TOPOLOGICAL SIMILARITY OF L-RELATIONS

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Abstract. L-fuzzy rough sets are extensions of the classical rough sets by relaxing the equivalence relations to L-relations. The topological structures induced by L-fuzzy rough sets have opened up the way for applications of topological facts and methods in granular computing. In this paper, we firstly prove that each arbitrary L-relation can generate an Alexandrov L-topology. Based on this fact, we introduce the topological similarity of L-relations, denote it by T-similarity, and we give intuitive characterization of T-similarity. Then we introduce the variations of a given L-relation and investigate the relationship among them. Moreover, we prove that each L-relation is uniquely topological similar to an L-preorder. Finally, we investigate the related algebraic structures of different sets of L-relations on the universe.

1. Introduction

The concept of rough set proposed by Pawlak [17] is a mathematical tool to deal with intelligent systems characterized by insufficient and incomplete information. As a new method of soft computing, rough set theory has been received much attention in the past decades due to its successful applications in various practical problems.

In rough set theory, the basic structure is approximation space constituted by the lower and upper approximation operators. Through the two operators, the hidden information in the data table could be revealed and expressed in the form of decision rules. Equivalence relation is a key concept in rough set theory and the equivalence classes are the building blocks for the lower and upper approximations. However, the equivalence relation is too restrictive to application, different generalizations have been presented by replacing the equivalence relation with general relations or coverings [12, 13, 23, 26, 31, 32]. With the development of rough set theory, the researchers have proposed various fuzzy rough approximations [5, 6, 19, 21, 30] as generalizations of classical ones to meet more general application environment. There are two basic approaches [31] to the development of rough set theory both in classical cases and fuzzy cases. One is the constructive approach [4, 19, 29, 30] in which lower and upper approximation operators are constructed from (fuzzy) relations, (fuzzy) coverings and so on. The other one is the axiomatic approach [16, 29], in which the lower and upper approximation operators satisfied a set of axioms are the same as those constructed ones.

Received: June 2016; Revised: September 2016; Accepted: December 2016

Key words and phrases: L-fuzzy rough set, L-relation, Alexandrov L-topology, L-preorder, Topological similarity.

Topology is an important mathematical tool to investigate information systems and rough sets, so discussing the topological properties of rough sets is an interesting research topic in the literature [2, 9, 11, 18]. Fuzzy rough sets based on residuated lattice have been proposed by Radzikowska and Kerre in [21]. L-fuzzy rough set based on residuated lattice was studied by an axiomatic approach in [22] and was deeply investigated in [27]. Hao and Li considered the relationship between L-fuzzy rough sets and L-topologies in [8]. Ma and Hu studied the relationship between upper sets and lower L-fuzzy approximation operators and applying the upper sets to study L-fuzzy rough set theory in [15]. Li and Cui introduced T-similarity of fuzzy relations by means of fuzzy rough sets and studied related algebraic structures.

In this paper, we generalize the notion of topological similarity to describe the connections among L-relations on a fixed universe. Given an L-fuzzy rough approximation space based on complete residuated lattice, we could construct an L-topology induced by the L-relation. Then we introduce several variations of the given L-relation and investigate the relationship between the variations and the original L-relation. By means of the induced L-topology, we define the concept of topological similarity of L-relations, which is indeed the core of this paper. And we also intuitively characterize the notion of topological similarity. We believe that topological methods are useful tools for the study of fuzzy rough set theory. From topological points of view, we could take a deep insight into the essence of fuzzy rough sets and make our discussions concise and profound.

The rest of the paper is organized as follows. In Section 2, we recall some fundamental properties of residuated lattice and L-topology. Some basic properties of L-fuzzy rough sets are given in Section 3. In Section 4, we introduce the L-topology generated by an arbitrary L-relation and prove that it is an Alexandrov L-topology. Topological similarity (T-similarity, for short) of L-relations have been proposed in Section 5. Then we give the intuitive characterization of T-similarity. This is indeed the core concept of the paper. Drawing on it, the topological similar equivalence relation is introduced to investigate the different variations of a given L-relation and their relationship. Moreover, we prove that each L-relation is uniquely topological similar to an L-preorder, which is the largest one in the associated equivalence class. We also give two intuitive constructions of the L-preorder in this section. In Section 6, the algebraic structures of different sets of L-relations based on T-similarity are investigated. Finally, we conclude the paper with a perspective of further research.

2. Preliminaries

A residuated lattice [1, 28] $\mathbf{L}$ is a structure $(L, *, \to, \vee, \wedge, 0, 1)$, where $(L, \vee, \wedge, 0, 1)$ is a bounded lattice with the greatest element 1 and the smallest element 0; $(L, *, 1)$ is a commutative monoid and $*$ is isotonic at both arguments; and $x * y \leq z$ if and only if $x \leq y \to z$ for all $x, y, z \in L$. A residuated lattice is said to be complete if the underlying lattice is complete.

In what follows, $*$ is sometimes called a generalized triangular norm and the implicator $\to$ is called the residuum of $*$. An implicator I is called left monotonic (respectively right monotonic) if $I(\cdot, a)$ is decreasing for every $a \in L$ (respectively

$I(a,\cdot)$ is increasing). If I is both left monotonic and right monotonic, then it is called hybrid monotonic.

Throughout this paper, L denotes a complete residuated lattice. The precomplement on L is the mapping $\neg : L \to L$ defined by $\neg a = a \to 0$ for every $a \in L$. Some basic properties of complete residuated lattices are listed as follows. More properties about complete residuated lattices can be referred to [1, 22, 28].

(1) $1 \to a = a$;

(2) $(a \to b) * (b \to c) \leq a \to c$;

(3) $a \to (b \to c) = b \to (a \to c) = (a * b) \to c$;

(4) $a \leq (b \to a * b)$, $a * (a \to b) \leq b$;

(5) $a * (\bigvee_{i\in I} b_i) = \bigvee_{i\in I} (a * b_i)$;

(6) $a \to (\bigwedge_{i\in I} b_i) = \bigwedge_{i\in I} (a \to b_i)$, $\quad (\bigvee_{i\in I} a_i) \to b = \bigwedge_{i\in I} (a_i \to b)$;

(7) $a \to (b \to a * b) = 1$, $\quad a \to (b \to a) = 1$;

(8) $b \to c \leq (a \to b) \to (a \to c)$, $\quad c \to b \leq (b \to a) \to (c \to a)$;

(9) $a \leq \neg\neg a$;

(10) $a \to \neg b = \neg(a * b)$, $\quad \neg(\bigvee_{i\in I} a_i) = \bigwedge_{i\in I} (\neg a_i)$.

If $\neg\neg a = a$ holds for every $a \in L$, then L is called involutive. In this case, the following conditions hold.

(11) $a \to b = (\neg b) \to (\neg a)$, $a * b = \neg(a \to (\neg b))$,
$a \to b = \neg(a * (\neg b))$;

(12) $\neg(\bigwedge_{i\in I} a_i) = \bigvee_{i\in I} (\neg a_i)$.

In [7], Goguen first introduced L-set as a generalization of Zadeh's fuzzy set where L is a complete residuated lattice. An L-set A on U is a mapping $A : U \to L$ and all the L-sets on U are denoted by L^U. For every $a \in L$, we use $\bar{a}$ to denote the constant L-set on U. For $A, B \in L^U$, we denote $A \subseteq B$ if $A(x) \leq B(x)$ for every $x \in U$.

Given two L-sets A and B, new L-sets can be induced as follows:

$$(A * B)(x) = A(x) * B(x);$$
$$(A \cap B)(x) = A(x) \wedge B(x);$$
$$(A \cup B)(x) = A(x) \vee B(x);$$
$$(A \to B)(x) = A(x) \to B(x);$$
$$(\neg A)(x) = A(x) \to 0.$$

An L-relation R on U is a mapping $R : U \times U \to L$. R is reflexive if $1 \leq R(x, x)$ for all $x \in U$; R is transitive if $\bigvee_{y\in U} R(x, y) * R(y, z) \leq R(x, z)$ for all $x, z \in U$;

R is symmetric if $R(x,y) = R(y,x)$ for all $x, y \in U$; R is antisymmetric if for all $x, y \in L$, $R(x,y) = R(y,x)$ implies $x = y$. R is called an L-preorder if it is reflexive and transitive.

A subset $\mathcal{T} \subseteq L^U$ is called an L-topology (proposed by Lowen) [14] if it satisfies:
(1) For each $a \in L$, $\bar{a} \in \mathcal{T}$;
(2) $\{A_i, i \in I\} \subseteq \mathcal{T}$ implies $\bigcup_{i \in I} \{A_i\} \in \mathcal{T}$;
(3) $A, B \in \mathcal{T}$ implies $A \cap B \in \mathcal{T}$.

An L-topology $\mathcal{T}$ is called Alexandrov L-topology [10] if it further satisfies the following:
(3′) $\{A_j, j \in J\} \subseteq \mathcal{T}$ implies $\bigcap_{j \in J} \{A_j\} \in \mathcal{T}$.
(4) $\overline{a} * A \in \mathcal{T}$ for all $A \in \mathcal{T}, a \in L$;
(5) $\overline{a} \to A \in \mathcal{T}$ for all $A \in \mathcal{T}, a \in L$.

It should be pointed out that if (1) is replaced by
(1′) $0_U, 1_U \in \mathcal{T}$.
Then it is the fuzzy topology in the sense of Chang [3]. We can see that a fuzzy topology in the sense of Lowen [14] must be a fuzzy topology in the sense of Chang. In this paper, we always consider the fuzzy topology in the sense of Lowen. We denote the interior operator and the closure operator of an L-topology by *int* and *cl* respectively.

3. L-fuzzy Rough Approximation Operators

The notion of fuzzy rough sets based on residuated lattice was proposed by Radzikowska and Kerre in [20]. By taking complete residuated lattices instead of $[0,1]$ as the truth value structures, it differs from the concept of fuzzy rough sets investigated in [5, 6, 16, 19, 24, 25, 29, 30].

Definition 3.1. [20, 22] Let $(L, *, \to, \vee, \wedge, 0, 1)$ be a complete residuated lattice, U a nonempty set and R an L-relation on U. Then (U, R) is called an L-fuzzy approximation space. For every $A \in L^U$ and every $x \in U$, $\overline{R}(A)$ and $\underline{R}(A)$ are defined as follows:

$$\overline{R}(A)(x) = \bigvee_{y \in U} (R(x,y) * A(y)), \tag{1}$$

$$\underline{R}(A)(x) = \bigwedge_{y \in U} (R(x,y) \to A(y)). \tag{2}$$

$\overline{R}(A)$ and $\underline{R}(A)$ are called upper and lower L-fuzzy rough approximations of A, respectively.

Let $R^{-1}(x,y) = R(y,x)$, we can also get another pair of rough approximation operators $(\overline{R^{-1}}, \underline{R^{-1}})$ as defined above by replacing $R(x,y)$ with $R(y,x)$. It is easy to check that fuzzy rough sets defined above is a wide generalization of $(\mathscr{I}, \mathscr{T})$-fuzzy rough sets [29, 30], where $\mathscr{T}$ is a t-norm and $\mathscr{I}$ is the residual implicator based on $\mathscr{T}$.

In the case when $L = \{0,1\}$, A and R can be reduced to crisp subsets of U and $U \times U$ respectively, $\overline{R}(A)$ and $\underline{R}(A)$ are precisely the corresponding concepts in classical rough set theory. Nevertheless, $\underline{R}(A) \subseteq \overline{R}(A)$, which is true in the classical case, is not always true in the fuzzy setting.

The following proposition provides basic properties of the lower and upper L-fuzzy rough approximation operators.

Proposition 3.2. [20, 22] *Let $(L, *, \to, \vee, \wedge, 0, 1)$ be a complete residuated lattice, (U, R) an L-fuzzy approximation space and $A, B \in L^U$, $A_i \in L^U (\forall i \in I)$. Then*

(1) $\overline{R}(\emptyset) = \emptyset,\ \underline{R}(U) = U$;

(2) *If* $A \subseteq B$, *then* $\underline{R}(A) \subseteq \underline{R}(B),\ \overline{R}(A) \subseteq \overline{R}(B)$;

(3) $\underline{R}(A) \subseteq \neg\overline{R}(\neg A),\ \overline{R}(A) \subseteq \neg\underline{R}(\neg A),\ \neg\overline{R}(A) = \underline{R}(\neg A)$;

(4) $\overline{R}(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} \overline{R}(A_i),\ \underline{R}(\bigcap_{i\in I}(A_i)) = \bigcap_{i\in I} \underline{R}(A_i)$;

(5) $\overline{R}(\bar{a} * A) = \bar{a} * \overline{R}(A),\ \underline{R}(\bar{a} \to A) = \bar{a} \to \underline{R}(A)$;

(6) $\overline{R}(\bar{a}) \subseteq \bar{a},\ \bar{a} \subseteq \underline{R}(\bar{a})$;

(7) $\overline{R}(\bar{a}) = \bar{a} \Leftrightarrow \overline{R}(U) = U,\ \bar{a} = \underline{R}(\bar{a}) \Rightarrow \underline{R}(\emptyset) = \emptyset$;

(8) $\overline{R}(1_y * \bar{a})(x) = R(x, y) * a,\ \underline{R}(1_y \to \bar{a}) = R(x, y) \to a.$

It is worthwhile to mention that when L is not involutive, $\underline{R}(A)$ and $\overline{R}(A)$ are not dual to each other. So many results of $\underline{R}(A)$ are not dual to $\overline{R}(A)$ as in many other kinds of fuzzy rough sets.

First of all, we recall some properties of L-relations and L-fuzzy approximation operators.

Proposition 3.3. [22] *Let R be an arbitrary L-relation on U. Then*
(1) *R is reflexive iff $\underline{R}(A) \subseteq A$ iff $A \subseteq \overline{R}(A)$ for every $A \in L^U$.*
(2) *R is transitive iff $\underline{R}(A) \subseteq \underline{R}(\underline{R}(A))$ iff $\overline{R}(\overline{R}(A)) \subseteq \overline{R}(A)$ for every $A \in L^U$.*

The following proposition investigates the relationship between the L-fuzzy rough sets induced by given L-relations and the new constructed L-relations.

Proposition 3.4. *Let R_1, R_2 be L-relations on U. Then for $A \in L^U$,*

(1) $\overline{(R_1 \cup R_2)}(A) = \overline{R_1}(A) \cup \overline{R_2}(A)$;

(2) $\underline{(R_1 \cup R_2)}(A) = \underline{R_1}(A) \cap \underline{R_2}(A)$.

Proof. (1) For every $x \in U$,

$$\begin{aligned}
\overline{(R_1 \cup R_2)}(A)(x) &= \bigvee_{y\in U} ((R_1 \cup R_2)(x, y) * A(y)) \\
&= \bigvee_{y\in U} ((R_1(x, y) \vee R_2(x, y)) * A(y)) \\
&= \bigvee_{y\in U} ((R_1(x, y) * A(y)) \vee (R_2(x, y) * A(y))) \\
&= \bigvee_{y\in U} (R_1(x, y) * A(y)) \vee \bigvee_{y\in U} (R_2(x, y) * A(y))
\end{aligned}$$

$$= \overline{R_1}(A)(x) \vee \overline{R_2}(A)(x) = (\overline{R_1}(A) \cup \overline{R_2}(A))(x).$$

(2) For every $x \in U$,

$$\begin{aligned}
\underline{(R_1 \cup R_2)}(A)(x) &= \bigwedge_{y \in U} ((R_1 \cup R_2)(x,y) \to A(y)) \\
&= \bigwedge_{y \in U} ((R_1(x,y) \vee R_2(x,y)) \to A(y)) \\
&= \bigwedge_{y \in U} ((R_1(x,y) \to A(y)) \wedge (R_2(x,y) \to A(y))) \\
&= \bigwedge_{y \in U} (R_1(x,y) \to A(y)) \wedge \bigwedge_{y \in U} (R_2(x,y) \to A(y)) \\
&= \underline{R_1}(A)(x) \wedge \underline{R_2}(A)(x) = (\underline{R_1}(A) \cap \underline{R_2}(A))(x).
\end{aligned}$$

□

The following proposition is obvious, so we omit the proof.

Proposition 3.5. *Let R_1, R_2 be L-relations on U and $R_1 \subseteq R_2$. Then for $A \in L^U$,*
(1) $\overline{R_1}(A) \subseteq \overline{R_2}(A)$;
(2) $\underline{R_2}(A) \subseteq \underline{R_1}(A)$.

Let R be an L-relations on U, the variations of R are defined as follows:

$$R^0(x,y) = \begin{cases} 0, & \text{if } x = y; \\ R(x,y), & \text{otherwise.} \end{cases}$$

$$R^r(x,y) = \begin{cases} 1, & \text{if } x = y; \\ R(x,y), & \text{otherwise.} \end{cases}$$

The next proposition investigates the relationship between the approximation operators of a given L-relation and its variations.

Proposition 3.6. *Let R be an L-relation on U. Then for $A \in L^U$,*
(1) $(i) A \subseteq \underline{R}(A) \Longleftrightarrow (ii) A = \underline{R^r}(A) \Longleftrightarrow (iii) A \subseteq \underline{R^0}(A)$;
(2) $(i) \overline{R}(A) \subseteq A \Longleftrightarrow (ii) A = \overline{R^r}(A) \Longleftrightarrow (iii) \overline{R^0}(A) \subseteq A$.

Proof. (1) For every $x \in U$,

$$\underline{R}(A)(x) = (R(x,x) \to A(x)) \wedge (\bigwedge_{y \in U - \{x\}} (R(x,y) \to A(y))); \tag{3}$$

$$\underline{R^r}(A)(x) = A(x) \wedge (\bigwedge_{y \in U - \{x\}} (R(x,y) \to A(y))); \tag{4}$$

$$\underline{R^0}(A)(x) = \bigwedge_{y \in U - \{x\}} (R(x,y) \to A(y)). \tag{5}$$

(i)⇒(ii) Since $A(x) \leq \underline{R}(A)(x) \leq \bigwedge_{y \in U - \{x\}} (R(x,y) \to A(y))$,

$$\text{we have } A(x) = A(x) \wedge (\bigwedge_{y \in U - \{x\}} (R(x,y) \to A(y))) = \underline{R^r}(A).$$

(ii)⇒(iii). Due to $R^0 \subseteq R^r$, $A = \underline{R^r}(A) \subseteq \underline{R^0}(A)$.

(iii)$\Rightarrow$(i). According to $A \subseteq \underline{R^0}(A)$, $A(x) = A(x) \wedge \underline{R^0}(A)(x) \leq \underline{R}(A)(x)$. Hence $A \subseteq \underline{R}(A)$.

(2) For every $x \in U$,

$$\overline{R}(A)(x) = (R(x,x) * A(x)) \vee (\bigvee_{y \in U - \{x\}} (R(x,y) * A(y))); \tag{6}$$

$$\overline{R^r}(A)(x) = A(x) \vee (\bigvee_{y \in U - \{x\}} (R(x,y) * A(y))); \tag{7}$$

$$\overline{R^0}(A)(x) = \bigvee_{y \in U - \{x\}} (R(x,y) * A(y)). \tag{8}$$

(i)$\Rightarrow$(ii). According to (i),

$$\begin{aligned} A(x) &= A(x) \vee \overline{R}(A)(x) \\ &= A(x) \vee (R(x,x) * A(x)) \vee (\bigvee_{y \in U - \{x\}} (R(x,y) * A(y))) \\ &= \overline{R^r}(A)(x). \end{aligned}$$

(ii)$\Rightarrow$(iii). Since $A = \overline{R^r}(A)$ and $R^0 \subseteq R^r$, we have $\overline{R^0}(A) \subseteq \overline{R^r}(A) = A$.

(iii)$\Rightarrow$(i). Because $\overline{R^0}(A) \subseteq A$, $A(x) = A(x) \vee \underline{R^0}(A)(x) \geq \overline{R}(A)(x)$. That is to say, $\overline{R}(A) \subseteq A$. □

The next proposition illustrates the correspondence between the approximation operators and the topology operators.

Proposition 3.7. [22] *Let (U, R) be an L-fuzzy approximation space. Then $\underline{R}$ is an L-fuzzy interior operator ($\overline{R}$ is an L-fuzzy closure operator, respectively) if and only if R is an L-preorder.*

Remark 3.8. Let R be an L-preorder on U, then $\underline{R}$ is an L-fuzzy interior operator and $\overline{R}$ is an L-fuzzy closure operator. However, they may not generate the same L-topology unless L is an involutive complete residuated lattice.

4. L-topology Induced by L-relation

In the previous section, we discussed the properties of the rough approximation operators $\overline{R}$, $\underline{R}$ in the L-fuzzy power setting. In the following, we investigate the relationship between L-topologies (generated in different ways) and L-fuzzy rough sets based on arbitrary L-relations.

Let (U, R) be an L-fuzzy approximation space, we define 3 types of families of L-fuzzy sets as follows:

$$\begin{aligned} \sigma_R &:= \{A \in L^U | A \subseteq \underline{R}(A)\}; \\ \theta_R &:= \{\underline{R}(A) | A \in L^U\}; \\ \tau_R &:= \{A \in L^U | \underline{R}(A) = A\}. \end{aligned}$$

In general, $\tau_R \subseteq \sigma_R, \tau_R \subseteq \theta_R$. Furthermore,
(1) If R is reflexive, then $\sigma_R = \tau_R \subseteq \theta_R$;
(2) If R is transitive, then $\tau_R \subseteq \theta_R \subseteq \sigma_R$.

Proposition 4.1. [8] *Let (U, R) be an L-fuzzy approximation space, then*
(1) *If R is reflexive, then τ_R is an L-topology on U.*
(2) *If R is reflexive and transitive, then $\sigma_R = \tau_R = \theta_R$.*

Let $\mathcal{F} \subseteq L^U$, we define an L-relation

$$R_{\mathcal{F}}(x, y) = \bigwedge_{A \in \mathcal{F}} (A(x) \to A(y)). \tag{9}$$

Obviously, $R_{\mathcal{F}}$ is an L-preorder on U. The following proposition presents the connections between L-preorders and Alexandrov L-topologies.

Proposition 4.2. [8] *Let R be an L-preorder on U, τ be Alexandrov L-topology on U. Then $R_{\tau_R} = R$ and $\tau_{R_\tau} = \tau$.*

The above 2 propositions have shown that L-preorder is a much better L-relation in some sense, which would lead to the same L-topology through different ways. So in the rest of the paper, the relationship between general L-relations and L-preorders is established via L-fuzzy rough set theory.

Theorem 4.3. *Let (U, R) be an L-fuzzy approximation space, then*
(1) *σ_R is an Alexandrov L-topology on U;*
(2) *$\sigma_{R^0} = \sigma_R = \sigma_{R^r}$;*
(3) *$int_{\sigma_R}(A) \subseteq \underline{R}(A)$ and $\overline{R}(A) \subseteq cl_{\sigma_R}(A)$ holds for each $A \in L^U$;*
(4) *Let $(\sigma_R)^c = \{\neg B : B \in \sigma_R\}$, then $A \in (\sigma_R)^c \Longrightarrow \overline{R}(A) \subseteq A$.*
(5) *Moreover, if L is an involutive complete residuated lattice, then*

$$\overline{R}(A) \subseteq A \Longleftrightarrow A \in (\sigma_R)^c.$$

Proof. (1) First, for each $a \in L$, $\bar{a} \subseteq \underline{R}(\bar{a})$ in terms of Proposition 3.2, so $\bar{a} \in \sigma_R$. Second, let $\{A_i, i \in I\} \subseteq \sigma_R$, then $\bigcup\limits_{i \in I} A_i \subseteq \bigcup\limits_{i \in I} \underline{R}(A_i) \subseteq \underline{R}(\bigcup\limits_{i \in I} A_i)$ and $\bigcap\limits_{i \in I} A_i \subseteq \bigcap\limits_{i \in I} \underline{R}(A_i) = \underline{R}(\bigcap\limits_{i \in I} A_i)$. Hence $\bigcup\limits_{i \in I} A_i$, $\bigcap\limits_{i \in I} A_i \in \sigma_R$.
Third, for each $a \in L, A \in \sigma_R$ and $x \in U$, we have

$$\begin{aligned}(\bar{a} * A)(x) &\leq (\bar{a} * \underline{R}(A))(x) \\ &= a * \bigwedge_{y \in U} (R(x, y) \to A(y)) \\ &\leq \bigwedge_{y \in U} a * (R(x, y) \to A(y)) \\ &\leq \bigwedge_{y \in U} (R(x, y) \to a * A(y)) \\ &= \underline{R}(\bar{a} * A)(x)\end{aligned}$$

and

$$\begin{aligned}(\bar{a} \to A)(x) &\le (\bar{a} \to \underline{R}(A))(x) \\ &= a \to \bigwedge_{y \in U} (R(x,y) \to A(y)) \\ &= \bigwedge_{y \in U} (a \to (R(x,y) \to A(y))) \\ &= \bigwedge_{y \in U} (R(x,y) \to (a \to A(y))) \\ &= \underline{R}(\bar{a} \to A))(x).\end{aligned}$$

Hence $\bar{a} * A,\ \bar{a} \to A \in \sigma_R$. In conclusion, σ_R is an Alexandrov L-topology.

(2) According to Proposition 3.6, $\sigma_{R^0} = \sigma_R = \sigma_{R^r}$.

(3) For each $A \in L^U$,

$$\begin{aligned}int_{\sigma_R}(A) &= \bigcup\{B \in \sigma_R : B \subseteq A\} \\ &\subseteq \bigcup\{\underline{R}(B) : B \in \sigma_R \text{ and } B \subseteq A\} \\ &\subseteq \underline{R}(A).\end{aligned}$$

According to Proposition 3.2 (3), $\underline{R}(B) \subseteq \neg\overline{R}(\neg B)$ for $B \in \sigma_R$, so we have $B \subseteq \neg\overline{R}(\neg B)$. Thus $\overline{R}(\neg B) \subseteq \neg\neg\overline{R}(\neg B) \subseteq \neg B$ and

$$\begin{aligned}cl_{\sigma_R}(A) &= \bigcap\{\neg B : A \subseteq \neg B \text{ and } B \in \sigma_R\} \\ &\supseteq \bigcap\{\overline{R}(\neg B) : A \subseteq \neg B \text{ and } B \in \sigma_R\} \\ &\supseteq \overline{R}(A).\end{aligned}$$

(4) This is easy to verify by the proof of (3).

(5) If L is an involutive complete residuated lattice, $A = \neg\neg A$ for each $A \in L^U$. So

$$\begin{aligned}A \in (\sigma_R)^c &\Longleftrightarrow \exists B \in \sigma_R, A = \neg B \\ &\Longleftrightarrow B = \neg A \text{ and } B \subseteq \underline{R}(B) \\ &\Longleftrightarrow \neg A \subseteq \underline{R}(\neg A) \\ &\Longleftrightarrow \overline{R}(A) \subseteq \neg\underline{R}(\neg A) \subseteq A.\end{aligned}$$

□

Remark 4.4. (1) $A \in (\sigma_R)^c \Longrightarrow \overline{R}(A) \subseteq A$. The converse is not true in general, we only could obtain $\neg\neg A \in (\sigma_R)^c$ on the hypothesis that $\overline{R}(A) \subseteq A$.

(2) If L is an involutive complete residuated lattice, by Proposition 3.2 (6) and the above proof, $\bar{a} \in (\sigma_R)^c$ for each $a \in L$.

Theorem 4.5. *Let R_1 and R_2 be L-relations on U.*

(1) *If $R_1 \subseteq R_2$, then $\sigma_{R_2} \subseteq \sigma_{R_1}$.*

(2) *If R_1 and R_2 are L-preorders, then $\sigma_{R_1} = \sigma_{R_2} \Longleftrightarrow R_1 = R_2$.*

Proof. (1) It is obvious by Proposition 3.5.

(2) We only need to prove the sufficiency. Since R_1 and R_2 are L-preorders, according to Proposition 4.2, $R_1 = R_{\sigma_{R_1}}$ and $R_2 = R_{\sigma_{R_2}}$. So $R_1 = R_2$ in terms of $\sigma_{R_1} = \sigma_{R_2}$. □

Theorem 4.6. *Let $\{R_j, j \in J\}$ be a family of L-relations on U, then*

$$\sigma_{\bigcup_{j\in J} R_j} = \bigcap_{j\in J} \sigma_{R_j}.$$

Proof. On one hand, $\sigma_{\bigcup_{j\in J} R_j} \subseteq \bigcap_{j\in J} \sigma_{R_j}$ by (1) of Theorem 4.5.
On the other hand, for each $A \in \bigcap_{j\in J} \sigma_{R_j}$, $A \subseteq \underline{R_j}(A)$ for all $j \in J$. So we have $A \subseteq \bigcap_{j\in J} \underline{R_j}(A) = \underline{(\bigcup_{j\in J} R_j)}(A)$ by Proposition 3.4, i.e. $A \in \sigma_{\bigcup_{j\in J} R_j}$. Hence $\sigma_{\bigcup_{j\in J} R_j} = \bigcap_{j\in J} \sigma_{R_j}$. □

5. Topological Similarity of L-relations

In this section, we propose the concept of topological similarity of L-relations and investigate the corresponding properties.

Definition 5.1. Let R_1 and R_2 be L-relations on U. If $\sigma_{R_1} = \sigma_{R_2}$, then we call R_1 is topological similar (T-similar, for short) to R_2 and we denote it by $R_1 \sim_T R_2$.

Remark 5.2. Obviously, $\sim_T$ is an equivalence relation on L^U.
Let $R \in L^{U\times U}$, we denote $[R]_T = \{Q \in L^{U\times U} : Q \sim_T R\}$. For convenience, we use $[R]$ to denote $[R]_T$.

The following theorem discusses the topological similar equivalence class of a given L-relation.

Theorem 5.3. *Let R be an L-relation on U. Then*
(1) *$R^0, R^r \in [R]$;*
(2) *If $\{R_j, j \in J\} \subseteq [R]$, then $\bigcup_{j\in J} R_j \in [R]$;*
(3) *If $R_1,\ R_2 \in [R]$ and $R_1 \subseteq Q \subseteq R_2$, then $Q \in [R]$;*
(4) *If R_1 and R_2 are L-preorders, then $R_1 \in [R_2]$ if and only if $R_1 = R_2$.*

Proof. (1) It is easy to verify.
(2) It is obvious in terms of Theorem 4.6.
(3) Since $R_1 \subseteq Q \subseteq R_2$, we have $\sigma_{R_2} \subseteq \sigma_Q \subseteq \sigma_{R_1}$. According to $R_1,\ R_2 \in [R]$, $\sigma_{R_1} = \sigma_{R_2}$. Therefore, $\sigma_{R_2} = \sigma_Q = \sigma_{R_1}$, i.e. $Q \in [R]$.
(4) By Theorem 4.5, it follows immediately. □

Definition 5.4. [27] Let R be an L-relation on U and $A \in L^U$.
(1) A is said to be a generalized L-fuzzy upper set if $A(x) * R(x, y) \leq A(y)$ for all $x, y \in U$;
(2) A is said to be a generalized L-fuzzy lower set if $A(y) * R(x, y) \leq A(x)$ for all $x, y \in U$.

Proposition 5.5. [27] *Let R be an L-relation on U and $A \in L^U$.*
(1) *A is a generalized L-fuzzy upper set iff $A \subseteq \underline{R}(A)$;*
(2) *A is a generalized L-fuzzy lower set iff $\overline{R}(A) \subseteq A$.*

Remark 5.6. According to Definition 5.4 and Proposition 5.5,

$$\sigma_R = \{A \in L^U : \ A(x) * R(x,y) \leq A(y), \text{ for all } x, y \in U\}.$$

Theorem 5.7. *Let R be an L-relation on U and $A \in L^U$, then $R \subseteq R_{\sigma_R}$. Moreover, $R^{rt} = R_{\sigma_R}$, where R^{rt} is the reflexive and transitive closure of R.*

Proof. On one hand, for all $x, y \in U$,

$$\begin{aligned} A \in \sigma_R &\Longleftrightarrow A(x) * R(x,y) \leq A(y) \\ &\Longleftrightarrow R(x,y) \leq A(x) \to A(y). \end{aligned}$$

Since $\quad R_{\sigma_R}(x,y) = \bigwedge_{A \in \sigma_R} (A(x) \to A(y)), \quad R(x,y) \leq R_{\sigma_R}(x,y).$

On the other hand, according to the definition of R^{rt}, we have $R \subseteq R^{rt} \subseteq R_{\sigma_R}$. This implies $\sigma_{R_{\sigma_R}} \subseteq \sigma_{R^{rt}} \subseteq \sigma_R$. Moreover, since R_{σ_R} is an L-preorder, $\sigma_{R_{\sigma_R}} = \sigma_R$, so we have $\sigma_{R^{rt}} = \sigma_R$. In addition, R_{σ_R} and R^{rt} are L-preorders, hence $R_{\sigma_R} = R^{rt}$ by Theorem 4.5. □

The above theorem illustrates that the topology induced by an arbitrary L-relation R could be seen as the topology induced by its transitive and reflexive closure, i.e., by an L-preorder. And we denote this L-preorder by R^p. In other words, each L-relation R is topological similar to R^p.

Corollary 5.8. *Let R be an L-relation on U, then R^p is the largest L-relation in $[R]$.*

Let R be an L-relation on U, we borrow the method in [11] to compute R^p step by step. At first, we define R^{uv}, R^{uvw} as follows: for $x, y, u, v, w \in U$,

$$R^{uv}(x,y) = \begin{cases} R(u,v) \vee (R \circ R)(u,v), & \text{if}(x,y) = (u,v); \\ R(x,y), & \text{otherwise.} \end{cases}$$

$$R^{uvw}(x,y) = \begin{cases} R(u,v) \vee (R(u,w) * R(w,v)), & \text{if}(x,y) = (u,v); \\ R(x,y), & \text{otherwise.} \end{cases}$$

Obviously, $R^{uv} = \bigcup_{w \in U} R^{uvw}$.

Theorem 5.9. *Let R be an L-relation on U, then R^{uvw}, $R^{uv} \in [R]$ for $u, v, w \in U$.*

Proof. We firstly prove $R^{uvw} \in [R]$ for $u, v, w \in U$, then $R^{uv} \in [R]$ by Theorem 5.3.
(i) If $u = v$, then $R^0 \subseteq R^{uvw} \subseteq R^r$, so $R^{uvw} \in [R]$.
(ii) If $u \neq v$, we need to prove $R^{uvw} \in [R]$, i.e. , $\sigma_R = \sigma_{R^{uvw}}$. On one hand, since $R \subseteq R^{uvw}$, we have $\sigma_{R^{uvw}} \subseteq \sigma_R$. On the other hand, we need to show that $\sigma_R \subseteq$

$\sigma_{R^{uvw}}$. For every $A \in \sigma_R$, when $x \neq u$, we have $\underline{R^{uvw}}(A)(x) = \underline{R}(A)(x) \supseteq A(x)$. When $x = u$,

$$\begin{aligned}\underline{R^{uvw}}(A)(u) =& \bigwedge_{y\in U}(R^{uvw}(u,y) \to A(y)),\\ =&(\bigwedge_{y\in U-\{v\}}(R(u,y)\to A(y)))\wedge(R^{uvw}(u,v)\to A(v)),\\ =&(\bigwedge_{y\in U-\{v\}}(R(u,y)\to A(y)))\wedge((R(u,v)\vee(R(u,w)*R(w,v)))\to A(v)).\end{aligned}$$

If $R(u,v) \geq R(u,w) * R(w,v)$, then $\underline{R^{uvw}}(A)(u) = \underline{R}(A)(u) \geq A(u)$.
If $R(u,v) < R(u,w) * R(w,v)$, since

$$\begin{aligned}(R(u,w)*R(w,v))\to A(v)) =& R(u,w)\to(R(w,v)\to A(v)),\\ \geq& R(u,w)\to A(w)\geq A(u),\end{aligned}$$

we have

$$\begin{aligned}\underline{R^{uvw}}(A)(u) =& (\bigwedge_{y\in U-\{v\}}(R(u,y)\to A(y)))\wedge((R(u,w)*R(w,v))\to A(v)),\\ \geq& (\bigwedge_{y\in U-\{v\}}(R(u,y)\to A(y)))\wedge A(u)\\ \geq& \underline{R}(A)(u)\wedge A(u) = A(u).\end{aligned}$$

Thus $A \subseteq \underline{R^{uvw}}(A)$. □

Next, we define a sequence of L-relations based on R:

$$R_0 = R,$$
$$R_{n+1} = R_n \circ R_n^r,\ \forall n \in \mathbb{N}.$$

It is not difficult to verify that

$$R_{n+1} = R_n \bigcup R_n \circ R_n \text{ and } R_{n+1} = \bigcup_{u,v\in U} R_n^{uv}, \text{ where}$$

$$R_n^{uv}(x,y) = \begin{cases} R_n(u,v) \vee (R_n \circ R_n)(u,v), & \text{if } (x,y) = (u,v);\\ R_n(x,y), & \text{otherwise.}\end{cases}$$

Remark 5.10. Denote $R^* = \lim_{n\to\infty} R_n$. Then
(1) The sequence $\{R_n, n \in \mathbb{N}\}$ is increasing;
(2) $R^* = \bigcup_{n=0}^{\infty} R_n$;
(3) $R_n \in [R],\ \forall n \in \mathbb{N}$;
(4) For an L-relation Q, if $Q \subseteq R$, then $Q_n \subseteq R_n$.

Theorem 5.11. *Let R be an L-relation on U, then R is transitive if and only if $R = R_1$.*

Proof. Necessity is obvious. As for the sufficiency, since $R = R_1 = R \cup (R \circ R) \supseteq R \circ R$, R is transitive. □

The following corollary is a direct consequence of the above theorem.

Corollary 5.12. *Let R be an L-relation on U. If R is transitive, then R_n is transitive for each $n \in \mathbb{N}$.*

Theorem 5.13. *Let R, P, Q be L-relations on U. If $P, Q \in [R]$, then $P \circ Q^r$, $P^r \circ Q$, $P^r \circ Q^r \in [R]$.*

Proof. Since $P, Q \in [R]$, we immediately have $P \cup Q \in [R]$ and $(P \cup Q)_1 \in [R]$ in terms of Remark 5.10. Moreover, $P \subseteq P \circ Q^r \subseteq (P \cup Q)_1 = (P \cup Q) \circ (P \cup Q)^r$, we have $P \circ Q^r \in [R]$. Similarly, we can prove $P^r \circ Q$, $P^r \circ Q^r \in [R]$. □

Theorem 5.14. *Let R be an L-relation on U, then R^* is transitive.*

Proof. For $n \in N$, we denote $R^{n+1} = R^n \circ R$. Since

$$\begin{aligned} R^* &= \bigcup_{n=0}^{\infty} R_n = R \cup R_1 \cup \cdots \\ &= R \cup (R \cup (R \circ R)) \cup \cdots \\ &= \bigcup_{n=1}^{\infty} R^n. \end{aligned}$$

This is just the transitive closure of R, it follows immediately that R^* is transitive. □

Theorem 5.15. *Let R be an L-relation on U, then there exists a unique L-preorder $Q \in [R]$. Moreover, $Q = R^p$.*

Proof. Let $Q = R^r \cup R^* = R^r \cup R^t$, then Q is an L-preorder in terms of Theorem 5.14. Since $\sigma_{R^r} = \sigma_R = \sigma_{R^*}$, we have $Q \in [R]$. If there is another L-preorder $P \in [R]$, then $\sigma_P = \sigma_R = \sigma_Q$ and $P = Q$ by Theorem 4.5. Furthermore, according to the construction of Q, $Q = R^p$. □

Corollary 5.16. *Let R be an L-relation on U, then the unique L-preorder*

$$R^p = \bigcup_{P \in [R]} P = \bigcap \{Q : \ Q \text{ is } L\text{-preorder and } R \subseteq Q\}$$

6. Algebraic Structures Based on T-similarity of L-relations

Let R be an L-relation on U, we denote

$$\begin{aligned} [R]^r &= \{P^r : P \in [R]\}, \\ [R]^0 &= \{P^0 : P \in [R]\}, \\ [R]^t &= \{P \in [R] : P \text{ is transitive}\}. \end{aligned}$$

Theorem 6.1. *Let R be an L-relation on U, then*
(1) $([R], \cup)$ is a commutative semigroup.
(2) $([R]^r, \cup)$ and $([R]^0, \cup)$ are sub-semigroups of $([R], \cup)$.

According to Theorem 5.15, we have the following theorem .

Theorem 6.2. *Let R be an L-relation on U, then $([R]^r, \circ)$ is a commutative semigroup.*

Next we give an intuitive characterization of $[R]^t$ and $\bigcap[R]^t$.

Theorem 6.3. *Let R be an L-relation on U, then*

(1) $\forall P \in [R]^t,\ (R^p)^0 \subseteq P \subseteq R^p$.

(2) *a)* $\bigcap[R]^t \in [R]^t$.

b) $$(\bigcap[R]^t)(x,y) = \begin{cases} R^p(x,y), & x \neq y; \\ \bigvee\limits_{z\neq x}(R^p(x,z) * R^p(z,y)), & x = y. \end{cases}$$

c) $\bigcap[R]^t = \bigcap\{P : P$ *is transitive and* $R^0 \subseteq P\}$.

d) $\bigcap[R]^t \subseteq R^*$.

(3) $[R]^t = \{P : \bigcap[R]^t \subseteq P \subseteq R^p\}$.

Proof. (1) According to Corollary 5.8, for each $P \in [R]^t$, $P \subseteq R^p$ and P is transitive. Moreover, P^r is an L-preorder and $P^r \in [R]$. Assume there exists $x, y \in U$ and $x \neq y$ such that $P(x,y) < (R^p)^0(x,y)$, we have $P(x,y) < R^p(x,y)$. Consequently, $P^r(x,y) < R^p(x,y)$. However, since P^r is an L-preorder, $P^r(x,y) = R^p(x,y)$ by (4) of Theorem 5.3. This is a contradiction. Hence $(R^p)^0 \subseteq P$.

(2) a) Since $(R^p)^0 \subseteq P \subseteq R^p$ for any $P \in [R]^t$, $(R^p)^0 \subseteq \bigcap[R]^t \subseteq R^p$ and $\bigcap[R]^t$ is transitive. Because $(R^p)^0,\ R^p \in [R]$, we have $\bigcap[R]^t \in [R]$. Hence $\bigcap[R]^t \in [R]^t$.

b) Let

$$Q(x,y) = \begin{cases} R^p(x,y), & x \neq y; \\ \bigvee\limits_{z\neq x}(R^p(x,z) * R^p(z,y)), & x = y. \end{cases}$$

Through a direct computing, we obtain that Q is transitive and $(R^p)^0 \subseteq Q \subseteq R^p$, so $Q \in [R]^t$. In the following, we prove that $Q = \bigcap[R]^t$.
If $x \neq y$, $Q(x,y) = R^p(x,y)$. By the proof of (1), $(R^p)^0 \subseteq P \subseteq R^p$ for any $P \in [R]^t$. Therefore, $Q(x,y) = R^p(x,y) = (\bigcap[R]^t)(x,y)$.
If $x = y$, then for any $P \in [R]^t$,

$$Q(x,x) = \bigvee_{z\neq x}(R^p(x,z) * R^p(z,x)) \leq \bigvee_{z\neq x} P(x,z) * P(z,x)) \leq P(x,x),$$

so we have $Q(x,x) \leq (\bigcap[R]^t)(x,x)$. In addition, $(\bigcap[R]^t)(x,x) \leq Q(x,x)$ in terms of $Q \in [R]^t$. Hence $Q(x,x) = (\bigcap[R]^t)(x,x)$.

c) It is obvious.

d) According to the construction of R^*, $\bigcap[R]^t \subseteq R^*$.

(3) On one hand, since for any $P \in [R]^t$, $(R^p)^0 \subseteq P \subseteq R^p$. Hence $\bigcap[R]^t \subseteq P \subseteq R^p$, i.e. $[R]^t \subseteq \{P : \bigcap[R]^t \subseteq P \subseteq R^p\}$. On the other hand, for any $P \in \{P : \bigcap[R]^t \subseteq P \subseteq R^p\}$, we have $P \in [R]$. For all $x, y, z \in U$, we need to prove that

$P(x,z) \geq P(x,y) * P(y,z)$. Because

$$P(x,z) \geq (\bigcap [R]^t)(x,z) = \begin{cases} \bigvee_{u \neq x} (R^p(x,u) * R^p(u,z)), & \text{if } x = z \\ R^p(x,z), & \text{if } x \neq z. \end{cases}$$

When $x = z$ and $x \neq y$, we have

$$P(x,z) \geq R^p(x,y) * R^p(y,z) = P(x,y) * P(y,z).$$

When $x \neq z$, $x \neq y$ and $y \neq z$, we have

$$P(x,z) = R^p(x,z) \geq R^p(x,y) * R^p(y,z) = P(x,y) * P(y,z)$$

For the remaining cases, when $x = y$ or $y = z$, it is obvious.
Therefore, $[R]^t = \{P : \bigcap [R]^t \subseteq P \subseteq R^p\}$. □

Theorem 6.4. *Let R be an L-relation on U, then $([R]^t, \subseteq, \cup, \cap)$ is a complete lattice with $1_{[R]^t} = R^p$ and $0_{[R]^t} = \bigcap [R]^t$.*

Proof. According to Theorem 6.3, $1_{[R]^t} = R^p$ and $0_{[R]^t} = \bigcap [R]^t$. For any family of L-relations $\{R_i, i \in I\} \subseteq [R]^t$, we have $\bigcap [R]^t \subseteq R_i \subseteq R^p, \forall i \in I$. Hence $\bigcap [R]^t \subseteq \bigcup_{i \in I} R_i \subseteq R^p$ and $\bigcap [R]^t \subseteq \bigcap_{i \in I} R_i \subseteq R^p$. Therefore, $([R]^t, \subseteq, \cup, \cap)$ is a complete lattice. □

7. Concluding Remarks

In this paper, we propose the notion of topological similarity of L-relations based on L-topologies induced by L-fuzzy rough sets, which is in fact an equivalence relation. We firstly discuss the variations of a given L-relation and prove that each L-relation can induce an Alexandrov L-topology. Then we show that each L-relation is uniquely T-similar to an L-preorder, which is the largest one in the corresponding topological similar equivalence class. And we give intuitive characterization of the unique L-preorder similar to the given L-relation. As an application of T-similarity, related algebraic structures are studied finally. We believe that the notion of T-similarity will be helpful in investigating L-relations and through which we could take a deep insight into their structures from the topological point of view. In future work, we will generalize the results of this paper to generalized residuated lattices.

Acknowledgements. The authors would like to express their thanks to anonymous referees for their valuable comments and useful recommendations which helped to improve the paper.

This work is supported by NNSF of China (No. 11601132) and NNSF of Henan Province (No. 122300410381).

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