A COMMON FRAMEWORK FOR LATTICE-VALUED, PROBABILISTIC AND APPROACH UNIFORM (CONVERGENCE) SPACES

G. JÄGER

Abstract. We develop a general framework for various lattice-valued, probabilistic and approach uniform convergence spaces. To this end, we use the concept of s-stratified LM-filter, where L and M are suitable frames. A stratified LMN-uniform convergence tower is then a family of structures indexed by a quantale N. For different choices of L, M and N we obtain the lattice-valued, probabilistic and approach uniform convergence spaces as examples. We show that the resulting category $sLMN\text{-}UCTS$ is topological, well-fibred and Cartesian closed. We furthermore define stratified LMN-uniform tower spaces and show that the category of these spaces is isomorphic to the subcategory of stratified LMN-principal uniform convergence tower spaces. Finally we study the underlying stratified LMN-convergence tower spaces.

1. **Introduction**

Over the last decades many generalizations of Cook and Fischer's uniform convergence spaces [5] were introduced such as, in particular, probabilistic uniform convergence spaces [24, 2], approach uniform convergence spaces [21] and lattice-valued uniform convergence spaces [20, 6]. Similarly, generalizations of uniform spaces [3] like probabilistic uniform spaces [9, 2], approach uniform spaces [21] and lattice-valued uniform spaces [11, 12] were studied. All these generalizations have big similarities and it seems therefore desirable to establish one common framework, in which all of them can be studied simultaneously. This is the purpose of the present paper. Based on the concept of s-stratified LM-filters, that contains many existing concepts of (lattice-valued) filters, we define what we call s-stratified LMN-uniform convergence tower spaces. The category of these spaces is shown to be well-fibred, topological and Cartesian closed, i.e. it has nice properties. For suitable choices of the lattices L, M and the quantale N we obtain as special instances the categories of uniform convergence spaces mentioned above. Likewise, the concept of an s-stratified uniform tower space encompasses as special cases the uniform spaces mentioned above. Furthermore, we study in our general framework the underlying convergence spaces. These can be described with the general framework of s-stratified LMN-convergence tower spaces introduced recently [18].

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The paper is organised as follows. In a preliminary section we describe the lattice background that we use and introduce basic notations about L-sets. The third section is devoted to a review of s-stratified LM-filters. We introduce in particular LM-filters on $X \times X$. The fourth section then describes the category $sLMN$-$UCTS$ of s-stratified LMN-uniform convergence tower spaces, gives many examples and states the important categorical properties. Section 5 then studies the category of s-stratified LMN-uniform tower spaces and its relationship to the category $sLMN$-$UCTS$. The sixth section is devoted to the underlying s-stratified LMN-convergence tower spaces and finally, in Section 7, we draw some conclusions.

2. **Preliminaries**

We consider in the paper frames $L = (L, \wedge, \vee)$, i.e. complete lattices that satisfy the distributive law $\bigvee_{i\in J}(\alpha \wedge \beta_i) = \alpha \wedge \bigvee_{i\in J} \beta_i$ for all $\alpha, \beta_i \in L$, $(i \in J)$. In a complete lattice L we can define the *wedge-below relation* $\alpha \lhd \beta$ if for all $D \subseteq L$ with $\beta \leq \bigvee D$ there is $\delta \in D$ such that $\alpha \leq \delta$. A complete lattice is called *completely distributive*, if for each $\alpha \in L$, we have $\alpha = \bigvee\{\beta \in L \ : \ \beta \lhd \alpha\}$. For more results on lattices see e.g. [10].

The triple $(L, \leq, *)$, where $(L, \leq)$ is a complete lattice, is called a *quantale* if $(L, *)$ is a semigroup, and $*$ is distributive over arbitrary joins, i.e.

$$(\bigvee_{i\in J} \alpha_i) * \beta = \bigvee_{i\in J} (\alpha_i * \beta) \quad \text{and} \quad \beta * (\bigvee_{i\in J} \alpha_i) = \bigvee_{i\in J} (\beta * \alpha_i).$$

A quantale $(L, *)$ is called *commutative* if $(L, *)$ is a commutative semigroup and it is called *integral* if the top element of L acts as the unit, i.e. if $\alpha * \top = \top * \alpha = \alpha$ for all $\alpha \in L$. We consider in this paper only commutative and integral quantales.

Example 2.1. A *triangular norm* or *t-norm* is a binary operation $*$ on the unit interval $[0, 1]$ which is associative, commutative, non-decreasing in each argument and which has 1 as the unit. The triple $([0, 1], \leq, *)$ can be considered as a quantale if the t-norm is left-continuous. The three most commonly used (left-continuous) t-norms are:

- the minimum t-norm: $\alpha * \beta = \alpha \wedge \beta$,
- the product t-norm: $\alpha * \beta = \alpha \cdot \beta$,
- the Lukasiewicz t-norm: $\alpha * \beta = (\alpha + \beta - 1) \vee 0$.

Example 2.2. The interval $[0, \infty]$ with the opposite order and addition as the quantale operation $\alpha * \beta = \alpha + \beta$ (extended by $\alpha + \infty = \infty + a = \infty$ for all $\alpha, \beta \in [0, \infty]$) is a quantale.

Example 2.3. A function $\varphi : [0, \infty] \longrightarrow [0, 1]$, which is non-decreasing, left-continuous on $(0, \infty)$ and satisfies $\varphi(0) = 0$ and $\varphi(\infty) = 1$ is called a *distance distribution function* [26]. The set of all distance distribution functions is denoted by Δ^+. For example, for each $0 \leq a < \infty$ the functions

$$\varepsilon_a(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq a \\ 1 & \text{if } a < x \leq \infty \end{cases} \qquad \text{and} \qquad \varepsilon_\infty(x) = \begin{cases} 0 & \text{if } 0 \leq x < \infty \\ 1 & \text{if } x = \infty \end{cases}$$

are in Δ^+. The set Δ^+ is ordered pointwise, i.e. for $\varphi, \psi \in \Delta^+$ we define $\varphi \leq \psi$ if for all $x \geq 0$ we have $\varphi(x) \leq \psi(x)$. The bottom element of Δ^+ is then ε_∞ and the top element is ε_0. The set Δ^+ with this order then becomes a complete lattice. A binary operation, $* : \Delta^+ \times \Delta^+ \longrightarrow \Delta^+$, which is commutative, associative, non-decreasing in each place and that satisfies the boundary condition $\varphi * \varepsilon_0 = \varphi$ for all $\varphi \in \Delta^+$, is called a *triangle function* [26]. A triangle function is called *sup-continuous* [26], if $(\bigvee_{i\in I} \varphi_i) * \psi = \bigvee_{i\in I} (\varphi_i * \psi)$ for all $\varphi_i, \psi \in \Delta^+$, $(i \in I)$, i.e. if $(\Delta^+, \leq, *)$ is a quantale.

For a complete lattice L with top element $\top^L$ and bottom element $\bot^L$ and a set X, we denote the power set of X by $\mathbb{P}(X)$ and the set of all L-sets on X, $a, b, c, ... : X \longrightarrow L$, by L^X. A constant L-set with value $\alpha \in L$ is denoted by α_X. In particular, we write $\top^L_X$, resp. $\bot^L_X$, for the constant L-sets with value $\top^L$, resp. $\bot^L$. The lattice operations are extended pointwise from L to L^X, i.e. we define $(a \wedge b)(x) = a(x) \wedge b(x)$, $(a \vee b)(x) = a(x) \vee b(x)$, $(\bigwedge_{i\in J} a_i)(x) = \bigwedge_{i\in J}(a_i(x))$, $(\bigvee_{i\in J} a_i)(x) = \bigvee_{i\in J}(a_i(x))$. For a mapping $f : X \longrightarrow Y$ and $a \in L^X$ and $b \in L^Y$ the image of a under f, $f(a) \in L^Y$, is defined by $f(a)(y) = \bigvee_{f(x)=y} a(x)$ and the preimage of b under f, $f^\leftarrow(b) \in L^X$, is defined by $f^\leftarrow(b)(x) = b(f(x))$. If $a \in L^X$ and $s : L \longrightarrow M$ is a mapping between the frames L, M, then we define $s(a) \in M^X$ by $s(a)(x) = s(a(x))$, $x \in X$.

For notions from category theory, we refer to the textbook [1].

3. Stratified LM-filters

In this section, we review and extend results from [17].

Let L, M be frames. We consider a mapping $s : L \longrightarrow M$ with the properties (S1) $s(\bot^L) = \bot^M$; (S2) $s(\top^L) = \top^M$ and (S3) $s(\alpha \wedge \beta) = s(\alpha) \wedge s(\beta)$ for all $\alpha, \beta \in L$. We call such a mapping a *stratification mapping*. Note that, by the property (S3), a stratification mapping is non-decreasing.

Example 3.1. (1) Every frame morphism $s : L \longrightarrow M$ is a stratification mapping.

(2) The pointwisely smallest stratification mapping is given by
$s_0(\alpha) = \begin{cases} \top^M & \text{if } \alpha = \top^L \\ \bot^M & \text{if } \alpha \neq \top^L \end{cases}$.

(3) If $\bot^L$ is prime, then the pointwisely largest stratification mapping is given by $s_1(\alpha) = \begin{cases} \bot^M & \text{if } \alpha = \bot^L \\ \top^M & \text{if } \alpha \neq \bot^L \end{cases}$.

In the sequel, we fix a stratification mapping $s : L \longrightarrow M$.

Definition 3.2. A mapping $\mathcal{F} : L^X \longrightarrow M$ is an *s-stratified LM-filter on X* if
(F1) $\mathcal{F}(\bot^L_X) = \bot^M$ and $\mathcal{F}(\top^L_X) = \top^M$;
(F2) $\mathcal{F}(a) \leq \mathcal{F}(b)$ whenever $a \leq b$;
(F3) $\mathcal{F}(a) \wedge \mathcal{F}(b) \leq \mathcal{F}(a \wedge b)$ for all $a, b \in L^X$;
(Fs) $s(\alpha) \leq \mathcal{F}(\alpha_X)$ for all $\alpha \in L$.

Note that the *stratification condition* (Fs) is equivalent to $s(\alpha)\wedge\mathcal{F}(a)\leq\mathcal{F}(\alpha_X\wedge a)$ for all $\alpha\in L$, $a\in L^X$. We denote the set of all s-stratified LM-filters on X by $\mathcal{F}^s_{LM}(X)$.

Example 3.3. (1) We define, for $a\in L^X$, $[x]_s(a)=s(a(x))$. Then $[x]_s\in\mathcal{F}^s_{LM}(X)$ is called the *s-stratified point LM-filter of x.* Note that in the case $X=\{x\}$, $\mathcal{F}\in\mathcal{F}^s_{LM}(X)$ implies $[x]_s(a)=s(a(x))\leq\mathcal{F}(a(x))=\mathcal{F}(a)$ for $a\in L^X$, i.e. $[x]_s\leq\mathcal{F}$. If the stratification mapping s is clear, we also simply write $[x]$ for $[x]_s$.

(2) $L=M$, $s=id_L$. A stratified L-filter [14] is an id_L-stratified LL-filter.

(3) $L=M=\{0,1\}$. A filter $\mathbb{F}\in\mathbb{F}(X)$ can be identified with an s_0-stratified LM-filter. (In this case, $s_0=id_{\{0,1\}}$ is the only possible stratification mapping.)

(4) $L=\{0,1\}$. An s-stratified LM-filter is an M-filter of ordinary subsets, [13]. (The property (M3) is always true for any mapping s that satisfies (M1) and (M2).)

(5) $L=[0,1]$, $M=\{0,1\}$. An s-stratified LM-filter can be identified with a prefilter [22]. Note that only $s=s_0$ is possible in order that all prefilters are s-stratified. As a consequence we have to define the point prefilter by $[x]_{s_0}=\{a\in[0,1]^X\ :\ a(x)=1\}$.

For a mapping $f:X\longrightarrow Y$ and $\mathcal{F}\in\mathcal{F}^s_{LM}(X)$ we define $f(\mathcal{F})\in\mathcal{F}^s_{LM}(Y)$ by $f(\mathcal{F})(b)=\mathcal{F}(f^{\leftarrow}(b))$.

We consider now a further stratification mapping, $t:M\longrightarrow L$.

For $\mathcal{F}_j\in\mathcal{F}^s_{LM}(X)$ ($j\in J$), consider for $a\in L^X$ the M-set on J, $\mathcal{F}_{(\cdot)}(a)$ with $\mathcal{F}_{(\cdot)}(a)(j)=\mathcal{F}_j(a)$. Define further $t(\mathcal{F}_{(\cdot)}(a))(j):=t(\mathcal{F}_j(a))\in L$. Then $t(\mathcal{F}_{(\cdot)}(a))\in L^J$. For $\mathcal{G}\in\mathcal{F}^s_{LM}(J)$ we can then form $\mathcal{G}(\mathcal{F}_{(\cdot)})(a):=\mathcal{G}(t(\mathcal{F}_{(\cdot)})(a))$.

Lemma 3.4. [18] *Let $s:L\longrightarrow M$ and $t:M\longrightarrow L$ be stratification mappings with $s\circ t\circ s\geq s$. For sets J,X and $\mathcal{G}\in\mathcal{F}^s_{LM}(J)$ and $\mathcal{F}_j\in\mathcal{F}^s_{LM}(X)$ ($j\in J$), then $\mathcal{G}(\mathcal{F}_{(\cdot)})\in\mathcal{F}^s_{LM}(X)$.*

We call $\mathcal{G}(\mathcal{F}_{(\cdot)})$ the *s-stratified LM-diagonal filter* of $(\mathcal{G},(\mathcal{F}_j)_{j\in J})$, see [16, 18].

For two s-stratified LM-filters $\mathcal{F}\in\mathcal{F}^s_{LM}(X),\mathcal{G}\in\mathcal{F}^s_{LM}(Y)$ we define their product, $\mathcal{F}\times\mathcal{G}:L^{X\times Y}\longrightarrow M$, by

$$\mathcal{F}\times\mathcal{G}(a)=\bigvee\{\mathcal{F}(f)\wedge\mathcal{G}(g)\ :\ f\times g\leq a\},\qquad (a\in L^{X\times Y})$$

with $f\times g(x,y)=f(x)\wedge g(y)$ for $f\in L^X$ and $g\in L^Y$, see [15, 18].

Lemma 3.5. [18] *Let $s:L\longrightarrow M$ and $t:M\longrightarrow L$ be stratification mappings such that $s\circ t\geq id_M$ and $t\circ s\geq id_L$. If $\mathcal{F}\in\mathcal{F}^s_{LM}(X),\mathcal{G}\in\mathcal{F}^s_{LM}(Y)$ then $\mathcal{F}\times\mathcal{G}\in\mathcal{F}^s_{LM}(X\times Y)$.*

We denote s-stratified LM-filters on $X\times X$ by $\Phi,\Psi:L^{X\times X}\longrightarrow M$. For $\Phi,\Psi\in\mathcal{F}^s_{LM}(X\times X)$ we define Φ^{-1} by $\Phi^{-1}(a)=\Phi(a^{-1})$ with $a^{-1}(x,y)=a(y,x)$ for all $(x,y)\in X\times X$. Further, we define $\Phi\circ\Psi$ by $\Phi\circ\Psi(a)=\bigvee\{\Phi(b)\wedge\Psi(c)\ :\ b\circ c\leq a\}$ with $b\circ c(x,y)=\bigvee_{z\in X}b(x,z)\wedge c(z,y)$ for all $(x,y)\in X\times X$. Finally, we denote $[\Delta]=[\Delta_X]=\bigwedge_{x\in X}[(x,x)]\in\mathcal{F}^s_{LM}(X\times X)$.

Lemma 3.6. *Let* $\Phi, \Psi \in \mathcal{F}^s_{LM}(X \times X)$*. Then*

(1) $\Phi^{-1} \in \mathcal{F}^s_{LM}(X \times X)$*;*
(2) $\Phi \circ \Psi \in \mathcal{F}^s_{LM}(X \times X)$ *if and only if* $b \circ c = \perp^L_{X \times X}$ *implies* $\Phi(b) \wedge \Psi(c) = \perp^M$*.*

The proof in [20] can easily be adapted. Also almost all other properties of these constructions that are shown in [20] carry over to our more general situation. We will later explicitely use the following results.

Lemma 3.7. *Let* L, M *be frames and let* $s : L \longrightarrow M$ *be a stratification mapping. Then for all* $x, y, z \in X$

(i) $[(x, y)]^{-1} = [(y, x)]$*;*
(ii) $[(x, y)] \circ [(y, z)] \leq [(x, z)]$*;*

Proof. (1) is easy and left for the reader. We prove (2). Let $d \in L^{X \times X}$. Then $[(x, y)] \circ [(y, z)](d) = \bigvee_{a \circ b \leq d} s(a(x, y)) \wedge s(b(y, z)) = \bigvee_{a \circ b \leq d} s(a(x, y) \wedge b(y, z)) \leq \bigwedge_{a \circ b \leq d} s(a \circ b(x, z)) \leq s(d(x, z)) = [(x, z)](d)$. □

Let X be a set, $\mathcal{F} \in \mathcal{F}^s_{LM}(X), x \in X$. We define $\mathcal{F}_x : L^{X \times X} \longrightarrow M$ by

$$\mathcal{F}_x(d) = \mathcal{F}\big(d(\cdot, x)\big), \text{ for } d \in L^{X \times X},$$

see [6]. Then $\mathcal{F}_x \in \mathcal{F}^s_{LM}(X \times X)$.

Lemma 3.8. [6] *Let* $x, y \in X$ *and* $\mathcal{F}, \mathcal{G} \in \mathcal{F}^s_{LM}(X)$ *and let* $f : X \longrightarrow Y$*. Then*

(i) $[x]_y = [(x, y)]$,
(ii) $\mathcal{F}_x \wedge \mathcal{G}_x = (\mathcal{F} \wedge \mathcal{G})_x$,
(iii) $(f \times f)(\mathcal{F}_x) = f(\mathcal{F})_{f(x)}$.

Let $\Psi \in \mathcal{F}^s_{LM}(X \times X)$ and let $x \in X$. We define $\Psi(x) : L^X \longrightarrow M$ by $\Psi(x)(a) = \bigvee\{\Psi(d) : d(\cdot, x) \leq a\}$. Then $\Psi(x) \in \mathcal{F}^s_{LM}(X)$ if and only if $\Psi(d) = \perp^M$ whenever $d(\cdot, x) = \perp^L_X$, see [19]. We note that if $\Psi \leq [\Delta_X]$, then $d(\cdot, x) = \perp^L_X$ implies $\Psi(d) \leq \bigwedge_{y \in X} s(d(y, y)) \leq s(d(x, x)) = s(\perp^L) = \perp^M$. Hence, in this case, $\Psi(x) \in \mathcal{F}^s_{LM}(X)$.

Lemma 3.9. *Let* $\Phi, \Psi \in \mathcal{F}^s_{LM}(X \times X)$*,* $\mathcal{F} \in \mathcal{F}^s_{LM}(X)$*,* $x \in X$ *and let* $f : X \longrightarrow X'$*. The following hold.*

(1) *If* $\Phi \leq \Psi$*, then* $\Phi(x) \leq \Psi(x)$*,*
(2) $(\Phi \wedge \Psi)(x) \leq \Phi(x) \wedge \Psi(x)$*,*
(3) $[\Delta_X](x) = [x]$*,*
(4) $\mathcal{F} \geq \Phi(x) \iff \mathcal{F}_x \geq \Phi$*,*
(5) $\Phi \leq (\Phi(x))_x$*,*
(6) $(\mathcal{F}_x)(x) \leq \mathcal{F}$*,*
(7) $(f \times f)(\Phi)(f(x)) \leq f(\Phi(x))$*.*

Proof. (1), (2) and (3) are easy, see [19]. The properties (5) and (6) follow directly from (4). For (4), let first $d \in L^{X \times X}$. Then $\Phi(d) \leq \bigvee\{\Phi(e) : e(\cdot, x) \leq d(\cdot, x)\} = \Phi(x)(d(\cdot, x)) \leq \mathcal{F}(d(\cdot, x)) = \mathcal{F}_x(d)$. Conversely, let $a \in L^X$. Then $\Phi(x)(a) = \bigvee\{\Phi(d) : d(\cdot, x) \leq a\} \leq \{\mathcal{F}(d(\cdot, x)) : d(\cdot, x) \leq a\} \leq \mathcal{F}(a)$. For property (7) let $c \in L^{X' \times X'}$ and $b \in L^{X'}$. If $c(\cdot, f(x)) \leq b$, then for all $y \in X$ we have $(f \times$

$f)^{\leftarrow}(c)(y,x) = c(f(y), f(x)) \leq b(f(y)) = f^{\leftarrow}(b)(y)$, i.e. we have $(f\times f)^{\leftarrow}(c)(\cdot,x) \leq f^{\leftarrow}(b)$. Hence we obtain

$$\begin{aligned}(f\times f)(\Phi)(f(x))(b) &= \bigvee\{(f\times f)(\Phi)(c) \ : \ c(\cdot, f(x)) \leq b\}\\ &\leq \bigvee\{\Phi((f\times f)^{\leftarrow}(c)) \ : \ (f\times f)^{\leftarrow}(c)(\cdot,x) \leq f^{\leftarrow}(b)\}\\ &\leq \bigvee\{\Phi(d) \ : \ d(\cdot,x) \leq f^{\leftarrow}(b)\}\\ &= \Phi(x)(f^{\leftarrow}(b)) \ = \ f(\Phi(x))(b).\end{aligned}$$

□

In [18] we gave the following definition.

Definition 3.10. [18] Let L, M be frames, let N be a complete lattice and let $s : L \longrightarrow M$ be a stratification mapping. A pair $(X, \overline{q})$, of a set X and $\overline{q} = (q_\alpha : \mathcal{F}^s_{LM}(X) \longrightarrow \mathbb{P}(X))_{\alpha\in N}$, is an *$s$-stratified LMN-convergence tower space* if

(CT1) $x \in q_\alpha([x]_s)$ for all $x \in X$, $\alpha \in N$;
(CT2) $q_\alpha(\mathcal{F}) \subseteq q_\alpha(\mathcal{G})$ whenever $\mathcal{F} \leq \mathcal{G}$;
(CT3) $q_\beta(\mathcal{F}) \subseteq q_\alpha(\mathcal{F})$ whenever $\alpha \leq \beta$;
(CT4) $q_{\perp^N}(\mathcal{F}) = X$ for all $\mathcal{F} \in \mathcal{F}^s_{LM}(X)$.

A mapping $f : (X, \overline{q}) \longrightarrow (X', \overline{q'})$ between two s-stratified LMN-convergence tower spaces is called *continuous* if $f(q_\alpha(\mathcal{F})) \subseteq q'_\alpha(f(\mathcal{F}))$ for all $\alpha \in N$, $\mathcal{F} \in \mathcal{F}^s_{LM}(X)$. We denote the category with objects the s-stratified LMN-convergence tower spaces and morphisms the continuous mappings by $sLMN$-CTS.

A space $(X, \overline{q}) \in |sLMN\text{-}CTS|$ is called *left-continuous* if

(CTL) $\bigcap_{\beta\in A} q_\beta(\mathcal{F}) \subseteq q_{\bigvee A}(\mathcal{F})$ whenever $A \subseteq N$.

4. The Category $sLMN$-$UCTS$

Definition 4.1. Let L, M be frames, let $(N, *)$ be a quantale and let $s : L \longrightarrow M$ be a stratification mapping. A pair $(X, \overline{\Lambda})$, of a set X and $\overline{\Lambda} = (\Lambda_\alpha)_{\alpha\in N}$, $\Lambda_\alpha \subseteq \mathcal{F}^s_{LM}(X\times X)$, is an *$s$-stratified LMN-uniform convergence tower space* if
(UCT1) $[(x,x)] \in \Lambda_\alpha$ for all $x \in X$, $\alpha \in N$;
(UCT2) $\Psi \in \Lambda_\alpha$ whenever $\Phi \leq \Psi$ and $\Phi \in \Lambda_\alpha$;
(UCT3) $\Phi^{-1} \in \Lambda_\alpha$ whenever $\Phi \in \Lambda_\alpha$;
(UCT4) $\Phi \wedge \Psi \in \Lambda_\alpha$ whenever $\Phi, \Psi \in \Lambda_\alpha$;
(UCT5) $\Phi \in \Lambda_\beta$ whenever $\Phi \in \Lambda_\alpha$ and $\beta \leq \alpha$;
(UCT6) $\Lambda_{\perp^N} = \mathcal{F}^s_{LM}(X\times X)$;
(UCT7) $\Phi \circ \Psi \in \Lambda_{\alpha*\beta}$ whenever $\Phi \in \Lambda_\alpha$, $\Psi \in \Lambda_\beta$ and $\Phi\circ\Psi \in \mathcal{F}^s_{LM}(X\times X)$.

A mapping $f : (X, \overline{\Lambda}) \longrightarrow (X', \overline{\Lambda'})$ is called *uniformly continuous* if $(f\times f)(\Phi) \in \Lambda'_\alpha$ whenever $\Phi \in \Lambda_\alpha$. We denote the category with objects the s-stratified LMN-uniform convergence tower spaces and morphisms the uniformly continuous mappings by $sLMN$-$UCTS$.

A space $(X, \overline{\Lambda}) \in |sLMN\text{-}UCTS|$ is called *left-continuous* if
(UCTL) $\Phi \in \Lambda_{\bigvee A}$ whenever $A \subseteq N$ and $\Phi \in \Lambda_\alpha$ for all $\alpha \in A$.

Note: Left-continuous s-stratified LMN-uniform convergence tower spaces can be identified with s-stratified LMN-uniform convergence spaces. To this end,

we define for a left-continuous s-stratified LMN-uniform convergence tower space $(X, \overline{\Lambda})$ the mapping $\Lambda : \mathcal{F}^s_{LM}(X \times X) \longrightarrow N$ by $\Lambda(\Phi) = \bigvee\{\alpha \in N \ : \ \Phi \in \Lambda_\alpha\}$. We then have the following properties:
(UC1) $\Lambda([(x,x)]) = \top^N$ for all $x \in X$;
(UC2) $\Lambda(\Phi) \leq \Lambda(\Psi)$ whenever $\Phi \leq \Psi$;
(UC3) $\Lambda(\Phi) \leq \Lambda(\Phi^{-1})$;
(UC4) $\Lambda(\Phi) \wedge \Lambda(\Psi) \leq \Lambda(\Phi \wedge \Psi)$;
(UC5) $\Lambda(\Phi) * \Lambda(\Psi) \leq \Lambda(\Phi \circ \Psi)$ whenever $\Phi \circ \Psi \in \mathcal{F}^s_{LM}(X \times X)$.
The pair (X, Λ) is then called an *s-stratified LMN-uniform convergence space*, see also [6, 20].

Example 4.2. Let $L = M = N = \{0,1\}$. Then an s-stratified LMN-uniform convergence tower space is a uniform convergence space in the definition of Cook and Fischer [5], as improved by Wyler [27].

Example 4.3. Let $L = M = N$. Then a left-continuous id_L-stratified LMN-uniform convergence tower space is a stratified L-uniform convergence space in the definition of Jäger and Burton [20]. If we consider $N = (L, *)$ with a quantale operation, then a left-continuous id_L-stratified LLN-uniform convergence space is a stratified L-uniform convergence space in the definition of Craig and Jäger [6] with the enriched cl-premonoid $(L, \wedge, *)$.

Example 4.4. Let $L = M = \{0,1\}$ and $N = [0,1]$ with a left-continuous t-norm $*$. Then an s-stratified LMN-uniform convergence tower space is a probabilitistic uniform convergence space in the definition of Nusser [24].

Example 4.5. Let $L = M = \{0,1\}$ and $N = \Delta^+$ the lattice of distance distribution functions with a sup-continuous triangle function $*$. Then an s-stratified LMN-uniform convergence tower space is a probabilistic uniform convergence space in the definition of Ahsanullah and Jäger [2].

Example 4.6. Let $L = M = \{0,1\}$ and let $N = [0,\infty]$ with the opposite order and $* = +$ the extended addition. Then a left-continuous s-stratified LMN-uniform convergence tower space is an approach uniform convergence space in the definition of Lee and Windels [21]. If we use $* = \vee$ as the quantale operation, then we obtain an ultra-approach uniform convergence space [21].

Theorem 4.7 (Categorical properties I)**.** *The category sLMN-UCTS is well-fibred and topological over SET, i.e. it is a topological category in the sense of Preuss* [25].

Proof. The proof is routine and we only present initial structures. For a source $(f_i : X \longrightarrow (X_i, \overline{\Lambda^i}))_{i \in J}$ the *initial structure* $\overline{\Lambda}$ *on* X is given by

$$\Phi \in \Lambda_\alpha \iff (f_i \times f_i)(\Phi) \in \Lambda^i_\alpha \ \forall i \in J.$$

Furthermore, as a consequence of the stratification axiom (LFS), we have $[(x,x)]_s \leq \Phi$ for all s-stratified LM-filters $\Phi \in \mathcal{F}^s_{LM}(\{x\} \times \{x\})$ on a one-point set. Hence, by (UCT1) and (UCT2), there is only one structure $\overline{\Lambda}$ on $\{x\}$. Clearly the class of all structures on a set X is a set. □

As a consequence, we have *subspaces* and *product spaces.*

Theorem 4.8 (Categorical properties II). *Let $s : L \longrightarrow M$ and $t : M \longrightarrow L$ be stratification mappings such that $s \circ t \geq id_M$ and $t \circ s \geq id_L$ and let N be a complete Heyting algebra, i.e. $* = \wedge$. Then the category sLMN-UCTS is Cartesian closed.*

Proof. The proof is a bit tedious but can in principle be adapted from the proof in [20]. We note that because *sLMN-UCTS* is a well-fibred topolocial category, the Cartesian closedness is equivalent to the existence of function spaces, see [1]. So we simply state these function space structures. We denote $UC(X, X') = \{f : (X, \overline{\Lambda}) \longrightarrow (X', \overline{\Lambda'}) : f \text{ uniformly continuous}\}$, the evaluation mapping $ev : UC(X, X') \times X \longrightarrow X'$ defined by $ev(f, x) = f(x)$ and $\eta : (UC(X, X') \times UC(X, X')) \times (X \times X) \longrightarrow (UC(X, X') \times X) \times (UC(X, X') \times X)$, defined by $\eta((f, g), (x, x')) = ((f, x), (g(x'))$. We then define the s-stratified LMN-uniform convergence tower space $(UC(X, X'), \overline{\Lambda^{uc}})$ by putting for $\Phi \in \mathcal{F}^s_{LM}(UC(X, X') \times UC(X, X'))$

$$\Phi \in \Lambda^{uc}_\alpha \iff (ev \times ev)(\eta(\Phi \times \mathcal{F})) \in \Lambda'_\beta \text{ whenever } \beta \leq \alpha \text{ and } \mathcal{F} \in \Lambda_\beta.$$

□

5. The Category *sLMN-UTS*

Let L, M be frames, let $(N, *)$ be a quantale and let $s : L \longrightarrow M$ be a stratification mapping. A pair $(X, \overline{\mathcal{U}})$, of a set X and $\overline{\mathcal{U}} = (\mathcal{U}_\alpha)_{\alpha \in N}, \mathcal{U}_\alpha \in \mathcal{F}^s_{LM}(X \times X)$, is an *s-stratified LMN-uniform tower space* if for all $\alpha, \beta \in N$
(UT1) $\mathcal{U}_\alpha \leq [\Delta]$, where $[\Delta](d) = \bigwedge_{x \in X} s(d(x, x))$;
(UT2) $\mathcal{U}_\alpha \leq \mathcal{U}_\alpha^{-1}$;
(UT3) $\mathcal{U}_{\alpha * \beta} \leq \mathcal{U}_\alpha \circ \mathcal{U}_\beta$;
(UT4) $\mathcal{U}_\alpha \leq \mathcal{U}_\beta$ whenever $\alpha \leq \beta$;
(UT5) $\mathcal{U}_{\perp^N} = \bigwedge \mathcal{F}^s_{LM}(X \times X)$.

A mapping $f : (X, \overline{\mathcal{U}}) \longrightarrow (X', \overline{\mathcal{U}'})$ is called *uniformly continuous* if $\mathcal{U}'_\alpha \leq (f \times f)(\mathcal{U}_\alpha)$ for all $\alpha \in N$. The category with objects the s-stratified LMN-uniform tower spaces and uniformly continuous mappings as morphisms is denoted by *sLMN-UTS*.

A space $(X, \overline{\mathcal{U}}) \in |sLMN\text{-}UTS|$ is called *left-continuous* if
(UTLC) $\mathcal{U}_{\bigvee A} \leq \bigvee_{\alpha \in A} \mathcal{U}_\alpha$ whenever $A \subseteq N$.

We note that if N is a complete Heyting algebra, i.e. if $* = \wedge$, then the axiom (UT3) is equivalent to the requirement $\mathcal{U}_\alpha \leq \mathcal{U}_\alpha \circ \mathcal{U}_\alpha$ for all $\alpha \in N$.

Example 5.1. Let $L = M = N = \{0, 1\}$, then an s-stratified LMN-uniform tower space is a uniform space in the definition of Bourbaki [3].

Example 5.2. Let $L = M = \{0, 1\}$ and $N = [0, 1]$ with a left-continuous t-norm. Then an s-stratified LMN-uniform tower space is a probabilistic uniform space in the definition of Florescu [9]. If the t-norm is the minimum, then these spaces can be identified with the generalized uniform spaces of Burton et al. [4].

Example 5.3. Let $L = M = \{0, 1\}$ and let $N = [0, \infty]$ with the opposite order and with the (extended) addition as quantale operation. Then a left-continuous

s-stratified LMN-uniform tower space is an approach uniform space [23, 21]. If the quantale operation is the maximum, then the resulting approach uniform spaces are called *level-uniform* in [21].

Example 5.4. Let $L = M = \{0,1\}$ and let $N = \Delta^+$ be the set of distance distribution functions with a sup-continuous triangle function as quantale operation. Then an s-stratified LMN-uniform tower space is a probabilistic uniform space in the definition of Ahsanullah and Jäger [2].

Example 5.5. Let $L = M$ be frames and $N = \{0,1\}$. Then an id_L-stratified LMN-uniform tower space is a stratified L-uniform space in the definition of Gutiérrez García [11, 12].

Example 5.6 (L-continuity space)**.** A *value quantale* [8] is a completely distributive lattice $(L, \le)$ with a quantale operation $*$ such that $\bot \lhd \top$ and $\alpha \vee \beta \lhd \top$ whenever $\alpha, \beta \lhd \top$. Examples for value quantales are $([0,\infty], +)$ or $(\Delta^+, *)$ with a sup-continuous triangle function, see [8]. It should be noted that Flagg [8] uses the opposite order. For a value quantale L, an L*-continuity space* [8] is a pair (X, d) of a set X and a mapping $d : X \times X \longrightarrow L$ which is *reflexive*, i.e. $d(x,x) = \top$ for all $x \in X$, and *transitive*, i.e. $d(x,y) * d(y,z) \le d(x,z)$ for all $x, y, z \in X$. We additionally demand that d is *symmetric*, i.e. that $d(x,y) = d(y,x)$ for all $x, y \in X$.

In case $L = [0,\infty]$ with the opposite order and extended addition as quantale operation, a symmetric L-continuity space is a pseudometric space. If $L = \Delta^+$ and $*$ is a sup-continuous triangle function, a symmetric L-continuity space is a probabilistic pseudometric space, see [8].

For a symmetric L-continuity space (X, d) we define now, for $\epsilon \in L$, the set $U_\epsilon = \{(x,y) \in X \times X \,:\, d(x,y) \ge \epsilon\}$. It is then not difficult to show that $\Delta \subseteq U_\epsilon$, that $U_\epsilon \subseteq U_\delta$, whenever $\delta \le \epsilon$, that $U_\epsilon \cap U_\delta = U_{\epsilon\vee\delta}$, that $U_\epsilon = U_\epsilon^{-1}$ and that $U_\epsilon \circ U_\delta \subseteq U_{\epsilon*\delta}$. Hence, if we define $\mathcal{U}^d_\alpha = [\{U_\epsilon \;:\; \epsilon \le \alpha\}]$, i.e. $\mathcal{U}^d_\alpha$ is the filter on $X \times X$ generated by the sets U_ϵ with $\epsilon \le \alpha$, then $(X, \overline{\mathcal{U}^d})$ is an s_0-stratified $\{0,1\}\{0,1\}L$-uniform tower space.

We are now going to show that the category of s-stratified LMN-uniform tower spaces is isomorphic to a subcategory of the category of s-stratified LMN-uniform convergence tower spaces.

Let first $(X, \overline{\mathcal{U}}) \in |sLMN\text{-}UTS|$. If we define $\Phi \in \Lambda^{\overline{\mathcal{U}}}_\alpha$ if $\Phi \ge \mathcal{U}_\alpha$, then $(X, \overline{\Lambda^{\overline{\mathcal{U}}}}) \in |sLMN\text{-}PUCTS|$. It is also clear that uniformly continuous mappings between s-stratified LMN-uniform tower spaces are uniformly continuous as mappings between the corresponding s-stratified LMN-uniform convergence tower spaces.

Let $(X, \overline{\Lambda}) \in |sLMN\text{-}UCTS|$. We define, for $\alpha \in N$, the *s-stratified α-LM-entourage filter* by $\mathcal{U}^{\overline{\Lambda}}_\alpha = \bigwedge_{\Phi\in\Lambda_\alpha} \Phi$. We call $(X, \overline{\Lambda})$ a *principal s-stratified LMN-uniform convergence tower space* if the axiom

(PUCT) $\Phi \in \Lambda_\alpha$ if and only if $\Phi \ge \mathcal{U}^{\overline{\Lambda}}_\alpha$

is satisfied. The subcategory of $sLMN\text{-}UCTS$ consisting of the principal s-stratified LMN-uniform convergence tower space is denoted by $sLMN\text{-}PUCTS$.

Lemma 5.7. *Let* $(X,\overline{\Lambda}) \in |sLMN\text{-}PUCTS|$*. Then* $(X,\overline{\mathcal{U}^{\overline{\Lambda}}}) \in |sLMN\text{-}UTS|$*.*

Proof. (UT1) We have $[(x,x)] \in \Lambda_\alpha$ for all $x \in X$ and hence $\mathcal{U}^{\overline{\Lambda}}_\alpha \leq [(x,x)]$ for all $x \in X$. Consequently, by (PUCT), $\mathcal{U}^{\overline{\Lambda}}_\alpha \leq \bigwedge_{x\in X}[(x,x)] = [\Delta]$. (UT2) We have $\mathcal{U}^{\overline{\Lambda}}_\alpha \in \Lambda_\alpha$ and hence, by (UCT3), $(\mathcal{U}^{\overline{\Lambda}}_\alpha)^{-1}_\alpha \in \Lambda_\alpha$, from which, again using (PUCT), $\mathcal{U}^{\overline{\Lambda}}_\alpha \leq (\mathcal{U}^{\overline{\Lambda}}_\alpha)^{-1}_\alpha$ follows. For (UT3) we note that because of (UT1) $\mathcal{U}^{\overline{\Lambda}}_\alpha \circ \mathcal{U}^{\overline{\Lambda}}_\beta$ always exists. Hence, using (PUCT) and (UCT7), it follows that $\mathcal{U}^{\overline{\Lambda}}_\alpha \circ \mathcal{U}^{\overline{\Lambda}}_\beta \in \Lambda_{\alpha*\beta}$ and therefore $\mathcal{U}^{\overline{\Lambda}}_{\alpha*\beta} \leq \mathcal{U}^{\overline{\Lambda}}_\alpha \circ \mathcal{U}^{\overline{\Lambda}}_\beta$. Similarly, from $\mathcal{U}^{\overline{\Lambda}}_\beta \in \Lambda_\beta$ and $\alpha \leq \beta$, we conclude with (UCT3) that $\mathcal{U}^{\overline{\Lambda}}_\beta \in \Lambda_\alpha$ and again (PUCT) leads to $\mathcal{U}^{\overline{\Lambda}}_\alpha \leq \mathcal{U}^{\overline{\Lambda}}_\beta$, i.e. (UT4) is valid. (UT5) finally is a direct consequence of (UCT6). □

Lemma 5.8. *Let* $(X,\overline{\Lambda}),(X',\overline{\Lambda'}) \in |sLMN\text{-}PUCTS|$ *and let* $f : (X,\overline{\Lambda}) \longrightarrow (X',\overline{\Lambda'})$ *be uniformly continuous. Then* $f : (X,\overline{\mathcal{U}^{\overline{\Lambda}}}) \longrightarrow (X',\overline{\mathcal{U}^{\overline{\Lambda'}}})$ *is uniformly continuous.*

Proof. We have $(f \times f)(\mathcal{U}^{\Lambda}_\alpha) = (f \times f)(\bigwedge_{\Phi\in\Lambda_\alpha}\Phi) = \bigwedge_{\Phi\in\Lambda_\alpha}(f \times f)(\Phi) \geq \bigwedge_{(f\times f)(\Phi)\in\Lambda'_\alpha}(f \times f)(\Phi) \geq \mathcal{U}^{\Lambda'}_\alpha$. □

Theorem 5.9. *The categories sLMN-UTS and sLMN-PUCTS are isomorphic.*

Proof. We define two functors.

$$A : \begin{cases} sLMN\text{-}PUCTS & \longrightarrow & sLMN\text{-}UTS \\ (X,\overline{\Lambda}) & \longmapsto & (X,\overline{\mathcal{U}^{\overline{\Lambda}}}) \\ f & \longmapsto & f \end{cases}$$

and

$$B : \begin{cases} sLMN\text{-}UTS & \longrightarrow & sLMN\text{-}PUCTS \\ (X,\overline{\mathcal{U}}) & \longmapsto & (X,\overline{\Lambda^{\overline{\mathcal{U}}}}) \\ f & \longmapsto & f \end{cases}.$$

Because $\mathcal{U}^{(\Lambda^{\overline{\mathcal{U}}})}_\alpha = \bigwedge_{\Phi\in\Lambda^{\overline{\mathcal{U}}}_\alpha}\Phi = \bigwedge_{\Phi\geq\mathcal{U}_\alpha}\Phi = \mathcal{U}_\alpha$ and $\Phi \in \Lambda^{(\mathcal{U}^{\overline{\Lambda}})}_\alpha \iff \Phi \geq \mathcal{U}^{\overline{\Lambda}}_\alpha \iff \Phi \in \Lambda_\alpha$ (because $(X,\overline{\Lambda})$ is principal), these functors are isomorphism functors. □

Lemma 5.10. *Let* L, M, N *be frames and let* $s : L \longrightarrow M$ *be a stratification mapping. Then sLMN-PUCTS is a reflective subcategory of sLMN-UCTS.*

Proof. For $(X,\overline{\Lambda}) \in |sLMN\text{-}UCTS|$ and for $\alpha \in N$ we denote $\mathcal{U}^{\overline{\Lambda}}_\alpha = \bigwedge_{\Phi\in\Lambda_\alpha}\Phi$. Then $\mathcal{U}^{\overline{\Lambda}}_\alpha \leq [\Delta]$, $\mathcal{U}^{\overline{\Lambda}}_\alpha \leq (\mathcal{U}^{\overline{\Lambda}}_\alpha)^{-1}$ and $\alpha \leq \beta$ implies $\mathcal{U}^{\overline{\Lambda}}_\alpha \leq \mathcal{U}^{\overline{\Lambda}}_\beta$. We define $\mathcal{J}^{\overline{\Lambda}}_\alpha = \{\Phi \in \mathcal{F}^s_{LM}(X \times X) : \Phi \leq \mathcal{U}^{\overline{\Lambda}}_\alpha,\ \Phi \leq \Phi \circ \Phi\}$. Then $[X \times X] \in \mathcal{J}^{\overline{\Lambda}}_\alpha$, i.e. $\mathcal{J}^{\overline{\Lambda}}_\alpha$ is not empty. Furthermore $\mathcal{U}^*_\alpha = \bigwedge_{\Phi\in\mathcal{J}^{\overline{\Lambda}}_\alpha}\Phi \in \mathcal{F}^s_{LM}(X \times X)$. It is not difficult to show that $\mathcal{U}^*_\alpha \in \mathcal{J}^{\overline{\Lambda}}_\alpha$ and hence $\mathcal{U}^*_\alpha \leq \mathcal{U}^*_\alpha \circ \mathcal{U}^*_\alpha$. Furthermore, $\mathcal{U}^*_\alpha \leq \mathcal{U}^{\overline{\Lambda}}_\alpha \leq [\Delta]$ and also $\mathcal{U}^*_\alpha = (\mathcal{U}^*_\alpha)^{-1}$. We define now $\Phi \in \Lambda^*_\alpha$ if $\Phi \geq \mathcal{U}^*_\alpha$. Clearly then $(X,\overline{\Lambda^*}) \in |sLMN\text{-}PUCTS|$ and for $\Phi \in \Lambda_\alpha$ we have $\Phi \geq \mathcal{U}^{\overline{\Lambda}}_\alpha \geq \mathcal{U}^*_\alpha$, i.e. $\Phi \in \Lambda^*_\alpha$. Hence, the identity mapping $id_X : (X,\overline{\Lambda}) \longrightarrow (X,\overline{\Lambda^*})$ is uniformly continuous. If $f : (X,\overline{\Lambda}) \longrightarrow (X',\overline{\Lambda'})$ is uniformly continuous in $sLMN\text{-}PUCTS$, then $f : (X,\overline{\Lambda^*}) \longrightarrow (X',\overline{\Lambda'^*})$ is also uniformly continuous as a mapping in *sLMN-PUCTS*. In fact, let $\Phi \leq \mathcal{U}_\alpha^{\overline{\Lambda'}}$ such that $\Phi \leq \Phi \circ \Phi$. Then $\Phi \leq (f \times f)(\mathcal{U}^{\overline{\Lambda}} a_\alpha)$ and hence $(f \times f)^{-1}(\Phi)$ exists and $(f \times f)^{-1}(\Phi) \leq \mathcal{U}_\alpha^{\overline{\Lambda}}$. As $(f \times f)^{-1}(\Phi) \leq (f \times f)^{-1}(\Phi \circ \Phi) \leq (f \times f)^{-1}(\Phi) \circ (f \times f)^{-1}(\Phi)$ we see that $(f \times f)^{-1}(\Phi) \in \mathcal{J}^{\overline{\Lambda}} a_\alpha$ and therefore $(f \times f)^{-1}(\Phi) \leq \mathcal{U}_\alpha^*$. We conclude from this that $\Phi \leq (f \times f)(\mathcal{U}_\alpha^*)$ and because $\Phi \in \mathcal{J}_\alpha^{\overline{\Lambda'}}$ was arbitrary, we conclude $(f \times f)(\mathcal{U}_\alpha^*) \geq \mathcal{U}_\alpha^{'*}$. Hence we can define a functor

$$K : \begin{cases} sLMN\text{-}UCTS & \longrightarrow & sLMN\text{-}PUCTS \\ (X, \overline{\Lambda}) & \longmapsto & (X, \overline{\Lambda^*}) \\ f & \longmapsto & f \end{cases}.$$

If we denote the embedding functor $E : sLMN\text{-}PUCTS \longrightarrow sLMN\text{-}UCTS$, then for $(X, \overline{\Lambda}) \in |sLMN\text{-}PUCTS|$ we have $\Lambda_\alpha = \Lambda_\alpha^*$ for all $\alpha \in N$. This follows from the idempotency of $\wedge$ as in this case $\mathcal{U}_\alpha^{\overline{\Lambda}} \in \mathcal{J}_\alpha^{\overline{\Lambda}}$. Hence $K \circ E = id_{sLMN\text{-}PUCTS}$. We have seen above that $E \circ K \geq id_{sLMN\text{-}UCTS}$ and hence the claim follows. □

We can state the last result in the following form:

Theorem 5.11. *If $(N, \wedge)$ is a complete Heyting algebra, then sLMN-UTS is isomorphic to a reflective subcategory of sLMN-UCTS.*

6. The Underlying *sLMN*-convergence Tower Space

We define for $(X, \overline{\Lambda}) \in |sLMN\text{-}UCTS|$, $x \in q_\alpha^{\overline{\Lambda}}(\mathcal{F}) \iff \mathcal{F}_x \in \Lambda_\alpha$. The following result is straightforward.

Theorem 6.1. *Let $(X, \overline{\Lambda}), (X', \overline{\Lambda'}) \in |sLMN\text{-}UCTS|$. Then*

- (i) *$(X, \overline{q^{\overline{\Lambda}}})$ is an s-stratified LMN-convergence tower space;*
- (ii) *$(X, \overline{q^{\overline{\Lambda}}})$ is left-continuous whenever $(X, \overline{\Lambda})$ is left-continuous;*
- (iii) *$f : (X, \overline{q^{\overline{\Lambda}}}) \longrightarrow (X', \overline{q^{\overline{\Lambda'}}})$ is continuous whenever $f : (X, \overline{\Lambda}) \longrightarrow (X', \overline{\Lambda'})$ is uniformly continuous.*

Hence we have a forgetful functor

$$F : sLMN\text{-}UCTS \longrightarrow sLMN\text{-}CTS.$$

Theorem 6.2. *The functor F preserves initial constructions.*

Proof. Let $(f_i : X \longrightarrow (X_i, \overline{\Lambda^i}))_{i \in J}$ be a source and the initial space $(X, \overline{\Lambda})$, i.e. $\Phi \in \Lambda_\alpha$ if $(f_i \times f_i)(\Phi) \in \Lambda_\alpha$ for all $i \in J$. For the source $(f_i : X \longrightarrow (X_i, \overline{q^{\overline{\Lambda^i}}}))_{i \in J}$ and the initial space $(X, \overline{q})$ we then have $x \in q_\alpha(\mathcal{F})$ if and only if $f_i(x) \in q_\alpha^{\overline{\Lambda^i}}(f_i(\mathcal{F}))$ for all $i \in J$, if and only if $(f_i \times f_i)(\mathcal{F}_x) = (f_i(\mathcal{F}))_{f_i(x)} \in \Lambda_\alpha^i$ for all $i \in J$ if and only if $\mathcal{F}_x \in \Lambda_\alpha$. This is equivalent to $x \in q_\alpha^{\overline{\Lambda}}(\mathcal{F})$. □

The s-stratified LMN-convergence tower space $(X, \overline{q^{\overline{\Lambda}}})$ underlying an s-stratified LMN-uniform convergence tower space $(X, \overline{\Lambda})$ has some strong properties. We call $(X, \overline{q}) \in |sLMN\text{-}CTS|$ *symmetric* if, for all $x, y \in X$, $y \in q_\alpha([x])$ whenever $x \in q_\alpha([y])$. It is called, for a quantale operation $*$ on N, $*$-*transitive* if, for all $x, y, z \in X$, $x \in c_{\alpha * \beta}([z])$ whenever $x \in c_\alpha([y])$ and $y \in c_\beta([z])$.

Lemma 6.3. *Let L, M be frames, $s : L \longrightarrow M$ a stratification mapping and let $(N, *)$ be a quantale. Let $(X, \overline{\Lambda})$ be an s-stratified LMN-uniform convergence tower space.*

(i) $(X, \overline{q^{\overline{\Lambda}}})$ *satisfies the axiom* (CTLIM) $q_\alpha^{\overline{\Lambda}}(\mathcal{F}) \cap q_\alpha^{\overline{\Lambda}}(\mathcal{G}) \subseteq q_\alpha^{\overline{\Lambda}}(\mathcal{F} \wedge \mathcal{G})$*;*

(ii) $(X, \overline{q^{\overline{\Lambda}}})$ *is symmetric;*

(iii) $(X, \overline{q^{\overline{\Lambda}}})$ *is transitiv.*

Proof. Property (i) follows from (UCT3), property (ii) follows from (UCT3) and $[(x, y)]^{-1} = [(y, x)]$ and property (iii) follows from (UCT2), (UCT5) and $[(x, y)] \circ [(y, z)] \leq [(x, z)]$. □

Let now $(X, \overline{\mathcal{U}}) \in |sLMN\text{-}UTS|$. We define the *$\alpha$-$LM$-neighbourhood filter of* $x \in X$ by $\mathcal{U}_\alpha^x = \mathcal{U}_\alpha(x)$. Then, for $a \in L^X$ we have $\mathcal{U}_\alpha^x(a) = \bigvee\{\mathcal{U}_\alpha(d) \ : \ d(\cdot, x) \leq a\}$.

Proposition 6.4. *Let $(X, \overline{\mathcal{U}}) \in |sLMN\text{-}UTS|$ and for $\alpha \in N$ and $x \in X$ define $\mathcal{U}_\alpha^x = \mathcal{U}_\alpha(x)$. Then*

(LNT0) $\mathcal{U}_\alpha^x \in \mathcal{F}_{LM}^s(X)$*;*
(LNT1) $\mathcal{U}_\alpha^x \leq [x]$*;*
(LNT2) $\mathcal{U}_\alpha^x \leq \mathcal{U}_\beta^x$ *whenever* $\alpha \leq \beta$*;*
(LNT3) $\mathcal{U}_\alpha^x \leq \mathcal{U}_\alpha^x(\mathcal{U}_\alpha^{(\cdot)})$ *in case $(N, \wedge)$ is a complete Heyting algebra and if there is a stratification mapping $t : M \longrightarrow L$ with $s \circ t \circ s \geq s$.*

Proof. (LNT0) and (LNT2) are easy and not presented. For (LNT1) we remark that from $d(y, x) \leq a(y)$ for all $y \in X$ it follows that $s(d(x, x)) \leq s(a(x))$ and hence, with $\mathcal{U}_\alpha \leq [\Delta]$, we obtain $\mathcal{U}_\alpha^x(a) \leq \bigvee\{s(d(x, x)) \ : \ d(\cdot, x) \leq a\} \leq s(a(x)) = [x](a)$. For (LNT3) let $t(d(\cdot, x)) \leq \mathcal{U}_\alpha^{(\cdot)}(a)$. If $b, c \in L^{X \times X}$ such that $b \circ c \leq d$, then $\bigvee_{z \in X} b(\cdot, z) \wedge c(z, x) \leq d(\cdot, x)$ and in particular $b(\cdot, x) \wedge c(x, x) \leq d(\cdot, x)$. We conclude from this

$$\begin{aligned} \mathcal{U}_\alpha(d) &\leq \mathcal{U}_\alpha \circ \mathcal{U}_\alpha(d) \\ &\leq \bigvee_{b(\cdot,x) \wedge c(x,x) \leq d(\cdot,x)} \mathcal{U}_\alpha(b) \wedge \mathcal{U}_\alpha(c) \end{aligned}$$

Using $\mathcal{U}_\alpha(c) \leq [\Delta](c) \leq s(c(x, x))$, we obtain

$$\begin{aligned} \mathcal{U}_\alpha(d) &\leq \bigvee_{b(\cdot,x) \wedge c(x,x) \leq d(\cdot,x)} \mathcal{U}_\alpha(b) \wedge s(c(x, x)) \\ &\leq \bigvee_{b(\cdot,x) \wedge c(x,x) \leq d(\cdot,x)} \mathcal{U}_\alpha(b \wedge (c(x, x))_X) \\ &\leq \bigvee_{e(\cdot,x) \leq d(\cdot,x)} \mathcal{U}_\alpha(e) \\ &\leq \bigvee_{t(e(\cdot,x)) \leq t(d(\cdot,x))} \mathcal{U}_\alpha(e) \\ &\leq \bigvee_{t(e(\cdot,x)) \leq \mathcal{U}_\alpha^{(\cdot)}(a)} \mathcal{U}_\alpha(e) \\ &= \mathcal{U}_\alpha^x(t(\mathcal{U}_\alpha^{(\cdot)}(a))). \end{aligned}$$

□

Proposition 6.5. *$(X,\overline{\mathcal{U}}), (X',\overline{\mathcal{U}'}) \in |sLMN\text{-}UTS|$ and let $f : (X,\overline{\mathcal{U}}) \longrightarrow (X',\overline{\mathcal{U}'})$ be uniformly continuous. Then for all $\alpha \in N$ and all $x \in X$ we have $\mathcal{U}_\alpha^{'f(x)} \le f(\mathcal{U}_\alpha^x)$.*

Proof. We have $\mathcal{U}'_\alpha \le (f\times f)(\mathcal{U}_\alpha)$ and hence, by Lemma 3.9(7), we obtain $\mathcal{U}_\alpha^{'f(x)} = \mathcal{U}'_\alpha(f(x)) \le (f\times f)(\mathcal{U}_\alpha)(f(x)) \le f(\mathcal{U}_\alpha(x)) = f(\mathcal{U}_\alpha^x)$. □

We call a pair $(X, (\overline{\mathcal{U}^x})_{x\in X})$ with $\overline{\mathcal{U}^x} = (\mathcal{U}_\alpha^x)_{\alpha\in N}$ that satisfies the axioms (NT0), (NT1) and (NT2) an *s-stratified LMN-neighbourhood tower space.* If $* = \wedge$ and additionally (LNT3) is satisfied, then we call $(X, (\overline{\mathcal{U}^x})_{x\in X})$ *topological.* (For $* \neq \wedge$ we have to formulate (LNT3) differently, however, the paper [18] does not contain this general set-up.) For an s-stratified LMN-neighbourhood tower space $(X, (\overline{\mathcal{U}^x})_{x\in X})$ we define $x \in q_\alpha^{(\overline{\mathcal{U}^x})_{x\in X}}(\mathcal{F}) \iff \mathcal{F} \ge \mathcal{U}_\alpha^x$. If the s-stratified LMN-neighbourhood space is clear, we simply write $q_\alpha(\mathcal{F})$ for $q_\alpha^{(\overline{\mathcal{U}^x})_{x\in X}}(\mathcal{F})$. It is clear then that $(X, \overline{q})$ is an s-stratified LMN-pretopological convergence tower space and that, in case that $(X, (\overline{\mathcal{U}^x})_{x\in X})$ satisfies the axiom (LNT3) and $(N, \wedge)$ is a complete Heyting algebra, $(X, \overline{q})$ is then an s-stratified LMN-topological convergence tower space, i.e. that it satisfies the Kowalsky axiom, see [18]. If we define morphisms as in Proposition 6.5, then we can define the category $sLMN\text{-}NTS$ with objects the s-stratified LMN-neighbourhood tower spaces. It is not difficult to show that this category is isomoprhic to the category of pretopological $sLMN$-convergence tower spaces.

Remark 6.6. We now have two ways for defining an s-stratified LMN-convergence tower space for an s-stratified LMN-uniform tower space $(X,\overline{\mathcal{U}})$. We can first go to the induced s-stratified LMN-uniform convergence tower space $(X, \overline{\Lambda^{\overline{\mathcal{U}}}})$ and then consider the underlying s-stratified LMN-convergence tower space $(X, \overline{q^{\overline{\Lambda^{\overline{\mathcal{U}}}}}})$. Alternatively, we can go to the induced s-stratfied LMN-neighbourhood tower space $(X, (\overline{\mathcal{U}^x})_{x\in X})$ and then consider the generated s-stratified LMN-pretopological convergence tower space $(X, \overline{q^{(\overline{\mathcal{U}^x})_{x\in X}}})$. We have $x \in q_\alpha^{\Lambda^{\overline{\mathcal{U}}}}(\mathcal{F})$ if and only if $\mathcal{F}_x \ge \mathcal{U}_\alpha$. Using Lemma 3.9(4) this is equivalent to $\mathcal{F} \ge \mathcal{U}_\alpha^x$, i.e. to $x \in q^{(\overline{\mathcal{U}^x})_{x\in X}}(\mathcal{F})$. So we obtain in both ways the same s-stratified LMN-convergence tower space.

Example 6.7. Let $L = M = N$ be completely distributive and consider the stratification mapping $s = id_L$. We consider an s-stratified LMN-uniform tower space $(X,\overline{\mathcal{U}})$ that satisfies additionally the axioms

(LUT0) $\mathcal{U}_\top = [\Delta]$;

(LUTRC) $\bigwedge_{j\in J} \mathcal{U}_{\alpha_j} \le \mathcal{U}_{\bigwedge_{j\in J}\alpha_j}$.

We call the condition (LUTRC) the right-continuity condition. We consider further the opposite order on N, i.e. we consider $N = L^{op}$. Then the underlying s-stratified LMN-neighbourhood tower space satisfies the axioms

(N1) $\mathcal{U}_\alpha^x \in \mathcal{F}_{LL}^s(X)$;

(N2) $\alpha \le^{op} \beta$ implies $\mathcal{U}_\beta^x \le \mathcal{U}_\alpha^x$;

(N3) $\mathcal{U}_\alpha^x \le [x]$;

(N4) $\mathcal{U}_\alpha^x \le \mathcal{U}_\alpha^x(\mathcal{U}_\alpha^{(\cdot)})$;

(N5) $\mathcal{U}_{\perp^{op}}^x = [x]$;

(N6) $a(x) \le \mathcal{U}^x_{\alpha_i}(a)$ for all $x \in X$ and all $i \in J$ implies $a(x) \le \mathcal{U}^x_{\bigvee^{op}_{i\in J}\alpha_i}(a)$ for all $x \in X$.

We only need to prove (N5) and (N6).

For (N5), we have because $\mathcal{U}_\top = [\Delta]$ that $\mathcal{U}_{\perp^{op}}(a) = \bigvee\{\bigwedge_{y\in X} s(d(y,y)) \,:\, d(\cdot,x) \le a\}$. We define $d_x \in L^{X\times X}$ by $d_x(u,v) = a(x)$ if $u = v$ and $d_x(u,v) = \perp$ if $u \ne v$. Then $d(u,x) = a(x)$ if $u = x$ and $d(u,x) = \perp \le a(u)$ if $u \ne x$, i.e. we have $d(\cdot,x) \le a$ and hence $\mathcal{U}^x_{\perp^{op}}(a) \ge a(x) = [x](a)$.

For (N6) we use the complete distributivity of L. If $a(x) \le \mathcal{U}^x_{\alpha_i}(a)$ for all $i \in J$ and all $x \in X$, then let $\gamma \lhd a(x)$. Then for all $i \in J$ there is $d_i \in L^{X\times X}$ such that $d_i(\cdot,x) \le a$ and $\mathcal{U}_{\alpha_i}(d_i) \ge \gamma$. We define $d = \bigvee_{i\in J} d_i$. Then $d(\cdot,x) \le a$ and $\mathcal{U}_{\alpha_i}(d) \ge \gamma$ for all $i \in J$ and hence also $\bigwedge_{i\in J}\mathcal{U}_{\alpha_i}(d) \ge \gamma$. Using (LUTRC) then $\mathcal{U}_{\bigvee^{op}_{i\in J}\alpha_i}(d) = \mathcal{U}_{\bigwedge_{i\in J}\alpha_i}(d) \ge \gamma$. Hence $\gamma \le \bigvee\{\mathcal{U}_{\bigvee^{op}_{i\in J}\alpha_i}(d) \,:\, d(\cdot,x) \le a\} = \mathcal{U}^x_{\bigvee^{op}_{i\in J}\alpha_i}(a)$. As L is completely distributive, we obtain $a(x) \le \mathcal{U}^x_{\bigvee^{op}_{i\in J}\alpha_i}(a)$.

Defining $\mathcal{N}(x,a,\alpha) = \mathcal{U}^x_\alpha(a)$, this shows that for an id_L-stratified LLL-uniform tower space that satisfies (LUT0) and (LUTRC) the underlying id_L-stratified LLL^{op}-neighbourhood space is an enriched L-fuzzy topological space in the definition of Höhle and Šostak, see Definition 8.1.8 and Proposition 8.1.9 together with Theorem 8.1.2 in [14]. In this sense are id_L-stratified LLL-uniform tower spaces that satisfy the additional axioms (LUT0) and (LUTRC) natural candidates for uniform structures that belong to stratified L-fuzzy topological spaces.

7. **Conclusions**

We introduced in this paper s-stratified LMN-uniform convergence spaces. The category of these spaces is topological and Cartesian closed. Furthermore, for special choices of the lattices L, M and the quantale N, many existing concepts of uniform convergence spaces, like lattice-valued, probabilistic and approach uniform convergence spaces, are covered as examples. This shows that a theory of such spaces is rich in examples and has good properties. Rather than developping theories for the particular instances it seems desirable to develop the theory for these very general spaces and obtain the corresponding theories for the examples as subcases. So there are two routes that can be followed. Firstly, one can develop e.g. a theory of Cauchy filters and completions for these spaces or, secondly, one can look at suitable generalizations, like e.g. Cauchy tower spaces or semi-uniform convergence tower spaces. A class of spaces that is at present not contained in our general framework are suitable tower spaces that contain e.g. the stratified L-ordered semi-uniform convergence spaces of Fang [7] as examples.

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Gunther Jäger, School of Mechanical Engineering, University of Applied Sciences Stralsund, 18435 Stralsund, Germany

E-mail address: `gunther.jaeger@fh-stralsund.de`