Iranian Journal of Fuzzy Systems **Vol. 14, No. 3,** (2017) pp. 139-149

M-FUZZIFYING MATROIDS INDUCED BY M-FUZZIFYING CLOSURE OPERATORS

X. XIN AND S. J. YANG

Abstract. In this paper, the notion of closure operators of matroids is generalized to fuzzy setting which is called M-fuzzifying closure operators, and some properties of M-fuzzifying closure operators are discussed. The M-fuzzifying matroid induced by an M-fuzzifying closure operator can induce an M-fuzzifying closure operator. Finally, the characterizations of M-fuzzifying acyclic matroids are given.

1. Introduction

Matroids were first introduced by Whitney in 1935s. It can be characterized by independent sets, circuits, rank functions and so on.

The closure operators play an important role in matroids theory. The definition of closed set, open set, hyperplane and the relation between finite geometric lattices and matroids are all induced by closure operators.

Matroids were generalized to fuzzy setting. Shi first introduced the notion of M-fuzzifying matroids in [4] and (L, M)-fuzzy matroids in [5], where L, M are completely distributive lattices. The M-fuzzifying rank functions of M-fuzzifying matroids are also defined in [4]. Since then, there are some references to study the M-fuzzifying matroids [2, 4, 7, 9, 10, 11, 12, 13, 14, 15, 16, 17].

Wang and Shi [7, 10] gave two different characterizations of M-fuzzifying matroids by two M-fuzzifying closure operators. However, any of those two definitions is not a normal way to generalize the notion of closure operators since those two M-fuzzifying closure operators are not defined on 2^E. So the aim of this paper is to generalize the notion of closure operators of matroids to M-fuzzy setting by a normal way, and to present a characterization of M-fuzzifying matroids induced by M-fuzzifying closure operators.

2. Preliminaries

Let E be a non-empty finite set and $\mathcal{I} \subseteq 2^E$. $(E, \mathcal{I})$ is called a matroid if $\mathcal{I}$ satisfies

(1) $\emptyset \in \mathcal{I}$;
(2) for arbitrary $A, B \in 2^E$, $A \subseteq B$ and $B \in \mathcal{I}$ implies $A \in \mathcal{I}$;

Received: December 2015; Revised: August 2016; Accepted: November 2016

Key words and phrases: M-fuzzifying matroids, M-fuzzifying closure operators, M-fuzzifying exchange law.

(3) for arbitrary $A, B \in \mathcal{I}$, $|A| < |B|$ implies there exists $x \in B - A$ such that $A \cup \{x\} \in \mathcal{I}$.

Let $(E, \mathcal{I})$ be a matroid and $A \in 2^E$. $\mathcal{R}_{\mathcal{I}}(A) = \max\{|B| : B \subseteq A \text{ and } B \in \mathcal{I}\}$ is called the rank of A.

Let $(E, \mathcal{I})$ be a matroid where E is a non-empty finite set, and $\mathcal{R}_{\mathcal{I}}$ be the rank function of $(E, \mathcal{I})$. We define a mapping $cl_{\mathcal{I}}$: $2^E \to 2^E$ such that $cl_{\mathcal{I}}(A) = \{e \in E : \mathcal{R}_{\mathcal{I}}(A) = \mathcal{R}_{\mathcal{I}}(A \cup \{e\})\}$ for arbitrary $A \in 2^E$. It simply possible to check that $cl_{\mathcal{I}}$ is a closure operator which satisfies the following conditions.

(CL1) *$A \subseteq cl_{\mathcal{I}}(A)$ for arbitrary $A \in 2^E$.*

(CL2) *$A \subseteq B$ implies $cl_{\mathcal{I}}(A) \subseteq cl_{\mathcal{I}}(B)$ for arbitrary $A, B \in 2^E$.*

(CL3) *$cl_{\mathcal{I}}(cl_{\mathcal{I}}(A)) = cl_{\mathcal{I}}(A)$ for arbitrary $A \in 2^E$.*

(CL4) *For arbitrary $x, y \in E$, $A \in 2^E$, $x \in cl_{\mathcal{I}}(A \cup \{y\}) - cl_{\mathcal{I}}(A)$ implies $y \in cl_{\mathcal{I}}(A \cup \{x\})$.*

In particular, $e \in cl_{\mathcal{I}}(A)$ if and only if $e \in A$ or $B \cup \{e\} \notin \mathcal{I}$ for arbitrary $B \in \mathcal{B}(\mathcal{I}|A)$, where $(A, \mathcal{I}|A)$ is the matroid that $(E, \mathcal{I})$ restrict to A and $\mathcal{B}(\mathcal{I}|A)$ is the set of all bases of $\mathcal{I}|A$. In addition, $e \in cl_{\mathcal{I}}(A) - A$ is equivalent to that there exists a $B \in \mathcal{B}(\mathcal{I}|A)$ such that $B \cup \{e\} \notin \mathcal{I}$.

Conversely, if cl: $2^E \to 2^E$ is a closure operator, then there exists a matroid $(E, \mathcal{I}_{cl})$ such that $cl_{\mathcal{I}_{cl}} = cl$. In particular, $\mathcal{I}_{cl} = \{I \in 2^E : \forall x \in I, x \notin cl(I - x)\}$.

For more properties of matroids and corresponding closure operators, the readers can refer to [3].

A matroid $(E, \mathcal{I})$ is called a acyclic matroid if $\{e\} \in \mathcal{I}$ for arbitrary $e \in E$.

Throughout this paper, $(M, \vee, \wedge, \top, \bot)$ is a completely distributive lattice with the largest element $\top$ and the smallest element $\bot$. "$'$: $M \to M$" is an order-reversing involution mapping in M. The set of non-zero co-prime elements of M is denoted by $J(M)$ and the set of non-unit prime elements of M is denoted by $P(M)$.

The relation $\prec$ in L is defined as follows: for $a, b \in M$, $a \prec b$ if and only if for every $D \subseteq M$, $b \leq \bigvee D$ implies the existence of $d \in D$ with $a \leq d$. $\prec^{op}$ is denoted the dual relation of $\prec$. That is, for $a, b \in M$, $a \prec^{op} b$ if and only if for every $D \subseteq M$, $\bigwedge D \leq b$ implies the existence of $d \in D$ with $d \leq a$. We denote $\beta(a) = \{b \in M : b \prec a\}$ and $\alpha(a) = \{b \in M : b \prec^{op} a\}$ for $a \in M$. Let $\beta^*(a) = \beta(a) \cap J(M)$ and $\alpha^*(a) = \alpha(a) \cap P(M)$. Thus, a complete lattice M is completely distributive if and only if $a = \bigvee \beta(a) = \bigvee \beta^*(a) = \bigwedge \alpha(a) = \bigwedge \alpha^*(a)$ for arbitrary $a \in M$ [8]. M is a completely distributive lattice implies both β and β^* are union-preserving mappings [8].

Throughout this paper, let E be a non-empty finite set and M^E be the set of all M-fuzzy sets of E. We often do not distinguish a crisp subset A of E from its characteristic function χ_A. Let M be a completely distributive lattice and $A \in M^E$. For arbitrary $a \in M$, we can define

$$A_{[a]} = \{x \in E : A(x) \geq a\},\ A^{(a)} = \{x \in E : A(x) \not\leq a\}.$$

Lemma 2.1. [1] *Let E be a non-empty finite set and $A \in M^E$, we have:*

(1) $A = \bigvee_{a \in M}(a \wedge A_{[a]}) = \bigvee_{a \in J(M)}(a \wedge A_{[a]})$.

(2) $A = \bigwedge_{a\in M}(a \vee A^{(a)}) = \bigwedge_{a\in P(M)}(a \vee A^{(a)})$.
(3) $A_{[a]} = \bigcap_{b\in\beta(a)} A_{[b]} = \bigcap_{b\in\beta^*(a)} A_{[b]}$.
(4) $A^{(a)} = \bigcup_{b\in\alpha(a)} A^{(b)} = \bigcup_{b\in\alpha^*(a)} A^{(b)}$.

The notion of M-fuzzifying matroids was first given in [4]. It was revised as follows.

Definition 2.2. [5] Let $\mathcal{I}$: $2^E \to M$ be a mapping. If $\mathcal{I}$ satisfies the following conditions:
(MFI1) $\mathcal{I}(\emptyset) = \top$;
(MFI2) for arbitrary $A, B \in 2^E, A \supseteq B$ implies $\mathcal{I}(A) \leq \mathcal{I}(B)$;
(MFI3) for arbitrary $A, B \in 2^E$ and $|A| < |B|$, $\bigvee_{e\in B-A} \mathcal{I}(A\cup\{e\}) \geq \mathcal{I}(A)\wedge\mathcal{I}(B)$;
then the pair $(E, \mathcal{I})$ is called an M-fuzzifying matroid.

Lemma 2.3. [4, 5] *Let $\mathcal{I}$: $2^E \to M$ be a mapping. Then the following conditions are equivalent.*
(1) *$(E, \mathcal{I})$ is an M-fuzzifying matroid.*
(2) *For arbitrary $a \in J(M)$, $(E, \mathcal{I}_{[a]})$ is a crisp matroid.*
(3) *For arbitrary $a \in P(M)$, $(E, \mathcal{I}^{(a)})$ is a crisp matroid.*

Let $\mathbb{N}$ be the set of all natural numbers. A fuzzy natural number is an antitone map $\lambda : \mathbb{N} \to M$ satisfying $\lambda(0) = \top$ and $\bigwedge_{n\in\mathbb{N}} \lambda(n) = \perp$. The set of all fuzzy natural numbers is denoted by $\mathbb{N}(M)$. For arbitrary $m \in \mathbb{N}$, define $\underline{m} \in \mathbb{N}(M)$ such that $\underline{m}(t) = \top$ when $t \leq m$ and $\underline{m}(t) = \perp$ when $t \geq m + 1$[4]. In the sequel, we shall not distinguish m from $\underline{m}$. For more properties of fuzzy natural numbers, the readers can refer to [4].

Definition 2.4. [4, 5] Let $(E, \mathcal{I})$ be an M-fuzzifying matroid. The mapping $\mathcal{R}_\mathcal{I}$: $2^E \to \mathbb{N}(M)$ defined by $\mathcal{R}_\mathcal{I}(A)(n) = \bigvee\{\mathcal{I}(B) : B \subseteq A, |B| \geq n\}$ is called the M-fuzzifying rank function of $(E, \mathcal{I})$ and $\mathcal{R}_\mathcal{I}(A)$ is called the M-fuzzifying rank of A.

Let $(E, \mathcal{I})$ be an M-fuzzifying matroid and $\mathcal{R}_\mathcal{I}$ be the M-fuzzifying rank function. For arbitrary $a \in J(M)$ and $b \in P(M)$, let $\mathcal{R}_{[a]}$ and $\mathcal{R}^{(b)}$ be the rank functions of $(E, \mathcal{I}_{[a]})$ and $(E, \mathcal{I}^{(b)})$, respectively. Then $\mathcal{R}_{[a]}(A) = (\mathcal{R}_\mathcal{I}(A))_{[a]}$ [7] and $\mathcal{R}^{(b)}(A) = (\mathcal{R}_\mathcal{I}(A))^{(b)}$ [4].

3. Definition and Properties of M-fuzzifying Closure Operators

In this section ,we will generalize the notion of closure operators of matroids to fuzzy setting as follows.

Definition 3.1. A mapping cl: $2^E \to M^E$ is called an M-fuzzifying closure operator on E provided that it satisfies the following conditions:
(MFCL1) $\chi_A \leq cl(A)$ for arbitrary $A \in 2^E$.
(MFCL2) $A \subseteq B$ implies $cl(A) \leq cl(B)$.
(MFCL3) $cl(A)(x) = \bigwedge_{x\notin B, A\subseteq B} \bigvee_{y\notin B} cl(B)(y)$.

(MFCL4) $cl(A \cup \{x\})(y) \vee cl(A)(x) \geq cl(A \cup \{y\})(x)$. for arbitrary $A \in 2^E$ and $x, y \in E$.

Remark 3.2. It is proved in [6] that (MFCL3) is equivalent to $(cl((cl(A))_{[a]}))_{[a]} = (cl(A))_{[a]}$ for arbitrary $a \in M \setminus \{\perp\}$ if cl: $2^E \to M^E$ is a mapping satisfying (MFCL1) and (MFCL2). Also, this condition is equivalent to $(cl((cl(A))_{[a]}))_{[a]} = (cl(A))_{[a]}$ for arbitrary $a \in J(M)$.

Lemma 3.3. *Let cl: $2^E \to M^E$ be a mapping on E satisfying* (MFCL1) *and* (MFCL2). *Then* (MFCL3) *is equivalent to the following condition.*

(MFCL3*) $(cl((cl(A))^{(a)}))^{(a)} = (cl(A))^{(a)}$ *for arbitrary $a \in P(M)$.*

Proof. Suppose that cl satisfies (MFCL3). Let $A \in 2^E$, $a \in P(M)$ with $x \notin (cl(A))^{(a)}$. Thus $cl(A)(x) \leq a$. By (MFCL3), there exists $B \in 2^E$ such that $x \notin B$, $A \subseteq B$ and $\bigvee_{y \notin B} cl(B)(y) \leq a$ since E is a finite set. This shows $(cl(A))^{(a)} \subseteq (cl(B))^{(a)} = B$. Hence we get $cl((cl(A))^{(a)})(x) \leq cl(B)(x) \leq a$, which implies $(cl((cl(A))^{(a)}))^{(a)} \subseteq (cl(A))^{(a)}$. Then we obtain that $(cl((cl(A))^{(a)}))^{(a)} = (cl(A))^{(a)}$ by $(cl((cl(A))^{(a)}))^{(a)} \supseteq (cl(A))^{(a)}$.

Conversely, if cl satisfies (MFCL3*), then $cl(A)(x) \leq \bigwedge_{x \notin B, A \subseteq B} \bigvee_{y \notin B} cl(B)(y)$ is trivial. Suppose that $cl(A)(x) \leq b$ for $b \in M$. Let $a \in \alpha^*(b)$. Then we have $x \notin (cl(A))^{(a)} \supseteq A$. By (MFCL3*), we know that $cl((cl(A))^{(a)})(y) \leq a$ for arbitrary $y \notin (cl(A))^{(a)}$. This follows that $\bigvee_{y \notin (cl(A))^{(a)}} cl((cl(A))^{(a)})(y) \leq a$. Hence we obtain that $\bigwedge_{x \notin B, A \subseteq B} \bigvee_{y \notin B} cl(B)(y) \leq a$. Therefore we get $\bigwedge_{x \notin B, A \subseteq B} \bigvee_{y \notin B} cl(B)(y) \leq b$. It shows that $cl(A)(x) = \bigwedge_{x \notin B, A \subseteq B} \bigvee_{y \notin B} cl(B)(y)$. □

Lemma 3.4. *Let cl: $2^E \to M^E$ be a mapping. Define mappings $cl_{[a]}$: $2^E \to 2^E$ and $cl^{(a)}$: $2^E \to 2^E$ such that $cl_{[a]}(A) = (cl(A))_{[a]}$ and $cl^{(a)}(A) = (cl(A))^{(a)}$, respectively. Then the following conditions are equivalent.*

(1) *cl is an M-fuzzifying closure operator on E.*

(2) *$cl_{[a]}$ is a crisp closure operator on E for arbitrary $a \in J(M)$.*

(3) *$cl^{(a)}$ is a crisp closure operators on E for arbitrary $a \in P(M)$.*

Proof. ((1) ⇒ (2)) Suppose that cl is an M-fuzzifying closure operator and $a \in J(M)$. In order to prove $cl_{[a]}$ is a crisp closure operator on E, it suffices to prove (CL4) since (CL1)-(CL3) is trivial. Let $x, y \in E$ and $A \in 2^E$ such that $x \in cl_{[a]}(A \cup \{y\}) - cl_{[a]}(A)$. Then $cl(A \cup \{y\})(x) \geq a$ and $cl(A)(x) \ngeq a$, which implies $cl(A \cup \{x\})(y) \geq a$ by (MFCL4). Therefore $y \in cl_{[a]}(A \cup \{x\})$.

((2) ⇒ (1)) Suppose that $cl_{[a]}$ a crisp closure operator on E for arbitrary $a \in J(M)$. In order to prove cl is an M-fuzzifying closure operator on E, it suffices to prove (MFCL4) since (MFCL1)-(MFCL3) is trivial. Let $A \in 2^E$, $x, y \in E$ and denote $b = cl(A \cup \{y\})(x)$. For arbitrary $a \in \beta^*(b)$, we get $x \in cl_{[a]}(A \cup \{y\})$. If $x \in cl_{[a]}(A)$, then $cl(A \cup \{x\})(y) \vee cl(A)(x) \geq a$. Otherwise, we have $y \in cl_{[a]}(A \cup \{x\})$ by (CL4), which implies $cl(A \cup \{x\})(y) \geq a$ and $cl(A \cup \{x\})(y) \vee cl(A)(x) \geq a$. So $cl(A \cup \{x\})(y) \vee cl(A)(x) \geq \bigvee_{a \in \beta^*(b)} a = b = cl(A \cup \{y\})(x)$. This shows cl is an M-fuzzifying closure operator.

The proof of (1) ⇔ (3) is similar to the proof of (1) ⇔ (2). □

Lemma 3.5. *Let $cl\colon 2^E \to M^E$ be an M-fuzzifying closure operator on E . Then $cl(A \cup \{x\}) = cl(A)$ if and only if $cl(A)(x) = \top$.*

Proof. ($\Rightarrow$) If $cl(A \cup \{x\}) = cl(A)$, then it is obvious that $cl(A)(x) = \top$.

($\Leftarrow$) If $cl(A)(x) = \top$, then $A \cup \{x\} \subseteq (cl(A))_{[a]}$ for arbitrary $a \in M \setminus \{\bot\}$. This implies $(cl(A \cup \{x\}))_{[a]} \subseteq (cl(A))_{[a]}$. Thus we have $(cl(A \cup \{x\}))_{[a]} = (cl(A))_{[a]}$ since $(cl(A \cup \{x\}))_{[a]} \supseteq (cl(A))_{[a]}$. This follows $cl(A \cup \{x\}) = cl(A)$ by Lemma 2.1. □

Theorem 3.6. *Let $\{cl(a) : a \in J(M)\}$ be a family of crisp closure operators on E satisfying $cl(b)(A) = \bigcap_{a \in \beta^*(b)} cl(a)(A)$ for all $b \in J(M)$ and arbitrary $A \in 2^E$. Then there exists an M-fuzzifying closure operator cl such that $cl_{[b]} = cl(b)$ for arbitrary $b \in J(M)$. In addition, $cl(A) = \bigvee_{a \in J(M)}(a \wedge cl(a)(A))$.*

Proof. Let $A \in 2^E$ and $b \in J(M)$. Since $cl(b)(A) = \bigcap_{a \in \beta^*(b)} cl(a)(A)$ and β^* is a union-preserving mapping, we have $cl(b)(A) \subseteq cl(c)(A)$ for arbitrary $c \in J(M)$ with $c \leq b$. Denote $cl(A) = \bigvee_{a \in J(M)}(a \wedge cl(a)(A))$.

If $x \in cl(b)(A)$, then by the notion of cl, we get $cl(A)(x) \geq b$, which implies $x \in cl_{[b]}(A)$.

Conversely, If $x \in cl_{[b]}(A)$, then $cl(A)(x) \geq b$. Hence $\beta^*(cl(A)(x)) \supseteq \beta^*(b)$. So for arbitrary $a \in \beta^*(b)$, we have $a \in \beta^*(cl(A)(x))$, which means there exist some $c \in J(M)$ such that $c \wedge cl(c)(A)(x) \geq a$. This follows $c \geq a$ and $x \in cl(c)(A) \subseteq cl(a)(A)$. Since $cl(b)(A) = \bigcap_{a \in \beta^*(b)} cl(a)(A)$, we get $x \in cl(b)(A)$.

By Lemma 3.4, cl is an M-fuzzifying closure operator. □

Theorem 3.7. *Let $\{cl(a) : a \in P(M)\}$ be a family of crisp closure operators on E satisfying $cl(b)(A) = \bigcup_{a \in \alpha^*(b)} cl(a)(A)$ for all $b \in P(M)$ and arbitrary $A \in 2^E$. Then there exists an M-fuzzifying closure operator cl such that $cl^{(b)} = cl(b)$ for arbitrary $b \in P(M)$. In addition, $cl(A) = \bigwedge_{a \in P(M)}(a \vee cl(a)(A))$.*

Proof. This proof is similar to the proof of Theorem 3.6. □

4. M-fuzzifying Matroids Induced by M-fuzzifying Closure Operators

In this section, we will consider the relation between M-fuzzifying matroids and M-fuzzifying closure operators.

Lemma 4.1. *Let $cl\colon 2^E \to M^E$ be an M-fuzzifying closure operator on E. For arbitrary $A \in 2^E$, define a mapping $\mathcal{I}_{cl}\colon 2^E \to M$ as follows,*

$$\mathcal{I}_{cl}(A) = \bigwedge_{x \in A} (cl(A - \{x\})(x))'.$$

Then the following conditions hold.

(1) *$(\mathcal{I}_{cl})^{(a')} = \mathcal{I}_{cl_{[a]}}$ for arbitrary $a \in J(M)$.*
(2) *$(\mathcal{I}_{cl})_{[a']} = \mathcal{I}_{cl^{(a)}}$ for arbitrary $a \in P(M)$.*
(3) *$(E, \mathcal{I}_{cl})$ is an M-fuzzifying matroid, which is called an M-fuzzifying matroid induced by an M-fuzzifying closure operator.*
(4) *$\mathcal{I}_{cl}(A \cup \{e\}) \geq \mathcal{I}_{cl}(A) \wedge (cl(A)(e))'$ for arbitrary $e \in E$ and $A \in 2^E$.*

(5) (MFI4) *For arbitrary $a, b \in P(M)$, $a \leq b$ implies $cl_{(\mathcal{I}_{cl})^{(a)}}(A) \subseteq cl_{(\mathcal{I}_{cl})^{(b)}}(A)$ for arbitrary $A \in 2^E$, where $cl_{(\mathcal{I}_{cl})^{(a)}}$ and $cl_{(\mathcal{I}_{cl})^{(b)}}$ are the closure operators of crisp matroids $(E, (\mathcal{I}_{cl})^{(a)})$ and $(E, (\mathcal{I}_{cl})^{(b)})$, respectively.*

(6) (MFI4*) *For arbitrary $a, b \in J(M)$, $a \leq b$ implies $cl_{(\mathcal{I}_{cl})_{[a]}}(A) \subseteq cl_{(\mathcal{I}_{cl})_{[b]}}(A)$ for arbitrary $A \in 2^E$, where $cl_{(\mathcal{I}_{cl})_{[a]}}$ and $cl_{(\mathcal{I}_{cl})_{[b]}}$ are the closure operators of crisp matroids $(E, (\mathcal{I}_{cl})_{[a]})$ and $(E, (\mathcal{I}_{cl})_{[b]})$, respectively.*

Proof. (1) For arbitrary $A \in 2^E$ and $a \in J(M)$,

$$\begin{aligned} A \in \mathcal{I}_{cl_{[a]}} &\Leftrightarrow x \notin cl_{[a]}(A - \{x\}) \text{ for arbitrary } x \in A \\ &\Leftrightarrow cl(A - \{x\})(x) \not\geq a \text{ for arbitrary } x \in A \\ &\Leftrightarrow (cl(A - \{x\})(x))' \not\leq a' \text{ for arbitrary } x \in A \\ &\Leftrightarrow \bigwedge_{x \in A} (cl(A - \{x\})(x))' \not\leq a' \text{ (since } A \text{ is a finite set)} \\ &\Leftrightarrow A \in (\mathcal{I}_{cl})^{(a')}. \end{aligned}$$

(2) This proof is similar to the proof of (1).

(3) By (1) and (2), this proof is trivial.

(4) For arbitrary $e \in E$ and $A \in 2^E$. If $e \in A$, then we have $cl(A)(e) = \top$ and $(cl(A)(e))' = \bot$. This follows that $\mathcal{I}_{cl}(A \cup \{e\}) \geq \mathcal{I}_{cl}(A) \wedge (cl(A)(e))'$. If $e \notin A$, then

$$\begin{aligned} \mathcal{I}_{cl}(A \cup \{e\}) &= \bigwedge_{x \in A} (cl(A \cup \{e\} - \{x\})(x))' \wedge (cl(A)(e))' \\ &= \bigwedge_{x \in A} (cl((A - \{x\}) \cup \{e\})(x))' \wedge (cl(A)(e))' \text{ (By (MFCL4))} \\ &\geq \bigwedge_{x \in A} ((cl((A - \{x\}) \cup \{x\})(e))' \wedge (cl(A - \{x\})(x))') \wedge (cl(A)(e))' \\ &= \bigwedge_{x \in A} (cl(A - \{x\})(x))' \wedge (cl(A)(e))' \\ &= \mathcal{I}_{cl}(A) \wedge (cl(A)(e))'. \end{aligned}$$

(5) According to (1), $cl_{(\mathcal{I}_{cl})^{(a)}} = cl_{[a']}$ and $cl_{(\mathcal{I}_{cl})^{(b)}} = cl_{[b']}$ for arbitrary $a, b \in P(M)$. It is obvious that $cl_{(\mathcal{I}_{cl})^{(a)}} \subseteq cl_{(\mathcal{I}_{cl})^{(b)}}$ when $a \leq b$.

(6) According to (2), $cl_{(\mathcal{I}_{cl})_{[a]}} = cl^{(a')}$ and $cl_{(\mathcal{I}_{cl})_{[b]}} = cl^{(b')}$ for arbitrary $a, b \in J(M)$. It is obvious that $cl_{(\mathcal{I}_{cl})_{[a]}} \subseteq cl_{(\mathcal{I}_{cl})_{[b]}}$ when $a \leq b$.

□

Remark 4.2. There exists an M-fuzzifying matroid that does not satisfy (MFI4) and (MFI4*).

Example 4.3. Let $M = [0, 1]$, $E = \{x, y, z\}$ and $\mathcal{I} \in [0, 1]^{2^E}$, where

$$\mathcal{I}(\emptyset) = 1;\ \mathcal{I}(\{x\}) = 1;\ \mathcal{I}(\{y\}) = 1;\ \mathcal{I}(\{z\}) = 1;$$
$$\mathcal{I}(\{x, y\}) = 0.8;\ \mathcal{I}(\{y, z\}) = 0.8;\ \mathcal{I}(\{x, z\}) = 0.6;\ \mathcal{I}(E) = 0.$$

It is obvious that $(E,\mathcal{I})$ is a $[0,1]$-fuzzifying matroid. However,

$$\begin{aligned}
&\mathcal{I}_{[0.7]} = \mathcal{I}^{(0.7)} = \{\emptyset, \{x\}, \{y\}, \{z\}, \{x,y\}, \{y,z\}\};\\
&cl_{\mathcal{I}_{[0.7]}}(\{x,z\}) = cl_{\mathcal{I}^{(0.7)}}(\{x,z\}) = \{x,z\};\\
&\mathcal{I}_{[0.5]} = \mathcal{I}^{(0.5)} = \{\emptyset, \{x\}, \{y\}, \{z\}, \{x,y\}, \{y,z\}, \{x,z\}\};\\
&cl_{\mathcal{I}_{[0.5]}}(\{x,z\}) = cl_{\mathcal{I}^{(0.5)}}(\{x,z\}) = E.
\end{aligned}$$

Thus, the $[0,1]$-fuzzifying matroid $(E,\mathcal{I})$ does not satisfy (MFI4) and (MFI4*).

Lemma 4.4. *Let $(E,\mathcal{I})$ be an M-fuzzifying matroid.*

(1) *If $(E,\mathcal{I})$ satisfies* (MFI4)*, then for arbitrary $a \in P(M)$ and $A \in 2^E$, the following condition holds.*

$$cl_{\mathcal{I}^{(a)}}(A) = \bigcap_{b\in\alpha^*(a)} cl_{\mathcal{I}^{(b)}}(A).$$

(2) *If $(E,\mathcal{I})$ satisfies* (MFI4*)*, then for arbitrary $a \in J(M)$ and $A \in 2^E$, the following condition holds.*

$$cl_{\mathcal{I}_{[a]}}(A) = \bigcup_{b\in\beta^*(a)} cl_{\mathcal{I}_{[b]}}(A).$$

Proof. (1) By (MFI4), we know that $cl_{\mathcal{I}^{(a)}}(A) \subseteq \bigcap_{b\in\alpha^*(a)} cl_{\mathcal{I}^{(b)}}(A)$. In order to prove $cl_{\mathcal{I}^{(a)}}(A) \supseteq \bigcap_{b\in\alpha^*(a)} cl_{\mathcal{I}^{(b)}}(A)$, it suffices to prove $x \notin \bigcap_{b\in\alpha^*(a)} cl_{\mathcal{I}^{(b)}}(A)$ for arbitrary $x \notin cl_{\mathcal{I}^{(a)}}(A)$. Since $x \notin cl_{\mathcal{I}^{(a)}}(A)$, there exists a $B \in \mathcal{B}(\mathcal{I}^{(a)}|A)$ such that $B \cup \{x\} \in \mathcal{I}^{(a)}$, where $\mathcal{B}(\mathcal{I}^{(a)}|A)$ is the set of all bases of $\mathcal{I}^{(a)}|A$. By $\mathcal{I}^{(a)} = \bigcup_{b\in\alpha^*(a)} \mathcal{I}^{(b)}$, there exists a $b \in \alpha^*(a)$ such that $B \in \mathcal{B}(\mathcal{I}^{(b)}|A)$ and $B \cup \{x\} \in \mathcal{I}^{(b)}$, which implies $x \notin cl_{\mathcal{I}^{(b)}}(A)$. Thus $x \notin \bigcap_{b\in\alpha^*(a)} cl_{\mathcal{I}^{(b)}}(A)$. This shows $cl_{\mathcal{I}^{(a)}}(A) = \bigcap_{b\in\alpha^*(a)} cl_{\mathcal{I}^{(b)}}(A)$.

(2) This proof is similar to the proof of (1). □

Theorem 4.5. *Let $(E,\mathcal{I})$ be an M-fuzzifying matroid and $\mathcal{R}_\mathcal{I}$ be the M-fuzzifying rank function of $(E,\mathcal{I})$.*

(1) *Suppose that $(E,\mathcal{I})$ satisfies* (MFI4)*. Define a mapping $cl_\mathcal{I}$: $2^E \to M^E$ such that for arbitrary $A \in 2^E$ and $x \in E$,*

$$cl_\mathcal{I}(A)(x) = \bigvee\{a \in J(M) : (\mathcal{R}_\mathcal{I}(A\cup\{x\}))^{(a')} = (\mathcal{R}_\mathcal{I}(A))^{(a')}\}.$$

Then $cl_\mathcal{I}$ is an M-fuzzifying closure operator on E.

(2) *Suppose that $(E,\mathcal{I})$ satisfies* (MFI4*)*. Define a mapping $cl^*_\mathcal{I}$: $2^E \to M^E$ such that for arbitrary $A \in 2^E$ and $x \in E$,*

$$cl^*_\mathcal{I}(A)(x) = \bigwedge\{a \in P(M) : (\mathcal{R}_\mathcal{I}(A\cup\{x\}))_{[a']} = (\mathcal{R}_\mathcal{I}(A))_{[a']} + 1\}.$$

Then $cl^*_\mathcal{I}$ is an M-fuzzifying closure operator on E.

Proof. (1) For an arbitrary $a \in J(M)$, since $(\mathcal{R}_{\mathcal{I}}(A))^{(a')} = \mathcal{R}_{\mathcal{I}^{(a')}}(A)$, where $\mathcal{R}_{\mathcal{I}^{(a')}}$ is the rank function of classical matroid $(E, \mathcal{I}^{(a')})$, we get

$$\begin{aligned} cl_{\mathcal{I}}(A)(x) &= \bigvee \{a \in J(M) : (\mathcal{R}_{\mathcal{I}}(A \cup \{x\}))^{(a')} = (\mathcal{R}_{\mathcal{I}}(A))^{(a')}\} \\ &= \bigvee \{a \in J(M) : \mathcal{R}_{\mathcal{I}^{(a')}}(A \cup \{x\}) = \mathcal{R}_{\mathcal{I}^{(a')}}(A)\} \\ &= \bigvee \{a \in J(M) : x \in cl_{\mathcal{I}^{(a')}}(A)\} \\ &= \bigvee_{a \in J(M)} (a \wedge cl_{\mathcal{I}^{(a')}}(A)(x)). \end{aligned}$$

Denote $cl(a)(A) = cl_{\mathcal{I}^{(a')}}(A)$ for arbitrary $a \in J(M)$. Then $cl(a)$ is a crisp closure operator on E and $cl_{\mathcal{I}}(A) = \bigvee_{a \in J(M)} (a \wedge cl(a)(A))$. Since $a \in \beta^*(b)$ if and only if $a' \in \alpha^*(b')$, we have $cl(b)(A) = \bigcap_{a \in \beta^*(b)} cl(a)(A)$ by Lemma 4.4. Thus according to theorem 3.6, $cl_{\mathcal{I}}$ is an M-fuzzifying closure operator on E.

(2) This proof is similar to the proof of (1). □

Corollary 4.6. *Let $(E, \mathcal{I})$ be an M-fuzzifying matroid.*

(1) *If $(E, \mathcal{I})$ satisfies* (MFI4)*, then $(cl_{\mathcal{I}})_{[a]} = cl_{\mathcal{I}^{(a')}}$ for an arbitrary $a \in J(M)$.*

(2) *If $(E, \mathcal{I})$ satisfies* (MFI4*)*, then $(cl^*_{\mathcal{I}})^{(a)} = cl_{\mathcal{I}_{[a']}}$ for arbitrary $a \in P(M)$.*

Proof. According to the proof of Theorem 4.5, this proof is trivial. □

Theorem 4.7. *Let $(E, \mathcal{I})$ be an M-fuzzifying matroid.*

(1) *If $(E, \mathcal{I})$ satisfies* (MFI4)*, then $\mathcal{I}_{cl_{\mathcal{I}}} = \mathcal{I}$.*

(2) *If $(E, \mathcal{I})$ satisfies* (MFI4*)*, then $\mathcal{I}_{cl^*_{\mathcal{I}}} = \mathcal{I}$.*

Proof. (1) In order to prove $\mathcal{I}_{cl_{\mathcal{I}}} = \mathcal{I}$, it suffices to prove $(\mathcal{I}_{cl_{\mathcal{I}}})^{(a)} = (\mathcal{I})^{(a)}$ for an arbitrary $a \in P(M)$. Then

$$\begin{aligned} A \in (\mathcal{I}_{cl_{\mathcal{I}}})^{(a)} &\Rightarrow \mathcal{I}_{cl_{\mathcal{I}}}(A) \not\leq a \\ &\Rightarrow \bigwedge_{x \in A} (cl_{\mathcal{I}}(A - \{x\})(x))' \not\leq a \\ &\Rightarrow (cl_{\mathcal{I}}(A - \{x\})(x))' \not\leq a \text{ for arbitrary } x \in A \\ &\Rightarrow cl_{\mathcal{I}}(A - \{x\})(x) \not\geq a' \text{ for arbitrary } x \in A \\ &\Rightarrow (\mathcal{R}_{\mathcal{I}}(A))^{(a)} \neq (\mathcal{R}_{\mathcal{I}}(A - \{x\}))^{(a)} \text{ for arbitrary } x \in A \\ &\Rightarrow \mathcal{R}_{\mathcal{I}^{(a)}}(A) \neq \mathcal{R}_{\mathcal{I}^{(a)}}(A - \{x\}) \text{ for arbitrary } x \in A \\ &\Rightarrow x \notin cl_{\mathcal{I}^{(a)}}(A - \{x\}) \text{ for arbitrary } x \in A \\ &\Rightarrow A \in (\mathcal{I})^{(a)}. \end{aligned}$$

Conversely, $A \notin (\mathcal{I}_{cl_{\mathcal{I}}})^{(a)}$ implies $\bigwedge_{x \in A} (cl_{\mathcal{I}}(A - \{x\})(x))' \leq a$. So there exists an element $x \in A$ such that $(cl_{\mathcal{I}}(A - \{x\})(x))' \leq a$ since A is a finite set. Then $cl_{\mathcal{I}}(A - \{x\})(x) \geq a'$. For arbitrary $b \in \alpha^*(a)$, $b' \in \beta^*(cl_{\mathcal{I}}(A - \{x\})(x))$ since $b' \in \beta^*(a')$. So there exists an element $c \in P(M)$ such that $b' \leq c'$ and

$(\mathcal{R}_{\mathcal{I}}(A))^{(c)} = (\mathcal{R}_{\mathcal{I}}(A-\{x\}))^{(c)}$. Thus $x \in cl_{\mathcal{I}^{(c)}}(A-\{x\}) \subseteq cl_{\mathcal{I}^{(b)}}(A-\{x\})$ and $A \notin (\mathcal{I})^{(b)}$, which implies $A \notin (\mathcal{I})^{(a)}$ by $(\mathcal{I})^{(a)} = \bigcup_{b\in\alpha^*(a)}(\mathcal{I})^{(b)}$.

This follows $\mathcal{I}_{cl_{\mathcal{I}}} = \mathcal{I}$.

(2) This proof is similar to the proof of (1). □

Corollary 4.8. *Let $(E,\mathcal{I})$ be an M-fuzzifying matroid. Then $(E,\mathcal{I})$ satisfies* (MFI4) *if and only if $(E,\mathcal{I})$ satisfies* (MFI4*). *In addition, $cl_{\mathcal{I}} = cl^*_{\mathcal{I}}$.*

Proof. According to Lemma 4.1 and Theorem 4.7, this proof is trivial. □

By Corollary 4.8, now we do not distinguish (MFI4) from (MFI4*) and $cl_{\mathcal{I}}$ from $cl^*_{\mathcal{I}}$.

Corollary 4.9. *Let $(E,\mathcal{I})$ be an M-fuzzifying matroid induced by an M-fuzzifying closure operator and $a \in P(M)$. Then $\bigwedge_{x\in A}(cl_{\mathcal{I}}(A-\{x\})(x))' \not\leq a$ if and only if $\{x \in A : (\mathcal{R}_{\mathcal{I}}(A))^{(a)} = (\mathcal{R}_{\mathcal{I}}(A-\{x\}))^{(a)}\} = \emptyset$.*

Proof. According to the Theorem 4.7, this proof is trivial. □

Theorem 4.10. *Let $cl : 2^E \to M^E$ be an M-fuzzifying closure operator. Then $cl_{\mathcal{I}_{cl}} = cl$.*

Proof. We have $\mathcal{I}_{cl_{\mathcal{I}_{cl}}} = \mathcal{I}_{cl}$ by Theorem 4.7. Let $a \in J(M)$. According to Lemma 4.1, we get $\mathcal{I}_{cl_{[a]}} = (\mathcal{I}_{cl})^{(a')} = (\mathcal{I}_{cl_{\mathcal{I}_{cl}}})^{(a')} = \mathcal{I}_{(cl_{\mathcal{I}_{cl}})_{[a]}}$. This follows $cl_{\mathcal{I}_{cl}} = cl$. □

Definition 4.11. Let $(E,\mathcal{I})$ be an M-fuzzifying matroid. $(E,\mathcal{I})$ is called an M-fuzzifying acyclic matroid if $\mathcal{I}(\{e\}) = \top$ for an arbitrary $e \in E$.

Similar to Lemma 2.3, we have the following property.

Lemma 4.12. *Let $(E,\mathcal{I})$ be an M-fuzzifying matroid. Then the following conditions are equivalent.*

(1) *$(E,\mathcal{I})$ is an M-fuzzifying acyclic matroid.*
(2) *$\mathcal{R}_{\mathcal{I}}(\{e\}) = \underline{1}$ for arbitrary $e \in E$.*
(3) *$(E,\mathcal{I}_{[a]})$ is a acyclic matroid for arbitrary $a \in J(M)$.*
(4) *$(E,\mathcal{I}^{(a)})$ is a acyclic matroid for arbitrary $a \in P(M)$.*

Proof. This proof is trivial. □

Theorem 4.13. *Let $(E,\mathcal{I})$ be an M-fuzzifying acyclic matroid induced by an M-fuzzifying closure operator. Then $cl_{\mathcal{I}}$ satisfies:*

(MFCL5) $cl_{\mathcal{I}}(\emptyset) = \emptyset$.

Proof. If $(E,\mathcal{I})$ is acyclic, then $cl_{\mathcal{I}}(\emptyset)(x) = \bigvee\{a \in J(M) : (\mathcal{R}_{\mathcal{I}}(\{x\})^{(a')} = (\mathcal{R}_{\mathcal{I}}(\emptyset))^{(a')}\} = \bigvee\emptyset = \perp$ for arbitrary $x \in E$. This implies $cl_{\mathcal{I}}(\emptyset) = \emptyset$. □

Theorem 4.14. *Let cl: $2^E \to M^E$ be an M-fuzzifying closure operator on E satisfying* (MFCL5). *Then $(E,\mathcal{I}_{cl})$ is an M-fuzzifying acyclic matroid.*

Proof. If cl satisfies (MFCL5), then $\mathcal{I}_{cl}(\{e\}) = (cl(\emptyset)(e))' = \top$ for arbitrary $e \in E$. □

5. Conclusions

In this paper, we first define the notion of M-fuzzifying closure operators and discuss the properties of M-fuzzifying closure operators. M-fuzzifying closure operators can induce M-fuzzifying matroids and those M-fuzzifying matroids also can induce M-fuzzifying closure operators by two equivalent forms. Finally, we give the notion of M-fuzzifying acyclic matroids and discuss the M-fuzzifying closure operators induced by M-fuzzifying acyclic matroids.

Acknowledgements. The authors would like to thank the valuable reviews and also appreciate the constructive suggestions from the anonymous referees. The project is supported by the National Natural Science Foundation of China (11371002, 11601388) and Specialized Research Fund for the Doctoral Program of Higher Education (20131101110048).

References

[1] H. L. Huang and F. G. Shi, *L-fuzzy numbers and their properties*, Information Sciences, **178** (2008), 1141–1151.
[2] H. Lian and X. Xin, *The nullities for M-fuzzifying matroids*, Applied Mathematics Letters, **25(3)** (2012), 279–286.
[3] J. Oxley, Matroid Theory, Oxford University Press, 1992.
[4] F. G. Shi, *A new approach to fuzzification of matroids*, Fuzzy Sets and Systems, **160(5)** (2009), 696–705.
[5] F. G. Shi, *(L,M)-fuzzy matroids*, Fuzzy Sets and Systems, **160(16)** (2009), 2387–2400.
[6] F. G. Shi and B. Pang, *Categories isomorphic to the category of L-fuzzy closure system spaces*, Iranian Journal of Fuzzy Systems, **10(5)** (2013), 127–146.
[7] F. G. Shi and L. Wang, *Characterizations and applications of M-fuzzifying matroids*, Journal of Intelligent and Fuzzy Systems, **25** (2013), 919–930.
[8] G. J. Wang, *Theory of topological molecular lattices*, Fuzzy Sets and Systems, **47(3)** (1992), 351–376.
[9] L. Wang and F. G. Shi, *Characterization of L-fuzzifying matroids by M-fuzzifying families of α-flats*, Advances of Fuzzy Sets and Systems, **2** (2009), 203–213.
[10] L. Wang and F. G. Shi, *Characterization of L-fuzzifying matroids by L-fuzzifying closure operators*, Iranian Journal of Fuzzy Systems, **7(1)** (2010), 47–58.
[11] L. Wang and Y. P. Wei, *M-fuzzifying P-closure operators*, Advances in Intelligent and Soft Computing, **62** (2009), 547–554.
[12] X. Xin and F. G. Shi, *M-fuzzifying bases*, Proyecciones, **28(3)** (2009), 271–283.
[13] X. Xin and F. G. Shi, *Categories of bi-fuzzy pre-matroids*, Computer and Mathematics with Applications, **59** (2010), 1548–1558.
[14] X. Xin and F. G. Shi, *Rank functions for closed and perfect [0,1]-matroids*, Hacettepe Journal of Mathematics and Statistics, **39(1)** (2010), 31–39.
[15] X. Xin, F. G. Shi and S. G. Li, *M-fuzzifying derived operators and difference derived operators*, Iranian Journal of Fuzzy Systems, **7(2)** (2010), 71–81.
[16] Z. Y. Xiu and F. G. Shi, *M-fuzzifying submodular functions*, Journal of Intelligent and Fuzzy Systems, **27** (2014), 1243–1255.
[17] W. Yao and F. G. Shi, *Base axioms and circuits axioms for fuzzifying matroids*, Fuzzy Sets and Systems, **161** (2010), 3155–3165.

Xiu Xin, Department of Mathematics, Tianjin University of Technology, Tianjin 300384, P.R.China
E-mail address: xinxiu518@163.com

Shao-Jun Yang*, School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 100081, P.R.China and Beijing Key Laboratory on MCAACI, Beijing Institute of Technology, Beijing 100081, P.R.China

E-mail address: shaojunyang@outlook.com

*Corresponding author