SOME FIXED POINT RESULTS FOR ADMISSIBLE GERAGHTY CONTRACTION TYPE MAPPINGS IN FUZZY METRIC SPACES

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ABSTRACT. In this paper, we introduce the notions of fuzzy α-Geraghty contraction type mapping and fuzzy β-φ-contractive mapping and establish some interesting results on the existence and uniqueness of fixed points for these two types of mappings in the setting of fuzzy metric spaces and non-Archimedean fuzzy metric spaces. The main results of our work generalize and extend some known comparable results in the literature. Furthermore, several illustrative examples are given to support the usability of our obtained results.

1. Introduction

It is well known that the fuzzy set concept plays an important role in many scientific and engineering applications. The fuzziness appears when we need to perform, on manifold, calculations with imprecision variables. The concept of fuzzy sets was introduced initially by Zadeh [25] in 1965. The contraction type mappings in fuzzy metric spaces play a crucial role in fixed point theory. In 1988, Grabiec [11] defined the Banach contraction in a fuzzy metric space (in the sense of Kramosil and Michálek [13]) and extended fixed point theorems of Banach and Edelstein to fuzzy metric spaces. Successively, George and Veeramani [7] slightly modified the notion of a fuzzy metric space introduced by Kramosil and Michálek and then defined a Hausdorff and first countable topology on it.

In 2002, Gregori and Sapena [12] introduced the notion of fuzzy contractive mapping and gave some fixed point theorems for complete fuzzy metric spaces in the sense of George and Veeramani, and also for Kramosil and Michálek's fuzzy metric spaces which are complete in Grabiec's sense. Soon after, Mihet [15] proposed a fuzzy fixed point theorem for a (weak) Banach contraction in M-complete fuzzy metric spaces. In this direction, Mihet [16, 17, 18] further extended the fixed point theory in fuzzy metric spaces besides introducing some new notions of contraction mappings such as Edelstein fuzzy contractive mappings, fuzzy ψ-contractive mappings, fuzzy contractive mappings of (ε-λ) type, etc. For more details on fixed point theory for contraction type mappings in fuzzy metric spaces, we refer the interested reader to [1, 2, 5, 10, 19, 20, 24, 23] and the references cited therein.

The applications of fixed point theorems are remarkable in different disciplines of mathematics, engineering and economics in dealing with problems arising in

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approximation theory, game theory and many others. Consequently, many researchers, following the Banach contraction principle, investigated the existence of weaker contractive conditions or extended previous results under relatively weak hypotheses on the metric space. One of the interesting results which generalizes the Banach contraction principle was given by Geraghty [9] in the setting of complete metric spaces by considering an auxiliary function. Later, Amini-Harandi and Emami [3] characterized the result of Geraghty in the context of a partially ordered complete metric space. Recently, Samet et al. [21] obtained remarkable fixed point results by defining the notion of α-ψ-contractive mappings via admissible mappings.

Motivated and inspired by Samet et al. [21], we introduce the new concepts of fuzzy α-Geraghty contraction type mapping and fuzzy β-φ-contractive mapping via triangular α and β-admissible mappings, respectively. Subsequently, we derive several sufficient conditions which ensure the existence and uniqueness of fixed points for these classes of mappings in the setup of complete fuzzy metric spaces and complete non-Archimedean fuzzy metric spaces. Our main results substantially generalize and extend some known results in the existing literature. Meantime, we provide some illustrative examples in support of our new results where results from current literature are not applicable.

2. **Preliminaries**

In this section, we briefly recall some known definitions and terminologies from the theory of fuzzy metric spaces which will be needed in the sequel.

Definition 2.1. (Schweizer and Sklar [22]) A binary operation $\star : [0,1]\times[0,1] \to [0,1]$ is called a continuous triangular norm (in short, continuous t-norm) if it satisfies the following assertions:

(i) $\star$ is commutative and associative;
(ii) $\star$ is continuous;
(iii) $a \star 1 = a$ for all $a \in [0,1]$;
(iv) $a \star b \leq c \star d$ whenever $a \leq c$ and $b \leq d$ for all $a,b,c,d \in [0,1]$.

Four basic examples of the continuous t-norm are $a \star_1 b = \min\{a,b\}$, $a \star_2 b = \frac{ab}{\max\{a,b,\lambda\}}$ for $\lambda \in (0,1)$, $a \star_3 b = ab$, and $a \star_4 b = \max\{a+b-1, 0\}$ for all $a,b \in [0,1]$.

Definition 2.2. (Kramosil and Michálek [13]) A fuzzy metric space is a triple $(X, M, \star)$ such that X is an arbitrary nonempty set, $\star$ is a continuous t-norm and M is a fuzzy set on $X \times X \times [0, +\infty)$ satisfying, for all $x, y \in X$, the following properties:

(KM1) $M(x,y,0) = 0$;
(KM2) $M(x,y,t) = 1$ for all $t > 0$ if and only if $x = y$;
(KM3) $M(x,y,t) = M(y,x,t)$ for all $t > 0$;
(KM4) $M(x,y,\cdot) : [0,+\infty) \to [0,1]$ is left continuous;
(KM5) $M(x,z,t+s) \geq M(x,y,t) \star M(y,z,s)$ for all $z \in X$ and for all $t, s > 0$.

If we replace the triangular inequality (KM5) by

(NA) $M(x,z,\max\{t,s\}) \geq M(x,y,t) \star M(y,z,s)$ for all $x,y,z \in X$ and for all $t,s > 0$, then the triple $(X, M, \star)$ is called a non-Archimedean fuzzy metric space.

It is easy to check that (NA) implies (KM5). Hence, each non-Archimedean fuzzy metric space is a fuzzy metric space.

In order to introduce a Hausdorff topology on the fuzzy metric space, George and Veeramani [8] modified the above definition as follows.

Definition 2.3. (George and Veeramani [8]) A fuzzy metric space is a triple $(X, M, *)$ such that X is an arbitrary nonempty set, $\star$ is a continuous t-norm and M is a fuzzy set on $X \times X \times (0, +\infty)$ satisfying, for all $x, y \in X$, the following properties:

(GV1) $M(x, y, t) > 0$ for all $t > 0$;
(GV2) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$;
(GV3) $M(x, y, t) = M(y, x, t)$ for all $t > 0$;
(GV4) $M(x, y, \cdot) : (0, +\infty) \to [0, 1]$ is continuous;
(GV5) $M(x, z, t + s) \geq M(x, y, t) \star M(y, z, s)$ for all $z \in X$ and for all $t, s > 0$.

Example 2.4. [8] Let (X, d) be a metric space. Then the triple $(X, M, \star)$ is a fuzzy metric space, where $a \star b = ab$ for all $a, b \in [0, 1]$ and $M(x, y, t) = \frac{t}{t+d(x,y)}$ for all $x, y \in X$ and for all $t > 0$. We call this M as the standard fuzzy metric induced by the metric d. Even if we define $a \star b = \min\{a, b\}$ for all $a, b \in [0, 1]$, then the triple $(X, M, \star)$ will be a fuzzy metric space.

From now on, we will work in fuzzy metric spaces on the sense of George and Veeramani.

Definition 2.5. [7] Let $(X, M, \star)$ be a fuzzy metric space (or a non-Archimedean fuzzy metric space).

(i) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$, denoted by $x_n \to x$ as $n \to +\infty$, if and only if $\lim_{n\to+\infty} M(x_n, x, t) = 1$ for all $t > 0$, i.e. for each $r \in (0, 1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x, t) > 1 - r$ for all $n \geq n_0$.

(ii) A sequence $\{x_n\}$ in X is a Cauchy sequence if and only if for all $\varepsilon \in (0, 1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all $m, n \geq n_0$.

(iii) The fuzzy metric space (or the non-Archimedean fuzzy metric space) is called complete if every Cauchy sequence is convergent.

Definition 2.6. (Di Bari and Vetro [6]) Let $(X, M, \star)$ be a fuzzy metric space. The fuzzy metric M is said to be triangular whenever,

$$\left(\frac{1}{M(x, y, t)} - 1\right) \leq \left(\frac{1}{M(x, z, t)} - 1\right) + \left(\frac{1}{M(y, z, t)} - 1\right) \tag{1}$$

for all $x, y, z \in X$ and any $t > 0$.

Finally, let X be a nonempty set. If $(X, M, \star)$ is a fuzzy metric space and $(X, \preceq)$ is a partially ordered set, then $(X, M, \star, \preceq)$ is called an ordered fuzzy metric space. Also, $x, y \in X$ are called comparable if $x \preceq y$ or $y \preceq x$ holds. Let $(X, \preceq)$ be a partially ordered set and $T : X \to X$ be a mapping. T is called a nondecreasing mapping if $Tx \preceq Ty$ whenever $x \preceq y$ for all $x, y \in X$.

3. Main Results

We begin this section by introducing the new notion of triangular α-admissible mappings in fuzzy metric spaces as follows.

Definition 3.1. Let $(X, M, \star)$ be a fuzzy metric space and $T : X \to X$ be a given mapping. We say that T is a triangular α-admissible mapping if there exists a function $\alpha : X \times X \times (0, +\infty) \to (-\infty, +\infty)$ such that

(T_{α_1}) $\alpha(x, y, t) \geq 1$ implies $\alpha(Tx, Ty, t) \geq 1$ for all $x, y \in X$ and any $t > 0$;
(T_{α_2}) $\alpha(x, z, t) \geq 1$ and $\alpha(z, y, t) \geq 1$ imply $\alpha(x, y, t) \geq 1$ for all $x, y, z \in X$ and any $t > 0$.

We now prove the following important lemma that will be used for proving our first theorem.

Lemma 3.2. *Let $(X, M, \star)$ be a fuzzy metric space and $T : X \to X$ be a triangular α-admissible mapping. Assume that there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0, t) \geq 1$. Define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$. Then*

$$\alpha(x_m, x_n, t) \geq 1 \qquad \text{for all } m, n \in \mathbb{N} \text{ with } m < n.$$

Proof. Since there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0, t) \geq 1$, it follows that $\alpha(x_0, x_1, t) = \alpha(x_0, Tx_0, t) \geq 1$. Now, by using T_{α_1} of Definition 3.1, we obtain

$$\alpha(x_1, x_2, t) = \alpha(Tx_0, Tx_1, t) \geq 1 \quad \Longrightarrow \quad \alpha(x_2, x_3, t) = \alpha(Tx_1, Tx_2, t) \geq 1.$$

By continuing the process as above, we get

$$\alpha(x_n, x_{n+1}, t) \geq 1 \qquad \text{for all } n \in \mathbb{N} \cup \{0\}.$$

Let $m, n \in \mathbb{N}$ with $m < n$. Because $\alpha(x_m, x_{m+1}, t) \geq 1$ and $\alpha(x_{m+1}, x_{m+2}, t) \geq 1$, it follows by T_{α_2} that $\alpha(x_m, x_{m+2}, t) \geq 1$. Again, since $\alpha(x_m, x_{m+2}, t) \geq 1$ and $\alpha(x_{m+2}, x_{m+3}, t) \geq 1$, by applying T_{α_2}, we have $\alpha(x_m, x_{m+3}, t) \geq 1$. By continuing this process inductively, we obtain $\alpha(x_m, x_n, t) \geq 1$. □

As mentioned before, in order to generalize the Banach contraction principle, Geraghty [9] used the following class of functions.

Let Ψ denote the class of all functions $\psi : [0, +\infty) \to [0, 1)$ satisfying the following condition:

$$\psi(t_n) \to 1 \quad \text{as} \quad n \to +\infty \quad \text{implies} \quad t_n \to 0 \quad \text{as} \quad n \to +\infty.$$

We now introduce the concept of fuzzy α-Geraghty contraction type mappings and prove the fixed point theorems for such mappings.

Definition 3.3. Let $(X, M, \star)$ be a fuzzy metric space. A mapping $T : X \to X$ is said to be a fuzzy α-Geraghty contraction type mapping if there exist two functions $\alpha : X \times X \times (0, +\infty) \to (-\infty, +\infty)$ and $\psi \in \Psi$ such that

$$\alpha(x, y, t)\left(\frac{1}{M(Tx, Ty, t)} - 1\right) \leq \psi\left(\frac{1}{M(x, y, t)} - 1\right)\left(\frac{1}{M(x, y, t)} - 1\right) \tag{2}$$

for all $x, y \in X$ and any $t > 0$.

Remark 3.4. It is interesting to remark at this point that due to property of $\psi \in \Psi$, it follows that

$$\begin{aligned}\alpha(x,y,t)\left(\frac{1}{M(Tx,Ty,t)}-1\right) &\leq \psi\left(\frac{1}{M(x,y,t)}-1\right)\left(\frac{1}{M(x,y,t)}-1\right)\\ &< \left(\frac{1}{M(x,y,t)}-1\right)\end{aligned}$$

for all $x, y \in X$ with $x \neq y$ and any $t > 0$.

Remark 3.5. If $\alpha(x,y,t) = 1$ for all $x, y \in X$ and any $t > 0$ and $\psi(r) = \lambda$ for all $r > 0$ and for some $\lambda \in (0,1)$, then Definition 3.3 reduces to the definition of fuzzy contractive mapping given by Gregori and Sapena [12]. Hence, a fuzzy contractive mapping is a fuzzy α-Geraghty contraction type mapping, but the converse is not necessarily true (see Example 3.9).

Our first result is an existence theorem for fixed points of α-Geraghty contraction type mapping.

Theorem 3.6. *Let $(X, M, \star)$ be a complete fuzzy metric space such that M be triangular. Suppose that $T : X \to X$ be a self-mapping satisfying the following assertions:*

(i) *T is a fuzzy α-Geraghty contraction type mapping;*
(ii) *T is a triangular α-admissible mapping;*
(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0, t) \geq 1$ for all $t > 0$;*
(iv) *T is continuous.*

Then T has a fixed point, that is, there exists $x^* \in X$ such that $Tx^* = x^*$.

Proof. By assumption (iii), there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0, t) \geq 1$ for all $t > 0$. We define a sequence $\{x_n\}$ in X by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. Suppose that $x_n = x_{n+1}$ for some $n \in \mathbb{N} \cup \{0\}$. Then, in this case, the proof is completed since $x^* = x_n = x_{n+1} = Tx_n = Tx^*$. Hence, throughout the proof, we assume that $x_n \neq x_{n-1}$ for all $n \in \mathbb{N}$.

By virtue of Lemma 3.2, we have

$$\alpha(x_{n-1}, x_n, t) \geq 1 \qquad \text{for all } n \in \mathbb{N} \text{ and for all } t > 0. \tag{3}$$

By applying the inequality (2) with $x = x_{n-1}$ and $y = x_n$ and thanks to (3), we obtain

$$\begin{aligned}\left(\frac{1}{M(x_n,x_{n+1},t)}-1\right) &= \left(\frac{1}{M(Tx_{n-1},Tx_n,t)}-1\right)\\ &\leq \alpha(x_{n-1},x_n,t)\left(\frac{1}{M(Tx_{n-1},Tx_n,t)}-1\right)\\ &\leq \psi\left(\frac{1}{M(x_{n-1},x_n,t)}-1\right)\left(\frac{1}{M(x_{n-1},x_n,t)}-1\right)\\ &< \left(\frac{1}{M(x_{n-1},x_n,t)}-1\right).\end{aligned} \tag{4}$$

Thus, we conclude that $M(x_n, x_{n+1}, t) > M(x_{n-1}, x_n, t)$ for all $n \in \mathbb{N}$. Hence, $\{M(x_{n-1}, x_n, t)\}$ is an increasing sequence of positive real numbers in $[0, 1]$. So, there exists $\ell(t) \in [0, 1]$ such that $\lim_{n\to+\infty} M(x_{n-1}, x_n, t) = \ell(t)$ for all $t > 0$. We will prove that $\ell(t) = 1$ for all $t > 0$. Suppose to the contrary that there exists $t_0 > 0$ such that $\ell(t_0) < 1$. By taking the limit as $n \to +\infty$ in the inequality (4), we get

$$\lim_{n\to+\infty} \psi\Big(\frac{1}{M(x_{n-1}, x_n, t)} - 1\Big) = 1.$$

Due to property of $\psi \in \Psi$, we have

$$\lim_{n\to+\infty} \Big(\frac{1}{M(x_{n-1}, x_n, t)} - 1\Big) = 0,$$

so equivalently, $\lim_{n\to+\infty} M(x_{n-1}, x_n, t) = 1$, which is a contradiction. Hence, we conclude that

$$\lim_{n\to+\infty} M(x_{n-1}, x_n, t) = 1 \qquad \text{for all } t > 0. \tag{5}$$

Now, we assert that $\{x_n\}$ is a Cauchy sequence. Arguing by contradiction, we suppose

$$\lambda := \limsup_{m,n\to+\infty} M(x_m, x_n, t) < 1. \tag{6}$$

By applying the inequality (2) and Lemma 3.2, we have

$$\begin{aligned}\Big(\frac{1}{M(x_{m+1}, x_{n+1}, t)} - 1\Big) &= \Big(\frac{1}{M(Tx_m, Tx_n, t)} - 1\Big)\\ &\leq \alpha(x_m, x_n, t)\Big(\frac{1}{M(Tx_m, Tx_n, t)} - 1\Big)\\ &\leq \psi\Big(\frac{1}{M(x_m, x_n, t)} - 1\Big)\Big(\frac{1}{M(x_m, x_n, t)} - 1\Big).\end{aligned}$$

By taking limit supremum as $m, n \to +\infty$ in the above inequality, we obtain

$$\begin{aligned}\limsup_{m,n\to+\infty} \Big(\frac{1}{M(x_{m+1}, x_{n+1}, t)} - 1\Big) &\leq \limsup_{m,n\to+\infty} \psi\Big(\frac{1}{M(x_m, x_n, t)} - 1\Big)\\ &\quad \limsup_{m,n\to+\infty} \Big(\frac{1}{M(x_m, x_n, t)} - 1\Big)\\ &= \Big(\frac{1}{\lambda} - 1\Big) \limsup_{m,n\to+\infty} \psi\Big(\frac{1}{M(x_m, x_n, t)} - 1\Big).\end{aligned} \tag{7}$$

On the other hand, we have

$$\begin{aligned}\Big(\frac{1}{M(x_m, x_n, t)} - 1\Big) &\leq \Big(\frac{1}{M(x_m, x_{m+1}, t)} - 1\Big) + \Big(\frac{1}{M(x_{m+1}, x_n, t)} - 1\Big)\\ &\leq \Big(\frac{1}{M(x_m, x_{m+1}, t)} - 1\Big) + \Big(\frac{1}{M(x_{m+1}, x_{n+1}, t)} - 1\Big)\\ &\quad + \Big(\frac{1}{M(x_{n+1}, x_n, t)} - 1\Big).\end{aligned}$$

Again, by taking limit supremum as $m,n\to+\infty$ in the above inequality and using (5) and (7), we get

$$\begin{aligned}\left(\frac{1}{\lambda}-1\right) &\le \limsup_{m,n\to+\infty}\left(\frac{1}{M(x_{m+1},x_{n+1},t)}-1\right)\\ &\le \left(\frac{1}{\lambda}-1\right)\limsup_{m,n\to+\infty}\psi\left(\frac{1}{M(x_m,x_n,t)}-1\right),\end{aligned}$$

which implies that

$$\limsup_{m,n\to+\infty}\psi\left(\frac{1}{M(x_m,x_n,t)}-1\right)=1.$$

Owing to the fact that $\psi\in\Psi$, we deduce that

$$\limsup_{m,n\to+\infty}\left(\frac{1}{M(x_m,x_n,t)}-1\right)=0.$$

This yields to $\limsup_{m,n\to+\infty} M(x_m,x_n,t)=\ell(t_0)=1$, which is a contradiction. Hence, $\{x_n\}$ is a Cauchy sequence. Since $(X,M,\star)$ is a complete fuzzy metric space, it follows that the sequence $\{x_n\}$ converges to some $x^*\in X$, that is, $x_n\to x^*$ as $n\to+\infty$. Now, the continuity of T implies that $Tx_n\to Tx^*$ as $n\to+\infty$ and so $\lim_{n\to+\infty} M(Tx_n,Tx^*,t)=1$ for all $t>0$. It follows that

$$\lim_{n\to+\infty} M(x_{n+1},Tx^*,t)=\lim_{n\to+\infty} M(Tx_n,Tx^*,t)=1$$

for all $t>0$, that is, $x_n\to Tx^*$ as $n\to+\infty$. By the uniqueness of the limit, we get $x^*=Tx^*$, i.e. x^* is a fixed point of T. □

In the next theorem, we establish a fixed point result without any continuity assumption on the mapping T.

Theorem 3.7. *Let $(X,M,\star)$ be a complete fuzzy metric space such that M be triangular. Suppose that $T:X\to X$ be a self-mapping satisfying the following assertions:*

(i) *T is a fuzzy α-Geraghty contraction type mapping;*
(ii) *T is a triangular α-admissible mapping;*
(iii) *there exists $x_0\in X$ such that $\alpha(x_0,Tx_0,t)\ge 1$ for all $t>0$;*
(iv) *if $\{x_n\}$ is a sequence such that $\alpha(x_n,x_{n+1},t)\ge 1$ for all $n\in\mathbb{N}$ and $x_n\to x$ as $n\to+\infty$, then $\alpha(x_n,x,t)\ge 1$ for all $n\in\mathbb{N}$.*

Then T has a fixed point.

Proof. Following the same lines in the proof of Theorem 3.6, we get that the sequence $\{x_n\}$ defined by the schema $x_n=Tx_{n-1}$ for all $n\in\mathbb{N}$ which converging to some $x^*\in X$. Regarding (3) together with the condition (iv), we have

$$\alpha(x_n,x^*,t)\ge 1 \qquad \text{for all } n\in\mathbb{N} \text{ and for all } t>0. \tag{8}$$

By applying the inequality (2) and using (1) and (8) and in view of (GV3), we obtain

$$\begin{aligned}\left(\frac{1}{M(Tx^*,x^*,t)}-1\right) &\le \left(\frac{1}{M(Tx^*,Tx_n,t)}-1\right)+\left(\frac{1}{M(Tx_n,x^*,t)}-1\right)\\ &= \left(\frac{1}{M(Tx^*,Tx_n,t)}-1\right)+\left(\frac{1}{M(x_{n+1},x^*,t)}-1\right)\\ &\le \alpha(x_n,x^*,t)\left(\frac{1}{M(Tx_n,Tx^*,t)}-1\right)+\left(\frac{1}{M(x_{n+1},x^*,t)}-1\right)\\ &\le \psi\left(\frac{1}{M(x_n,x^*,t)}-1\right)\left(\frac{1}{M(x_n,x^*,t)}-1\right)\\ &\quad+\left(\frac{1}{M(x_{n+1},x^*,t)}-1\right).\end{aligned}$$

By taking the limit as $n\to+\infty$ in the above inequality, we get

$$\left(\frac{1}{M(Tx^*,x^*,t)}-1\right)=0.$$

Hence, $Tx^*=x^*$, i.e. x^* is a fixed point of T. □

Now, we present some examples to illustrate the usefulness of the proposed theoretical results.

Example 3.8. Let $X=[0,1]$, $a\star b=\min\{a,b\}$ for all $a,b\in[0,1]$ and $M(x,y,t)=\frac{t}{t+|x-y|}$ for all $x,y\in X$ and for all $t>0$. Obviously, $(X,M,\star)$ is a complete fuzzy metric space.

Consider the mapping $T:X\to X$ by

$$Tx=\begin{cases}\frac{1}{4}(1-x), & x\in[0,\frac{1}{3})\cup(\frac{1}{3},1],\\ \frac{1}{2}, & x=\frac{1}{3}\end{cases}$$

and the function $\alpha: X\times X\times(0,+\infty)\to(-\infty,+\infty)$ defined as

$$\alpha(x,y,t)=\begin{cases}1, & x,y\in[0,\frac{1}{3})\cup(\frac{1}{3},1],\\ 0, & \text{otherwise}\end{cases}$$

for all $t>0$. It is easy to check that T is a fuzzy α-Geraghty contraction type mapping with $\psi(s)=\frac{1}{2}$. In fact, if at least one between x and y is equal to $\frac{1}{3}$, then $\alpha(x,y,t)=0$ and so (2) holds trivially. Otherwise, if both x and y are in $(0,\frac{1}{3})\cup(\frac{1}{3},1]$, then $\alpha(x,y,t)=1$ and hence (2) becomes

$$\left(\frac{1}{M(Tx,Ty,t)}-1\right)\le\frac{1}{2}\left(\frac{1}{M(x,y,t)}-1\right).$$

Now, let $x,y\in X$ such that $\alpha(x,y,t)\ge 1$ for all $t>0$. This implies that $x,y\in[0,\frac{1}{3})\cup(\frac{1}{3},1]$ and by the definitions of T and α, we have $Tx,Ty\in[0,\frac{1}{3})\cup(\frac{1}{3},1]$. Thus, $\alpha(Tx,Ty,t)=1$ for all $t>0$. Also, let $x,y,z\in X$ such that $\alpha(x,z,t)\ge 1$ and $\alpha(z,y,t)\ge 1$ for all $t>0$. This implies that $x,y,z\in[0,\frac{1}{3})\cup(\frac{1}{3},1]$ and so $\alpha(x,y,t)\ge 1$ for all $t>0$. Hence, T is triangular α-admissible. Moreover, there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0, t) \geq 1$ for all $t > 0$. Indeed, for $x_0 = 1$, we have $\alpha(1, T1, t) = 1$.

Finally, let $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}, t) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to +\infty$. By the definition of the function α, it follows that $\{x_n\} \in [0, \frac{1}{3}) \cup (\frac{1}{3}, 1]$ for all $n \in \mathbb{N}$ and so $x \in [0, \frac{1}{3}) \cup (\frac{1}{3}, 1]$. Thus, $\alpha(x_n, x, t) = 1$ for all $n \in \mathbb{N}$. Therefore, all the required hypotheses of Theorem 3.7 are satisfied and hence T has a fixed point. Here $\frac{1}{5}$ is the fixed point of T.

The next example shows that our results generalize the corresponding classical concepts in the classical metric space.

Example 3.9. Let $X = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0, 3\}$, $a \star b = ab$ for all $a, b \in [0, 1]$ and $M(x, y, t) = \frac{t}{t+|x-y|}$ for all $x, y \in X$ and for all $t > 0$. Clearly, $(X, M, \star)$ is a complete fuzzy metric space.

Define the mapping $T : X \to X$ by

$$Tx = \begin{cases} \frac{x^2}{9}, & x \in X \setminus \{3\}, \\ 3, & x = 3 \end{cases}$$

and the function $\alpha : X \times X \times (0, +\infty) \to (-\infty, +\infty)$ as

$$\alpha(x, y, t) = \begin{cases} 1, & x, y \in X \setminus \{3\}, \\ 0, & \text{otherwise} \end{cases}$$

for all $t > 0$. It is elementary to check that T is a fuzzy α-Geraghty contraction type mapping by considering $\psi(s) = \frac{1}{3}$. In fact, if at least one between x and y is equal to 3, then $\alpha(x, y, t) = 0$ and so (2) holds trivially. Otherwise, if both x and y are in $X \setminus \{3\}$, then $\alpha(x, y, t) = 1$ and hence (2) becomes

$$\left(\frac{1}{M(Tx, Ty, t)} - 1\right) \leq \frac{1}{3}\left(\frac{1}{M(x, y, t)} - 1\right)$$

that is always true since $x + y \leq 3$. By the similar method in the proof of Example 3.8, we can show that all the required conditions of Theorem 3.7 hold and hence T has a fixed point. Indeed, 0 and 3 are two fixed points of T.

However, T is not a fuzzy contractive mapping in the sense of Gregori and Sapena [12]. To see this, take the points $x = 3$ and $y = 1$ and so

$$\left(\frac{1}{M(Tx, Ty, t)} - 1\right) = \frac{13}{9t} \not\leq \frac{\lambda}{t} = \lambda\left(\frac{1}{M(x, y, t)} - 1\right),$$

since $\lambda \in (0, 1)$.

For an improvement of the above results, we consider some extra sufficient conditions to establishing the uniqueness of the fixed point in Theorems 3.6 and 3.7. One of these conditions can be defined as follows.

(U_{α_1}) For all $x, y \in \mathrm{Fix}(T)$, we have $\alpha(x, y, t) \geq 1$ for all $t > 0$.

Alternatively, instead of the above condition, the following one can be used.

(U_{α_2}) For all $x, y \in \mathrm{Fix}(T)$, there exists $z \in X$ such that $\alpha(x, z, t) \geq 1$ and $\alpha(y, z, t) \geq 1$ for all $t > 0$.

Theorem 3.10. *Adding the condition* (U_{α_1}) *to the hypotheses of Theorem 3.6 (resp. Theorem 3.7), we obtain the uniqueness of the fixed point of* T.

Proof. The existence of a fixed point is obvious from the proof of Theorem 3.6 (resp. Theorem 3.7). To prove the uniqueness, suppose that x^* and y^* be any two fixed points of T with $x^* \neq y^*$. Thus, the condition U_{α_1} implies $\alpha(x^*, y^*, t) \geq 1$ for all $t > 0$. Now, by using the inequality (2), we have

$$\begin{aligned}\left(\frac{1}{M(x^*, y^*, t)} - 1\right) &= \left(\frac{1}{M(Tx^*, Ty^*, t)} - 1\right)\\ &\leq \alpha(x^*, y^*, t)\left(\frac{1}{M(Tx^*, Ty^*, t)} - 1\right)\\ &\leq \psi\left(\frac{1}{M(x^*, y^*, t)} - 1\right)\left(\frac{1}{M(x^*, y^*, t)} - 1\right)\\ &< \left(\frac{1}{M(x^*, y^*, t)} - 1\right),\end{aligned}$$

which is a contradiction. Hence, $x^* = y^*$. □

Theorem 3.11. *Adding the condition* (U_{α_2}) *to the hypotheses of Theorem 3.6 (resp. Theorem 3.7), we obtain the uniqueness of the fixed point of* T.

Proof. The existence of a fixed point is obvious from the proof of Theorem 3.6 (resp. Theorem 3.7). To prove the uniqueness, assume that x^* and y^* be any two fixed points of with $x^* \neq y^*$. Thus, by using the condition U_{α_2}, there exists $z \in X$ such that

$$\alpha(x^*, z, t) \geq 1 \quad \text{and} \quad \alpha(y^*, z, t) \geq 1 \quad \text{for all } t > 0.$$

Define the sequence $\{z_n\}$ in X by $z_0 = z$ and $z_{n+1} = Tz_n$ for all $n \in \mathbb{N} \cup \{0\}$. Due to T_{α_1} of Definition 3.1, we get

$$\alpha(x^*, z_n, t) \geq 1 \quad \text{and} \quad \alpha(y^*, z_n, t) \geq 1 \quad \text{for all } n \in \mathbb{N} \cup \{0\} \text{ and for all } t > 0. \tag{9}$$

Now, by applying the inequality (2) and thanks to (9), we obtain

$$\begin{aligned}\left(\frac{1}{M(x^*, z_{n+1}, t)} - 1\right) &= \left(\frac{1}{M(Tx^*, Tz_n, t)} - 1\right)\\ &\leq \alpha(x^*, z_n, t)\left(\frac{1}{M(Tx^*, Tz_n, t)} - 1\right)\\ &\leq \psi\left(\frac{1}{M(x^*, z_n, t)} - 1\right)\left(\frac{1}{M(x^*, z_n, t)} - 1\right)\\ &< \left(\frac{1}{M(x^*, z_n, t)} - 1\right).\end{aligned} \tag{10}$$

Hence, we deduce that $M(x^*, z_{n+1}, t) > M(x^*, z_n, t)$ for all $n \in \mathbb{N} \cup \{0\}$. Thus, the sequence $\{M(x^*, z_n, t)\}$ is an increasing sequence of positive real numbers in

$[0,1]$. So, there exists $\tau(t) \in [0,1]$ such that $\lim_{n\to+\infty} M(x^*, z_n, t) = \tau(t)$ for all $t > 0$. By taking the limit as $n \to +\infty$ in the inequality (10), we get

$$\lim_{n\to+\infty} \psi\left(\frac{1}{M(x^*, z_n, t)} - 1\right) = 1.$$

Since $\psi \in \Psi$, it follows that

$$\lim_{n\to+\infty} \left(\frac{1}{M(x^*, z_n, t)} - 1\right) = 0,$$

so equivalently, $\lim_{n\to+\infty} M(x^*, z_n, t) = 1$. Hence, $\lim_{n\to+\infty} z_n = x^*$.

In a similar way, one can show that $\lim_{n\to+\infty} z_n = y^*$. By the uniqueness of the limit, we get $x^* = y^*$. □

As immediate consequences of our results, we deduce versions of the Geraghty's fixed point theorem in the setup of fuzzy metric spaces and partially ordered fuzzy metric spaces.

Theorem 3.12. *Let $(X, M, \star)$ be a complete fuzzy metric space such that M be triangular and $T : X \to X$ be a self-mapping satisfying*

$$\left(\frac{1}{M(Tx, Ty, t)} - 1\right) \leq \psi\left(\frac{1}{M(x, y, t)} - 1\right)\left(\frac{1}{M(x, y, t)} - 1\right)$$

for all $x, y \in X$ and any $t > 0$, where $\psi \in \Psi$. Then T has a unique fixed point.

Proof. To prove the result, it suffices to take the function $\alpha : X \times X \times (0, +\infty) \to (-\infty, +\infty)$ by $\alpha(x, y, t) = 1$ for all $x, y \in X$ and any $t > 0$ in either Theorem 3.10 or Theorem 3.11. □

Theorem 3.13. *Let $(X, M, \star, \preceq)$ be a complete ordered fuzzy metric space such that M be triangular and $T : X \to X$ be a self-mapping satisfying the following conditions:*

(i) *there exists $\psi \in \Psi$ such that*

$$\left(\frac{1}{M(Tx, Ty, t)} - 1\right) \leq \psi\left(\frac{1}{M(x, y, t)} - 1\right)\left(\frac{1}{M(x, y, t)} - 1\right)$$

for all $x, y \in X$ with $x \preceq y$ and any $t > 0$;

(ii) *T is a nondecreasing mapping with respect to $\preceq$;*

(iii) *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$ with $M(x_0, Tx_0, t) > 0$ for all $t > 0$;*

(iv) *either T is continuous or if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x \in X$ as $n \to +\infty$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.*

Then T has a fixed point. Moreover, if

(v) *for all $x, y \in \mathrm{Fix}(T)$ either x and y are comparable, or there exists $z \in X$ which is comparable to x and y,*

then the fixed point of T is unique.

Proof. Consider the function $\alpha : X \times X \times (0, +\infty) \to (-\infty, +\infty)$ defined by

$$\alpha(x, y, t) = \begin{cases} 1, & x \preceq y, \\ 0, & \text{otherwise} \end{cases}$$

for all $t > 0$. The reader can show easily that T is a fuzzy α-Geraghty contraction type mapping, because of $\alpha(x, y, t) = 1$ for all $x, y \in X$ and for all $t > 0$ with $x \preceq y$ and $\alpha(x, y, t) = 0$ with $x \not\preceq y$. If $\alpha(x, y, t) \geq 1$ for all $t > 0$, then $x \preceq y$. As T is a nondecreasing mapping, we have $Tx \preceq Ty$. Thus, $\alpha(Tx, Ty, t) \geq 1$ for all $t > 0$. Also, if $\alpha(x, z, t) \geq 1$ and $\alpha(z, y, t) \geq 1$ for all $t > 0$, then $x \preceq z$ and $z \preceq y$. So, from transitivity, we have $x \preceq y$. Hence, $\alpha(x, y, t) \geq 1$ for all $t > 0$. Therefore, T is a triangular α-admissible mapping. In view of hypothesis (iii), there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$ which implies $\alpha(x_0, Tx_0, t) \geq 1$ for all $t > 0$. Precisely, in the case that T is a continuous mapping, the existence of a fixed point x^* of T is an immediate consequence of our Theorem 3.6.

On the other hand, define a sequence $\{x_n\}$ in X such that $\alpha(x_n, x_{n+1}, t) \geq 1$ for all $n \in \mathbb{N}$ and for all $t > 0$ and $x_n \to x \in X$ as $n \to +\infty$. Now, in view of the definition of function α, we have $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$. Hence, by hypothesis (iv), we get $x_n \preceq x$ for all $n \in \mathbb{N}$ and so $\alpha(x_n, x, t) \geq 1$ for all $n \in \mathbb{N}$ and for all $t > 0$. Thus, the existence of a fixed point x^* of T is a consequence of our Theorem 3.7.

Notice that the same considerations show that (v) of this theorem implies either Theorem 3.10 or Theorem 3.11. □

Next, we present the new notion of triangular β-admissible mappings in fuzzy metric spaces as follows.

Definition 3.14. Let $(X, M, \star)$ be a fuzzy metric space and $T : X \to X$ be a given mapping. We say that T is a triangular β-admissible mapping if there exists a function $\beta : X \times X \times (0, +\infty) \to (0, +\infty)$ such that

(T_{β_1}) $\beta(x, y, t) \leq 1$ implies $\beta(Tx, Ty, t) \leq 1$ for all $x, y \in X$ and for all $t > 0$;
(T_{β_2}) $\beta(x, z, t) \leq 1$ and $\beta(z, y, t) \leq 1$ imply $\beta(x, y, t) \leq 1$ for all $x, y, z \in X$ and for all $t > 0$.

We will require the following lemma which we use in the next theorem.

Lemma 3.15. *Let $(X, M, \star)$ be a fuzzy metric space and $T : X \to X$ be a triangular β-admissible mapping. Assume that there exists $x_0 \in X$ such that $\beta(x_0, Tx_0, t) \leq 1$. Define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$. Then*

$$\beta(x_m, x_n, t) \leq 1 \qquad \textit{for all } m, n \in \mathbb{N} \textit{ with } m < n.$$

Proof. The proof can be completed using a similar technique as given in the proof of Lemma 3.2. Therefore, to avoid repetitions, we omit the details. □

Let Φ denote the class of all functions $\varphi : [0, 1] \to [1, +\infty)$ satisfying the following condition:

$$\varphi(t_n) \to 1 \quad \text{as} \quad n \to +\infty \quad \text{implies} \quad t_n \to 1 \quad \text{as} \quad n \to +\infty.$$

We now introduce the concept of fuzzy β-φ-contractive mappings and prove the fixed point theorems for such mappings.

Definition 3.16. Let $(X, M, \star)$ be a fuzzy metric space. A mapping $T : X \to X$ is said to be a fuzzy β-φ-contractive mapping if there exist two functions $\beta : X \times X \times (0, +\infty) \to (0, +\infty)$ and $\varphi \in \Phi$ such that $M(x, y, t) > 0$ implies that

$$\beta(x, y, t)M(Tx, Ty, t) \geq \varphi\big(M(x, y, t)\big)M(x, y, t) \tag{11}$$

for all $x, y \in X$ with $x \neq y$ and any $t > 0$.

Theorem 3.17. *Let $(X, M, \star)$ be a complete non-Archimedean fuzzy metric space and $T : X \to X$ be a self-mapping satisfying the following assertions:*

(i) *T is a fuzzy β-φ-contractive mapping;*
(ii) *T is a triangular β-admissible mapping;*
(iii) *there exists $x_0 \in X$ such that $\beta(x_0, Tx_0, t) \leq 1$ for all $t > 0$;*
(iv) *if $\{x_n\}$ is a sequence such that $\beta(x_n, x_{n+1}, t) \leq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to +\infty$, then $\beta(x_n, x, t) \leq 1$ for all $n \in \mathbb{N}$.*

Then T has a fixed point.

Proof. Following (iii), there exists $x_0 \in X$ such that $\beta(x_0, Tx_0, t) \leq 1$ for all $t > 0$. Define a sequence $\{x_n\}$ in X by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N} \cup \{0\}$, then $x^* = x_n$ is a fixed point of T and the result is proved. Hence, we suppose that $x_n \neq x_{n-1}$ for all $n \in \mathbb{N}$.

Due to Lemma 3.15, we have

$$\beta(x_{n-1}, x_n, t) \leq 1 \qquad \text{for all } n \in \mathbb{N} \text{ and for all } t > 0. \tag{12}$$

By applying the inequality (11) with $x = x_{n-1}$ and $y = x_n$ and takes to (12), we obtain

$$\begin{aligned} M(x_n, x_{n+1}, t) &= M(Tx_{n-1}, Tx_n, t) \\ &\geq \beta(x_{n-1}, x_n, t)M(Tx_{n-1}, Tx_n, t) \\ &\geq \varphi\big(M(x_{n-1}, x_n, t)\big)M(x_{n-1}, x_n, t) \\ &\geq M(x_{n-1}, x_n, t). \end{aligned} \tag{13}$$

Thus, $\{M(x_{n-1}, x_n, t)\}$ is an increasing sequence in $(0, 1]$. Hence, there exists $\tau(t) \in (0, 1]$ such that $\lim_{n\to+\infty} M(x_{n-1}, x_n, t) = \tau(t)$ for all $t > 0$. We will prove that $\tau(t) = 1$ for all $t > 0$. Taking (13) into account, we have

$$\frac{M(x_n, x_{n+1}, t)}{M(x_{n-1}, x_n, t)} \geq \varphi\big(M(x_{n-1}, x_n, t)\big) \geq 1,$$

which implies that $\lim_{n\to+\infty} \varphi(M(x_{n-1}, x_n, t)) = 1$. Regarding the property of the function φ, we conclude that

$$\lim_{n\to+\infty} M(x_n, x_{n+1}, t) = 1.$$

Now, we shall prove that $\{x_n\}$ is a Cauchy sequence. Arguing by contradiction, we assume that $\{x_n\}$ is not a Cauchy sequence. Thus, there exist $\varepsilon \in (0, 1)$ and $t_0 > 0$ such that for all $k \in \mathbb{N}$ there exist $n(k), m(k) \in \mathbb{N}$ with $m(k) > n(k) \geq k$ and

$$M(x_{m(k)}, x_{n(k)}, t_0) \leq 1 - \varepsilon.$$

Assume that $m(k)$ is the least integer exceeding $n(k)$ satisfying the above inequality. Equivalently,

$$M(x_{m(k)-1}, x_{m(k)}, t_0) > 1 - \varepsilon \tag{14}$$

and so, for all $k \in \mathbb{N}$, we get

$$\begin{aligned} 1 - \varepsilon &\geq M(x_{m(k)}, x_{n(k)}, t_0) \\ &\geq M(x_{m(k)-1}, x_{m(k)}, t_0) \star M(x_{m(k)-1}, x_{n(k)}, t_0) \\ &> \tau(t_0) \star (1 - \varepsilon). \end{aligned} \tag{15}$$

By taking the limit as $n \to +\infty$ in (15), we deduce that

$$\lim_{n \to +\infty} M(x_{m(k)}, x_{n(k)}, t_0) = 1 - \varepsilon. \tag{16}$$

From

$$\begin{aligned} &M(x_{m(k)+1}, x_{n(k)+1}, t_0) \\ &\geq M(x_{m(k)+1}, x_{m(k)}, t_0) \star M(x_{m(k)}, x_{n(k)}, t_0) \star M(x_{n(k)}, x_{n(k)+1}, t_0) \end{aligned}$$

and

$$\begin{aligned} &M(x_{m(k)}, x_{n(k)}, t_0) \\ &\geq M(x_{m(k)+1}, x_{m(k)}, t_0) \star M(x_{m(k)+1}, x_{n(k)+1}, t_0) \star M(x_{n(k)}, x_{n(k)+1}, t_0), \end{aligned}$$

we get

$$\lim_{n \to +\infty} M(x_{m(k)+1}, x_{n(k)+1}, t_0) = 1 - \varepsilon.$$

In view of Lemma 3.15, we have $\beta(x_{m(k)}, x_{n(k)}, t) \leq 1$. By applying the inequality (11) with $x = x_{m(k)}$ and $y = x_{n(k)}$, we have

$$\begin{aligned} M(x_{m(k)+1}, x_{n(k)+1}, t_0) &= M(Tx_{m(k)}, Tx_{n(k)}, t_0) \\ &\geq \beta(x_{m(k)}, x_{n(k)}, t_0) M(Tx_{m(k)}, Tx_{n(k)}, t_0) \\ &\geq \varphi\big(M(x_{m(k)}, x_{n(k)}, t_0)\big) M(x_{m(k)}, x_{n(k)}, t_0), \end{aligned}$$

which yields that

$$\frac{M(x_{m(k)+1}, x_{n(k)+1}, t_0)}{M(x_{m(k)}, x_{n(k)}, t_0)} \geq \varphi\big(M(x_{m(k)}, x_{n(k)}, t_0)\big) \geq 1.$$

By taking the limit as $k \to +\infty$ in the above inequality, we get

$$\lim_{k \to +\infty} \varphi\big(M(x_{m(k)}, x_{n(k)}, t_0)\big) = 1.$$

Regarding the property of the function φ, we conclude that

$$\lim_{k \to +\infty} M(x_{m(k)}, x_{n(k)}, t_0) = 1,$$

which implies by (16) that

$$1 - \varepsilon = \lim_{k \to +\infty} M(x_{m(k)}, x_{n(k)}, t_0) = 1,$$

a contradiction since $\varepsilon = 0$. This means that $\{x_n\}$ is a Cauchy sequence. As $(X, M, \star)$ is complete, it follows that the sequence $\{x_n\}$ converges to some $x^* \in X$, that is, $x_n \to x^*$ as $n \to +\infty$. Now, by using Lemma 3.15 together with the condition (iv), we have

$$\beta(x_n, x^*, t) \leq 1 \qquad \text{for all } n \in \mathbb{N} \text{ and for all } t > 0. \tag{17}$$

By applying (NA) and using (11) and (17), we obtain

$$\begin{aligned} M(Tx^*, x^*, t) &\geq M(Tx^*, Tx_n, t) \star M(x_{n+1}, x^*, t) \\ &\geq \beta(x_n, x^*, t) M(Tx_n, Tx^*, t) \star M(x_{n+1}, x^*, t) \\ &\geq \varphi\big(M(x_n, x^*, t)\big) M(x_n, x^*, t) \star M(x_{n+1}, x^*, t) \\ &\geq M(x_n, x^*, t) \star M(x_{n+1}, x^*, t). \end{aligned}$$

On taking the limit as $n \to +\infty$ in the above inequality, we conclude that $Tx^* = x^*$, i.e. x^* is a fixed point of T. □

Next, we provide an example showing how Theorem 3.17 can be used.

Example 3.18. Let $X = [1, +\infty)$, $a \star b = \min\{a, b\}$ and $M(x, y, t) = \frac{\min\{x,y\}}{\max\{x,y\}}$ for all $t > 0$. Clearly, $(X, M, \star)$ is a complete non-Archimedean fuzzy metric space.

Define the mapping $T : X \to X$ by

$$Tx = \begin{cases} 2x, & x \in [1, 3], \\ 4, & x \in (3, +\infty), \end{cases}$$

and the function $\beta : X \times X \times (0, +\infty) \to (-\infty, +\infty)$ as

$$\beta(x, y, t) = \begin{cases} 1, & x, y \in [1, 3], \\ 2, & \text{otherwise} \end{cases}$$

for all $t > 0$. It is easy to see that T is a fuzzy β-φ contractive mapping with $\varphi(r) = 1$ for all $r \in [0, 1]$. In fact, let $x, y \in (0, 1]$ and $x < y$. Then

$$\beta(x, y, t) M(Tx, Ty, t) = \frac{x}{y} \geq \frac{x}{y} = \varphi\big(M(x, y, t)\big) M(x, y, t)$$

for all $t > 0$. Otherwise, $\beta(x, y, t) = 2$ and so

$$\beta(x, y, t) M(Tx, Ty, t) = 2 \geq \frac{x}{y} = \varphi\big(M(x, y, t)\big) M(x, y, t)$$

for all $t > 0$. Obviously, T is triangular β-admissible. Further, there exists $x_0 \in X$ such that $\beta(x_0, Tx_0, t) \leq 1$ for all $t > 0$. Indeed, for $x_0 = 1$, we have $\beta(1, T1, t) = 1$.

Finally, let $\{x_n\}$ be a sequence in X such that $\beta(x_n, x_{n+1}, t) \leq 1$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to +\infty$. By the definition of the function β, it follows that $x_n \in [1, 3]$ for all $n \in \mathbb{N}$. Now, if $x > 3$, we get $M(x_n, x, t) = \frac{\min\{x_n,x\}}{\max\{x_n,x\}} = \frac{x_n}{x} \leq \frac{3}{x} < 1$ which contradicts (i) of Definition 2.5, since $\lim_{n \to +\infty} M(x_n, x, t) = 1$ for all $t > 0$. Hence, we deduce that $x \in [1, 3]$. Therefore, $\beta(x_n, x, t) = 1$ for all $n \in \mathbb{N}$. Thus, all the required hypotheses of Theorem 3.17 are satisfied and hence T has a fixed point. In fact, 4 is the fixed point of T.

Now, we give some sufficient conditions to obtain the uniqueness of the fixed point in the previous theorem. Precisely, we consider the following hypothesis:

(U_{β_1}) For all $x, y \in \mathrm{Fix}(T)$, we have $\beta(x, y, t) \leq 1$ for all $t > 0$.

(U_{β_2}) For all $x, y \in \mathrm{Fix}(T)$, there exists $z \in X$ such that $\beta(x, z, t) \leq 1$ and $\beta(y, z, t) \leq 1$ for all $t > 0$.

Theorem 3.19. *Adding the condition* (U_{β_1}) *to the hypotheses of Theorem 3.17, we obtain the uniqueness of the fixed point of* T.

Proof. The proof of this theorem can obtained by using similar arguments as given in the proof of Theorem 3.10. So we omit the proof. □

Theorem 3.20. *Adding the condition* (U_{β_2}) *to the hypotheses of Theorem 3.17, we obtain the uniqueness of the fixed point of* T.

Proof. This theorem can be proved by the same method as was employed in Theorem 3.11. So we omit the proof. □

4. **Conclusion**

In view of their interesting applications, searching for fixed point theorems in fuzzy and non-Archimedean fuzzy metric spaces has received considerable attention through the last decades. In particular, researchers are currently focusing on weaker form of contractive conditions. In the present work, we introduced two new notions called fuzzy α-Geraghty contraction type mapping via triangular α-admissible mapping and fuzzy β-φ-contractive mapping via triangular β-admissible mapping by using Geraghty type contractive conditions. Subsequently, we proved some interesting results which guarantee the existence and uniqueness of fixed points for these new types of contractive mappings in the setting of complete fuzzy metric spaces and complete non-Archimedean fuzzy metric spaces. As a result, we also generalized and extended the well known generalizations [3, 4, 14] of Geraghty's theorem [9]. Further, the attached examples illustrate the validity of the obtained results. The new concepts lead to further investigations and applications and our work enriches our knowledge of fixed points in fuzzy metric spaces.

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