ON $(\alpha, \beta)$-FUZZY $H_v$-IDEALS OF $H_v$-RINGS

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Abstract. Using the notion of “belongingness ($\in$)” and “quasi-coincidence ($q$)” of fuzzy points with fuzzy sets, we introduce the concept of an $(\alpha, \beta)$-fuzzy $H_v$-ideal of an $H_v$-ring, where $\alpha, \beta$ are any two of $\{\in, q, \in \lor q, \in \land q\}$ with $\alpha \neq \in \land q$. Since the concept of $(\in, \in \lor q)$-fuzzy $H_v$-ideals is an important and useful generalization of ordinary fuzzy $H_v$-ideals, we discuss some fundamental aspects of $(\in, \in \lor q)$-fuzzy $H_v$-ideals. A fuzzy subset $A$ of an $H_v$-ring $R$ is an $(\in, \in \lor q)$-fuzzy $H_v$-ideal if and only if an $A_t$, level cut of $A$, is an $H_v$-ideal of $R$, for all $t \in (0, 0.5]$. This shows that an $(\in, \in \lor q)$-fuzzy $H_v$-ideal is a generalization of the existing concept of fuzzy $H_v$-ideal. Finally, we extend the concept of a fuzzy subgroup with thresholds to the concept of a fuzzy $H_v$-ideal with thresholds.

1. Introduction

The theory of algebraic hyperstructures which is a generalization of the concept of ordinary algebraic structures was first introduced by Marty [15] and since then many researchers have developed this theory, a short review of which appears in [4]. In a recent book [5] Corsini and Leoreanu have presented some of the numerous applications of algebraic hyperstructures, especially those from the last fifteen years, in the following fields: geometry, hypergraphs, binary relations, lattices, fuzzy sets and rough sets, automata, cryptography, codes, median algebras, relation algebras, artificial intelligence and probabilities. Vougiouklis in the fourth AHA congress (1999) [19] introduced a new class of hyperstructures so-called $H_v$-structure, and Davvaz [7] surveyed the theory of $H_v$-structures. The $H_v$-structures are hyperstructures where equality is replaced by non-empty intersection. In this paper, we deal with $H_v$-rings. $H_v$-rings are the largest class of algebraic systems that satisfy ring-like axioms. After the introduction of fuzzy sets by Zadeh [21], reconsideration of the concept of classical mathematics began. On the other hand, because of the importance of group theory in mathematics, as well as its many areas of application, the notion of fuzzy subgroup was defined by Rosenfeld [17] and its structure was investigated. This subject has been studied further by many mathematicians. Liu [14] introduced the notion of fuzzy subgroups and ideals. Using the notion of “belongingness ($\in$)” and “quasi-coincidence ($q$)” of fuzzy points with fuzzy sets, the concept of $(\alpha, \beta)$-fuzzy subgroup where $\alpha, \beta$ are any two of $\{\in, q, \in \lor q, \in \land q\}$ with $\alpha \neq \in \land q$ was introduced in [1]. The most viable generalization of Rosenfeld’s fuzzy subgroup is the notion of $(\in, \in \lor q)$-fuzzy subgroups, the detailed study of which may be found in [3]. The concept of an $(\in, \in \lor q)$-fuzzy subring and ideal of
a ring have been introduced in [2] and the concept of \((\in, \in \vee q)\)-fuzzy subnear-ring and ideal of a near-ring have been introduced in [6]. Fuzzy sets and hyperstructures introduced by Zadeh and Marty, respectively, are now studied both from the theoretical point of view and for their many applications. The relations between fuzzy sets and hyperstructures have been already considered by many authors. In [8,9,11], Davvaz applied the concept of fuzzy sets to the theory of algebraic hyperstructures and defined fuzzy \(H_v\)-subgroups, fuzzy \(H_v\)-ideals and fuzzy \(H_v\)-submodules, which are generalizations of the concepts of Rosenfeld’s fuzzy subgroups, fuzzy ideals and fuzzy submodules. The concept of a fuzzy \(H_v\)-ideal and \(H_v\)-subring has been studied further in [10,12]. In Section 2, we recall some basic definitions and results about \(H_v\)-structures. In Section 3, we introduce the concept of \((\alpha, \beta)\)-fuzzy \(H_v\)-ideal of an \(H_v\)-ring and investigate related results. Since the concept of \((\in, \in \vee q)\)-fuzzy \(H_v\)-ideal is an important and useful generalization of ordinary fuzzy \(H_v\)-ideals, some fundamental aspects of \((\in, \in \vee q)\)-fuzzy \(H_v\)-ideals have been discussed in Section 4. A fuzzy subset \(A\) of an \(H_v\)-ring \(R\) is an \((\in, \in \vee q)\)-fuzzy \(H_v\)-ideal if and only if \(A_t\), level cut of \(A\), is an \(H_v\)-ideal of \(R\) for all \(t \in (0, 0.5]\). This shows that an \((\in, \in \vee q)\)-fuzzy \(H_v\)-ideal is a generalization of the existing concept of fuzzy \(H_v\)-ideal. Finally, based on [20], we extend the concept of a fuzzy subgroup with thresholds to the concept of fuzzy \(H_v\)-ideal with thresholds.

2. \(H_v\)-structures

A hypergroupoid \((H, \circ)\) is a non-empty set \(H\) with a hyperoperation \(\circ\) defined on \(H\), i.e. a mapping of \(H \times H\) into the family of non-empty subsets of \(H\). If \((x,y) \in H \times H\), its image under \(\circ\) is denoted by \(x \circ y\). If \(A, B \subseteq H\), then \(A \circ B\) is given by \(A \circ B = \bigcup \{x \circ y \mid x \in A, y \in B\}\). \(x \circ A\) is used for \(\{x\} \circ A\) and \(A \circ x\) for \(A \circ \{x\}\).

A hypergroupoid \((H, \circ)\) is called an \(H_v\)-group if for all \(x, y, z \in H\) the following two conditions hold:

(i) \(x \circ (y \circ z) \cap (x \circ y) \circ z \neq \emptyset\),

(ii) \(x \circ H = H \circ x = H\).

The second condition, called the reproducibility condition, means that for any \(x, y \in H\) there exist \(u, v \in H\) such that \(y \in x \circ u\) and \(y \in v \circ x\). If \((H, \circ)\) satisfies only the first condition, then it is called an \(H_v\)-semigroup.

An \(H_v\)-ring [18] is a multi-valued system \((R, +, \cdot)\) which satisfies the following ring-like axioms:

(i) \((R, +)\) is an \(H_v\)-group,

(ii) \((R, \cdot)\) is an \(H_v\)-semigroup,

(iii) \((\cdot)\) is weak distributive with respect to \((+)\), i.e., for all \(x, y, z \in R\) we have

\[
\begin{align*}
(x + y) \cdot z &\cap ((x \cdot z) + (y \cdot z)) \neq \emptyset, \\
(x + y) \cdot z &\cap ((x \cdot z) + (y \cdot z)) \neq \emptyset.
\end{align*}
\]

Let \(R\) be an \(H_v\)-ring. A non-empty subset \(I\) of \(R\) is called a left (right) \(H_v\)-ideal if the following conditions hold:

(i) \((I, +)\) is an \(H_v\)-subgroup of \((R, +)\),

(ii) \((I, \cdot)\) is an \(H_v\)-submodule of \((R, \cdot)\),
(ii) $R \cdot I \subseteq I$ ($I \cdot R \subseteq I$).

A mapping $A : X \rightarrow [0, 1]$, where $X$ is an ordinary non-empty set, is called a fuzzy subset of $X$. In 1971, Rosenfeld [17] applied the concept of fuzzy sets to the theory of groups and studied fuzzy subgroups of a group. Since then many papers concerning various fuzzy algebraic structures have appeared in the literature. Liu [14] introduced and studied the notions of fuzzy subrings and fuzzy ideals. In [7-12], Davvaz applied the concept of fuzzy set theory to algebraic hyperstructures and, in particular [11], defined the concept of fuzzy $H_{v}$-ideal of an $H_{v}$-ring which is a generalization of the concept of fuzzy ideal.

**Definition 2.1.** (Davvaz [11]). Let $R$ be an $H_{v}$-ring and $A$ be a fuzzy subset of $R$. Then $A$ is said to be a fuzzy left (right) $H_{v}$-ideal of $R$ if the following axioms hold:

(i) $A(x) \land A(y) \leq \bigwedge_{\alpha \in x+y} A(\alpha)$, for all $x, y \in R$,

(ii) for all $x, a \in R$ there exists $y \in R$ such that $x \in a + y$ and $A(a) \land A(x) \leq A(y)$,

(iii) for all $x, a \in R$ there exists $z \in R$ such that $x \in z + a$ and $A(a) \land A(x) \leq A(z)$,

(iv) $A(y) \leq \bigwedge_{z \in x \cdot y} A(z)$, for all $x, y \in R$.

$A(x) \leq \bigwedge_{z \in x \cdot y} A(z)$, for all $x, y \in R$.

(ii) is called the left fuzzy reproduction axiom and (iii) is called the right fuzzy reproduction axiom.

In this paper we present all the proofs for left $H_{v}$-ideals. Similar results hold for right $H_{v}$-ideals.

Let $A$ be a fuzzy subset of a non-empty set $X$ and let $t \in (0, 1]$. The set $A_{t} = \{ x \in X | A(x) \geq t \}$ is called a level cut of $A$.

**Theorem 2.2.** (cf. [11]). Let $R$ be an $H_{v}$-ring and $A$ a fuzzy subset of $R$. Then $A$ is a fuzzy left (right) $H_{v}$-ideal of $R$ if and only if for every $t \in (0, 1]$, $A_{t}$ ($\neq \emptyset$) is a left (right) $H_{v}$-ideal of $R$.

When $A$ is a fuzzy $H_{v}$-ideal of $R$, $A_{t}$ is called a level $H_{v}$-ideal of $R$. The concept of level $H_{v}$-ideals has been used extensively to characterize various properties of fuzzy $H_{v}$-ideals.

### 3. $(\alpha, \beta)$-fuzzy $H_{v}$-ideals

A fuzzy subset $A$ of $R$ of the form

$$A(y) = \begin{cases} t(\neq 0) & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}$$
is said to be a fuzzy point with support \( x \) and value \( t \) and is denoted by \( x_t \). A fuzzy point \( x_t \) is said to belong to (resp. be quasi-coincident with) a fuzzy set \( A \), written as \( x_t \in A \) (resp. \( x_t.q.A \)) if \( A(x) \geq t \) (resp. \( A(x) + t > 1 \)). If \( x_t \in A \) or \( x_t.q.A \), then we write \( x_t \in \vee q.A \). The symbol \( \nabla \vee q \) means either \( \in \) or \( q \) hold. In what follows, unless otherwise specified, \( \alpha \) and \( \beta \) will denote any one of \( \varepsilon, q, \in \) or \( \vee q \). The idea of quasi-coincidence of a fuzzy point with a fuzzy set, which is mentioned in [16], play a vital role in generating another type of fuzzy subgroups, called \((\alpha, \beta)\)-fuzzy subgroups, introduced by Bhakat and Das [1]. Based on [3], we can extend the concept of \((\alpha, \beta)\)-fuzzy subgroups to the concept of \((\alpha, \beta)\)-fuzzy \( H_v \)-ideals.

**Definition 3.1.** Let \( R \) be an \( H_v \)-ring. A fuzzy subset \( A \) of \( R \) is said to be an \((\alpha, \beta)\)-fuzzy left (right) \( H_v \)-ideal of \( R \) if for all \( t, r \in [0, 1] \),

1. Suppose that \( x, y \in R \) and \( t, r \in [0, 1] \) be such that \( x_t, y_r \in A \). Then \( x_t \in \vee q.A \), and so \( z_{t,r} \in \vee q.A \), for all \( z \in x + y \).
2. Now, let \( x, a \in R \) and \( t, r \in (0, 1] \) be such that \( x_t, a_r \in A \). Then \( x_t, a_r \in \vee q.A \) which implies \( y_{t,r} \in \vee q.A \), for some \( y \in R \) with \( x \in a + y \).
3. The third condition is similarly verified.
4. Finally, let \( x, y \in R \) and \( t \in (0, 1] \) be such that \( y \in A \). Then \( y_t \in \vee q.A \) which implies \( z_t \in \vee q.A \), for all \( z \in x \cdot y \).

**Proposition 3.2.** Let \( R \) be an \( H_v \)-ring. Every \((\varepsilon, q)\)-fuzzy left (right) \( H_v \)-ideal of \( R \) is an \((\varepsilon, q)\)-fuzzy left (right) \( H_v \)-ideal of \( R \).

**Proof.** Let \( A \) be an \((\varepsilon, q)\)-fuzzy left \( H_v \)-ideal of \( R \).

(i) Suppose that \( x, y \in R \) and \( t, r \in [0, 1] \) be such that \( x_t, y_r \in A \). Then \( x_t \in \vee q.A \), and so \( z_{t,r} \in \vee q.A \), for all \( z \in x + y \).

(ii) Now, let \( x, a \in R \) and \( t, r \in (0, 1] \) be such that \( x_t, a_r \in A \). Then \( x_t, a_r \in \vee q.A \) which implies \( y_{t,r} \in \vee q.A \), for some \( y \in R \) with \( x \in a + y \).

(iii) The third condition is similarly verified.

(iv) Finally, let \( x, y \in R \) and \( t \in (0, 1] \) be such that \( y \in A \). Then \( y_t \in \vee q.A \) which implies \( z_t \in \vee q.A \), for all \( z \in x \cdot y \).

**Proposition 3.3.** Let \( R \) be an \( H_v \)-ring. Every \((\varepsilon, q)\)-fuzzy left (right) \( H_v \)-ideal of \( R \) is an \((\varepsilon, q)\)-fuzzy left (right) \( H_v \)-ideal of \( R \).

**Proof.** Straightforward.

**Lemma 3.4.** If \( A \) is a fuzzy left (right) \( H_v \)-ideal of \( R \), then the characteristic function \( \chi_A \) of \( A \) is an \((\varepsilon, q)\)-fuzzy left (right) \( H_v \)-ideal of \( R \).

Now, we give the main result on general \((\alpha, \beta)\)-fuzzy left (right) \( H_v \)-ideals of \( H_v \)-rings.

**Theorem 3.5.** Let \( A \) be a non-zero \((\alpha, \beta)\)-fuzzy left (right) \( H_v \)-ideal of \( R \). Then the set \( \text{Supp}A = \{x \in R|A(x) > 0\} \) is a left (right) \( H_v \)-ideal of \( R \).
Proof. The proof is a simple modification of the proofs of Theorems 3.6, 3.7, 3.8 and Corollary 3.10 in [13].

A fuzzy subset \(A\) of an \(H_v\)-ring \(R\) is said to be proper if \(\text{Im}A\) has at least two elements. Two fuzzy subsets are said to be equivalent if they have same family of level subsets. Otherwise, they are said to be non-equivalent.

**Theorem 3.6.** Let \(R\) have proper \(H_v\)-ideals. A proper \((\varepsilon, \varepsilon)\)-fuzzy \(H_v\)-ideal \(A\) of \(R\) such that card \(\text{Im}A \geq 3\), can be expressed as the union of two proper non-equivalent \((\varepsilon, \varepsilon)\)-fuzzy \(H_v\)-ideals of \(R\).

**Proof.** The proof is a modification of the proof of Theorem 3.17 in [3]. □

4. \((\varepsilon, \varepsilon)\)-fuzzy \(H_v\)-ideals

In this section, we consider a special case of \((\alpha, \beta)\)-fuzzy \(H_v\)-ideals. An \((\varepsilon, \varepsilon)\)-fuzzy \(H_v\)-ideal is an important and useful generalization of ordinary fuzzy \(H_v\)-ideal.

**Definition 4.1.** Let \(R\) be an \(H_v\)-ring. A fuzzy subset \(A\) of \(R\) is said to be an \((\varepsilon, \varepsilon)\)-fuzzy left (right) \(H_v\)-ideal of \(R\) if for all \(t, r \in (0, 1]\),

(i) \(x_t \in A, y_t \in A\) implies \(z_{t\land r} \in \varepsilon \text{Im}A\), for all \(z \in x + y\);

(ii) \(x_t \in A, a_t \in A\) implies \(y_{t\land r} \in \varepsilon \text{Im}A\), for some \(y \in R\) with \(x \in a + y\);

(iii) \(x_t \in A, a_t \in A\) implies \(z_{t\land r} \in \varepsilon \text{Im}A\), for some \(z \in R\) with \(x \in z + a\);

(iv) \(y_t \in A\) and \(x_t \in R\) imply \(z_t \in \varepsilon \text{Im}A\) for all \(z \in x\).

\[(x_t \in A\) and \(y_t \in R\) imply \(z_t \in \varepsilon \text{Im}A\), for all \(z \in x\cdot y).\]

It is easy to see that for any subset \(A\) of \(R\), \(\chi_A\) is an \((\varepsilon, \varepsilon)\)-fuzzy left (right) \(H_v\)-ideal of \(R\) if and only if \(A\) is a left (right) \(H_v\)-ideal of \(R\).

**Example 4.2.** Let \(R = \{a, b, c, d\}\) be a set, and consider addition and multiplication tables as follows:

<table>
<thead>
<tr>
<th>+</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>a</td>
<td>b</td>
<td>c</td>
<td>d</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>{a, b}</td>
<td>d</td>
<td>c</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>d</td>
<td>{a, c}</td>
<td>b</td>
</tr>
<tr>
<td>d</td>
<td>d</td>
<td>c</td>
<td>b</td>
<td>{a, d}</td>
</tr>
</tbody>
</table>

Then we can easily see that \((R, +, \cdot)\) is an \(H_v\)-ring. Let \(A: R \longrightarrow [0, 1]\) be defined by

\[A(a) = 0.6, \quad A(b) = A(c) = A(d) = 0.8.\]

Then \(A\) is an \((\varepsilon, \varepsilon)\)-fuzzy \(H_v\)-ideal of \(R\) but not an ordinary fuzzy \(H_v\)-ideal.

**Proposition 4.3.** Conditions (i)-(iv) in Definition 4.1, are respectively equivalent to the following:

1. \(A(x) \land A(y) \land 0.5 \leq \bigwedge_{z \in x + y} A(z), \) for all \(x, y \in R;\)
2. for all \(x, a \in R\) there exists \(y \in R\) such that \(x \in a + y\) and
   \[A(a) \land A(x) \land 0.5 \leq A(y);\]
(3) for all \( x, a \in R \) there exists \( z \in R \) such that \( x \in z + a \) and
\[
A(a) \land A(x) \land 0.5 \leq A(z);
\]
(4) \( A(y) \land 0.5 \leq \bigwedge_{x \in x \cdot y} A(z) \), for all \( x, y \in R \)
\[
(A(x) \land 0.5 \leq \bigwedge_{z \in x \cdot y} A(z), \text{ for all } x, y \in R).
\]

Proof. (i \( \implies \) 1): Suppose that \( x, y \in R \). We consider the following cases:

(a) \( A(x) \land A(y) < 0.5 \),
(b) \( A(x) \land A(y) \geq 0.5 \).

Case a: Assume that there exists \( z \in x + y \) such that \( A(z) < A(x) \land A(y) \land 0.5 \), which implies \( A(z) < A(x) \land A(y) \). Choose \( t \) such that \( A(z) < t < A(x) \land A(y) \). Then \( x_1, y_1 \in A, \) but \( z \not\in \overline{qA} \) which contradicts (i).

Case b: Assume that for all \( z \in x + y \), we have \( A(y) < A(x) \land A(a) \). Choose \( t \) such that \( A(y) < t < A(x) \land A(a) \) and \( t + A(y) < 1 \). Then \( x_1, a_1 \in A, \) but \( z \not\in \overline{qA} \), which contradicts (ii).

Case b: Assume that for all \( z \in x + y \), we have
\[
A(y) < A(x) \land A(a) \land 0.5.
\]
Then \( x_0.5, a_0.5 \in A, \) but \( y \not\in \overline{qA} \), which contradicts (ii).

Hence (2) holds.

(ii \( \implies \) 2): Suppose that \( x, a \in R \). We consider the following cases:

(a) \( A(x) \land A(a) < 0.5 \),
(b) \( A(x) \land A(a) \geq 0.5 \).

Case a: Assume that for all \( y \) with \( x \in a + y \), we have \( A(y) < A(x) \land A(a) \). Choose \( t \) such that \( A(y) < t < A(x) \land A(a) \) and \( t + A(y) < 1 \). Then \( x_1, a_1 \in A, \) but \( z \not\in \overline{qA} \), which contradicts (ii).

Case b: Assume that for all \( y \) with \( x \in a + y \), we have
\[
A(y) < A(x) \land A(a) \land 0.5.
\]
Then \( x_0.5, a_0.5 \in A, \) but \( y \not\in \overline{qA} \), which contradicts (ii).

Hence (2) holds.

(iii \( \implies \) 3): The proof is similar to (ii \( \implies \) 2).

(iv \( \implies \) 4): Suppose \( x, y \in R \). We consider the following cases:

(a) \( A(y) < 0.5 \),
(b) \( A(y) \geq 0.5 \).

Case a: Assume that there exists \( z \in x \cdot y \) such that \( A(z) < A(y) \land 0.5 \), which implies \( A(z) < A(y) \). Choose \( t \) such that \( A(z) < t < A(y) \). Then \( y \not\in A, \) but \( z \not\in \overline{qA} \), which contradicts (iv).

Case b: Assume that \( A(z) < 0.5 \) for some \( z \in x \cdot y \). Then \( y \not\in A, \) but \( z \not\in \overline{qA} \), which contradicts (iv).

Then \( z \not\in \overline{qA} \), which contradicts (iv).

Hence (4) holds.

(1 \( \implies \) i): Let \( x_1, y_r \in A \). Then \( A(x) \geq t \) and \( A(y) \geq r \). For every \( z \in x + y \) we have
\[
A(z) \geq A(x) \land A(y) \land 0.5 \geq t \land r \land 0.5.
\]
If \( t \land r > 0.5 \), then \( A(z) \geq 0.5 \) which implies \( A(z) + t \land r > 1 \).

If \( t \lor r \leq 0.5 \), then \( A(z) \geq t \lor r \).

Therefore \( z \not\in \overline{qA} \) for all \( z \in x + y \).
(2 $\implies$ ii): Let $x_t, a_r \in A$. Then $A(x) \geq t$ and $A(a) \geq r$. Now, for some $y$ with $x \in a + y$ we have
$$A(y) \geq A(a) \wedge A(x) \wedge 0.5 \geq t \wedge r \wedge 0.5.$$  
If $t \wedge r > 0.5$, then $A(y) \geq 0.5$ which implies $A(y) + t \wedge r > 1$.
If $t \vee r \leq 0.5$, then $A(y) \geq t \wedge r$.
Therefore $y, x \in \mathcal{Q}A$. Hence (ii) holds.
(3 $\implies$ iii): The proof is similar to (2 $\implies$ ii).
(4 $\implies$ iv): Let $y_t \in A$ and $x \in R$. Then $A(y) \geq t$. For every $z \in x \cdot y$ we have
$$A(z) \geq A(y) \wedge 0.5 \geq t \wedge 0.5.$$  
If $t > 0.5$, then $A(z) \geq 0.5$ which implies $A(z) + t > 1$.
If $t \leq 0.5$, then $A(z) \geq t$.
Therefore $z_t \in \mathcal{Q}A$ for all $z \in x \cdot y$.

By Definition 4.1 and Proposition 4.3, we immediately get:

**Corollary 4.4.** A fuzzy subset $A$ of an $H_v$-ring $R$ is an $(\varepsilon, \in \mathcal{Q})$-fuzzy left (right) $H_v$-ideal of $R$ if and only if the conditions (i)-(iv) in Proposition 4.3 hold.

Now, we characterize $(\varepsilon, \in \mathcal{Q})$-fuzzy left (right) $H_v$-ideals by their level $H_v$-ideals.

**Theorem 4.5.** Let $R$ be an $H_v$-ring and $A$ a fuzzy subset of $R$. If $A$ is an $(\varepsilon, \in \mathcal{Q})$-fuzzy left (right) $H_v$-ideal of $R$, then for all $0 < t \leq 0.5$, $A_t$ is an empty set or a left (right) $H_v$-ideal of $R$. Conversely, if $A_t$ is a left (right) $H_v$-ideal of $R$ for all $0 < t \leq 0.5$, then $A$ is an $(\varepsilon, \in \mathcal{Q})$-fuzzy left (right) $H_v$-ideal of $R$.

**Proof.** Let $A$ be an $(\varepsilon, \in \mathcal{Q})$-fuzzy left $H_v$-ideal of $R$ and $0 < t \leq 0.5$. Let $x, y \in A_t$. Then $A(x) \geq t$ and $A(y) \geq t$. Now
$$\bigwedge_{z \in x + y} A(z) \geq A(x) \wedge A(y) \wedge 0.5 \geq t \wedge 0.5 = t.$$  
Therefore for every $z \in x + y$ we have $A(z) \geq t$ or $z \in A_t$, so $x + y \subseteq A_t$. Hence for every $a \in A_t$ we have $a + A_t \subseteq A_t$. Now, let $x, a \in A_t$. Then there exists $y \in R$ such that $x \in a + y$ and $A(a) \wedge A(x) \wedge 0.5 \leq A(y)$. From $x, a \in A_t$, we have $A(x) \geq t$ and $A(a) \geq t$, and so
$$t = t \wedge t \wedge 0.5 \leq A(a) \wedge A(x) \wedge 0.5 \leq A(y).$$  
Hence $y \in A_t$, and this proves that $A_t \subseteq a + A_t$.

Now, let $y \in A_t$ and $x \in R$. Then $A(y) \geq t$ and so
$$\bigwedge_{z \in x \cdot y} A(z) \geq A(y) \wedge 0.5 \geq t \wedge 0.5 = t.$$  
Therefore for every $z \in x \cdot y$ we have $A(z) \geq t$ or $z \in A_t$, so $x \cdot y \subseteq A_t$.

Conversely, let $A$ be a fuzzy subset of $R$ such that $A_t$ is a left $H_v$-ideal of $R$ for all $0 < t \leq 0.5$. For every $x, y \in R$, we can write
$$A(x) \geq A(x) \wedge A(y) \wedge 0.5 = t_0,$$
$$A(y) \geq A(x) \wedge A(y) \wedge 0.5 = t_0,$$
then \( x \in A_{t_0} \) and \( y \in A_{t_0} \), so \( x + y \subseteq A_{t_0} \). Therefore for every \( z \in x + y \) we have \( A(z) \geq t_0 \) which implies
\[
\bigwedge_{z \in x + y} A(z) \geq t_0,
\]
and hence condition (1) of Proposition 4.3 is verified. To verify the second condition, for every \( a, x \in R \), we put \( t_1 = A(a) \land A(x) \land 0.5 \). Then \( x \in A_{t_1} \) and \( a \in A_{t_1} \). So there exists \( y \in A_{t_1} \), such that \( x \in a + y \). Since \( y \in A_{t_1} \), we have \( A(y) \geq t_1 \) or
\[
A(y) \geq A(a) \land A(x) \land 0.5.
\]
The third condition is similarly verified.

Now, let \( x, y \in R \). We can write
\[
A(y) \geq A(y) \land 0.5 = t_0.
\]
Then \( y \in A_{t_0} \) and so \( x \cdot y \subseteq A_{t_0} \). Therefore for every \( z \in x \cdot y \) we have \( A(z) \geq t_0 \) which implies
\[
\bigwedge_{z \in x \cdot y} A(z) \geq t_0,
\]
and hence condition (4) of Proposition 4.3 is verified.

Naturally, a corresponding result is true when \( A_t \) is a left \( H_o \)-ideal of \( R \) for all \( t \in (0, 1] \).

**Theorem 4.6.** Let \( R \) be an \( H_o \)-ring and \( A \) a fuzzy subset of \( R \). Then \( A_t \) (\( \not= \emptyset \)) is a left (right) \( H_o \)-ideal of \( R \) for all \( t \in (0, 1] \) if and only if

1. \( A(x) \land A(y) \leq \bigwedge_{z \in x + y} (A(z) \lor 0.5), \) for all \( x, y \in R \);
2. for all \( x, a \in R \) there exists \( y \in R \) such that \( x \in a + y \) and \( A(a) \land A(x) \leq A(y) \lor 0.5 \);
3. for all \( x, a \in R \) there exists \( z \in R \) such that \( x \in z + a \) and \( A(a) \land A(x) \leq A(z) \lor 0.5 \);
4. \( A(y) \leq \bigwedge_{z \in x \cdot y} (A(z) \lor 0.5), \) for all \( x, y \in R \).

**Proof.** (\( \Rightarrow \)): If there exist \( x, y, z \in R \) with \( z \in x + y \) such that
\[
A(z) \lor 0.5 < A(x) \land A(y) = t,
\]
then \( t \in (0.5, 1] \), \( A(z) < t \), \( x \in A_t \), and \( y \in A_t \). Since \( x, y \in A_t \) and \( A_t \) is a left \( H_o \)-ideal, so \( x + y \subseteq A_t \) and \( A(z) \geq t \), for all \( z \in x + y \), which is in contradiction with \( A(z) < t \). Therefore
\[
A(x) \land A(y) \geq A(z) \lor 0.5, \text{ for all } x, y, z \in R \text{ with } z \in x + y,
\]
which implies

\[ A(x) \land A(y) \geq \bigwedge_{z \in x+y} (A(z) \lor 0.5), \text{ for all } x, y \in R. \]

Hence (1) holds.

Now, assume that there exist \( x_0, a_0 \in R \) such that for all \( y \in R \) with \( x_0 \in a_0+y \), the following inequality holds:

\[ A(y) \lor 0.5 < A(a_0) \land A(x_0) = t. \]

Then \( t \in (0.5, 1] \), \( x_0 \in A_t \), \( a_0 \in A_t \) and \( A(y) < t \). Since \( x_0, a_0 \in A_t \) and \( A_t \) is a left \( H_{\ell} \)-ideal, there exists \( y_0 \in A_t \) such that \( x_0 \in a_0+y_0 \). From \( y_0 \in A_t \), we get \( A(y_0) \geq t \), which is in contradiction with \( A(y_0) < t \). Therefore for all \( x, a \in R \) there exists \( y \in R \) such that \( x \in a+y \) and

\[ A(a) \land A(x) \leq A(y) \lor 0.5. \]

Hence (2) holds.

The proof of the third condition is similar.

Now, if there exist \( x, y \in R \) with \( z \in x \cdot y \) such that

\[ A(z) \lor 0.5 < A(y) = t, \]

then \( t \in (0.5, 1] \), \( A(z) < t \), \( y \in A_t \). Since \( y \in A_t \) and \( A_t \) is a left \( H_{\ell} \)-ideal, \( x \cdot y \subseteq A_t \) and \( A(z) \geq t \) for all \( z \in x \cdot y \), which is in contradiction with \( A(z) < t \). Therefore

\[ A(y) \geq A(z) \lor 0.5 \text{ for all } y \in R \text{ with } z \in x \cdot y, \]

which implies

\[ A(y) \geq \bigwedge_{z \in x+y} (A(z) \lor 0.5), \text{ for all } x, y \in R. \]

Hence (4) holds.

\([\Leftarrow]\): Assume that \( t \in (0.5, 1] \) and \( x, y \in A_t \). Then

\[ 0.5 < t \leq A(x) \land A(y) \leq \bigwedge_{z \in x+y} (A(z) \lor 0.5). \]

It follows that for every \( z \in x+y \), \( 0.5 < t \leq A(z) \lor 0.5 \) and so \( t \leq A(z) \), which implies \( z \in A_t \). Hence \( x+y \subseteq A_t \).

Now, we prove the reproducibility rule. Let \( x, a \in A_t \). Then by condition (2), there exists \( y \in R \) such that \( x \in a+y \) and

\[ A(a) \land A(x) \leq A(y) \lor 0.5. \]

We show that \( y \in A_t \). We have

\[ 0.5 < t \leq A(x) \leq A(a) \land A(x) \leq A(y) \lor 0.5. \]

It follows that \( 0.5 \leq A(y) \) and so \( y \in A_t \). Therefore \( A_t = a+A_t \), for all \( a \in A_t \).

Similarly, we have \( A_t = A_t + a \), for all \( a \in A_t \).

Now, assume that \( t \in (0.5, 1] \), \( y \in A_t \) and \( x \in R \). Then

\[ 0.5 < t \leq A(y) \leq \bigwedge_{z \in x+y} (A(z) \lor 0.5). \]
It follows that for every \( z \in x \cdot y, 0.5 < t \leq A(z) \vee 0.5 \) and so \( t \leq A(z) \), which implies \( z \in A_t \). Hence \( x \cdot y \subseteq A_t \). Therefore \( A_t \) is a left \( H_v \)-ideal of \( R \) for all \( t \in (0.5, 1] \).

Let \( A \) be a fuzzy subset of an \( H_v \)-ring \( R \) and \( J = \{ t \mid t \in (0, 1] \text{ and } A_t \) is either an empty set or a left (right) \( H_v \)–ideal of \( R \} \).

When \( J = (0, 1] \), then \( A \) is an ordinary fuzzy left (right) \( H_v \)-ideal of the \( H_v \)-ring \( R \) (Theorem 4.2). When \( J = (0, 0.5] \), \( A \) is an \((\varepsilon, \in \vee q)\)-fuzzy left (right) \( H_v \)-ideal of \( R \) (Theorem 4.5).

In [20], Yuan, Zhang and Ren gave the definition of a fuzzy subgroup with thresholds which is a generalization of Rosenfeld’s fuzzy subgroup, and Bhakat and Das’s fuzzy subgroup. Based on [20], we can extend the concept of a fuzzy subgroup with thresholds to the concept of fuzzy \( H_v \)-ideal with thresholds as follows:

**Definition 4.7.** Let \( r, s \in [0, 1] \) and \( r < s \). Let \( A \) be a fuzzy subset of an \( H_v \)-ring \( R \). Then \( A \) is called a fuzzy left (right) \( H_v \)-ideal with thresholds \((r, s)\) of \( R \) if

1. \( A(x) \wedge A(y) \wedge s \leq \bigwedge_{z \in x+y} (A(z) \vee r) \), for all \( x, y \in R \);
2. for all \( x, a \in R \) there exists \( y \in R \) such that \( x \in a + y \) and \( A(a) \wedge A(x) \wedge s \leq A(y) \vee r \);
3. for all \( x, a \in R \) there exists \( z \in R \) such that \( x \in z + a \) and \( A(a) \wedge A(x) \wedge s \leq A(z) \vee r \);
4. \( A(y) \wedge s \leq \bigwedge_{z \in x+y} (A(z) \vee r) \), for all \( x, y \in R \).

If \( A \) is a fuzzy left (right) \( H_v \)-ideal with thresholds of \( R \), then we can conclude that \( A \) is an ordinary fuzzy left (right) \( H_v \)-ideal when \( r = 0, s = 1 \) and \( A \) is an \((\varepsilon, \in \vee q)\)-fuzzy left (right) \( H_v \)-ring when \( r = 0, s = 0.5 \).

Now, we characterize fuzzy left (right) \( H_v \)-ideals with thresholds by their level left (right) \( H_v \)-ideals.

**Theorem 4.8.** A fuzzy subset \( A \) of an \( H_v \)-ring \( R \) is a fuzzy left (right) \( H_v \)-ideal with thresholds \((r, s)\) of \( R \) if and only if \( A_t \) (\( \neq \emptyset \)) is a left (right) \( H_v \)-ideal of \( R \) for all \( t \in (r, s] \).

**Proof.** Let \( A \) be a fuzzy left \( H_v \)-ideal with thresholds of \( R \) and \( t \in (r, s] \). Let \( x, y \in A_t \). Then \( A(x) \geq t \) and \( A(y) \geq t \). Now

\[
\bigwedge_{z \in x+y} (A(z) \vee r) \geq A(x) \wedge A(y) \wedge s \geq t \wedge s \geq t > r.
\]

So for every \( z \in x+y \) we have \( A(z) \vee r \geq t > r \) which implies \( A(z) \geq t \) and \( z \in A_t \).

Hence \( x + y \subseteq A_t \). Now, let \( x, a \in A_t \), then there exists \( y \in R \) such that \( x \in a + y \) and \( A(a) \wedge A(x) \wedge s \leq A(y) \vee r \). From \( x, a \in A_t \), we have \( A(x) \geq t \) and \( A(a) \geq t \),...
and so
\[ r < t \leq t \land s \leq A(a) \land A(x) \land s \leq A(y) \lor r, \]
which implies \( A(y) \geq t \), and so \( y \in A_t \). Therefore we have \( A_t = a + A_t \) for all \( a \in A_t \). Similarly we get \( A_t + a = A_t \) for all \( a \in A_t \).

Now, let \( y \in A_t \) and \( x \in R \). Then \( A(x) \geq t \), and so
\[
\bigwedge_{z \in x \cdot y} (A(z) \lor r) \geq A(x) \land s \geq t \land s \geq t > r.
\]
So for every \( z \in x \cdot y \) we have \( A(z) \lor r \geq t > r \) which implies \( A(z) \geq t \) and \( z \in A_t \).

Hence the second condition of Definition 4.7 holds. The proof of third condition is similar.

If there exist \( x, z \in R \) with \( z \in x \cdot y \) such that
\[ A(z) \lor r < A(x) \land A(y) \land s = t, \]
then \( t \in (r, s] \), \( A(z) < t \), \( x \in A_t \) and \( y \in A_t \). Since \( A_t \) is a left \( H_v \)-ideal of \( R \) and \( x, y \in A_t \), so \( x + y \subseteq A_t \). Hence \( A(z) \geq t \) for all \( z \in x + y \). This is in contradiction with \( A(z) < t \). Therefore
\[ A(x) \land A(y) \land s \leq A(z) \lor r, \quad \text{for all } x, y, z \in R \text{ with } z \in x + y, \]
which implies
\[ A(x) \land A(y) \land s \leq \bigwedge_{z \in x + y} (A(z) \lor r), \quad \text{for all } x, y \in R. \]
Hence condition (1) of Definition 4.7 holds.

Now, assume that there exist \( x_0, a_0 \in R \) such that for all \( y \in R \) which satisfies \( x_0 \in a_0 + y \), the following inequality holds:
\[ A(y) \lor r < A(a_0) \land A(x_0) \land s = t. \]
Then \( t \in (r, s] \), \( x_0 \in A_t \), \( a_0 \in A_t \) and \( A(y) < t \). Since \( x_0, a_0 \in A_t \) and \( A_t \) is a left \( H_v \)-ideal, so there exists \( y_0 \in A_t \) such that \( x_0 \in a_0 + y_0 \). From \( y_0 \in A_t \), we get \( A(y_0) \geq t \). This is in contradiction with \( A(y_0) < t \). Therefore
\[ A(a) \land A(x) \land s \leq A(y) \lor r. \]
Hence the second condition of Definition 4.7 holds. The proof of third condition is similar.
then $t \in (r, s]$, $A(z) < t$, $y \in A_t$. Since $A_t$ is a left $H_v$-ideal of $R$ and $x \in A_t$, so $x \cdot y \subseteq A_t$. Hence $A(z) \geq t$ for all $z \in x \cdot y$. This is in contradiction with $A(z) < t$.

Therefore

\[ A(y) \land s \leq A(z) \lor r, \quad \text{for all } x, z \in R, \]

which implies

\[ A(y) \land s \leq \bigwedge_{z \in x \cdot y} (A(z) \lor r), \quad \text{for all } x \in R. \]

Hence condition (4) of Definition 4.7 holds.

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References


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