SOME RESULTS OF INTUITIONISTIC FUZZY WEAK DUAL HYPER K-IDEALS

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Abstract. In this note we consider the notion of intuitionistic fuzzy (weak) dual hyper K-ideals and obtain related results. Then we classify this notion according to level sets. After that we determine the relationships between intuitionistic fuzzy (weak) dual hyper K-ideals and intuitionistic fuzzy (weak) hyper K-ideals. Finally, we define the notion of the product of two intuitionistic fuzzy (weak) dual hyper K-ideals and prove several Decomposition Theorems.

1. Introduction

Hyperalgebraic structure theory was introduced by F. Marty [7] in 1934. Imai and Iseki [5] introduced the notion of a BCK-algebra in 1966. Borzooei, Jun, Hasankhani and Zahedi et.al. [3] applied hyperstructures to BCK-algebras and introduced the concept of hyper K-algebras which are a generalization of BCK-algebras. The idea of “intuitionistic fuzzy set” was first introduced by Atanassov [1] as a generalization of fuzzy sets. In this note, we consider intuitionistic fuzzification of the notion of (weak) dual hyper K-ideals obtain related results.

2. Preliminaries

Definition 2.1. [3] Let $H$ be a nonempty set and “$\circ$” be a hyperoperation on $H$, i.e. “$\circ$” is a function from $H \times H$ to $P^*(H) = P(H) - \{\emptyset\}$. Then $(H, \circ, 0)$ is called a hyper $K$-algebra if it contains a constant “0” and satisfies the following axioms:

(HK1) $(x \circ z) \circ (y \circ z) < x \circ y$,
(HK2) $(x \circ y) \circ z = (x \circ z) \circ y$,
(HK3) $x < x$,
(HK4) $x < y, y < x \Rightarrow x = y$,
(HK5) $0 < x$.

for all $x, y, z \in H$, where the relation $x < y$ is defined by $0 \in x \circ y$. For every $A, B \subseteq H$, $A < B$ is defined by $\exists a \in A, \exists b \in B$ such that $a < b$.

Note that if $A, B \subseteq H$, then by $A \circ B$ we mean the subset $\bigcup_{a \in A \land b \in B} a \circ b$ of $H$.

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Theorem 2.2. [3] Let \((H, \circ, 0)\) be a hyper K-algebra. Then for all \(x, y, z \in H\) and for all non-empty subsets \(A, B\) and \(C\) of \(H\) the following statements hold:

(i) \(x \circ y < z \iff x \circ z < y\),
(ii) \(x \circ (x \circ y) < y\),
(iii) \(x \circ y < x\),
(iv) \(x \in x \circ 0\),
(v) \(A \circ B \subset A\)

Definition 2.3. [3] Let \((H, \circ, 0)\) be a hyper K-algebra. If there exists an element \(1 \in H\) such that \(1 < x\) for all \(x \in H\), then \(H\) is called a bounded hyper K-algebra and \(1\) is said to be the unit of \(H\). In a bounded hyper K-algebra, we denote \(1 \circ x\) by \(Nx\).

Definition 2.4. [10] Let \(D\) be a nonempty subset of a hyper K-algebra \((H, \circ, 0)\) and \(1 \in D\). Then,

(i) \(D\) is called a weak dual hyper K-ideal of \(H\), if \(N(Nx \circ Ny) \subseteq D\) and \(y \in D\) imply that \(x \in D\).
(ii) \(D\) is called a dual hyper K-ideal of \(H\), if \(N(Nx \circ Ny) \cap D \neq \emptyset\) and \(y \in D\) imply that \(x \in D\).

Definition 2.5. [3] Let \((H, \circ, 0)\) be a hyper K-algebra. An element \(a \in H\) is called a left (resp. right) scalar if \(|a \circ x| = 1\) (resp. \(|x \circ a| = 1\)) for all \(x \in H\).

Theorem 2.6. [11] Let \((H, \circ, 0)\) be a bounded hyper K-algebra and \(NNx = x\), for all \(x \in H\). Then:

(i) \(1\) is a left scalar,
(ii) \(0\) is a right scalar,
(iii) If \(x < y\), then \(Ny < Nx\).

Theorem 2.7. [10] Let \((H, \circ, 0)\) be a bounded hyper K-algebra and let \(NNx = x\), for all \(x \in H\) and \(\emptyset \neq D \subseteq H\). Then \(D\) is a (weak) dual hyper K-ideal if and only if \(ND\) is a (weak) hyper K-ideal.

Definition 2.8. [3] Let \((H_1, \circ_1, 0_1)\) and \((H_2, \circ_2, 0_2)\) be two hyper K-algebras. Then a function \(f : H_1 \longrightarrow H_2\) is called a homomorphism if \(\forall x, y \in H_1, f(x \circ_1 y) = f(x) \circ_2 f(y)\) and \(f(0_1) = 0_2\).

Definition 2.9. [13] Let \(\mu\) be a fuzzy set of a nonempty set \(H\) and \(t \in [0, 1]\). Then the set

\[ U(\mu; t) = \{ x \in H \mid \mu(x) \geq t \} \]

(resp. \(L(\mu; t) = \{ x \in H \mid \mu(x) \leq t \} \))

is called an upper (resp. lower) level set of \(\mu\).

Definition 2.10. Let \(\mu\) and \(\nu\) be fuzzy sets of \(X\) and \(Y\), respectively. Then the fuzzy sets \(\mu \times \nu\) and \(\mu \circ \nu\) of \(X \times Y\), which are called the product and anti-product of \(\mu\) and \(\nu\), respectively, are defined by

\[(\mu \times \nu)(x, y) = \min\{\mu(x), \nu(y)\}\]

\[(\mu \circ \nu)(x, y) = \max\{\mu(x), \nu(y)\}\]
Definition 2.11. [1] An intuitionistic fuzzy set (briefly, IFS) $A$ on a nonempty set $X$ is an object having the form

$$A = \{(x, \mu_A(x), \gamma_A(x)) | x \in X\}$$

where the function $\mu_A : X \to [0, 1]$ and $\gamma_A : X \to [0, 1]$ denote the degree of membership and the degree of nonmembership, respectively, and $x$ for all $\mu$ be identified with an ordered pair $(\mu, \gamma)$ in $I^X \times I^X$. For the sake of simplicity, we shall use the symbol $A = (\mu_A, \gamma_A)$ for the IFS $A = \{(x, \mu_A(x), \gamma_A(x)) | x \in X\}$.

Definition 2.12. [12] Let $f : X \to Y$ be a function. An intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ of $X$ is said to be $f$-invariant, if $f(x) = f(y)$ implies that $\mu_A(x) = \mu_A(y)$ and $\gamma_A(x) = \gamma_A(y)$ for all $x, y \in H$.

Definition 2.13. [12] Let $f : X \to Y$ be a function and $A$ be an intuitionistic fuzzy set of $X$. Then the intuitionistic fuzzy set $f(A) = (f(\mu_A), f(\gamma_A))$ of $Y$ is defined by:

$$f(\mu_A)(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \mu_A(x) & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

$$f(\gamma_A)(y) = \begin{cases} \inf_{x \in f^{-1}(y)} \gamma_A(x) & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}.$$

Definition 2.14. Let $f : X \to Y$ be a function and $B$ be a fuzzy set of $Y$. Then the fuzzy set $f^{-1}(B)$ of $X$ is defined by:

$$f^{-1}(B)(x) = B(f(x)).$$

Definition 2.15. An IFS $A = (\mu_A, \gamma_A)$ of $H$ is said to satisfy the sup-inf property if for any subset $T$ of $H$ there exist $x_0, y_0 \in T$ such that $\mu_A(x_0) = \sup_{x \in T} \mu_A(x)$ and $\gamma_A(y_0) = \inf_{y \in T} \gamma_A(y)$.

Definition 2.16. [12] Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy set of $(H, \circ, 0)$. Then,

(i) $A$ is called an intuitionistic fuzzy week hyper $K$-ideal of $H$ if

$$\mu_A(0) \geq \mu_A(x) \geq \min\{\inf_{a \in x \circ y} \mu_A(a), \mu_A(y)\}$$

and

$$\gamma_A(0) \leq \gamma_A(x) \leq \max\{\sup_{b \in x \circ y} \gamma_A(b), \gamma_A(y)\}$$

for all $x, y \in H$.

(ii) $A$ is called an intuitionistic fuzzy hyper $K$-ideal of $H$ if

$$\mu_A(0) \geq \mu_A(x) \geq \min\{\sup_{a \in x \circ y} \mu_A(a), \mu_A(y)\}$$
and
\[ \gamma_A(0) \leq \gamma_A(x) \leq \max \left( \inf_{b \in \mathbb{N} \times \mathbb{N}} \gamma_A(b), \gamma_A(y) \right) \]
for all \( x, y \in H \).

**Theorem 2.17.** \[12\] Let \( A = (\mu_A, \gamma_A) \) be an IFS of \((H, \circ, 0)\) which satisfy the sup-inf property. Then \( A \) is an intuitionistic fuzzy hyper \( K \)-ideal if and only if for all \( s, t \in [0, 1] \) the nonempty level sets \( U(\mu_A; t) \) and \( L(\gamma_A; s) \) are hyper \( K \)-ideals of \( H \).

**Theorem 2.18.** \[12\] Let \( A = (\mu_A, \gamma_A) \) be an IFS of \((H, \circ, 0)\). Then \( A \) is an intuitionistic fuzzy weak hyper \( K \)-ideal if and only if for all \( s, t \in [0, 1] \) the nonempty level sets \( U(\mu_A; t) \) and \( L(\gamma_A; s) \) are weak hyper \( K \)-ideals of \( H \).

**Definition 2.19.** \[12\] Let \( A = (\mu_A, \gamma_A) \) be an intuitionistic fuzzy set of \((H, \circ, 0)\). Then, (i) \( A \) satisfies the additive condition, whenever for all \( x, y \in H \), \( x < y \) implies that \( \mu_A(x) \geq \mu_A(y) \) and \( \gamma_A(x) \leq \gamma_A(y) \). (ii) \( A \) satisfies the anti-additive condition if \( x < y \) implies that \( \mu_A(x) \leq \mu_A(y) \) and \( \gamma_A(x) \geq \gamma_A(y) \).

### 3. Intuitionistic Fuzzy (Weak) Dual Hyper \( K \)-ideals

In what follows let \( H \) denote a bounded hyper \( K \)-algebra.

**Definition 3.1.** An IFS \( A = (\mu_A, \gamma_A) \) of \( H \) is called an intuitionistic fuzzy weak dual hyper \( K \)-ideal of \( H \) if it satisfies the following conditions:
\[ \mu_A(1) \geq \mu_A(x) \geq \min \left( \inf_{a \in \mathbb{N} \times \mathbb{N}} \mu_A(a), \mu_A(y) \right) \]
and
\[ \gamma_A(1) \leq \gamma_A(x) \leq \max \left( \sup_{b \in \mathbb{N} \times \mathbb{N}} \gamma_A(b), \gamma_A(y) \right) \]
for all \( x, y \in H \).

**Example 3.2.** Let \( H = \{0, 1, 2\} \) be a hyper \( K \)-algebra with the following table.

<table>
<thead>
<tr>
<th>( \circ )</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>{0}</td>
<td>{0, 1, 2}</td>
<td>{0, 1, 2}</td>
</tr>
<tr>
<td>1</td>
<td>{1}</td>
<td>{0}</td>
<td>{2}</td>
</tr>
<tr>
<td>2</td>
<td>{2}</td>
<td>{0, 1, 2}</td>
<td>{0, 1, 2}</td>
</tr>
</tbody>
</table>

Define the IFS \( A = (\mu_A, \gamma_A) \) on \( H \) as follows:
\( \mu_A(2) = 0.2, \ \mu_A(0) = \mu_A(1) = 0.5, \ \gamma_A(2) = 0.6, \ \gamma_A(0) = \gamma_A(1) = 0.3 \)

Then \( A = (\mu_A, \gamma_A) \) is an intuitionistic fuzzy weak dual hyper \( K \)-ideal.

**Definition 3.3.** An IFS \( A = (\mu_A, \gamma_A) \) of \( H \) is called an intuitionistic fuzzy dual hyper \( K \)-ideal of \( H \) if it satisfies the following conditions:
\[ \mu_A(1) \geq \mu_A(x) \geq \min \left( \sup_{a \in \mathbb{N} \times \mathbb{N}} \mu_A(a), \mu_A(y) \right) \]
\[ \gamma_A(1) \leq \gamma_A(x) \leq \max(\inf_{b \in N(N \times N)} \gamma_A(b), \gamma_A(y)) \]

for all \( x, y \in H \).

**Example 3.4.** The following table shows a hyper \( K \)-algebra structure on \( H = \{0, 1, 2, 3\} \).

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>{0}</td>
<td>{0}</td>
<td>{0}</td>
<td>{0}</td>
</tr>
<tr>
<td>1</td>
<td>{1}</td>
<td>{0}</td>
<td>{3}</td>
<td>{1, 2}</td>
</tr>
<tr>
<td>2</td>
<td>{2}</td>
<td>{0}</td>
<td>{0}</td>
<td>{2}</td>
</tr>
<tr>
<td>3</td>
<td>{3}</td>
<td>{0}</td>
<td>{1, 2, 3}</td>
<td>{0, 3}</td>
</tr>
</tbody>
</table>

Define the IFS \( A = (\mu_A, \gamma_A) \) on \( H \) as follows:

\[ \mu_A(0) = \mu_A(3) = 0.2, \mu_A(1) = \mu_A(2) = 0.5, \gamma_A(0) = \gamma_A(3) = 0.6, \gamma_A(1) = \gamma_A(2) = 0.4 \]

Then \( A \) is an intuitionistic fuzzy dual hyper \( K \)-ideal.

**Theorem 3.5.** Every intuitionistic fuzzy dual hyper \( K \)-ideal is an intuitionistic fuzzy weak dual hyper \( K \)-ideal.

**Proof.** Let \( A = (\mu_A, \gamma_A) \) be an intuitionistic fuzzy dual hyper \( K \)-ideal. Then for all \( x, y \in H \)

\[ \mu_A(1) \geq \mu_A(x) \geq \min(\sup_{a \in N(N \times N)} \mu_A(a), \mu_A(y)) \]

and

\[ \gamma_A(1) \leq \gamma_A(x) \leq \max(\inf_{b \in N(N \times N)} \gamma_A(b), \gamma_A(y)). \]

Since \( \sup_{a \in N(N \times N)} \mu_A(a) \geq \inf_{a \in N(N \times N)} \mu_A(a) \) and \( \inf_{b \in N(N \times N)} \gamma_A(b) \leq \sup_{b \in N(N \times N)} \gamma_A(b) \),

hence

\[ \mu_A(1) \geq \mu_A(x) \geq \min(\inf_{a \in N(N \times N)} \mu_A(a), \mu_A(y)) \]

and

\[ \gamma_A(1) \leq \gamma_A(x) \leq \max(\sup_{b \in N(N \times N)} \gamma_A(b), \gamma_A(y)) \]

for all \( x, y \in H \). It follows that \( A = (\mu_A, \gamma_A) \) is an intuitionistic fuzzy weak dual hyper \( K \)-ideal. \( \square \)

Example 3.2 shows that the converse of the above theorem is not true in general.

**Theorem 3.6.** Let \( A = (\mu_A, \gamma_A) \) be an IFS of \( H \) which satisfies the sup-inf property. Then \( A \) is an intuitionistic fuzzy dual hyper \( K \)-ideal if and only if for all \( s, t \in [0, 1] \) the nonempty level sets \( U(\mu_A; t) \) and \( L(\gamma_A; s) \) are dual hyper \( K \)-ideals of \( H \).
Proof. Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy dual hyper $K$-ideal and $U(\mu_A; t) \neq \emptyset \neq L(\gamma_A; s)$. By Definition 3.3, it is clear that $1 \in U(\mu_A; t) \cap L(\gamma_A; s)$. Let $N(Nx \circ Ny) \cap U(\mu_A; t) \neq \emptyset$ and $y \in U(\mu_A; t)$. Then $\mu_A(y) \geq t$ and there exists $r \in N(Nx \circ Ny)$ such that $\mu_A(r) \geq t$. Thus

$$\mu_A(x) \geq \min(\sup_{a \in N(Nx \circ Ny)} \mu_A(a), \mu_A(y)) \geq \min(\mu_A(r), \mu_A(y)) \geq t$$

and so $x \in U(\mu_A; t)$. Now let $N(Nx \circ Ny) \cap L(\gamma_A; s) \neq \emptyset$ and $y \in L(\gamma_A; s)$. Then by a similar argument we can get that $x \in L(\gamma_A; s)$.

Conversely, let for all $s, t \in [0, 1]$ the nonempty level sets $U(\mu_A; t)$ and $L(\gamma_A; s)$ be dual hyper $K$-ideals. Let $x, y \in H$, $\mu_A(x) = t$ and $\gamma_A(y) = s$. Since $x \in U(\mu_A; t)$ and $y \in L(\gamma_A; s)$, hence by hypothesis we get that $1 \in U(\mu_A; t) \cap L(\gamma_A; s)$. It follows that $\mu_A(1) \geq \mu_A(x)$ and $\gamma_A(1) \leq \gamma_A(y)$. Now let $k = \min(\sup_{a \in N(Nx \circ Ny)} \mu_A(a), \mu_A(y))$ and $h = \max(\inf_{b \in N(Nx \circ Ny)} \gamma_A(b), \gamma_A(y))$. Since $A = (\mu_A, \gamma_A)$ satisfies the sup-inf property, there exist $x_0, y_0 \in N(Nx \circ Ny)$ such that

$$\mu_A(x_0) = \sup_{a \in N(Nx \circ Ny)} \mu_A(a) \geq \min(\sup_{a \in N(Nx \circ Ny)} \mu_A(a), \mu_A(y)) = k$$

and

$$\gamma_A(y_0) = \inf_{b \in N(Nx \circ Ny)} \gamma_A(b) \leq \max(\inf_{b \in N(Nx \circ Ny)} \gamma_A(b), \gamma_A(y)) = h$$

, we have $x_0 \in U(\mu_A; k)$ and $y_0 \in L(\gamma_A; h)$. Since $x_0 \in U(\mu_A; k) \cap N(Nx \circ Ny)$ and $y_0 \in L(\gamma_A; h) \cap N(Nx \circ Ny)$, then $N(Nx \circ Ny) \cap U(\mu_A; k) \neq \emptyset$ and $N(Nx \circ Ny) \cap L(\gamma_A; h) \neq \emptyset$. Also $\mu_A(y) \geq k$ and $\gamma_A(y) \leq h$ imply that $y \in U(\mu_A; k) \cap L(\gamma_A; h)$. Thus, by hypothesis, we get that $x \in U(\mu_A; k) \cap L(\gamma_A; h)$. So for all $x, y \in H$

$$\mu_A(x) \geq k \geq \min(\sup_{a \in N(Nx \circ Ny)} \mu_A(a), \mu_A(y))$$

and

$$\gamma_A(x) \leq h \geq \max(\inf_{b \in N(Nx \circ Ny)} \gamma(b), \gamma(y))$$

for all $x, y \in H$. Therefore $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy dual hyper $K$-ideal. \hfill \Box

Theorem 3.7. Let $A = (\mu_A, \gamma_A)$ be an IFS of $H$. Then $A$ is an intuitionistic fuzzy weak dual hyper $K$-ideal if and only if for all $s, t \in [0, 1]$ the nonempty level sets $U(\mu_A; t)$ and $L(\gamma_A; s)$ are weak dual hyper $K$-ideals of $H$.

Proof. The proof is similar to the proof of Theorem 3.6. \hfill \Box

Theorem 3.8. Let $NNx = x$ for all $x \in H$ and $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy dual hyper $K$-ideal. Then $A$ satisfies the anti-additive condition.
Let \( x < y \). Then \( Ny < Nx \), by Theorem 2.6 and so \( 0 \in Ny \circ Nx \). Thus
\[
\sup_{a \in N(Ny \circ Nx)} \mu_A(a) \geq \mu_A(1) \quad \text{and} \quad \inf_{b \in N(Ny \circ Nx)} \gamma_A(b) \leq \gamma_A(1).
\]
Hence, by hypothesis, we get that
\[
\mu_A(y) \geq \min(\sup_{a \in N(Ny \circ Nx)} \mu_A(a), \mu_A(x)) \geq \min(\mu_A(1), \mu_A(x)) = \mu_A(x)
\]
and
\[
\gamma_A(y) \leq \max(\inf_{b \in N(Ny \circ Nx)} \gamma_A(b), \gamma_A(x)) \leq \max(\gamma_A(1), \gamma_A(x)) = \gamma_A(x). \quad \square
\]

The following example shows that the above theorem is not true for intuitionistic fuzzy weak dual hyper \( K \)-ideals.

**Example 3.9.** In Example 3.2, \( A = (\mu_A, \gamma_A) \) is an intuitionistic fuzzy weak dual hyper \( K \)-ideal, \( NNx = x \) for all \( x \in H \) and \( 0 < 2 \) while \( 0.5 = \mu_A(0) \leq \mu_A(2) = 0.2 \) and \( 0.3 = \gamma_A(0) \geq \gamma_A(2) = 0.6 \).

**Theorem 3.10.** If \( A = (\mu_A, \gamma_A) \) is an intuitionistic fuzzy (weak) dual hyper \( K \)-ideal, then the set
\[
H_A = \{ x \in H | \mu_A(x) = \mu_A(1), \quad \gamma_A(x) = \gamma_A(1) \}
\]
is a (weak) dual hyper \( K \)-ideal.

**Proof.** Obviously \( 1 \in H_A \). Let \( N(Nx) \cap H_A \neq \emptyset \) and \( y \in H_A \). Then \( \mu_A(y) = \mu_A(1), \gamma_A(y) = \gamma_A(1) \) and there exists \( s \in N(Nx) \) such that \( \mu_A(s) = \mu_A(1) \) and \( \gamma_A(s) = \gamma_A(1) \). So, by hypothesis, we have
\[
\mu_A(1) \geq \mu_A(x) \geq \min(\sup_{a \in N(Nx \circ Ny)} \mu_A(a), \mu_A(y)) \geq \min(\mu_A(s), \mu_A(y)) = \mu_A(1)
\]
and
\[
\gamma_A(1) \leq \gamma_A(x) \leq \max(\inf_{b \in N(Nx \circ Ny)} \gamma_A(b), \gamma_A(y)) \leq \max(\gamma_A(s), \gamma_A(y)) = \gamma_A(1).
\]
Therefore \( \mu_A(x) = \mu_A(1) \) and \( \gamma_A(x) = \gamma_A(1) \) i.e., \( x \in H_A \). \quad \square

**Theorem 3.11.** Let \( 1 \in 1 \circ x \) for all \( x \in H \) and \( A = (\mu_A, \gamma_A) \) be an intuitionistic fuzzy dual hyper \( K \)-ideal. Then \( H_A = H \).

**Proof.** By hypothesis we get that
\[
\mu_A(1) \geq \mu_A(x) \geq \min(\sup_{a \in N(Nx \circ N1)} \mu_A(a), \mu_A(1)) \geq \min(\mu_A(1), \mu_A(1)) = \mu_A(1)
\]
and
\[
\gamma_A(1) \leq \gamma_A(x) \leq \max(\inf_{b \in N(Nx \circ N1)} \gamma_A(b), \gamma_A(1)) \leq \max(\gamma_A(1), \gamma_A(1)) = \gamma_A(1)
\]
for all \( x \in H \). Therefore \( \mu_A(x) = \mu_A(1) \) and \( \gamma_A(x) = \gamma_A(1) \), for all \( x \in H \), i.e. \( H_A = H \). \quad \square
Theorem 3.12. Let $NNx = x$ for all $x \in H$. Then $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy weak dual hyper $K$-ideal if and only if $A^N = (\mu_A^N, \gamma_A^N)$ is an intuitionistic fuzzy weak hyper $K$-ideal, where for all $x \in H$, $\mu_A^N(x) = \mu_A(Nx)$ and $\gamma_A^N(x) = \gamma_A(Nx)$.

Proof. Since $NNx = x$, for all $x \in H$, hence, by Theorem 2.6 (i), $A^N = (\mu_A^N, \gamma_A^N)$ is an intuitionistic fuzzy weak hyper set.

Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy weak dual hyper $K$-ideal and $U(\mu_A^N, t) \neq \emptyset \neq L(\gamma_A^N; s)$ for $s, t \in [0, 1]$. Now we prove that $NU(\mu_A^N; t)$ and $NL(\gamma_A^N; s). Let x \in NU(\mu_A^N; t). Then \exists h \in U(\mu_A^N; t)$ such that $x \in 1 \circ h$. So $\mu_A(h) \geq t$ and $1 \circ x \subseteq 1 \circ (1 \circ h) = h$. Thus $\mu_A^N(x) = \mu_A(Nx) \geq t$. Therefore $x \in U(\mu_A^N; t)$ i.e., $NU(\mu_A^N; t) \subseteq U(\mu_A^N; t)$.

Now let $x \in U(\mu_A^N; t)$. Then $\mu_A(Nx) = \mu_A^N(x) \geq t$. Hence by Theorem 2.6 there exists $h \in H$ such that $Nh = x$. Thus $\mu_A(h) \geq t$ and so $h \in U(\mu_A^N; t)$. Since $Nh = NNx = x$, hence $x = Nh \subseteq NU(\mu_A^N; t)$. Therefore $U(\mu_A^N; t) \subseteq NU(\mu_A^N; t)$.

Similarly we prove that $NL(\mu_A^N; s) = L(\gamma_A^N; s)$ and $U(\mu_A^N; t)$ and $L(\gamma_A^N; s)$ are nonempty too. Thus by Theorem 3.7 $U(\mu_A^N; t)$ and $L(\gamma_A^N; s)$ are weak dual hyper $K$-ideals and so, by Theorem 2.7, $NU(\mu_A^N; t)$ and $NL(\gamma_A^N; s)$ are weak hyper $K$-ideals. Hence $U(\mu_A^N; t)$ and $L(\gamma_A^N; s)$ are also weak hyper $K$-ideals. Therefore, by Theorem 2.18, $A^N = (\mu_A^N, \gamma_A^N)$ is an intuitionistic fuzzy weak hyper $K$-ideal. The proof of the converse is similar to above if we invoke Theorems 2.7, 3.7 and 2.18. □

Theorem 3.13. Let $NNx = x$ for all $x \in H$ and suppose $A = (\mu_A, \gamma_A)$ satisfies the sup-inf property. Then $A$ is an intuitionistic fuzzy dual hyper $K$-ideal if and only if $A^N = (\mu_A^N, \gamma_A^N)$ is an intuitionistic fuzzy hyper $K$-ideal.

Proof. The proof is similar to the proof of Theorem 3.12. □

Theorem 3.14. Let $NNx = x$ for all $x \in H$. Then $A = (\mu_A, \gamma_A)$ satisfies the additive condition if and only if $A^N = (\mu_A^N, \gamma_A^N)$ satisfies the anti-additive condition.

Proof. Let $A$ satisfy the additive condition and $x < y$. Then by Theorem 2.6 $N\gamma < Nx$ and so $\mu_A(Nx) \leq \mu_A(N\gamma)$ and $\gamma_A(Nx) \geq \gamma_A(N\gamma)$. Thus $A^N(x) \leq A^N(y)$ and $A^N(x) \geq A^N(y)$. Therefore $A^N = (\mu_A^N, \gamma_A^N)$ satisfies the anti-additive condition. The proof of the converse is similar. □

4. Decomposition of Intuitionistic Fuzzy (Weak) Dual Hyper $K$-ideals

Definition 4.1. Let $A$ be a fuzzy set of $H$ and $A(1) \geq A(x)$ for all $x \in H$. Then

(i) $A$ is called a fuzzy dual hyper $K$-ideal, if

$$A(x) \geq \min(\sup_{a \in N(Nx \circ N\gamma)} A(a), A(y)),$$

(ii) $A$ is called a fuzzy weak dual hyper $K$-ideal, if

$$A(x) \geq \min(\inf_{b \in N(Nx \circ N\gamma)} A(b), A(y)).$$
Theorem 4.2. Let $D$ be a nonempty subset of $H$. Then
(i) $D$ is a dual hyper $K$-ideal if and only if $\chi_D$ is a fuzzy dual hyper $K$-ideal.
(ii) $D$ is a (weak) dual hyper $K$-ideal if and only if $\chi_D$ is a fuzzy (weak) dual hyper $K$-ideal.

Proof. The proof is easy. □

Theorem 4.3. Every (weak) dual hyper $K$-ideal of a bounded hyper $K$-algebra $H$ is a level set of a fuzzy (weak) dual hyper $K$-ideal.

Proof. Let $D$ be a dual hyper $K$-ideal of $H$ and $A$ be a fuzzy set on $H$ defined by
$$A(x) = \begin{cases} \alpha & \text{if } x \in D \\ 0 & \text{otherwise} \end{cases}$$
where $\alpha \in [0, 1]$. It is clear that $U(A, \alpha) = D$. Now we show that $A$ is a fuzzy dual hyper $K$-ideal. If $N(Nx \circ Ny) \cap D = \emptyset$ or $y \notin D$, then $A(x) \geq \min(\sup_{a \in N(Nx \circ Ny)} A(a), A(y)) = 0$. If $N(Nx \circ Ny) \cap D \neq \emptyset$ and $y \in D$, then by hypothesis we have $x \in D$. Thus $\alpha = A(x) = \min(\sup_{a \in N(Nx \circ Ny)} A(a), A(y)) = \alpha$. Therefore $A$ is a fuzzy dual hyper $K$-ideal. □

Theorem 4.4. An IFS $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy (weak) dual hyper $K$-ideal if and only if the fuzzy sets $\mu_A$ and $\bar{\gamma}_A = 1 - \gamma_A$ are fuzzy (weak) dual hyper $K$-ideals.

Proof. Assume that $A$ is an intuitionistic fuzzy dual hyper $K$-ideal. Obviously $\mu_A$ is a fuzzy dual hyper $K$-ideal. By hypothesis we have $\gamma_A(1) \leq \gamma_A(x) \leq \max(\inf_{b \in N(Nx \circ Ny)} \gamma_A(b), \gamma_A(y))$ for all $x, y \in H$. Thus,
$$1 - \gamma_A(1) \geq 1 - \gamma_A(x) \geq 1 - \max(\inf_{b \in N(Nx \circ Ny)} \gamma_A(b), \gamma_A(y))$$
$$\bar{\gamma}_A(1) \geq \bar{\gamma}_A(x) \geq 1 + \min(\sup_{b \in N(Nx \circ Ny)} -\gamma_A(b), -\gamma_A(y))$$
$$= \min(\sup_{b \in N(Nx \circ Ny)} 1 - \gamma_A(b), 1 - \gamma_A(y))$$
$$= \min(\sup_{b \in N(Nx \circ Ny)} \bar{\gamma}_A(b), \bar{\gamma}_A(y))$$
Therefore $\bar{\gamma}_A$ is a fuzzy dual hyper $K$-ideal.

Conversely, let $\mu_A$ and $\bar{\gamma}_A$ be fuzzy dual hyper $K$-ideals. Then for all $x, y \in H$,
$$\mu_A(1) \geq \mu_A(x) \geq \min(\sup_{a \in N(Nx \circ Ny)} \mu_A(a), \mu_A(y))$$
and
$$\bar{\gamma}_A(1) \geq \bar{\gamma}_A(x) \geq \min(\sup_{b \in N(Nx \circ Ny)} \bar{\gamma}_A(b), \bar{\gamma}_A(y)).$$
Thus,
\[
1 - \gamma_A(1) \geq 1 - \gamma_A(x) \geq \min\left( \sup_{b \in N(Nx \cap Ny)} 1 - \gamma_A(b), 1 - \gamma_A(y) \right) \\
= 1 - \max\left( \inf_{b \in N(Nx \cap Ny)} \gamma_A(b), \gamma_A(y) \right)
\]

So \( \gamma_A(1) \leq \gamma_A(x) \leq \max\left( \inf_{b \in N(Nx \cap Ny)} \gamma_A(b), \gamma_A(y) \right) \) for all \( x, y \in H \). Therefore \( A = (\mu_A, \gamma_A) \) is an intuitionistic fuzzy dual hyper \( K \)-ideal.

**Theorem 4.5.** Let \( A = (\mu_A, \gamma_A) \) be an IFS of \( H \). Then \( A \) is an intuitionistic fuzzy (weak) dual hyper \( K \)-ideal if and only if \( A_\mu = (\mu_A, \bar{\mu}_A) \) and \( A_\gamma = (\bar{\gamma}_A, \gamma_A) \) are intuitionistic fuzzy (weak) dual hyper \( K \)-ideals, where \( \gamma_A = 1 - \gamma_A \) and \( \bar{\mu}_A \) is the same as \( \gamma_A \).

**Proof.** The proof follows from Theorem 4.4.

**Theorem 4.6.** Let \((H_1, \circ_1, 0_1)\) and \((H_2, \circ_2, 0_2)\) be bounded hyper \( K \)-algebras, and let \( 1_{H_1} \circ_1 1_{H_1} = 0_1, 1_{H_2} \circ_2 1_{H_2} = 0_2 \), and \( \mu \) and \( \nu \) be fuzzy sets of \( H_1 \) and \( H_2 \), respectively. If \( \mu(1_{H_1}) = \nu(1_{H_2}), \mu(1) \geq \mu(x) \) and \( \nu(1) \geq \nu(y), \forall (x, y) \in H_1 \times H_2 \), then \( \mu \times \nu \) is a fuzzy (weak) dual hyper \( K \)-ideal of \( H_1 \times H_2 \) if and only if \( \mu \) and \( \nu \) are fuzzy (weak) dual hyper \( K \)-ideals of \( H_1 \) and \( H_2 \), respectively.

**Proof.** Let \( \mu \) and \( \nu \) be fuzzy dual hyper \( K \)-ideals of \( H_1 \) and \( H_2 \), respectively. Then for all \( (x, y) \in H_1 \times H_2 \):
\[
(\mu \times \nu)(1_{H_1}, 1_{H_2}) = \min\{\mu(1_{H_1}), \nu(1_{H_2})\} \geq \min\{\mu(x, y)\} = (\mu \times \nu)(x, y).
\]

Let \((x_1, y_1), (x_2, y_2) \in H_1 \times H_2 \). Then
\[
\min\left\{ \sup_{(a, b) \in N(N(x_1, y_1) \cap N(x_2, y_2))} (\mu \times \nu)(a, b), (\mu \times \nu)(x_2, y_2) \right\} \\
= \min\left\{ \sup_{a \in N_1(N_1, x_1, 0_1)} \min\{\mu(a), \nu(b)\}, \min\{\mu(x_2), \nu(y_2)\} \right\} \\
\leq \min\left\{ \sup_{a \in N_1(N_1, x_1, 0_1)} \mu(a), \mu(x_2) \right\}, \min\{\sup_{b \in N_2(N_2, y_2, 0_2)} \nu(b), \nu(y_2)\} \\
\leq \min(\mu(x_1), \nu(y_1)) = (\mu \times \nu)(x_1, y_1).
\]

Therefore \( \mu \times \nu \) is a fuzzy dual hyper \( K \)-ideal.

Conversely, let \( \mu \times \nu \) be a fuzzy dual hyper \( K \)-ideal of \( H_1 \times H_2 \) and \( (x, y) \in H_1 \times H_2 \). Then, by hypothesis, we get that \( \mu(1) \geq \mu(x) \) and \( \nu(1) \geq \nu(y) \). Let \( x, y \in H_1 \). By
hypothesis and since $\nu(1) = \mu(1)$, then
\[
\mu(x) = \min(\mu(x), \mu(1)) = \min(\mu(x), \nu(1)) = \mu \times \nu(x, 1)
\]
\[
\geq \min(\sup_{(a, b) \in N(N(x), \infty \times N(1, y))} (\mu \times \nu)(a, b), (\mu \times \nu)(y, 1))
\]
\[
= \min(\sup_{a \in N_1(N_1 \circ \infty N_1, y)} \min\{\mu(a), \nu(b)\}, \mu(y))
\]
\[
= \min(\sup_{a \in N_1(N_1 \circ \infty y)} \min\{\mu(a), \nu(1)\}, \mu(y))
\]
\[
= \min\{\mu(a), \mu(y)\}
\]
Therefore $\mu$ is a fuzzy dual hyper $K$-ideal. Similarly we can prove that $\nu$ is also a fuzzy dual hyper $K$-ideal. \hfill $\square$

**Theorem 4.7.** Let $(H_1, \circ_1, 0_1)$ and $(H_2, \circ_2, 0_2)$ be bounded hyper $K$-algebras, $1_{H_1} \circ_1 1_{H_1} = 0_1$, $1_{H_2} \circ_2 1_{H_2} = 0_2$ and suppose that $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ are two intuitionistic fuzzy sets of $H_1$ and $H_2$, respectively. If $\mu_A(1_{H_1}) = \mu_B(1_{H_2})$, $\gamma_A(1_{H_1}) = \gamma_B(1_{H_2})$, $\mu_A(1_{H_1}) \geq \mu_B(x)$, $\mu_B(1_{H_2}) \geq \mu_B(y)$, $\gamma_A(1_{H_1}) \leq \gamma_B(x)$ and $\gamma_B(1_{H_2}) \leq \gamma_B(y)$ for all $(x, y) \in H_1 \times H_2$, then $A \times B = (\mu_A \times \mu_B, \gamma_A \infty \gamma_B)$ is an intuitionistic fuzzy (weak) dual hyper $K$-ideal if and only if $A$ and $B$ are intuitionistic fuzzy (weak) dual hyper $K$-ideals, where $(\mu \infty \nu)(x, y) = \max\{\mu(x), \nu(y)\}$.

**Proof.** Let $A$ and $B$ be intuitionistic fuzzy dual hyper $K$-ideals. Then, by Theorem 4.4, we have $\mu_A$, $\mu_B$, $\gamma_A$ and $\gamma_B$ are fuzzy dual hyper $K$-ideals. On the other hand, it is easy to check that $\gamma_A \infty \gamma_B = \gamma_A \times \gamma_B$. Hence, by Theorem 4.6, $\mu_A \times \mu_B$ and $\gamma_A \infty \gamma_B$ are fuzzy dual hyper $K$-ideal. So, by Theorem 4.4, $A \times B = (\mu_A \times \mu_B, \gamma_A \infty \gamma_B)$ is an intuitionistic fuzzy dual hyper $K$-ideal. The proof of the converse is similar. \hfill $\square$

**Theorem 4.8.** Let $(H_1, \circ_1, 0_1)$ and $(H_2, \circ_2, 0_2)$ be two bounded hyper $K$-algebras, $N_1N_1 = x$, $N_2N_2 = y$ and $y \circ_2 y = \{0_2\}$, for all $(x, y) \in H_1 \times H_2$. If $\mu$ is a fuzzy dual hyper $K$-ideal of $H_1 \times H_2$, then there are fuzzy dual hyper $K$-ideals $\mu_1$ and $\mu_2$ of $H_1$ and $H_2$, respectively, for which $\mu = \mu_1 \times \mu_2$.

**Proof.** Define $\mu_1(x) = \mu(x, 1_{H_2})$ and $\mu_2(y) = \mu(1_{H_1}, y)$, $\forall (x, y) \in H_1 \times H_2$. Then, by a proof similar to that of Theorem 4.6, we can see that $\mu_1$ and $\mu_2$ are fuzzy dual hyper $K$-ideal. Now we show that $\mu = \mu_1 \times \mu_2$. By Theorem 3.8, $\mu$ satisfies the fuzzy anti-additive condition. Hence $(x, y) < (x, 1)$ and $(x, 1) \times (y, y)$ imply that $\mu(x, y) \leq \mu(x, 1) = \mu_1(x)$ and $\mu(x, y) \leq \mu(1, y) = \mu_2(y)$. Thus $\mu(x, y) \leq \min\{\mu_1(x), \mu_2(y)\} = \mu_1 \times \mu_2(x, y)$, for all $(x, y) \in H_1 \times H_2$. Let $(x, y) \in H_1 \times H_2$. Then:
\[
\mu(x, y) \geq \min\{\mu(a), \mu(1, y)\}
\]
\[
= \min\{\mu(a), \mu(1, y)\} = \min\{\mu(x, 1), \mu(1, y)\}
\]
\[
= \min\{\mu_1(x), \mu_2(y)\} = \mu_1 \times \mu_2(x, y)
\]
Theorem 4.9. Let \( (H_1, \circ_1, 0_1) \) and \( (H_2, \circ_2, 0_2) \) be two bounded hyper \( K \)-algebras, \( N_1 N_1 x = x, N_2 N_2 y = y \) and \( y \circ_2 y = \{0_2\} \) for all \((x, y) \in H_1 \times H_2\) and suppose that \( \mu \) is a fuzzy weak dual hyper \( K \)-ideal of \( H_1 \times H_2 \). If \( \mu \) satisfies the anti-additive condition, then there exist fuzzy weak dual hyper \( K \)-ideals \( \mu_1 \) and \( \mu_2 \) of \( H_1 \) and \( H_2 \), respectively, in which \( \mu = \mu_1 \times \mu_2 \).

Proof. The proof is similar to the proof of Theorem 4.8.

Theorem 4.10. (Decomposition Theorem 1) Let \( (H_1, \circ_1, 0_1) \) and \( (H_2, \circ_2, 0_2) \) be two bounded hyper \( K \)-algebras and \( N_1 N_1 x = x, N_2 N_2 y = y \) and \( y \circ_2 y = \{0_2\} \) for all \((x, y) \in H_1 \times H_2\). If \( A = (\mu_A, \gamma_A) \) is an intuitionistic fuzzy dual hyper \( K \)-ideal of \( H_1 \times H_2 \), then there exist intuitionistic fuzzy dual hyper \( K \)-ideals \( A_1 = (\mu_{A_1}, \gamma_{A_1}) \) and \( A_2 = (\mu_{A_2}, \gamma_{A_2}) \) of \( H_1 \) and \( H_2 \), respectively, in which \( A = A_1 \times A_2 \).

Proof. The proof follows from Theorems 4.8 and 4.4.

Theorem 4.11. (Decomposition Theorem 2) Let \( (H_1, \circ_1, 0_1) \) and \( (H_2, \circ_2, 0_2) \) be two bounded hyper \( K \)-algebras and \( N_1 N_1 x = x, N_2 N_2 y = y \) and \( y \circ_2 y = \{0_2\} \) for all \((x, y) \in H_1 \times H_2\). Suppose \( A = (\mu_A, \gamma_A) \) is an intuitionistic fuzzy weak dual hyper \( K \)-ideal. If \( A \) satisfies the anti-additive condition of \( H_1 \times H_2 \), then there exist intuitionistic fuzzy weak dual hyper \( K \)-ideals \( A_1 = (\mu_{A_1}, \gamma_{A_1}) \) and \( A_2 = (\mu_{A_2}, \gamma_{A_2}) \) of \( H_1 \) and \( H_2 \), respectively, for which \( A = A_1 \times A_2 \).

Proof. The proof follows from Theorems 4.9 and 4.4.

Theorem 4.12. Let \( (H, \circ, 0) \) and \( (H', \circ', 0') \) be two bounded hyper \( K \)-algebras and let \( f : H \rightarrow H' \) be an onto homomorphism. Then :
(i) \( f(1) = 1 \),
(ii) \( N(f(x)) = f(Nx) \).

Proof. It is clear that \( f(1) < 1 \). Now since \( f \) is onto, then there exists \( x \in H \) such that \( f(x) = 1 \). Thus \( 0 = f(0) \in f(x \circ 1) = f(x) \circ f(1) = 1 \circ f(1) \), i.e. \( 1 < f(1) \). Therefore \( f(1) = 1 \).
(ii) By (i) we have
\[ N(f(x)) = 1 \circ f(x) = f(1) \circ f(x) = f(1 \circ x) = f(Nx) \]

Theorem 4.13. Let \( (H_1, \circ_1, 0_1) \) and \( (H_2, \circ_2, 0_2) \) be two bounded hyper \( K \)-algebras and \( f : H_1 \rightarrow H_2 \) be an onto homomorphism. Then \( A = (\mu_A, \gamma_A) \) is an intuitionistic fuzzy (weak) dual hyper \( K \)-ideal of \( H_2 \) if and only if \( f^{-1}(A) = (f^{-1}(\mu_A), f^{-1}(\gamma_A)) \) is an intuitionistic fuzzy (weak) dual hyper \( K \)-ideal of \( H_1 \).

Proof. Let \( A = (\mu_A, \gamma_A) \) be an intuitionistic fuzzy dual hyper \( K \)-ideal. Then by hypothesis we have \( \mu_A(y) \leq \mu_A(1) \) and \( \gamma_A(1) \leq \gamma_A(y) \) for all \( y \in H_2 \). Hence
Let \( d \) be the dual hyper \( K \). By hypothesis and Theorem 4.12 we have
\[
\gamma_A(f(x)) = f^{-1}(\gamma_A(x)) = f^{-1}(\gamma_A(1)) = \gamma_A(f(1)) \leq f^{-1}(\mu_A(1)) = \mu_A(f(1)) = f^{-1}(\mu_A(f(x)))
\]
for all \( x \in H_1 \). Let \( x, y \in H_1 \). Then,
\[
\min(\sup_{a \in N_1(N_1x_0)} f^{-1}(\mu_A)(a), f^{-1}(\mu_A)(y)) = \min(\sup_{a \in N_1(N_1x_0)} \mu_A(f(a)), \mu_A(f(y))) = \min(\sup_{b \in N_2(N_2f(x_0)x_2f(y))} \mu_A(b), \mu_A(f(y))) \leq \mu_A(f(x)) = f^{-1}(\mu_A(x))
\]
Similarly we can prove that
\[
f^{-1}(\gamma_A(x)) = \max(\inf_{b \in N_1(N_1x_0)} f^{-1}(\gamma_A)(b), f^{-1}(\gamma_A)(y))
\]
for all \( x, y \in H_1 \). Therefore, \( f^{-1}(A) = (f^{-1}(\mu_A), f^{-1}(\gamma_A)) \) is an intuitionistic fuzzy dual hyper \( K \)-ideal.

Conversely, let \( f^{-1}(A) = (f^{-1}(\mu_A), f^{-1}(\gamma_A)) \) be an intuitionistic fuzzy dual hyper \( K \)-ideal. Since \( f \) is onto, it is easy to check that \( \mu_A(x) \leq \mu_A(1) \) and \( \gamma_A(x) \geq \gamma_A(1) \) for all \( x \in H_2 \). Let \( x, y \in H_2 \). Then there exists \( x', y' \in H_1 \) such that \( f(x') = x \) and \( f(y') = y \). Now by hypothesis we have:
\[
\min(\sup_{b \in N_2(N_2x_0N_2y)} \mu_A(b), \mu_A(y)) = \min(\sup_{b \in f(N_1(N_1x_0N_1y'))} \mu_A(b), \mu_A(f(y'))) = \min(\sup_{r \in N_1(N_1x_0N_1y')} \mu_A(f(r)), \mu_A(f(y'))) = \min(\sup_{r \in N_1(N_1x_0N_1y')} f^{-1}(\mu_A)(r), f^{-1}(\mu_A)(y')) \leq f^{-1}(\mu_A(x)) = \mu_A(f(x')) = \mu_A(x)
\]
Similarly, \( \gamma_A(x) \leq \max(\inf_{b \in N_2(N_2x_0N_2y)} \gamma_A(b), \gamma_A(y)) \) for all \( x, y \in H_2 \). Therefore \( A = (\mu_A, \gamma_A) \) is an intuitionistic fuzzy dual hyper \( K \)-ideal of \( H_2 \).

**Theorem 4.14.** Let \( f : H_1 \to H_2 \) be an onto homomorphism of two bounded hyper \( K \)-algebras. If \( A = (\mu_A, \gamma_A) \) is an \( f \)-invariant and an intuitionistic fuzzy (weak) dual hyper \( K \)-ideal of \( H_1 \), then \( f(A) = (f(\mu_A), f(\gamma_A)) \) is an intuitionistic fuzzy (weak) dual hyper \( K \)-ideal of \( H_2 \).

**Proof.** By hypothesis and Theorem 4.12 we have \( f(\mu_A)(1) = \mu_A(1) \) and \( f(\gamma_A)(1) = \gamma_A(1) \) and so \( f(\mu_A)(y) \leq f(\mu_A)(1) \) and \( f(\gamma_A)(y) \leq f(\gamma_A)(1) \) for all \( y \in H_2 \). Let \( x, y \in H_2 \). Then \( f(x') = x \) and \( f(y') = y \). Now by hypothesis we get that;
\[
\min(\sup_{a \in N_2(N_2x_0N_2y)} f(\mu_A)(a), f(\mu_A)(y)) = \min(\sup_{a \in f(N_1(N_1x_0N_1y'))} f(\mu_A)(a), f(\mu_A)(y))
\]
Consider the following sets
\[
T_1 = \{ f(\mu_A)(a) \mid a \in f(N_1(N_1x_0N_1y')) \}, \quad T_2 = \{ f(\mu_A)(r) \mid r \in N_1(N_1x_0N_1y') \}.
\]
Since $A$ is an $f$-invariant, we can see that $T_1 = T_2$. Also by hypothesis we get that
\[ f(\mu_A)(y) = \sup_{t \in f^{-1}(y)} \mu_A(t) = \sup_{t \in f^{-1}(f(y'))} \mu_A(t) = \mu_A(y'). \]
Thus \( (1) \) is equal to
\[ \min_{r \in N_1(N_1 x' o_1 N_1 y')} \mu_A(r), \mu_A(y') \leq \mu_A(x') = f(\mu_A)(x). \]
Similarly
\[ f(\gamma_A)(x) \leq \max_{b \in N_2(N_2 x' o_2 N_2 y)} f(\gamma_A)(b), f(\gamma_A)(y)) \]
for all $x, y \in H$. Therefore $f(A) = (f(\mu_A), f(\gamma_A))$ is an intuitionistic fuzzy dual hyper $K$-ideal. \( \square \)

**References**


