ON (α, β)-FUZZY H_v-IDEALS OF H_v-RINGS

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Abstract. Using the notion of "belongingness ($\in$)" and "quasi-coincidence (q)" of fuzzy points with fuzzy sets, we introduce the concept of an (α, β)-fuzzy H_v-ideal of an H_v-ring, where α, β are any two of $\{\in, q, \in \vee q, \in \wedge q\}$ with $\alpha \neq \in \wedge q$. Since the concept of $(\in, \in \vee q)$-fuzzy H_v-ideals is an important and useful generalization of ordinary fuzzy H_v-ideals, we discuss some fundamental aspects of $(\in, \in \vee q)$-fuzzy H_v-ideals. A fuzzy subset A of an H_v-ring R is an $(\in, \in \vee q)$-fuzzy H_v-ideal if and only if an A_t, level cut of A, is an H_v-ideal of R, for all $t \in (0, 0.5]$. This shows that an $(\in, \in \vee q)$-fuzzy H_v-ideal is a generalization of the existing concept of fuzzy H_v-ideal. Finally, we extend the concept of a fuzzy subgroup with thresholds to the concept of a fuzzy H_v-ideal with thresholds.

1. Introduction

The theory of algebraic hyperstructures which is a generalization of the concept of ordinary algebraic structures was first introduced by Marty [15] and since then many researchers have developed this theory, a short review of which appears in [4]. In a recent book [5] Corsini and Leoreanu have presented some of the numerous applications of algebraic hyperstructures, especially those from the last fifteen years, in the following fields: geometry, hypergraphs, binary relations, lattices, fuzzy sets and rough sets, automata, cryptography, codes, median algebras, relation algebras, artificial intelligence and probabilities. Vougiouklis in the fourth AHA congress (1999) [19] introduced a new class of hyperstructures so-called H_v-structure, and Davvaz [7] surveyed the theory of H_v-structures. The H_v-structures are hyperstructures where equality is replaced by non-empty intersection. In this paper, we deal with H_v-rings. H_v-rings are the largest class of algebraic systems that satisfy ring-like axioms. After the introduction of fuzzy sets by Zadeh [21], reconsideration of the concept of classical mathematics began. On the other hand, because of the importance of group theory in mathematics, as well as its many areas of application, the notion of fuzzy subgroup was defined by Rosenfeld [17] and its structure was investigated. This subject has been studied further by many mathematicians. Liu [14] introduced the notion of fuzzy subrings and ideals. Using the notion of "belongingness ($\in$)" and "quasi-coincidence (q)" of fuzzy points with fuzzy sets, the concept of (α, β)-fuzzy subgroup where α, β are any two of $\{\in, q, \in \vee q, \in \wedge q\}$ with $\alpha \neq \in \wedge q$ was introduced in [1]. The most viable generalization of Rosenfeld's fuzzy subgroup is the notion of $(\in, \in \vee q)$-fuzzy subgroups, the detailed study of which may be found in [3]. The concept of an $(\in, \in \vee q)$-fuzzy subring and ideal of

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a ring have been introduced in [2] and the concept of $(\in, \in \vee q)$-fuzzy subnear-ring and ideal of a near-ring have been introduced in [6]. Fuzzy sets and hyperstructures introduced by Zadeh and Marty, respectively, are now studied both from the theoretical point of view and for their many applications. The relations between fuzzy sets and hyperstructures have been already considered by many authors. In [8,9,11], Davvaz applied the concept of fuzzy sets to the theory of algebraic hyperstructures and defined fuzzy H_v-subgroups, fuzzy H_v-ideals and fuzzy H_v-submodules, which are generalizations of the concepts of Rosenfeld's fuzzy subgroups, fuzzy ideals and fuzzy submodules. The concept of a fuzzy H_v-ideal and H_v-subring has been studied further in [10,12]. In Section 2, we recall some basic definitions and results about H_v-structures. In Section 3, we introduce the concept of (α, β)-fuzzy H_v-ideal of an H_v-ring and investigate related results. Since the concept of $(\in, \in \vee q)$-fuzzy H_v-ideal is an important and useful generalization of ordinary fuzzy H_v-ideals, some fundamental aspects of $(\in, \in \vee q)$-fuzzy H_v-ideals have been discussed in Section 4. A fuzzy subset A of an H_v-ring R is an $(\in, \in \vee q)$-fuzzy H_v-ideal if and only if A_t, level cut of A, is an H_v-ideal of R for all $t \in (0, 0.5]$. This shows that an $(\in, \in \vee q)$-fuzzy H_v-ideal is a generalization of the existing concept of fuzzy H_v-ideal. Finally, based on [20], we extend the concept of a fuzzy subgroup with thresholds to the concept of fuzzy H_v-ideal with thresholds.

2. H_v-structures

A *hypergroupoid* $(H, \circ)$ is a non-empty set H with a *hyperoperation* $\circ$ defined on H, i.e. a mapping of $H \times H$ into the family of non-empty subsets of H. If $(x, y) \in H \times H$, its image under $\circ$ is denoted by $x \circ y$. If $A, B \subseteq H$, then $A \circ B$ is given by $A \circ B = \bigcup \{x \circ y \mid x \in A,\ y \in B\}$. $x \circ A$ is used for $\{x\} \circ A$ and $A \circ x$ for $A \circ \{x\}$.

A hypergroupoid $(H, \circ)$ is called an H_v*-group* if for all $x, y, z \in H$ the following two conditions hold:

(i) $x \circ (y \circ z) \cap (x \circ y) \circ z \neq \emptyset$,
(ii) $x \circ H = H \circ x = H$.

The second condition, called the *reproduciblity condition*, means that for any $x, y \in H$ there exist $u, v \in H$ such that $y \in x \circ u$ and $y \in v \circ x$. If $(H, \circ)$ satisfies only the first condition, then it is called an H_v-semigroup.

An H_v-ring [18] is a multi-valued system $(R, +, \cdot)$ which satisfies the following the ring-like axioms:

(i) $(R, +)$ is an H_v-group,
(ii) $(R, \cdot)$ is an H_v-semigroup,
(iii) $(\cdot)$ is weak distributive with respect to $(+)$, i.e., for all $x, y, z \in R$ we have

$$x \cdot (y + z) \cap ((x \cdot y) + (x \cdot z)) \neq \emptyset,$$

$$(x + y) \cdot z \cap ((x \cdot z) + (y \cdot z)) \neq \emptyset.$$

Let R be an H_v-ring. A non-empty subset I of R is called a left (right) H_v-ideal if the following conditions hold:

(i) $(I, +)$ is an H_v-subgroup of $(R, +)$,

(ii) $R \cdot I \subseteq I$ $(I \cdot R \subseteq I)$.

A mapping $A : X \longrightarrow [0,1]$, where X is an ordinary non-empty set, is called a *fuzzy subset* of X. In 1971, Rosenfeld [17] applied the concept of fuzzy sets to the theory of groups and studied fuzzy subgroups of a group. Since then many papers concerning various fuzzy algebraic structures have appeared in the literature. Liu [14] introduced and studied the notions of fuzzy subrings and fuzzy ideals. In [7-12], Davvaz applied the concept of fuzzy set theory to algebraic hyperstructures and, in particular [11], defined the concept of fuzzy H_v-ideal of an H_v-ring which is a generalization of the concept of fuzzy ideal.

Definition 2.1. (Davvaz [11]). Let R be an H_v-ring and A be a fuzzy subset of R. Then A is said to be a fuzzy left (right) H_v-ideal of R if the following axioms hold:

(i) $A(x) \wedge A(y) \leq \bigwedge\limits_{\alpha \in x+y} A(\alpha)$,for all $x, y \in R$,

(ii) for all $x, a \in R$ there exists $y \in R$ such that $x \in a + y$ and

$$A(a) \wedge A(x) \leq A(y),$$

(iii) for all $x, a \in R$ there exists $z \in R$ such that $x \in z + a$ and

$$A(a) \wedge A(x) \leq A(z),$$

(iv) $A(y) \leq \bigwedge\limits_{z \in x \cdot y} A(z)$, for all $x, y \in R$

$(A(x) \leq \bigwedge\limits_{z \in x \cdot y} A(z)$, for all $x, y \in R)$.

(ii) is called the left fuzzy reproduction axiom and (iii) is called the right fuzzy reproduction axiom.

In this paper we present all the proofs for left H_v-ideals. Similar results hold for right H_v-ideals.

Let A be a fuzzy subset of a non-empty set X and let $t \in (0,1]$. The set $A_t = \{x \in X |\ A(x) \geq t\}$ is called a level cut of A.

Theorem 2.2. (cf. [11]). *Let R be an H_v-ring and A a fuzzy subset of R. Then A is a fuzzy left (right) H_v-ideal of R if and only if for every $t \in (0,1]$, A_t $(\neq \emptyset)$ is a left (right) H_v-ideal of R.*

When A is a fuzzy H_v-ideal of R, A_t is called a *level H_v-ideal* of R. The concept of level H_v-ideals has been used extensively to characterize various properties of fuzzy H_v-ideals.

3. (α, β)-fuzzy H_v-ideals

A fuzzy subset A of R of the form

$$A(y) = \begin{cases} t(\neq 0) & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}$$

is said to be a *fuzzy point with support* x and *value* t and is denoted by x_t. A fuzzy point x_t is said to belong to (resp. be quasi-coincident with) a fuzzy set A, written as $x_t \in A$ (resp. $x_t qA$) if $A(x) \geq t$ (resp. $A(x) + t > 1$). If $x_t \in A$ or $x_t qA$, then we write $x_t \in \vee qA$. The symbol $\overline{\in \vee q}$ means either $\in$ or q hold. In what follows, unless otherwise specified, α and β will denote any one of $\in, q, \in \vee q$ or $\in \wedge q$. The idea of quasi-coincidence of a fuzzy point with a fuzzy set, which is mentioned in [16] play a vital role in generating another type of fuzzy subgroups, called (α, β)-*fuzzy subgroups*, introduced by Bhakat and Das [1]. Based on [3], we can extend the concept of (α, β)-fuzzy subgroups to the concept of (α, β)-fuzzy H_v-ideals.

Definition 3.1. Let R be an H_v-ring. A fuzzy subset A of R is said to be an (α, β)-*fuzzy left (right)* H_v-*ideal* of R if for all $t, r \in (0, 1]$,

(i) $x_t \alpha A, y_r \alpha A$ implies $z_{t\wedge r} \beta A$, for all $z \in x + y$;
(ii) $x_t \alpha A, a_r \alpha A$ implies $y_{t\wedge r} \beta A$, for some $y \in R$ with $x \in a + y$;
(iii) $x_t \alpha A, a_r \alpha A$ implies $z_{t\wedge r} \beta A$, for some $z \in R$ with $x \in z + a$;
(iv) $y_t \alpha A$ and $x \in R$ imply $z_t \beta A$, for all $z \in x \cdot y$
($x_t \alpha A$ and $y \in R$ imply $z_t \beta A$, for all $z \in x \cdot y$).

Let A be a fuzzy subset of R such that $A(x) \leq 0.5$ for all $x \in R$. Let $x \in R$ and $t \in (0, 1]$ be such that $x_t \in \wedge qA$. Then $A(x) \geq t$ and $A(x) + t > 1$. It follows that $1 < A(x) + t \leq A(x) + A(x) = 2A(x)$, which implies $A(x) > 0.5$. This means that $\{x_t | x_t \in \wedge qA\} = \emptyset$. Therefore the case $\alpha = \in \wedge q$ in Definition 3.1 will be omitted.

Proposition 3.2. *Let R be an H_v-ring. Every $(\in \vee q, \in \vee q)$-fuzzy left (right) H_v-ideal of R is an $(\in, \in \vee q)$-fuzzy left (right) H_v-ideal of R.*

Proof. Let A be an $(\in \vee q, \in \vee q)$-fuzzy left H_v-ideal of R.

(i) Suppose that $x, y \in R$ and $t, r \in [0, 1]$ be such that $x_t, y_r \in A$. Then $x_t, y_r \in \vee qA$, and so $z_{t\wedge r} \in \vee qA$, for all $z \in x + y$.
(ii) Now, let $x, a \in R$ and $t, r \in (0, 1]$ be such that $x_t, a_r \in A$. Then $x_t, a_r \in \vee qA$ which implies $y_{t\wedge r} \in \vee qA$, for some $y \in R$ with $x \in a + y$.
(iii) The third condition is similarly verified.
(iv) Finally, let $x, y \in R$ and $t \in (0, 1]$ be such that $y_t \in A$. Then $y_t \in \vee qA$ which implies $z_t \in \vee qA$, for all $z \in x \cdot y$.

□

Proposition 3.3. *Let R be an H_v-ring. Every $(\in, \in)$-fuzzy left (right) H_v-ideal of R is an $(\in, \in \vee q)$-fuzzy left (right) H_v-ideal of R.*

Proof. Straightforward. □

Lemma 3.4. *If A is a fuzzy left (right) H_v-ideal of R, then the characteristic function χ_A of A is an $(\in, \in)$-fuzzy left (right) H_v-ideal of R.*

Now, we give the main result on general (α, β)-fuzzy left (right) H_v-ideals of H_v-rings.

Theorem 3.5. *Let A be a non-zero (α, β)-fuzzy left (right) H_v-ideal of R. Then the set $SuppA = \{x \in R | A(x) > 0\}$ is a left (right) H_v-ideal of R.*

Proof. The proof is a simple modification of the proofs of Theorems 3.6, 3.7, 3.8 and Corollary 3.10 in [13]. □

A fuzzy subset A of an H_v-ring R is said to be *proper* if ImA has at least two elements. Two fuzzy subsets are said to be *equivalent* if they have same family of level subsets. Otherwise, they are said to be non-equivalent.

Theorem 3.6. *Let R have proper H_v-ideals. A proper $(\in,\in)$-fuzzy H_v-ideal A of R such that card $ImA \geq 3$, can be expressed as the union of two proper non-equivalent $(\in,\in)$-fuzzy H_v-ideals of R.*

Proof. The proof is a modification of the proof of Theorem 3.17 in [3].

□

4. $(\in,\in\vee q)$-fuzzy H_v-ideals

In this section, we consider a special case of (α,β)-fuzzy H_v-ideals. An $(\in,\in\vee q)$-fuzzy H_v-ideal is an important and useful generalization of ordinary fuzzy H_v-ideal.

Definition 4.1. Let R be an H_v-ring. A fuzzy subset A of R is said to be an $(\in,\in\vee q)$-fuzzy left (right) H_v-ideal of R if for all $t,r\in(0,1]$,

(i) $x_t\in A, y_r\in A$ implies $z_{t\wedge r}\in\vee qA$, for all $z\in x+y$;
(ii) $x_t\in A, a_r\in A$ implies $y_{t\wedge r}\in\vee qA$, for some $y\in R$ with $x\in a+y$;
(iii) $x_t\in A, a_r\in A$ implies $z_{t\wedge r}\in\vee qA$, for some $z\in R$ with $x\in z+a$;
(iv) $y_t\in A$ and $x\in R$ imply $z_t\in\vee qA$ for all $z\in x\cdot y$
($x_t\in A$ and $y\in R$ imply $z_t\in\vee qA$, for all $z\in x\cdot y$).

It is easy to see that for any subset A of R, χ_A is an $(\in,\in\vee q)$-fuzzy left (right) H_v-ideal of R if and only if A is a left (right) H_v-ideal of R.

Example 4.2. Let $R=\{a,b,c,d\}$ be a set, and consider addition and multiplication tables as follows:

$+$	a	b	c	d
a	a	b	c	d
b	b	$\{a,b\}$	d	c
c	c	d	$\{a,c\}$	b
d	d	c	b	$\{a,d\}$

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	b	b	b
c	a	c	c	c
d	a	d	d	d

Then we can easily see that $(R,+,\cdot)$ is an H_v-ring. Let $A:R\longrightarrow[0,1]$ be defined by

$$A(a)=0.6,\quad A(b)=A(c)=A(d)=0.8.$$

Then A is an $(\in,\in\vee q)$-fuzzy H_v-ideal of R but not an ordinary fuzzy H_v-ideal.

Proposition 4.3. *Conditions (i)-(iv) in Definition 4.1, are respectively equivalent to the following:*

(1) $A(x)\wedge A(y)\wedge 0.5\leq \bigwedge_{z\in x+y} A(z)$, *for all $x,y\in R$;*

(2) *for all $x,a\in R$ there exists $y\in R$ such that $x\in a+y$ and*

$$A(a)\wedge A(x)\wedge 0.5\leq A(y);$$

(3) *for all* $x, a \in R$ *there exists* $z \in R$ *such that* $x \in z + a$ *and*

$$A(a) \wedge A(x) \wedge 0.5 \leq A(z);$$

(4) $A(y) \wedge 0.5 \leq \bigwedge_{z \in x \cdot y} A(z),$ *for all* $x, y \in R$

$(A(x) \wedge 0.5 \leq \bigwedge_{z \in x \cdot y} A(z),$ *for all* $x, y \in R).$

Proof. ($\mathrm{i} \Longrightarrow 1$): Suppose that $x, y \in R$. We consider the following cases:

(a) $A(x) \wedge A(y) < 0.5$,
(b) $A(x) \wedge A(y) \geq 0.5$.

Case a: Assume that there exists $z \in x + y$ such that $A(z) < A(x) \wedge A(y) \wedge 0.5$, which implies $A(z) < A(x) \wedge A(y)$. Choose t such that $A(z) < t < A(x) \wedge A(y)$. Then $x_t, y_t \in A$, but $z_t \overline{\in \vee q A}$ which contradicts (i).

Case b: Assume that $A(z) < 0.5$ for some $z \in x + y$. Then $x_{0.5}, y_{0.5} \in A$, but $z_{0.5} \overline{\in \vee q A}$, a contradiction.

Hence (1) holds.

($\mathrm{ii} \Longrightarrow 2$): Suppose that $x, a \in R$. We consider the following cases:

(a) $A(x) \wedge A(a) < 0.5$,
(b) $A(x) \wedge A(a) \geq 0.5$.

Case a: Assume that for all y with $x \in a + y$, we have $A(y) < A(x) \wedge A(a)$. Choose t such that $A(y) < t < A(x) \wedge A(a)$ and $t + A(y) < 1$. Then $x_t, a_t \in A$, but $y_t \overline{\in \vee q A}$, which contradicts (ii).

Case b: Assume that for all y with $x \in a + y$, we have

$$A(y) < A(x) \wedge A(a) \wedge 0.5.$$

Then $x_{0.5}, a_{0.5} \in A$, but $y_{0.5} \overline{\in \vee q A}$, which contradicts (ii).

Hence (2) holds.

($\mathrm{iii} \Longrightarrow 3$): The proof is similar to ($\mathrm{ii} \Longrightarrow 2$).

($\mathrm{iv} \Longrightarrow 4$): Suppose $x, y \in R$. We consider the following cases:

(a) $A(y) < 0.5$,
(b) $A(y) \geq 0.5$.

Case a: Assume that there exists $z \in x \cdot y$ such that $A(z) < A(y) \wedge 0.5$, which implies $A(z) < A(y)$. Choose t such that $A(z) < t < A(y)$. Then $y_t \in A$, but $z_t \overline{\in \vee q A}$, which contradicts (iv).

Case b: Assume that $A(z) < 0.5$ for some $z \in x \cdot y$. Then $y_{0.5} \in A$, but $z_{0.5} \overline{\in \vee q A}$, a contradiction. Hence (4) holds.

($1 \Longrightarrow \mathrm{i}$): Let $x_t, y_r \in A$. Then $A(x) \geq t$ and $A(y) \geq r$. For every $z \in x + y$ we have

$$A(z) \geq A(x) \wedge A(y) \wedge 0.5 \geq t \wedge r \wedge 0.5.$$

If $t \wedge r > 0.5$, then $A(z) \geq 0.5$ which implies $A(z) + t \wedge r > 1$.

If $t \vee r \leq 0.5$, then $A(z) \geq t \wedge r$.

Therefore $z_{t \wedge r} \in \vee q A$ for all $z \in x + y$.

($2 \Longrightarrow$ ii): Let $x_t, a_r \in A$. Then $A(x) \geq t$ and $A(a) \geq r$. Now, for some y with $x \in a + y$ we have

$$A(y) \geq A(a) \wedge A(x) \wedge 0.5 \geq t \wedge r \wedge 0.5.$$

If $t \wedge r > 0.5$, then $A(y) \geq 0.5$ which implies $A(y) + t \wedge r > 1$.
If $t \vee r \leq 0.5$, then $A(y) \geq t \wedge r$.
Therefore $y_{t \wedge r} \in \vee q A$. Hence (ii) holds.
($3 \Longrightarrow$ iii): The proof is similar to ($2 \Longrightarrow$ ii).
($4 \Longrightarrow$ iv): Let $y_t \in A$ and $x \in R$. Then $A(y) \geq t$. For every $z \in x \cdot y$ we have

$$A(z) \geq A(y) \wedge 0.5 \geq t \wedge 0.5.$$

If $t > 0.5$, then $A(z) \geq 0.5$ which implies $A(z) + t > 1$.
If $t \leq 0.5$, then $A(z) \geq t$.
Therefore $z_t \in \vee q A$ for all $z \in x \cdot y$. □

By Definition 4.1 and Proposition 4.3, we immediately get:

Corollary 4.4. *A fuzzy subset A of an H_v-ring R is an $(\in, \in \vee q)$-fuzzy left (right) H_v-ideal of R if and only if the conditions (1)-(4) in Proposition 4.3 hold.*

Now, we characterize $(\in, \in \vee q)$-fuzzy left (right) H_v-ideals by their level H_v-ideals.

Theorem 4.5. *Let R be an H_v-ring and A a fuzzy subset of R. If A is an $(\in, \in \vee q)$-fuzzy left (right) H_v-ideal of R, then for all $0 < t \leq 0.5$, A_t is an empty set or a left (right) H_v-ideal of R. Conversely, if A_t $(\neq \emptyset)$ is a left (right) H_v-ideal of R for all $0 < t \leq 0.5$, then A is an $(\in, \in \vee q)$-fuzzy left (right) H_v-ideal of R.*

Proof. Let A be an $(\in, \in \vee q)$-fuzzy left H_v-ideal of R and $0 < t \leq 0.5$. Let $x, y \in A_t$. Then $A(x) \geq t$ and $A(y) \geq t$. Now

$$\bigwedge_{z \in x+y} A(z) \geq A(x) \wedge A(y) \wedge 0.5 \geq t \wedge 0.5 = t.$$

Therefore for every $z \in x + y$ we have $A(z) \geq t$ or $z \in A_t$, so $x + y \subseteq A_t$. Hence for every $a \in A_t$ we have $a + A_t \subseteq A_t$. Now, let $x, a \in A_t$. Then there exists $y \in R$ such that $x \in a + y$ and $A(a) \wedge A(x) \wedge 0.5 \leq A(y)$. From $x, a \in A_t$, we have $A(x) \geq t$ and $A(a) \geq t$, and so

$$t = t \wedge t \wedge 0.5 \leq A(a) \wedge A(x) \wedge 0.5 \leq A(y).$$

Hence $y \in A_t$, and this proves that $A_t \subseteq a + A_t$.
Now, let $y \in A_t$ and $x \in R$. Then $A(y) \geq t$ and so

$$\bigwedge_{z \in x \cdot y} A(z) \geq A(y) \wedge 0.5 \geq t \wedge 0.5 = t.$$

Therefore for every $z \in x \cdot y$ we have $A(z) \geq t$ or $z \in A_t$, so $x \cdot y \subseteq A_t$.
Conversely, let A be a fuzzy subset of R such that A_t $(\neq \emptyset)$ is a left H_v-ideal of R for all $0 < t \leq 0.5$. For every $x, y \in R$, we can write

$$A(x) \geq A(x) \wedge A(y) \wedge 0.5 = t_0,$$
$$A(y) \geq A(x) \wedge A(y) \wedge 0.5 = t_0,$$

then $x \in A_{t_0}$ and $y \in A_{t_0}$, so $x + y \subseteq A_{t_0}$. Therefore for every $z \in x + y$ we have $A(z) \geq t_0$ which implies

$$\bigwedge_{z \in x+y} A(z) \geq t_0,$$

and hence condition (1) of Proposition 4.3 is verified. To verify the second condition, for every $a, x \in R$, we put $t_1 = A(a) \wedge A(x) \wedge 0.5$. Then $x \in A_{t_1}$ and $a \in A_{t_1}$. So there exists $y \in A_{t_1}$ such that $x \in a + y$. Since $y \in A_{t_1}$, we have $A(y) \geq t_1$ or

$$A(y) \geq A(a) \wedge A(x) \wedge 0.5.$$

The third condition is similarly verified.

Now, let $x, y \in R$. We can write

$$A(y) \geq A(y) \wedge 0.5 = t_0.$$

Then $y \in A_{t_0}$ and so $x \cdot y \subseteq A_{t_0}$. Therefore for every $z \in x \cdot y$ we have $A(z) \geq t_0$ which implies

$$\bigwedge_{z \in x \cdot y} A(z) \geq t_0,$$

and hence condition (4) of Proposition 4.3 is verified

□

Naturally, a corresponding result is true when A_t is a left H_v-ideal of R for all $t \in (0.5, 1]$.

Theorem 4.6. *Let R be an H_v-ring and A a fuzzy subset of R. Then A_t $(\neq \emptyset)$ is a left (right) H_v-ideal of R for all $t \in (0.5, 1]$ if and only if*

(1) $A(x) \wedge A(y) \leq \bigwedge_{z \in x+y} (A(z) \vee 0.5)$, *for all $x, y \in R$;*

(2) *for all $x, a \in R$ there exists $y \in R$ such that $x \in a + y$ and*

$$A(a) \wedge A(x) \leq A(y) \vee 0.5;$$

(3) *for all $x, a \in R$ there exists $z \in R$ such that $x \in z + a$ and*

$$A(a) \wedge A(x) \leq A(z) \vee 0.5;$$

(4) $A(y) \leq \bigwedge_{z \in x \cdot y} (A(z) \vee 0.5)$, *for all $x, y \in R$.*

Proof. ($\Longrightarrow$): If there exist $x, y, z \in R$ with $z \in x + y$ such that

$$A(z) \vee 0.5 < A(x) \wedge A(y) = t,$$

then $t \in (0.5, 1]$, $A(z) < t$, $x \in A_t$, and $y \in A_t$. Since $x, y \in A_t$ and A_t is a left H_v-ideal, so $x + y \subseteq A_t$ and $A(z) \geq t$, for all $z \in x + y$, which is in contradiction with $A(z) < t$. Therefore

$$A(x) \wedge A(y) \geq A(z) \vee 0.5, \quad \text{for all } x, y, z \in R \text{ with } z \in x + y,$$

which implies

$$A(x) \wedge A(y) \geq \bigwedge_{z \in x+y} (A(z) \vee 0.5), \quad \text{for all } x, y \in R.$$

Hence (1) holds.

Now, assume that there exist $x_0, a_0 \in R$ such that for all $y \in R$ with $x_0 \in a_0 + y$, the following inequality holds:

$$A(y) \vee 0.5 < A(a_0) \wedge A(x_0) = t.$$

Then $t \in (0.5, 1]$, $x_0 \in A_t$, $a_0 \in A_t$ and $A(y) < t$. Since $x_0, a_0 \in A_t$ and A_t is a left H_v-ideal, there exists $y_0 \in A_t$ such that $x_0 \in a_0 + y_0$. From $y_0 \in A_t$, we get $A(y_0) \geq t$, which is in contradiction with $A(y_0) < t$. Therefore for all $x, a \in R$ there exists $y \in R$ such that $x \in a + y$ and

$$A(a) \wedge A(x) \leq A(y) \vee 0.5.$$

Hence (2) holds.

The proof of the third condition is similar.

Now, if there exist $x, y \in R$ with $z \in x \cdot y$ such that

$$A(z) \vee 0.5 < A(y) = t,$$

then $t \in (0.5, 1]$, $A(z) < t$, $y \in A_t$. Since $y \in A_t$ and A_t is a left H_v-ideal, $x \cdot y \subseteq A_t$ and $A(z) \geq t$ for all $z \in x \cdot y$, which is in contradiction with $A(z) < t$. Therefore

$$A(y) \geq A(z) \vee 0.5 \ \text{ for all } y \in R \ \text{ with } \ z \in x \cdot y,$$

which implies

$$A(y) \geq \bigwedge_{z \in x \cdot y} (A(z) \vee 0.5), \quad \text{for all } x, y \in R.$$

Hence (4) holds.

($\Longleftarrow$): Assume that $t \in (0.5, 1]$ and $x, y \in A_t$. Then

$$0.5 < t \leq A(x) \wedge A(y) \leq \bigwedge_{z \in x+y} (A(z) \vee 0.5).$$

It follows that for every $z \in x + y$, $0.5 < t \leq A(z) \vee 0.5$ and so $t \leq A(z)$, which implies $z \in A_t$. Hence $x + y \subseteq A_t$.

Now, we prove the reproducibility rule. Let $x, a \in A_t$. Then by condition (2), there exists $y \in R$ such that $x \in a + y$ and

$$A(a) \wedge A(x) \leq A(y) \vee 0.5.$$

We show that $y \in A_t$. We have

$$0.5 < t \leq A(x) \leq A(a) \wedge A(x) \leq A(y) \vee 0.5.$$

It follows that $0.5 \leq A(y)$ and so $y \in A_t$. Therefore $A_t = a + A_t$, for all $a \in A_t$. Similarly, we have $A_t = A_t + a$, for all $a \in A_t$.

Now, assume that $t \in (0.5, 1]$, $y \in A_t$ and $x \in R$. Then

$$0.5 < t \leq A(y) \leq \bigwedge_{z \in x \cdot y} (A(z) \vee 0.5).$$

It follows that for every $z \in x \cdot y$, $0.5 < t \leq A(z) \vee 0.5$ and so $t \leq A(z)$, which implies $z \in A_t$. Hence $x \cdot y \subseteq A_t$. Therefore A_t is a left H_v-ideal of R for all $t \in (0.5, 1]$. □

Let A be a fuzzy subset of an H_v-ring R and

$$J = \{t \mid t \in (0,1] \text{ and } A_t \text{ is either an empty set or a left (right) } H_v - \text{ideal of } R\}.$$

When $J = (0,1]$, then A is an ordinary fuzzy left (right) H_v-ideal of the H_v-ring R (Theorem 2.2). When $J = (0, 0.5]$, A is an $(\in, \in \vee q)$-fuzzy left (right) H_v-ideal of R (Theorem 4.5).

In [20], Yuan, Zhang and Ren gave the definition of a fuzzy subgroup with thresholds which is a generalization of Rosenfeld's fuzzy subgroup, and Bhakat and Das's fuzzy subgroup. Based on [20], we can extend the concept of a fuzzy subgroup with thresholds to the concept of fuzzy H_v-ideal with thresholds as follows:

Definition 4.7. Let $r, s \in [0,1]$ and $r < s$. Let A be a fuzzy subset of an H_v-ring R. Then A is called a *fuzzy left (right) H_v-ideal with thresholds (r,s) of R* if

(1) $A(x) \wedge A(y) \wedge s \leq \bigwedge_{z \in x+y} (A(z) \vee r)$, for all $x, y \in R$;

(2) for all $x, a \in R$ there exists $y \in R$ such that $x \in a + y$ and
$$A(a) \wedge A(x) \wedge s \leq A(y) \vee r;$$

(3) for all $x, a \in R$ there exists $z \in R$ such that $x \in z + a$ and
$$A(a) \wedge A(x) \wedge s \leq A(z) \vee r;$$

(4) $A(y) \wedge s \leq \bigwedge_{z \in x \cdot y} (A(z) \vee r)$, for all $x, y \in R$
$A(x) \wedge s \leq \bigwedge_{z \in x \cdot y} (A(z) \vee r)$, for all $x, y \in R$.

If A is a fuzzy left (right) H_v-ideal with thresholds of R, then we can conclude that A is an ordinary fuzzy left (right) H_v-ideal when $r = 0$, $s = 1$ and A is an $(\in, \in \vee q)$-fuzzy left (right) H_v-ring when $r = 0$, $s = 0.5$.

Now, we characterize fuzzy left (right) H_v-ideals with thresholds by their level left (right) H_v-ideals.

Theorem 4.8. *A fuzzy subset A of an H_v-ring R is a fuzzy left (right) H_v-ideal with thresholds (r,s) of R if and only if A_t $(\neq \emptyset)$ is a left (right) H_v-ideal of R for all $t \in (r,s]$.*

Proof. Let A be a fuzzy left H_v-ideal with thresholds of R and $t \in (r,s]$. Let $x, y \in A_t$. Then $A(x) \geq t$ and $A(y) \geq t$. Now

$$\bigwedge_{z \in x+y} (A(z) \vee r) \geq A(x) \wedge A(y) \wedge s \geq t \wedge s \geq t > r.$$

So for every $z \in x + y$ we have $A(z) \vee r \geq t > r$ which implies $A(z) \geq t$ and $z \in A_t$. Hence $x + y \subseteq A_t$. Now, let $x, a \in A_t$, then there exists $y \in R$ such that $x \in a + y$ and $A(a) \wedge A(x) \wedge s \leq A(y) \vee r$. From $x, a \in A_t$, we have $A(x) \geq t$ and $A(a) \geq t$, and so

$$r < t \leq t \wedge s \leq A(a) \wedge A(x) \wedge s \leq A(y) \vee r,$$

which implies $A(y) \geq t$, and so $y \in A_t$. Therefore we have $A_t = a + A_t$ for all $a \in A_t$. Similarly we get $A_t + a = A_t$ for all $a \in A_t$.

Now, let $y \in A_t$ and $x \in R$. Then $A(x) \geq t$, and so

$$\bigwedge_{z \in x \cdot y} (A(z) \vee r) \geq A(x) \wedge s \geq t \wedge s \geq t > r.$$

So for every $z \in x \cdot y$ we have $A(z) \vee r \geq t > r$ which implies $A(z) \geq t$ and $z \in A_t$. Hence $x \cdot y \subseteq A_t$.

Therefore A_t is a left H_v-ideal of R, for all $t \in (r, s]$.

Conversely, let A be a fuzzy subset of R such that A_t $(\neq \emptyset)$ is a left H_v-ideal of R for all $t \in (r, s]$. If there exist $x, y, z \in R$ with $z \in x + y$ such that

$$A(z) \vee r < A(x) \wedge A(y) \wedge s = t.$$

Then $t \in (r, s],\ A(z) < t,\ x \in A_t$ and $y \in A_t$. Since A_t is a left H_v-ideal of R and $x, y \in A_t$, so $x + y \subseteq A_t$. Hence $A(z) \geq t$ for all $z \in x + y$. This is in contradiction with $A(z) < t$. Therefore

$$A(x) \wedge A(y) \wedge s \leq A(z) \vee r, \quad \text{for all } x, y, z \in R \text{ with } z \in x + y,$$

which implies

$$A(x) \wedge A(y) \wedge s \leq \bigwedge_{z \in x+y} (A(z) \vee r), \ \text{for all } x, y \in R.$$

Hence condition (1) of Definition 4.7 holds.

Now, assume that there exist $x_0, a_0 \in R$ such that for all $y \in R$ which satisfies $x_0 \in a_0 + y$, the following inequality holds:

$$A(y) \vee r < A(a_0) \wedge A(x_0) \wedge s = t.$$

Then $t \in (r, s],\ x_0 \in A_t,\ a_0 \in A_t$ and $A(y) < t$. Since $x_0, a_0 \in A_t$ and A_t is a left H_v-ideal, so there exists $y_0 \in A_t$ such that $x_0 \in a_0 + y_0$. From $y_0 \in A_t$, we get $A(y_0) \geq t$. This is in contradiction with $A(y_0) < t$. Therefore

$$A(a) \wedge A(x) \wedge s \leq A(y) \vee r.$$

Hence the second condition of Definition 4.7 holds. The proof of third condition is similar.

If there exist $x, z \in R$ with $z \in x \cdot y$ such that

$$A(z) \vee r < A(x) \wedge A(y) \wedge s = t,$$

then $t \in (r, s]$, $A(z) < t$, $y \in A_t$. Since A_t is a left H_v-ideal of R and $x \in A_t$, so $x \cdot y \subseteq A_t$. Hence $A(z) \geq t$ for all $z \in x \cdot y$. This is in contradiction with $A(z) < t$. Therefore

$$A(y) \wedge s \leq A(z) \vee r, \quad \text{for all } x, z \in R \text{ with } z \in x \cdot y,$$

which implies

$$A(y) \wedge s \leq \bigwedge_{z \in x \cdot y} (A(z) \vee r), \text{ for all } x \in R.$$

Hence condition (4) of Definition 4.7 holds. □

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