)}, \frac{d(x, y)}{t + d(x, y)}\right),
\]
in which \(m > 1\). Then \((X, \mathcal{M}_{M,N}, T)\) is an intuitionistic fuzzy metric space.

**Example 1.9.** [22] Let \(X = \mathbb{N}\). Define \(T(a, b) = (\max(0, a_1 + b_1 - 1), a_2 + b_2 - a_2b_2)\) for all \(a = (a_1, a_2)\) and \(b = (b_1, b_2)\) in \(L^*\) and let \(M\) and \(N\) be fuzzy sets on \(X \times [0, \infty]\) defined as follows:
\[
\mathcal{M}_{M,N}(x, y, t) = (M(x, y, t), N(x, y, t)) = \begin{cases} 
\left(\frac{x}{y}, \frac{y-x}{y}\right) & \text{if } x \leq y \\
\left(\frac{y}{x}, \frac{x-y}{x}\right) & \text{if } y \leq x.
\end{cases}
\]
for all $x, y \in X$ and $t > 0$. Then $(X, \mathcal{M}_{\mathcal{M}, \mathcal{N}}, \mathcal{T})$ is an intuitionistic fuzzy metric space.

Let $(X, \mathcal{M}, \mathcal{T})$ be an $\mathcal{L}$-fuzzy metric space. For $t \in [0, +\infty[$, we define the open ball $B(x, r, t)$ with center $x \in X$ and radius $r \in L \setminus \{0_\mathcal{L}, 1_\mathcal{L}\}$, as

$$B(x, r, t) = \{y \in X : \mathcal{M}(x, y, t) > L \mathcal{N}(r)\}.$$ 

A subset $A \subseteq X$ is called open if for each $x \in A$, there exist $t > 0$ and $r \in L \setminus \{0_\mathcal{L}, 1_\mathcal{L}\}$ such that $B(x, r, t) \subseteq A$. Let $\tau_\mathcal{M}$ denote the family of all open subsets of $X$. Then $\tau_\mathcal{M}$ is called the topology induced by the $\mathcal{L}$-fuzzy metric $\mathcal{M}$.

**Lemma 1.10.** [9] Let $(X, \mathcal{M}, \mathcal{T})$ be an $\mathcal{L}$-fuzzy metric space. Then, $\mathcal{M}(x, y, t)$ is nondecreasing with respect to $t$, for all $x, y$ in $X$.

**Proof.** Let $t, s \in [0, +\infty[$ be such that $t < s$. Then $k = s - t > 0$ and

$$\mathcal{M}(x, y, t) = \mathcal{T}(\mathcal{M}(x, y, t), 1_\mathcal{L}) = \mathcal{T}(\mathcal{M}(x, y, t), \mathcal{M}(y, y, k)) \leq_{\mathcal{L}} \mathcal{M}(x, y, s).$$

\[\square\]

**Definition 1.11.** A sequence $\{x_n\}_{n \in \mathbb{N}}$ in an $\mathcal{L}$-fuzzy metric space $(X, \mathcal{M}, \mathcal{T})$ is called a Cauchy sequence, if for each $\varepsilon \in L \setminus \{0_\mathcal{L}\}$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $m \geq n \geq n_0$ ($n \geq m \geq n_0$),

$$\mathcal{M}(x_m, x_n, t) > L \mathcal{N}(\varepsilon).$$

The sequence $\{x_n\}_{n \in \mathbb{N}}$ is said to be convergent to $x \in X$ in the $\mathcal{L}$-fuzzy metric space $(X, \mathcal{M}, \mathcal{T})$ (denoted by $x_n \xrightarrow{\mathcal{M}} x$) if $\mathcal{M}(x_n, x, t) = \mathcal{M}(x, x_n, t) \rightarrow 1_\mathcal{L}$ whenever $n \rightarrow +\infty$ for every $t > 0$. A $\mathcal{L}$-fuzzy metric space is said to be complete if and only if every Cauchy sequence is convergent.

**Definition 1.12.** Let $(X, \mathcal{M}, \mathcal{T})$ be an $\mathcal{L}$-fuzzy metric space. $\mathcal{M}$ is said to be continuous on $X \times X \times [0, +\infty[$ if

$$\lim_{n \rightarrow \infty} \mathcal{M}(x_n, y_n, t_n) = \mathcal{M}(x, y, t)$$

whenever a sequence $\{(x_n, y_n, t_n)\}$ in $X \times X \times [0, +\infty[$ converges to a point $(x, y, t) \in X \times X \times [0, +\infty[$, i.e., $\lim_{n} \mathcal{M}(x_n, x, t) = \lim_{n} \mathcal{M}(y_n, y, t) = 1_\mathcal{L}$ and $\lim_{n} \mathcal{M}(x, y, t_n) = \mathcal{M}(x, y, t)$.

**Lemma 1.13.** Let $(X, \mathcal{M}, \mathcal{T})$ be an $\mathcal{L}$-fuzzy metric space. Then $\mathcal{M}$ is a continuous function on $X \times X \times [0, +\infty[$.

**Proof.** The proof is same as for fuzzy metric spaces (see Proposition 1 of [20]). \[\square\]

2. **The Main Results**

**Definition 2.1.** Let $f$ and $g$ be maps from an $\mathcal{L}$-fuzzy metric space $(X, \mathcal{M}, \mathcal{T})$ into itself. The maps $f$ and $g$ are said to be weakly commuting if

$$\mathcal{M}(fgx, gfx, t) \geq_{\mathcal{L}} \mathcal{M}(fx, gfx, t)$$

for each $x \in X$ and $t > 0$. 
**Definition 2.2.** Let $f$ and $g$ be maps from an $\mathcal{L}$-fuzzy metric space $(X, \mathcal{M}, T)$ into itself. The maps $f$ and $g$ are said to be $\psi$-weakly commuting if there exists a positive real function $\psi : (0, \infty) \to (0, \infty)$ such that

$$\mathcal{M}(fgx, gfx, t) \geq L \mathcal{M}(fx, gx, \psi(t))$$

for each $x$ in $X$ and $t > 0$.

Weak commutativity implies $\psi$-weak commutativity in $\mathcal{L}$-fuzzy metric spaces. However, $\psi$-weak commutativity implies weak commutativity only when $\psi(t) \geq t$.

**Example 2.3.** Let $X = \mathbb{R}$. Let $T(a, b) = (a_1b_1, \min(a_2 + b_2, 1))$ for all $a = (a_1, a_2), b = (b_1, b_2) \in L^2$ and let $\mathcal{M}_{M,N}$ be the intuitionistic fuzzy set on $X \times X \times [0, +\infty]$ defined as follows:

$$\mathcal{M}_{M,N}(x, y, t) = \left(\frac{\exp(|x - y|)}{t}, \frac{\exp(|x - y|) - 1}{\exp(|x - y|)}\right),$$

for all $t \in \mathbb{R}^+$. Then $(X, \mathcal{M}_{M,N}, T)$ is an intuitionistic fuzzy metric space. Define $f(x) = 2x - 1$ and $g(x) = x^2$. Then

$$\mathcal{M}_{M,N}(fgx, gfx, t) = \left(\frac{\exp(2|x - 1|^2)}{t}, \frac{\exp(2|x - 1|^2) - 1}{\exp(2|x - 1|^2)}\right)$$

$$= \left(\frac{\exp(|x - 1|^2)}{t/2}, \frac{\exp(|x - 1|^2) - 1}{\exp(|x - 1|^2)}\right) = \mathcal{M}_{M,N}(fx, gx, t/2)$$

$$< L^* \left(\frac{\exp(|x - 1|^2)}{t}, \frac{\exp(|x - 1|^2) - 1}{\exp(|x - 1|^2)}\right) = \mathcal{M}_{M,N}(fx, gx, t)$$

Therefore, for $\psi(t) = t/2$, $f$ and $g$ are $\psi$-weakly commuting. But $f$ and $g$ are not weakly commuting since the exponential function is strictly increasing.

**Theorem 2.4.** Let $(X, \mathcal{M}, T)$ be a left complete $\mathcal{L}$-fuzzy metric space and let $f$ and $g$ be $\psi$-weakly commuting self-mappings of $X$ satisfying the following conditions:

(a) $f(x) \subseteq g(x)$;

(b) Either $f$ or $g$ is continuous;

(c) $\mathcal{M}(fx, fy, t) \geq L \mathcal{C}(\mathcal{M}(gx, gy, t))$, where $\mathcal{C} : L \to L$ is a continuous function such that $\mathcal{C}(a) > L a$ for each $a \in L \setminus \{0_L, 1_L\}$.

Then $f$ and $g$ have a unique common fixed point.

**Proof.** Let $x_0$ be an arbitrary point in $X$. By (a), choose a point $x_1$ in $X$ such that $fx_0 = gx_1$. In general choose $x_{n+1}$ such that $fx_n = gx_{n+1}$. Then for $t > 0$,

$$\mathcal{M}(fx_n, fx_{n+1}, t) \geq L \mathcal{C}(\mathcal{M}(gx_n, gx_{n+1}, t)) = \mathcal{C}(\mathcal{M}(fx_{n-1}, fx_n, t))$$

$$> L \mathcal{M}(fx_n, fx_{n+1}, t)$$

Thus $\{\mathcal{M}(fx_n, fx_{n+1}, t); n \geq 0\}$ is an increasing sequence in $L$ and therefore, tends to a limit $a \leq L 1_L$. We claim that $a = 1_L$. For if $a < L 1_L$, when $n \to \infty$ in the
above inequality we get $a \geq_L C(a) >_L a$, a contradiction. Hence $a = 1_L$, i.e.,
\[
\lim_n \mathcal{M}(fx_n, fx_{n+1}, t) = 1_L.
\]

If we define
\[
(2.1) \quad c_n(t) = \mathcal{M}(fx_n, fx_{n+1}, t)
\]
then $\lim_{n \to \infty} c_n(t) = 1_L$. Now, we prove that $\{fx_n\}$ is a Cauchy sequence in $f(X)$. Suppose that $\{fx_n\}$ is not a Cauchy sequence in $f(X)$. For convenience, let $y_n = fx_n$ for $n = 1, 2, 3, \cdots$. Then there is an $\epsilon \in L \setminus \{0_L, 1_L\}$ such that for each integer $k$, there exist integers $m(k)$ and $n(k)$ with $m(k) > n(k) \geq k$ such that
\[
(2.2) \quad d_k(t) = \mathcal{M}(y_{m(k)}, y_{n(k)}, t) \leq \mathcal{N}(\epsilon) \quad \text{for} \quad k = 1, 2, \cdots.
\]

We may assume that
\[
(2.3) \quad \mathcal{M}(y_{n(k)}, y_{m(k)-1}, t) > \mathcal{N}(\epsilon),
\]
by choosing $m(k)$ to be the smallest number exceeding $n(k)$ for which (2.2) holds. Using (2.1), we have
\[
(2.4) \quad \mathcal{N}(\epsilon) \geq d_k(t) \geq T(\mathcal{M}(y_{m(k)}, y_{m(k)-1}, t/2), \mathcal{M}(y_{m(k)-1}, y_{m(k)}, t/2)) \geq T(c_k(t/2), \mathcal{N}(\epsilon))
\]
Hence, $d_k(t) \to \mathcal{N}(\epsilon)$ for every $t > 0$ as $k \to \infty$.

We note that
\[
d_k(t) = \mathcal{M}(y_{m(k)}, y_{m(k)}, t)
\]
\[
\geq T^2(\mathcal{M}(y_{m(k)+1}, y_{m(k)+1}, t/3), \mathcal{M}(y_{m(k)+1}, y_{m(k)}, t/3))
\]
\[
\geq T^2(c_k(t/3), C(\mathcal{M}(y_{m(k)}, y_{m(k)}, t/3), c_k(t/3)))
\]
\[
= T^2(c_k(t/3), C(d_k(t/3), c_k(t/3))).
\]
Thus, as $k \to \infty$ in the above inequality we have
\[
\mathcal{N}(\epsilon) \geq C(\mathcal{N}(\epsilon)) > \mathcal{N}(\epsilon)
\]
which is a contradiction. Thus, $\{fx_n\}$ is Cauchy and by the completeness of $X$, $\{fx_n\}$ converges to $z$ in $X$. Also $\{gx_n\}$ converges to $z$ in $X$. Let us suppose that the mapping $f$ is continuous. Then $\lim_n ffx_n = fz$ and $\lim_n fgx_n = fz$. Further we have since $f$ and $g$ are $\psi$-weakly commuting
\[
\mathcal{M}(ffx_n, fgx_n, t) \geq_L \mathcal{M}(fx_n, gx_n, \psi(t)).
\]
On letting $n \to \infty$ in the above inequality we get $\lim_n gf x_n = fz$, by Lemma 1.13. We now prove that $z = fz$. Suppose $z \neq fz$ then $\mathcal{M}(z, fz, t) <_L 1_L$. By (c)
\[
\mathcal{M}(fx_n, fx_n, t) \geq_L C(\mathcal{M}(gx_n, gx_n, t)).
\]
Letting $n \to \infty$ in the above inequality we get
\[
\mathcal{M}(z, fz, t) \geq_L C(\mathcal{M}(z, fz, t)) >_L \mathcal{M}(z, fz, t),
\]
a contradiction. Therefore, $z = fz$. Since $f(X) \subseteq g(X)$ we can find $z_1$ in $X$ such that $z = fz = g z_1$. Now,
\[
\mathcal{M}(f x_n, f z_1, t) \geq_L C(\mathcal{M}(g x_n, g z_1, t)).
\]
Taking limits as $n \to \infty$ we get
\[
\mathcal{M}(f z, f z_1, t) \geq_L C(\mathcal{M}(f z, g z_1, t)) = 1_L
\]
Since $C(1_L) = 1_L$, this implies that $fz = fz_1$, i.e., $z = fz = fz_1 = gz_1$. Also for any $t > 0$, 
\[ M(fz, gz, t) = M(gz_1, fz_1, t) \geq_L M(fz_1, gz_1, \psi(t)) = 1_L \]
which again implies that $fz = gz$. Thus $z$ is a common fixed point of $f$ and $g$.

Now, to prove uniqueness suppose $z' \neq z$ is another common fixed point of $f$ and $g$. Then there exists $t > 0$ such that $M(z, z', t) \leq 1_L$ and
\[ M(z, z', t) = M(fz, fz', t) \geq_L C(M(gz, gz', t)) = C(M(z, z', t)) \]
which is contradiction. Therefore, $z = z'$, i.e., $z$ is a unique common fixed point of $f$ and $g$. □

**Example 2.5.** Consider Example 1.8 in which $X = [0, 1]$. Define $f(x) = 1$ and
\[
g(x) = \begin{cases} 
1, & \text{if } x \text{ is rational,} \\
0, & \text{if } x \text{ is irrational,}
\end{cases}
\]
on $X$. It is evident that $f(X) \subseteq g(X)$, $f$ is continuous and $g$ is discontinuous. Define $C : L^* \rightarrow L^*$ by $C(a) = (\sqrt{a_1}, a_2^2)$, then $C(a) = (\sqrt{a_1}, a_2^2) >_L (a_1, a_2) = a$ for $0 < a_i < 1$, $i = 1, 2$ and
\[ M(fx, fy, t) \geq_L C(M(gx, gy, t)) \]
for all $x, y$ in $X$, $f$ and $g$ are $\psi$-weakly commuting. Thus all the conditions of last theorem are satisfied and $1$ is a common fixed point of $f$ and $g$.

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**References**


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