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STONE DUALITY FOR R_0-ALGEBRAS WITH INTERNAL STATES

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ABSTRACT. R_0-algebras, which were proved to be equivalent to Esteva and Godo's NM-algebras modelled by Fodor's nilpotent minimum t-norm, are the equivalent algebraic semantics of the left-continuous t-norm based fuzzy logic firstly introduced by Guo-jun Wang in the mid 1990s. In this paper, we first establish a Stone duality for the category of MV-skeletons of R_0-algebras and the category of three-valued Stone spaces. Then we extend Flaminio-Montagna internal states to R_0-algebras. Such internal states must be idempotent MV-endomorphisms of R_0-algebras. Lastly we present a Stone duality for the category of MV-skeletons of R_0-algebras with Flaminio-Montagna internal states and the category of three-valued Stone spaces with Zadeh type idempotent continuous endofunctions. These dualities provide a topological viewpoint for better understanding of the algebraic structures of R_0-algebras.

1. Introduction

With the rapid development of fuzzy control techniques in the 1990s, fuzzy logic, presented as the key part of fuzzy control, was often attacked because of the alleged lack of rigorous logic foundations [15]. In order to provide a logic foundation for fuzzy reasoning and to reduce the gap between fuzzy reasoning and fuzzy formal deduction, Wang [32, 31] modified Kleene-Dienes's implication operator $x \to y = \max\{1-x, y\}$ by setting $x \to y = 1$ if $x \le y$ and otherwise $x \to y = \max\{1-x, y\}$ for $x, y \in [0,1]$. The resulting operator is called R_0-implication[1], and then Wang proposed a fuzzy formal deductive system $\mathcal{L}^*$ (also known as R_0-logic) as formalization of R_0-implication. In order to prove the algebraic completeness of the logic $\mathcal{L}^*$, Wang also introduced R_0-algebras [31], and Pei et al. [29] proved later the standard completeness of $\mathcal{L}^*$ with respect to R_0-algebras. Hence, R_0-algebras are to $\mathcal{L}^*$ just what Boolean algebras are to the classical propositional logic. Besides their logical interest and contributions to fuzzy reasoning [40, 21, 35], R_0-algebras have many important algebraic properties [28, 37, 5]. Here it is worthy of noting that, by the adjointness between R_0-implication and Fodor's nilpotent minimum t-norm [18], the well-known nilpotent minimum logic introduced by Esteva and Godo [16] for capturing the tautologies of the nilpotent minimum t-norm and its residuum is a

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[1] The R_0-implication was also independently discovered by Fodor [18] as the residuum of a left-continuous t-norm, called nilpotent minimum t-norm.

logic system equivalent to the R_0-logic $\mathcal{L}^*$, and hence nilpotent minimum algebras (NM-algebras, for short) are equivalent to R_0-algebras, see [27, 22].

Stone representation theorem [30] is a theorem having a far-reaching influence in the history of development of mathematics, which shows that every Boolean algebra is isomorphic to the algebra of all clopen subsets of its Stone space consisting of all maximal filters of the Boolean algebra. This theorem is fundamental to the deeper understanding of Boolean algebras. Such idea of delineating relationships between posets and topological spaces has been extensively enhanced, see, e.g., [38, 2, 33, 7, 1]. In [43], we presented two analogues for an R_0-algebra of the Stone representation theorem for Boolean algebras, which state that the Boolean skeleton of an R_0-algebra is isomorphic to the algebra of all clopen subsets of its Stone space and the MV-skeleton of an R_0-algebra is isomorphic to the algebra of all clopen fuzzy subsets of its three-valued Stone space, respectively.

States on MV-algebras were first introduced by Mundici [25] as averaging process for truth values of propositions in Łukasiewicz infinite-valued propositional logic. In the last decade states have been deeply investigated by many authors and were proved to be closely related to other mathematical notions such as integral and de Finetti's coherence criterion [24]. Nowadays, states have been extended to different algebraic structures of many-valued logic [20, 8]. However, underlying logical algebras with a state are not universal algebras, and hence they do not provide an algebraizable logic in the sense of Blok and Pigozzi [3] for reasoning about probability of many-valued events. Keeping this situation in mind, Flaminio and Montagna [17] enlarged the language of MV-algebras by adding a unary internal operator equationally described so as to preserve the basic properties of a state in its original meaning. The resulting algebras are called MV-algebras with internal states or state MV-algebras. This topic has aroused considerable interest of researchers in the research communities of non-classical logics and of many-valued probability theory. Internal states have been extended to more general algebraic structures [14, 19, 26] and actually, a very general framework of states has been established in residuated lattices [10, 42, 9]. Among these results, Di Nola and Dvurečenskij [11, 4] introduced a stronger version of state algebras, called state-morphism algebras, i.e., the involved internal states are idempotent endomorphisms.

After having Stone representations of logical algebras, a natural question is to find the topological counterparts of internal states on the underlying logical algebras in their topological duals, which provides us a new viewpoint to deal with internal states on logical algebras in topological setting, besides the algebraic and logical settings proposed by Flaminio and Montagna [17]. Recently, Di Nola et al. [13] studied the topological counterparts of endomorphic internal states on weakly divisible σ-complete MV-algebras, and finally presented a Stone duality theorem for such state-morphism MV-algebras. Towards the same direction of the research on Stone dualities of logical algebras with internal states, the present paper aims to study the topological counterparts of internal states on R_0-algebras. Such dualities will provide us a topological viewpoint for understanding the behavior of internal states. To do so, we first study the topological counterparts of MV-homomorphism between R_0-algebras in the three-valued Stone dual spaces of R_0-algebras, and then investigate the topological counterparts of internal states. The main results of the paper are the following Stone duality theorems for R_0-algebras and for ones with internal states, which show that (i) the category of MV-skeletons of R_0-algebras with MV-homomorphisms as morphisms is dually equivalent to the category of three-valued Stone spaces with Zadeh type continuous functions as morphisms; (ii) the category of Boolean skeletons of R_0-algebras with Boolean homomorphisms as morphisms is dually equivalent to the category of Stone spaces with continuous functions as morphisms; (iii) the category of MV-skeletons of R_0-algebras with internal states is dual to the category of three-valued Stone spaces with Zadeh type idempotent continuous endofunctions; and (iv) the category of Boolean skeletons of R_0-algebras with internal states is dual to the category of Stone spaces with idempotent continuous endofunctions.

The rest of the paper is arranged as follows: In Section 2 we recall some necessary notions and results about R_0-algebras. In Section 3 we extend Stone representations of R_0-algebras obtained in [43] to a categorical duality. In Section 4 we first extend Flaminio-Montagna internal states to R_0-algebras and show that such internal states are actually MV-homomorphisms, and then, on the basis of results from Section 3, we establish two Stone duality theorems for R_0-algebras with internal states. Section 5 contains some concluding remarks and possible further research topics.

2. **Preliminaries**

In this section we will recapitulate some preliminaries on R_0-algebra [36, 39, 5].

Let M be a bounded distributive lattice with an order-reversing involution $\neg$, a binary operation $\to$ and the greatest element 1. Then $(M, \vee, \wedge, \neg, \to, 1)$ is called an R_0*-algebra* if it satisfies the following axioms (M1)-(M6):

(M1) $\neg x \to \neg y = y \to x$,
(M2) $1 \to x = x, \quad x \to x = 1$,
(M3) $y \to z \leq (x \to y) \to (x \to z)$,
(M4) $x \to (y \to z) = y \to (x \to z)$,
(M5) $x \to (y \vee z) = (x \to y) \vee (x \to z), \quad x \to (y \wedge z) = (x \to y) \wedge (x \to z)$,
(M6) $(x \to y) \vee ((x \to y) \to \neg x \vee y) = 1$.

Instead of immediately listing properties of R_0-algebras, we would like to recall definitions of NM-algebras and of other related algebras.

A *residuated lattice* $M = (M, \vee, \wedge, \otimes, \to, 0, 1)$ is an algebra of type $(2, 2, 2, 2, 0, 0)$ such that $(M, \vee, \wedge, 0, 1)$ is a bounded lattice, $(M, \otimes, 1)$ is a commutative monoid and the residuation law holds for all $x, y, z \in M$:

$$x \leq y \to z \text{ iff } x \otimes y \leq z.$$

In a residuated lattice M, we denote $\neg x = x \to 0$. We make the assumption that $\neg$ has the highest priority, $\otimes$ has higher priority than $\vee$ and $\wedge$, and $\to$ has the lowest priority.

A residuated lattice M is called an *MTL-algebra* [16] if it satisfies also the prelinearity equation:

$$\text{(PRL)} \quad (x \to y) \vee (y \to x) = 1.$$

An MTL-algebra is called an *IMTL-algebra* if it satisfies the involutivity equation:

$$\text{(INV)} \quad \neg\neg x = x.$$

An *NM-algebra* is an IMTL-algebra satisfying the additional equation:

$$\text{(WNM)} \quad (x \otimes y \to 0) \vee (x \wedge y \to x \otimes y) = 1.$$

In this paper, it is necessary to recall *MV-algebras*. MV-algebras were first introduced by Chang [6] as the algebraic counterpart of Łukasiewicz infinite-valued logic. Here we recall different, but equivalent characterizations of MV-algebras in the context of residuated lattices. An MV-algebra is an IMTL-algebra satisfying the divisibility equation:

$$\text{(DIV)} \quad x \otimes (x \to y) = x \wedge y.$$

An alternative characterization of MV-algebras is that a residuated lattice is an MV-algebra iff it satisfies the following axiom:

$$\text{(MV)} \quad (x \to y) \to y = (y \to x) \to x.$$

Lastly, a *Boolean algebra* is a complemented distributive lattice. Clearly, every Boolean algebra is an MV-algebra. A residuated lattice M is a Boolean algebra iff

$$x \vee \neg x = 1$$

for all $x \in M$.

We now turn to recall the equivalence between R_0-algebras and NM-algebras [27, 22]. Let M be an R_0-algebra, and define $0 = \neg 1$ and

$$x \otimes y = \neg(x \to \neg y)$$

for $x, y \in M$, then $(M, \vee, \wedge, \otimes, \to, 0, 1)$ is an NM-algebra. Conversely, it is easy to see that every NM-algebra is an R_0-algebra. Hence we can use the notion of R_0-algebra or that of NM-algebra, or even alternatively, without any confusion.

The class of R_0-algebras forms a variety, which is a proper subvariety of the variety of residuated lattices. Hence, the notions of chain, homomorphism, quotient, subalgebra, subdirect product and direct product for R_0-algebras are just the particular cases of the corresponding universal algebraic notions.

Since the prelinearity equation (PRL) holds in R_0-algebras, which is a necessary and sufficient condition for representing a residuated lattice as a subdirect product of a family of linearly ordered residuated lattices, the subdirect representation theorem holds for R_0-algebras. Thus, R_0-chains play a key role in the study of R_0-algebras. The most important example of R_0-chain is the *standard R_0-algebra* on the unit interval $[0, 1]$. Its operations are defined in the following way: $x \vee y = \max\{x, y\}$, $x \wedge y = \min\{x, y\}$, $\neg x = 1 - x$, and

$$x \to y = \begin{cases} 1, & x \leq y, \\ \neg x \vee y, & \text{otherwise}, \end{cases} \tag{1}$$

for all $x, y \in [0,1]$. This algebra is usually denoted by $[0,1]_{R_0}$. The t-norm $\otimes$ adjoint to the implication defined by (1) in $[0,1]_{R_0}$ is given by

$$x \otimes y = \neg(x \to \neg y) = \begin{cases} x \wedge y, & x > \neg y, \\ 0, & \text{otherwise.} \end{cases}$$

This t-norm was first proposed by Fodor [18] as an example of a non-continuous, but left-continuous t-norm, called nilpotent minimum t-norm. The name of NM-algebra originates from this fact. Define the same operations on $[0,1]^- = [0,1] - \{\frac{1}{2}\}$ as on $[0,1]_{R_0}$, then $[0,1]^-$ forms another example of R_0-chain. Moreover, for each natural number $n \geq 2$, $W_n = \left\{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\right\}$, equipped the operations inherited from $[0,1]_{R_0}$, is an example of n-element R_0-chain. By (M6) it is easy to see that in any R_0-chain the implication operator $\to$ is always given by (1). Thus, up to isomorphism, W_n is the unique R_0-chain with exactly n many elements.

Finally, let us recall the notion of *MV-skeleton* and that of *Boolean skeleton* of an R_0-algebra. It can be checked that, on the three-valued truth set $W_3 = \left\{0, \frac{1}{2}, 1\right\}$, the nilpotent minimum t-norm coincides with the Łukasiewicz t-norm

$$x \otimes y = \max\{x + y - 1, 0\}.$$

This fact shows that three-valued R_0-algebras and three-valued MV-algebras coincide; they form a proper subvariety of the variety of R_0-algebras, which can also be characterized by the equation (MV) in the context of R_0-algebras. We use the same notation $\mathcal{MV}_3$ as in [5] to denote this subvariety. Let now M be an R_0-algebra and define endofunctions $\triangle, \nabla, \phi : M \to M$ by

$$\triangle(x) = (\neg(\neg x)^2)^2, \quad \nabla(x) = \neg(\neg x^2)^2$$

and

$$\phi(x) = \triangle(x) \wedge (\nabla(x \vee \neg x) \vee x) \tag{2}$$

for $x \in M$, where $y^2 = y \otimes y$ for $y \in M$ has higher priority than $\neg$. Then ϕ is an idempotent endomorphism, and the image of ϕ over M, $MV_3(M) = \{\phi(x) \mid x \in M\}$, is called the *MV-skeleton* of M [5]. It was shown that $MV_3(M)$ with operations inherited from M is the largest subalgebra of M which is in $\mathcal{MV}_3$. Analogously, the *Boolean skeleton* of an R_0-algebra M, denoted by $B(M)$, is the subalgebra consisting of all Boolean elements of M, i.e.,

$$B(M) = \{x \in M \mid x \vee \neg x = 1, x \wedge \neg x = 0\}.$$

In the sequel, we recall two analogues of Stone representation theorem for Boolean algebras which were established in [43] for R_0-algebras, and in the next section we will extend them to categorical equivalences. We start from recalling maximal filters of R_0-algebras.

Let M be an R_0-algebra. A nonempty subset F of M is called a *filter* if $1 \in F$ and F is closed under the MP rule, i.e., $x, x \to y \in F$ implies $y \in F$. A filter F is said to be *proper* if $F \neq M$. Unless otherwise explicitly stated, filters are assumed to be proper. A filter is said to be *maximal* if it cannot be properly contained in any proper filters. It was proved in [43, 39] that a filter is maximal iff, for every $x \in M$, exactly one of $x \in F$, $\neg x \in F$ and $(\neg x^2) \otimes (\neg(\neg x)^2) \in F$ holds. Such a characterization implies a one-to-one correspondence between maximal filters and homomorphisms from R_0-algebras into the three-valued R_0-chain W_3, defined by $F \mapsto \xi_F$, where

$$\xi_F(x) = \begin{cases} 1, & x \in F, \\ \frac{1}{2}, & (\neg x^2) \otimes (\neg(\neg x)^2) \in F, \\ 0, & \neg x \in F. \end{cases} \tag{3}$$

For more properties of maximal filters in R_0-algebras, we refer to [43].

We now recall topologies on maximal filters in R_0-algebras. Let M be an R_0-algebra and $Max(M)$ denote the set of all maximal filters in M. $\forall x \in M$, define

$$\mathcal{V}(x) = \{F \in Max(M) \mid x \in F\}, \quad \mathcal{B} = \{\mathcal{V}(x) \mid x \in M\}. \tag{4}$$

Then $\mathcal{B}$ is a basis for a topology $\mathcal{T}$ on $Max(M)$, and moreover, we have

$$\{\mathcal{V}(x) \mid x \in M\} = \{\mathcal{V}(x) \mid x \in B(M)\},$$

and for $x, y \in \mathrm{B}(M)$,

$$\mathcal{V}(x) = \mathcal{V}(y) \text{ iff } x = y.$$

Theorem 2.1. [43]

(i) *$(Max(M), \mathcal{T})$ is a Stone space, i.e., a topological space which is compact, zero-dimensional and Hausdorff.*

(ii) *The Boolean skeleton $B(M)$ of M is isomorphic to the set of all clopen subsets of $(Max(M), \mathcal{T})$ under the set theoretic inclusion.*

Before recalling three-valued Stone representations of R_0-algebras, it is necessary to recall three-valued Stone topology on a nonempty set. A three-valued topology δ on a nonempty set X is a collection of three-valued sets of X (i.e., functions from X into $\{0, \frac{1}{2}, 1\}$) such that (i) $0_X, 1_X \in \delta$; (ii) δ is closed under finite intersections; and (iii) δ is closed under arbitrary unions. Elements of δ is said to be *open*. The pair (X, δ) is called a *three-valued topological space*. A three-valued topological space (X, δ) is said to be *zero-dimensional* if δ has a basis (i.e., a subset of δ such that each member of δ can be expressed as unions of its members) consisting of three-valued clopen sets. A three-valued set λ of X is said to be *compact* if $\lambda \leq \vee_{i \in I} \mu_i$ for some open sets μ_i of X implies $\lambda \leq \vee_{i \in J} \mu_i$ for some finite subset J of the index I. (X, δ) is said to be *compact* if 1_X is compact. (X, δ) is said to be *Hausdorff* if, for each pair x_1, x_2 of distinct elements of X, there are open three-valued sets μ_1 and μ_2 such that $\mu_1(x_1) = \mu_2(x_2) = 1$ and $\mu_1 \wedge \mu_2 = 0_X$. (X, δ) is called a *three-valued Stone space* if it is zero-dimensional, compact and Hausdorff.

Let M be an R_0-algebra, and define a mapping $s : M \to \{0, \frac{1}{2}, 1\}^{Max(M)}$ by

$$s(x)(F) = \xi_F(x) \tag{5}$$

for every $x \in M$ and for every $F \in Max(M)$, where ξ_F is defined by (3). Then one can show that

$$\{s(x) \mid x \in M\} = \{s(x) \mid x \in MV_3(M)\} \tag{6}$$

and for $x, y \in MV_3(M)$,

$$s(x) = s(y) \text{ iff } x = y. \tag{7}$$

(6) is obviously equivalent to the fact that $s = s \circ \phi$, where ϕ is defined by (2).

Let $\beta = \{s(x) \mid x \in M\}$. Then β is a basis of a three-valued topology, denoted by δ, on $Max(M)$.

Theorem 2.2. [43]

(i) *$(Max(M), \delta)$ is a three-valued Stone space, and $\beta = Clop(Max(M))$, the set of all clopen three-valued sets of $(Max(M), \delta)$.*

(ii) *β, which forms an MV-algebra in $\mathcal{MV}_3$ under the pointwise order, is isomorphic to $MV_3(M)$.*

3. Stone Duality for R_0-algebras

In this section we extend the Stone representations in Theorems 2.1 and 2.2 to categorical dualities. The key is to find the topological counterparts of homomorphisms between MV-skeletons of R_0-algebras in three-valued Stone spaces.

Theorem 2.2 (ii) can be stated as saying that the three-valued bidual of an R_0-algebra M, i.e., the dual three-valued algebra of the dual three-valued Stone space of M, is isomorphic to $MV_3(M)$. For finite spaces there is a topological dual of this fact above: every finite three-valued Stone space is homeomorphic to its bidual:

Theorem 3.1. *Every finite three-valued Stone space is homeomorphic to the three-valued Stone space of an R_0-algebra.*

Proof. Let (X, δ) be a finite three-valued Stone space, and $M = \text{Clop}(X)$. Define on M the operations $\neg, \vee$ and $\rightarrow$ as follows: $\forall \lambda, \mu \in M$, $\forall x \in X$,

$$(\neg\lambda)(x) = 1 - \lambda(x), \quad (\lambda \vee \mu)(x) = \max\{\lambda(x), \mu(x)\},$$

$$(\lambda \rightarrow \mu)(x) = \begin{cases} 1, & \lambda(x) \leq \mu(x), \\ (1 - \lambda(x)) \vee \mu(x), & \text{otherwise.} \end{cases}$$

Claim 1. M is an R_0-algebra.

It is obvious that $\neg$ and $\vee$ are well-defined on M. For the closedness of M under the implication, we note first that, in the three-valued R_0-chain $W_3 = \{0, \frac{1}{2}, 1\}$, $x \rightarrow y \neq (1-x) \vee y$ iff $x = y = \frac{1}{2}$. From this fact we have $\lambda \rightarrow \mu = (1_X - \lambda) \vee \mu \vee \{1_x \mid \lambda(x) = \mu(x) = \frac{1}{2}, x \in X\}$, where 1_x is the characteristic function of the singleton subset $\{x\}$ of X. Then it is sufficient to show that, for every $x \in X$, $1_x \in M$ whenever $\mu(x) = \frac{1}{2}$ for some $\mu \in M$. Indeed, $\forall x \in X$, it follows from the separation property of X that, for every $y \in X$ which is different from x, there is a $\mu_y \in M$ such that $\mu_y(x) = 1$ and $\mu_y(y) = 0$. Then $\wedge\{\mu_y \mid y \neq x\} = 1_x$. By the finiteness of X, we have $1_x \in M$. This shows that M is closed under the implication, and hence M is an R_0-algebra.

Claim 2. (X, δ) is homeomorphic to $(Max(M), \delta^{'})$, where $\delta^{'}$ is the induced three-valued Stone topology on $Max(M)$ by (5).

Define $t : X \rightarrow Max(M)$ by

$$t(x) = \{\mu \in M \mid \mu(x) = 1\}, \quad x \in X. \tag{8}$$

We claim that t is a homeomorphism. By [23, Proposition 2.4.7 (ii)], it is sufficient to show that t is bijective, and the Zadeh type extension $t^{\rightarrow}$ of t is continuous and clopen, where $t^{\rightarrow} : \left\{0, \frac{1}{2}, 1\right\}^X \to \left\{0, \frac{1}{2}, 1\right\}^{Max(M)}$, usually denoted simply by $t^{\rightarrow} : X \to Max(M)$, is defined by

$$t^{\rightarrow}(\mu)(F) = \vee\{\mu(x) \mid t(x) = F\}, \tag{9}$$

for $\mu \in \left\{0, \frac{1}{2}, 1\right\}^X$ and $F \in Max(M)$, and in addition, the reverse mapping $t^{\leftarrow} : \left\{0, \frac{1}{2}, 1\right\}^{Max(M)} \to \left\{0, \frac{1}{2}, 1\right\}^X$ of $t^{\rightarrow}$, similarly denoted $t^{\leftarrow} : Max(M) \to X$, is defined by

$$t^{\leftarrow}(\nu) = \nu \circ t \tag{10}$$

for $\nu \in \{0, \frac{1}{2}, 1\}^{Max(M)}$.

The remaining proof for Claim 2 is divided into the following four steps.

(i) *t is well-defined.*

$\forall x \in X, \forall \mu \in M$, it is trivial that exactly one of $\mu(x) = 1$, $(\neg\mu)(x) = 1$ and $[(\neg\mu^2) \otimes (\neg(\neg\mu)^2)](x) = (\neg(\mu(x))^2) \otimes (\neg(\neg(\mu(x)))^2) = 1$ holds. Hence, exactly one of μ, $\neg\mu$ and $(\neg\mu^2) \otimes (\neg(\neg\mu)^2)$ belongs to $t(x)$. This shows that $t(x)$ is a maximal filter of M.

(ii) *t is injective.*

Take any $x, y \in X$ with $x \neq y$. By the separation property of X, there are open sets $\mu_1, \mu_2 \in \{0, \frac{1}{2}, 1\}^X$ such that $\mu_1(x) = \mu_2(y) = 1$ and $\mu_1 \wedge \mu_2 = 0_X$. Hence $\mu_1 \in t(x)$, $\mu_2 \in t(y)$, but $\mu_1 \neq \mu_2$, showing that $t(x) \neq t(y)$.

(iii) *t is surjective.*

Take any $F \in Max(M)$. Now X is covering-compact and F is a family of closed three-valued sets of X with the finite square intersection property[2], then F has a non-void normal intersection, i.e., $\wedge F \neq 0_X$ and $(\wedge F)^{-1}(1) \neq \emptyset$, see, e.g., [34, Theorem 6.1.5]. Take any $x \in (\wedge F)^{-1}(1)$, then $F \subseteq t(x)$. By the maximality of F, we have $F = t(x)$.

(iv) *$t^{\rightarrow}$ is continuous and clopen.*

Take any $s(\mu) \in \mathrm{Clop}(Max(M))$, where $s(\mu)(F) = \xi_F(\mu)$, as defined by (5), for $\mu \in M = \mathrm{Clop}(X)$ and for $F \in Max(M)$. Then, $\forall x \in X$,

$$\begin{aligned} t^{\leftarrow}(s(\mu))(x) &= (s(\mu) \circ t)(x) \\ &= s(\mu)(t(x)) \\ &= \xi_{t(x)}(\mu) \\ &= \begin{cases} 1, & \mu \in t(x), \\ \frac{1}{2}, & (\neg\mu^2) \otimes (\neg(\neg\mu)^2) \in t(x), \\ 0, & \neg\mu \in t(x), \end{cases} \\ &= \mu(x). \end{aligned}$$

This shows that $t^{\leftarrow}(s(\mu)) = \mu$ is clopen, and thus $t^{\rightarrow}$ is continuous.

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[2]A subset A of an R_0-algebra is said to have the *finite square intersection property* [43] if, for arbitrary finitely many elements $x_1, \ldots, x_n \in A$, $x_1^2 \wedge \ldots \wedge x_n^2 > 0$.

Now, take any clopen set μ of (X, δ), i.e., $\mu \in M = \text{Clop}(X)$, then, $\forall F \in Max(M)$,

$$\begin{aligned} t^{\rightarrow}(\mu)(F) &= \vee\{\mu(x) \mid t(x) = F\} \\ &= \mu(t^{-1}(F)) \quad (t \text{ is bijective}) \\ &= \begin{cases} 1, & \mu \in F, \\ \frac{1}{2}, & (\neg\mu^2) \otimes (\neg(\neg\mu)^2) \in F, \\ 0, & \neg\mu \in F, \end{cases} \quad \text{(by (8))} \\ &= \xi_F(\mu) \quad \text{(see (3))} \\ &= s(\mu)(F) \quad \text{(see (5))} \end{aligned}$$

We have $t^{\rightarrow}(\mu) = s(\mu)$, showing that $t^{\rightarrow}$ maps clopen sets of X to clopen ones of $Max(M)$, and therefore $t^{\rightarrow}$ is clopen.

It follows from items (i) to (iv) that Claim 2 is true. □

Remark 3.2. (i) By the proof of Claim 1 of Theorem 3.1, we have that $1_x \in \delta$ for every $x \in X$, which implies that the characteristic function 1_A of each subset A of X is in δ, since δ is closed under arbitrary unions. Hence $\{1_A \mid A \subseteq X\} \subseteq \delta$. Unlike the ordinary Stone topology, δ is not necessarily discrete, i.e., it is not necessary that $\delta = \{0, \frac{1}{2}, 1\}^X$. For example, let X be an arbitrary finite set and $x_0 \in X$ a fixed element. Define

$$\delta = \{1_A \mid A \subseteq X\} \cup \left\{\mu : X \to \left\{0, \frac{1}{2}, 1\right\} \mid \mu^{-1}\left(\frac{1}{2}\right) = \{x_0\}\right\}.$$

It is easy to see that δ is a three-valued Stone topology on X which is not discrete.

(ii) Another difference from ordinary topologies is that a compact subset of a three-valued Stone space is not necessarily closed, because every three-valued subset of a finite space is compact.

We do not know, however, whether Theorem 3.1 holds for infinite three-valued Stone spaces, more precisely, we couldn't prove that $M = \text{Clop}(X)$ under the pointwise operations is an R_0-algebra. Hence, to extend Theorem 3.1 to general case, we say that a three-valued Stone space (X, δ) is of *R_0-type* if it has a clopen basis which, under the pointwise R_0-operations $\neg, \vee$ and $\rightarrow$, forms an R_0-algebra. Thanks to Theorem 3.1, every finite three-valued Stone space is of R_0-type.

Theorem 3.3. *Every R_0-type three-valued Stone space (i.e., having a clopen basis forming an R_0-algebra) is homeomorphic to the three-valued Stone space of an R_0-algebra.*

Proof. It is not difficult to check that the proofs for Claim 2 of Theorem 3.1 do work here. □

Example 3.4. (i) Let $X = \{x\}$ be a singleton set, and δ a discrete three-valued (Stone) topology on X, i.e., $\delta = \left\{0_X, \frac{1}{2}_X, 1_X\right\}$. Then (X, δ) is a finite three-valued Stone space and, as Theorem 3.1 states, $M = \text{Clop}(X) = \delta = \left\{0_X, \frac{1}{2}_X, 1_X\right\}$ is an R_0-algebra under the pointwise order. It is obvious that $Max(M) = \{\{1_X\}\}$, and the three-valued Stone topology δ' on $Max(M)$ generated through (3) and (5) is $\delta' = \left\{0_{Max(M)}, \frac{1}{2}_{Max(M)}, 1_{Max(M)}\right\}$. These two spaces (X,δ) and $(Max(M),\delta')$ are obviously homeomorphic under the Zadeh type extension $t^{\rightarrow}$ of $t : X \to Max(M)$ defined by $t(x) = \{1_X\}$ (using (9), one can check that $t^{\rightarrow}$ is defined by $0_X \mapsto 0_{Max(M)}$, $\frac{1}{2}_X \mapsto \frac{1}{2}_{Max(M)}$ and $1_X \mapsto 1_{Max(M)}$).

(ii) Let X be an $(n+1)$-element set, say $X = \{x_0, x_1, \dots, x_n\}$, and δ defined on X as in Remark 3.2, i.e.,

$$\delta = \{1_A \mid A \subseteq X\} \cup \left\{\mu : X \to \left\{0, \frac{1}{2}, 1\right\} \mid \mu^{-1}\left(\frac{1}{2}\right) = \{x_0\}\right\}.$$

As stated in Remark 3.2, (X,δ) is a finite three-valued Stone space. Then $M = \mathrm{Clop}(X) = \delta$ is an R_0-algebra under the pointwise order, and

$$Max(M) = \{F_0, F_1, \dots, F_n\}$$

where $F_i = \{\mu \in M \mid \mu \geq 1_{x_i}\}$ for all $i = 0, 1, \dots, n$. Let δ' be the three-valued Stone topology on $Max(M)$ generated through (3) and (5). Then the spaces (X,δ) and $(Max(M),\delta')$ are homeomorphic under the Zadeh type extension $t^{\rightarrow}$ of $t : X \to Max(M)$ which is defined by $t(x_i) = F_i$ for all $i = 0, 1, \dots, n$.

(iii) Let $X = \{0, \frac{1}{2}, 1\}^{\omega}$, $F(\omega)$ the free R_0-algebra with countably many generators, and δ the three-valued Stone topology on X generated by the basis $\beta = \{s(\alpha) \mid \alpha \in F(\omega)\}$ with $s(\alpha) : X \to \{0, \frac{1}{2}, 1\}$ defined by $s(\alpha)(\mathbf{x}) = \overline{\alpha}(\mathbf{x})$, where $\overline{\alpha}$ is the truth-function induced by α, for all $\mathbf{x} \in X$ and $\alpha \in F(\omega)$. Then $M = \mathrm{Clop}(X) = \beta$ is a free algebra in $\mathcal{MV}_3$, and $Max(M) = \{\{s(\alpha) \in M \mid s(\alpha)(\mathbf{x}) = 1, \alpha \in F(\omega)\} \mid \mathbf{x} \in X\}$. Let δ' be the three-valued Stone topology on $Max(M)$ generated through (3) and (5), then the two R_0-type three-valued Stone spaces (X,δ) and $(Max(M),\delta')$ are homeomorphic under the Zadeh type extension $t^{\rightarrow}$ of the function $t : X \to Max(M)$ defined by $t(\mathbf{x}) = \{s(\alpha) \in M \mid s(\alpha)(\mathbf{x}) = 1, \alpha \in F(\omega)\}$ for $\mathbf{x} \in X$.

In the following we extend the Stone duality above between MV-skeletons of R_0-algebras and R_0-type three-valued Stone spaces to a duality between MV-homomorphisms of R_0-algebras and Zadeh type continuous functions of R_0-type Stone spaces. Here, by an *MV-homomorphism* from an R_0-algebra M_1 into an R_0-algebra M_2 we mean an R_0-homomorphism from M_1 into M_2 which preserves the equation (MV), or equivalently, an R_0-homomorphism from M_1 into $MV_3(M_2)$.

The crucial idea for dualizing MV-homomorphisms and Zadeh type continuous functions is that, for any MV-homomorphism $h : M_1 \to M_2$ of R_0-algebras, the preimage $h^{-1}(F_2) = \{x_1 \in M_1 \mid h(x_1) \in F_2\}$ of a maximal filter F_2 of M_2 is a maximal filter of M_1. Dually, for any Zadeh type continuous map $f^{\rightarrow} : (X_1,\delta_1) \to (X_2,\delta_2)$ of R_0-type three-valued Stone spaces, the preimage $f^{\leftarrow}(\mu_2)$ of a clopen three-valued set μ_2 of X_2 is clopen in X_1.

Suppose that M is an R_0-algebra and (X,δ) is an R_0-type three-valued Stone space. Let

$$s_M : MV_3(M) \to \mathrm{Clop}(Max(M))$$

denote (the restriction to $MV_3(M)$ of) the Stone map (2) and

$$t_X^{\rightarrow} : X \to Max(\mathrm{Clop}(X))$$

the canonical homeomorphism (9).

For every MV-homomorphism $h : M_1 \to M_2$ of R_0-algebras, the dual of h is the map

$$h^d : Max(M_2) \to Max(M_1)$$

defined by

$$h^d = (h^{-1})^{\rightarrow}$$

where $h^{-1}(F_2) = \{x_1 \in M_1 \mid h(x_1) \in F_2\} \in Max(M_1)$ for $F_2 \in Max(M_2)$.

For every Zadeh type continuous map $f^{\rightarrow} : X_1 \to X_2$ of R_0-type three-valued Stone spaces, the dual of $f^{\rightarrow}$ is the map

$$(f^{\rightarrow})^d : \mathrm{Clop}(X_2) \to \mathrm{Clop}(X_1)$$

defined by $(f^{\rightarrow})^d = f^{\leftarrow}$.

Theorem 3.5. *Let $h : M_1 \to M_2$, $k : M_2 \to M_3$ be MV-homomorphisms of R_0-algebras, and $f^{\rightarrow} : X_1 \to X_2$, $g^{\rightarrow} : X_2 \to X_3$ Zadeh type continuous functions of R_0-type three-valued Stone spaces. Then:*

- (i) *$h^d : Max(M_2) \to Max(M_1)$ is a Zadeh type continuous function and $(f^{\rightarrow})^d : Clop(X_2) \to Clop(X_1)$ is an MV-homomorphism.*
- (ii) *$(id_M)^d = id^{\rightarrow}_{Max(M)}$ and $(id^{\rightarrow}_X)^d = id_{Clop(X)}$.*
- (iii) *$(k \circ h)^d = h^d \circ k^d$ and $(g^{\rightarrow} \circ f^{\rightarrow})^d = (f^{\rightarrow})^d \circ (g^{\rightarrow})^d$.*
- (iv) *$h^{dd} \circ s_{M_1} = s_{M_2} \circ h$ and $(f^{\rightarrow})^{dd} \circ t^{\rightarrow}_{X_1} = t^{\rightarrow}_{X_2} \circ f^{\rightarrow}$, i.e., each of the following two diagrams is commutative:*

$$\begin{array}{ccc} MV_3(M_1) & \xrightarrow{s_{M_1}} & Clop(Max(M_1)) \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h^{dd}} \\ MV_3(M_2) & \xrightarrow[s_{M_2}]{} & Clop(Max(M_2)) \end{array} \qquad \begin{array}{ccc} X_1 & \xrightarrow{t^{\rightarrow}_{X_1}} & Max(Clop(X_1)) \\ {\scriptstyle f^{\rightarrow}}\downarrow & & \downarrow{\scriptstyle (f^{\rightarrow})^{dd}} \\ X_2 & \xrightarrow[t^{\rightarrow}_{X_2}]{} & Max(Clop(X_2)) \end{array}$$

Proof. (i) For every clopen three-valued set $s_{M_1}(x)$ of $Max(M_1)$, where $x \in MV_3(M_1)$, and $\forall F_2 \in Max(M_2)$,

$$\begin{aligned} (h^{-1})^{\leftarrow}(s_{M_1}(x))(F_2) &= (s_{M_1}(x) \circ h^{-1})(F_2) \\ &= s_{M_1}(x)(h^{-1}(F_2)) \\ &= \xi_{h^{-1}(F_2)}(x) \\ &= \xi_{F_2}(h(x)) \quad \text{(since } h \text{ is a homomorphism)} \\ &= s_{M_2}(h(x))(F_2). \end{aligned}$$

This shows that $(h^{-1})^{\leftarrow}(s_{M_1}(x)) = s_{M_2}(h(x))$ is clopen, and so $h^d = (h^{-1})^{\rightarrow}$ is continuous. Recall that $\mathrm{Clop}(X_1)$ and $\mathrm{Clop}(X_2)$ are R_0-algebras in $\mathcal{MV}_3$. $\forall \mu_2, \lambda_2 \in$

$\mathrm{Clop}(X_2)$ and $\forall x_1 \in X_1$,

$$\begin{aligned}(f^{\rightarrow})^d(\neg\mu_2)(x_1) &= f^{\leftarrow}(\neg\mu_2)(x_1) = \neg\mu_2(f(x_1))\\ &= (\neg f^{\leftarrow}(\mu_2))(x_1)\\ &= [\neg((f^{\rightarrow})^d(\mu_2))](x_1);\end{aligned}$$

$$\begin{aligned}(f^{\rightarrow})^d(\mu_2 \vee \lambda_2)(x_1) &= f^{\leftarrow}(\mu_2 \vee \lambda_2)(x_1) = (\mu_2 \vee \lambda_2)(f(x_1))\\ &= \mu_2(f(x_1)) \vee \lambda_2(f(x_1))\\ &= f^{\leftarrow}(\mu_2)(x_1) \vee f^{\leftarrow}(\lambda_2)(x_1)\\ &= [(f^{\rightarrow})^d(\mu_2) \vee (f^{\rightarrow})^d(\lambda_2)](x_1);\end{aligned}$$

$$\begin{aligned}(f^{\rightarrow})^d(\mu_2 \rightarrow \lambda_2)(x_1) &= f^{\leftarrow}(\mu_2 \rightarrow \lambda_2)(x_1) = (\mu_2 \rightarrow \lambda_2)(f(x_1))\\ &= \mu_2(f(x_1)) \rightarrow \lambda_2(f(x_1))\\ &= f^{\leftarrow}(\mu_2)(x_1) \rightarrow f^{\leftarrow}(\lambda_2)(x_1)\\ &= [(f^{\rightarrow})^d(\mu_2) \rightarrow (f^{\rightarrow})^d(\lambda_2)](x_1).\end{aligned}$$

This shows that $(f^{\rightarrow})^d$ is an MV-homomorphism.

(ii) and (iii) are straightforward.

(iv) By the proof of (i), we have $h^{dd} \circ s_{M_1} = ((h^{-1})^{\rightarrow})^d \circ s_{M_1} = (h^{-1})^{\leftarrow} \circ s_{M_1} = s_{M_2} \circ h$. For the second identity we show first that, $\forall x_1 \in X_1$,

$$(f^{\leftarrow})^{-1}(t_{X_1}(x_1)) = t_{X_2}(f(x_1)). \tag{11}$$

Note first that both sides of (11) are maximal filters in $Max(\mathrm{Clop}(X_2))$. Hence, by the maximality, it remains to show

$$t_{X_2}(f(x_1)) \subseteq (f^{\leftarrow})^{-1}(t_{X_1}(x_1)). \tag{12}$$

Take any $\mu_2 \in t_{X_2}(f(x_1))$, then $\mu_2(f(x_1)) = 1$, and consequently,

$$f^{\leftarrow}(\mu_2)(x_1) = (\mu_2 \circ f)(x_1) = \mu_2(f(x_1)) = 1.$$

This shows that $f^{\leftarrow}(\mu_2) \in t_{X_1}(x_1)$, and so $\mu_2 \in (f^{\leftarrow})^{-1}(t_{X_1}(x_1))$. It follows from the arbitrariness of μ_2 that (12) holds, and finally we have (11).

Let now $\mu_1 \in \{0, \frac{1}{2}, 1\}^{X_1}$ and $F_2 \in Max(\mathrm{Clop}(X_2))$. Then

$$\begin{aligned}((f^{\rightarrow})^{dd} \circ t^{\rightarrow}_{X_1})(\mu_1)(F_2) &= [((f^{\leftarrow})^{-1})^{\rightarrow}(t^{\rightarrow}_{X_1}(\mu_1))](F_2)\\ &= \vee\{t^{\rightarrow}_{X_1}(\mu_1)(F_1) \mid (f^{\leftarrow})^{-1}(F_1) = F_2\}\\ &= \vee\{\mu_1(x_1) \mid t_{X_1}(x_1) = F_1, (f^{\leftarrow})^{-1}(F_1) = F_2\}\\ &= \vee\{\mu_1(x_1) \mid (f^{\leftarrow})^{-1}(t_{X_1}(x_1)) = F_2\}\\ &= \vee\{\mu_1(x_1) \mid t_{X_2}(f(x_1)) = F_2\} \quad \text{(by (3))}\\ &= \vee\{\mu_1(x_1) \mid f(x_1) = x_2, t_{X_2}(x_2) = F_2\}\\ &= \vee\{f^{\rightarrow}(\mu_1)(x_2) \mid t_{X_2}(x_2) = F_2\}\\ &= t^{\rightarrow}_{X_2}(f^{\rightarrow}(\mu_1))(F_2)\\ &= (t^{\rightarrow}_{X_2} \circ f^{\rightarrow})(\mu_1)(F_2).\end{aligned}$$

Thus, we get $(f^{\rightarrow})^{dd} \circ t^{\rightarrow}_{X_1} = t^{\rightarrow}_{X_2} \circ f^{\rightarrow}$. □

In the language of Category Theory, Theorems 2.2, 3.3 and 3.5 state that there are contravariant functors from the category $\mathcal{MV}_3(\mathcal{R}_0)$ of MV-skeletons of R_0-algebras as objects and (restrictions of) MV-homomorphisms as morphisms into the category $\mathcal{R}_0\mathcal{TSS}$ of R_0-type three-valued Stone spaces as objects and Zadeh type continuous maps as morphisms, and vice versa. Moreover, the composites of these two kinds of contravariant functors are naturally equivalent to identity functors. In short words, the categories $\mathcal{MV}_3(\mathcal{R}_0)$ and $\mathcal{R}_0\mathcal{TSS}$ are dually equivalent.

Corollary 3.6. *Let* $\varphi : \mathcal{MV}_3(\mathcal{R}_0) \to \mathcal{R}_0\mathcal{TSS}$ *be defined by*

$$\varphi(MV_3(M)) = (Max(M), \delta)$$

and

$$\varphi(h) = h^d$$

for every R_0-algebra M and for every MV-homo-morphism h of R_0-algebras, where δ is the three-valued Stone topology generated by (5), and the function $\psi : \mathcal{R}_0\mathcal{TSS} \to \mathcal{MV}_3(\mathcal{R}_0)$ *defined by*

$$\psi((X, \delta)) = Clop(X)$$

and

$$\psi(f^{\rightarrow}) = (f^{\rightarrow})^d$$

for $(X, \delta) \in \mathcal{R}_0\mathcal{TSS}$ *and for every Zadeh type continuous map $f^{\rightarrow}$. Then:*

(i) *φ and ψ are contravariant functors.*
(ii) $s : I_{\mathcal{MV}_3(\mathcal{R}_0)} \cong \psi \circ \varphi$ *and* $t^{\rightarrow} : I_{\mathcal{R}_0\mathcal{TSS}} \cong \varphi \circ \psi$ *are natural equivalences.*

Analogously, we can show that the category $\mathcal{B}(\mathcal{R}_0)$ of Boolean skeletons of R_0-algebras and Boolean homomorphisms is dually equivalent to the category $\mathcal{CSS}$ of (crisp) Stone spaces and continuous maps.

Corollary 3.7. *The categories $\mathcal{B}(\mathcal{R}_0)$ and $\mathcal{CSS}$ are dual.*

4. Stone Duality for R_0-algebras with Internal States

4.1. Internal States on R_0-algebras.

States, as many-valued analogue of probability measures on Boolean algebras, were first introduced by Mundici [25] for MV-algebras and then extended to more general logical algebras. In order to provide a unified treatment of probabilities of many-valued events in both algebraic and logical setting, Flaminio and Montagna [17] introduced the notion of internal state on MV-algebras. Nowadays, a very general notion of state has been established in residuated lattices where the domain and codomain of a state are allowed to be different residuated lattices [10, 42, 9, 39]. In this paper we limit ourselves to internal states on R_0-algebras.

Definition 4.1. Let M be an R_0-algebra. An endofunction $\sigma : M \to M$, with $\sigma(0) = 0$ and $\sigma(1) = 1$, is called an *internal state*, or a *state-operator* on M if it satisfies, for all $x, y \in M$,

(i) $\sigma(x \to y) \to \sigma(y) = \sigma(y \to x) \to \sigma(x)$.
(ii) $\sigma(\sigma(x) \to \sigma(y)) = \sigma(x) \to \sigma(y)$.

By Proposition 3.3 of [42], Definition 4.1 (i) is equivalent to each of the following conditions:

(a) $\sigma(x \vee y) = \sigma(x \leftrightarrow y) \to \sigma(x \wedge y)$, where $x \leftrightarrow y = (x \to y) \wedge (y \to x)$.
(b) $\sigma(x) = \sigma(x \to y) \to \sigma(x \wedge y)$.
(c) $\sigma(x) = \sigma(x \to y) \to \sigma(y)$ whenever $y \leq x$.
(d) $\sigma(x \vee y) = \sigma(x \to y) \to \sigma(y)$.

Since MV-homomorphisms between R_0-algebras are just R_0-homomorphisms preserving the equation (MV), every idempotent MV-endomorphism of R_0-algebras, particularly the ϕ defined by (2), is an internal state. More examples of internal state will be given after investigation of its properties. The main result of this subsection is to show that the converse of the above statement is true too, more precisely, to show that every internal state is an idempotent MV-endomorphism. Let M be an R_0-algebra and σ an internal state on M. Then the pair (M, σ) is called a *state R_0-algebra*.

Theorem 4.2. *Let M be an R_0-algebra, and σ an internal state on M. Then:*

(i) *σ is isotone.*
(ii) *σ is idempotent, i.e., $\sigma \circ \sigma = \sigma$.*
(iii) *$\sigma((x \to y) \to y) = \sigma((y \to x) \to x) = \sigma(x \vee y)$.*
(iv) *$\sigma(M)$ is a subalgebra of $MV_3(M)$.*
(v) *$\sigma = \sigma \circ \phi = \phi \circ \sigma$, where ϕ is defined by (2).*
(vi) *$\sigma\mid_{MV_3(M)}$ is an internal state on $MV_3(M)$ in sense of Flaminio and Montagna.*
(vii) *σ is an idempotent MV-endomorphism.*

Proof. (i) Let $x, y \in M$ with $x \leq y$. Then $\sigma(x \to y) = \sigma(1) = 1$, and by Definition 4.1 (i), $\sigma(y) = \sigma(x \to y) \to \sigma(y) = \sigma(y \to x) \to \sigma(x) \geq \sigma(x)$.

(ii) $\forall x \in M$, $\sigma(\sigma(x)) = \sigma(\sigma(1) \to \sigma(x)) = \sigma(1) \to \sigma(x) = \sigma(x)$. This shows the idempotency of σ.

(iii) This is a particular case of Theorem 2.6 in [42].

(iv) We show first that $\sigma(M) \subseteq MV_3(M)$. It is sufficient to show that the MV-equation (MV) holds in $\sigma(M)$, i.e., for all $x, y \in M$,

$$(\sigma(x) \to \sigma(y)) \to \sigma(y) = (\sigma(y) \to \sigma(x)) \to \sigma(x). \tag{13}$$

Indeed, by (iii) and by multiple applications of Definition 4.1 (ii), we have

$$\begin{aligned}(\sigma(x) \to \sigma(y)) \to \sigma(y) &= \sigma(\sigma(x) \to \sigma(y)) \to \sigma(y)\\ &= \sigma[\sigma(\sigma(x) \to \sigma(y)) \to \sigma(y)]\\ &= \sigma[(\sigma(x) \to \sigma(y)) \to \sigma(y)]\\ &= \sigma[(\sigma(y) \to \sigma(x)) \to \sigma(x)]\\ &= (\sigma(y) \to \sigma(x)) \to \sigma(x).\end{aligned}$$

Now, note first that $0 = \sigma(0) \in \sigma(M)$ and $1 = \sigma(1) \in \sigma(M)$. Definition 4.1 (ii) is just the closedness of $\sigma(M)$ under $\to$. Then it is left to show the closedness of $\sigma(M)$ under $\neg$ and $\vee$. $\forall x, y \in M$,

$$\neg\sigma(x) = \neg\sigma(\neg\neg x) = \sigma(\neg x \to 0) \to \sigma(0) = \sigma(0 \to \neg x) \to \sigma(\neg x) = \sigma(\neg x).$$

$$\begin{aligned}\sigma(x)\vee\sigma(y) &= (\sigma(x)\to\sigma(y))\to\sigma(y) \quad \text{(by (13))}\\ &= \sigma[(\sigma(x)\to\sigma(y))\to\sigma(y)] \quad \text{(see the proof of (13))}\\ &= \sigma(\sigma(x)\vee\sigma(y)).\end{aligned}$$

Therefore, $\sigma(M)$ is a subalgebra of $MV_3(M)$.

(v) By (iv) we have $\phi\circ\sigma=\sigma$. To show $\sigma=\sigma\circ\phi$, we recall first that a filter of an R_0-algebra M is called an *MV-filter* if it contains

$$((x\to y)\to y)\to((y\to x)\to x)$$

for all $x,y\in M$, and it was shown in [43, Theorem 5.18] that

$$\phi^{-1}(1)=\{x\in M\mid\phi(x)=1\}$$

is just the intersection of all MV-filters of M and in [42, Proposition 2.10] that $\sigma^{-1}(1)$ is an MV-filter. Thus, we have

$$\phi^{-1}(1)\subseteq\sigma^{-1}(1).$$

$\forall x\in M$, since $\phi(x\leftrightarrow\phi(x))=\phi(x)\leftrightarrow\phi(x)=1$, we have $\sigma(x\leftrightarrow\phi(x))=1$, and so $\sigma(x\to\phi(x))=\sigma(\phi(x)\to x)=1$ by the isotonicity of σ. Consequently,

$$\sigma(x)=\sigma(\phi(x)\to x)\to\sigma(x)=\sigma(x\to\phi(x))\to\sigma(\phi(x))=\sigma(\phi(x)).$$

This shows $\sigma=\sigma\circ\phi$.

(vi) Denote $\sigma\mid_{MV_3(M)}$ by σ, it is equivalent to show that $\sigma: MV_3(M)\to MV_3(M)$ satisfies, for all $x,y\in MV_3(M)$,

$$\sigma(\neg x)=\neg\sigma(x), \tag{14}$$

$$\sigma(x\oplus y)=\sigma(x)\oplus\sigma(y\otimes(\neg(x\otimes y))), \tag{15}$$

$$\sigma(\sigma(x)\oplus\sigma(y))=\sigma(x)\oplus\sigma(y), \tag{16}$$

where $x\oplus y=\neg x\to y$, $x\otimes y=\neg(x\to\neg y)$.

(14) is proved in the proof of (iv). For (15),

$$\begin{aligned}\sigma(x)\oplus\sigma(y\otimes\neg(x\otimes y)) &= \sigma(x)\oplus\sigma(\neg(y\to x\otimes y))\\ &= \sigma(x)\oplus\sigma[\neg(y\to\neg(x\to\neg y))]\\ &= \sigma(x)\oplus\sigma(\neg(x\vee\neg y))\\ &= \sigma(x)\oplus\sigma(\neg x\wedge y)\\ &= \sigma(\neg x)\to\sigma(\neg x\wedge y)\\ &= \sigma(\neg x\to y) \quad \text{(by (b))}\\ &= \sigma(x\oplus y).\end{aligned}$$

For (16),

$$\begin{aligned}\sigma(\sigma(x)\oplus\sigma(y)) &= \sigma(\sigma(\neg x)\to\sigma(y))=\sigma(\neg x)\to\sigma(y)\\ &= \sigma(x)\oplus\sigma(y).\end{aligned}$$

(vii) By (iv), (vi) and Theorem 6.8 of [12], $\sigma\mid_{MV_3(M)}$ is an endomorphism of $MV_3(M)$. By (v), σ is an idempotent MV-endomorphism of M. □

Example 4.3. (i) Let M be the 4-element Boolean algebra $\{0,a,b,1\}$ with $a\vee b=1$, $a\wedge b=0$, $\neg a=b$ and $\neg b=a$. Then M has just three internal states, namely, $\mathrm{id}_M=\phi$, σ_1 and σ_2, where $\sigma_1(1)=\sigma_1(a)=1,\sigma_1(0)=\sigma_1(b)=0$, and $\sigma_2(1)=\sigma_2(b)=1,\sigma_2(0)=\sigma_2(a)=0$.

(ii) Let $M = [0,1]_{R_0}$, then $MV_3(M) = \left\{0, \frac{1}{2}, 1\right\}$ and the ϕ on M defined by (2) reduces to:

$$\phi(x) = \begin{cases} 1, & x > \frac{1}{2}, \\ \frac{1}{2}, & x = \frac{1}{2}, \\ 0, & x < \frac{1}{2}, \end{cases} \quad x \in [0,1]. \tag{17}$$

It is obvious that ϕ is an internal state on M. Conversely, given an internal state σ on M, it follows from Theorem 4.2 (v) that $\sigma(x) = 1$ for $x > \frac{1}{2}$, and $\sigma(x) = 0$ for $x < \frac{1}{2}$. Moreover, by Definition 4.1 (i), one has that $1 - \sigma\left(\frac{1}{2}\right) = \sigma\left(\frac{1}{2} \to 0\right) \to \sigma(0) = \sigma\left(0 \to \frac{1}{2}\right) \to \sigma\left(\frac{1}{2}\right) = \sigma\left(\frac{1}{2}\right)$, which together with the above facts shows that $\sigma = \phi$. Therefore the standard R_0-algebra $[0,1]_{R_0}$ has a unique internal state ϕ.

(iii) Let M be an arbitrary R_0-algebra and the ϕ defined on M by (2). Consider now the direct product R_0-algebra $M \times M$, and define

$$\sigma_1(x,y) = (\phi(x), \phi(x))$$

and

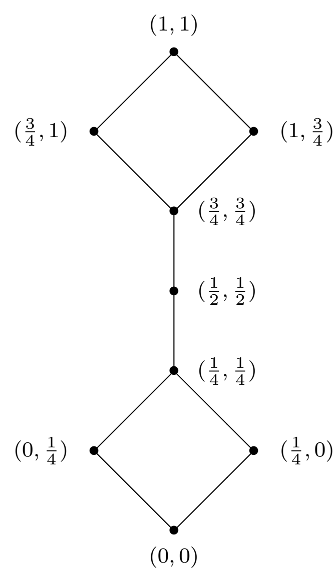
$$\sigma_2(x,y) = (\phi(y), \phi(y))$$

for all $(x,y) \in M \times M$. Then σ_1 and σ_2 are both idempotent MV-endomorphisms, and so they are internal states. Moreover, for every idempotent endomorphism σ on $\phi(M \times M) = \phi(M) \times \phi(M)$, $\sigma \circ \phi$ is an internal state on $M \times M$, where ϕ is extended pointwisely on $M \times M$.

(iv) Let $M = W_3 \times W_3$ be the direct product of the 3-element R_0-algebra $W_3 = \left\{0, \frac{1}{2}, 1\right\}$. Then M has just three internal states: $\mathrm{id}_M = \phi$, σ_1 and σ_2, where $\sigma_1(x,y) = (x,x)$ and $\sigma_2(x,y) = (y,y)$, respectively, for all $(x,y) \in M$.

(v) Consider the subdirect product M of R_0-chains M_1 and M_2, where $M_1 = M_2 = W_5 = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$, given by the following picture. Then M has a unique internal state ϕ, defined by

$$\phi(x,y) = \begin{cases} (1,1), & (x,y) \geq \left(\frac{3}{4}, \frac{3}{4}\right), \\ \left(\frac{1}{2}, \frac{1}{2}\right), & (x,y) = \left(\frac{1}{2}, \frac{1}{2}\right), \\ (0,0), & \text{otherwise}, \end{cases} \quad (x,y) \in M.$$



4.2. **Stone Duality Between Internal States and Idempotent Continuous Endofunctions of Zadeh Type.**

In this subsection we show that there is a duality between internal states of R_0-algebras and idempotent Zadeh type continuous functions. Combining the duality between MV-skeletons of R_0-algebras and R_0-type three-valued Stone spaces we finally obtain a duality between state MV-skeletons of state R_0-algebras and R_0-type three-valued Stone state spaces, i.e., spaces with a Zadeh type idempotent continuous endofunction, denoted by $((X,\delta), f^{\to})$, sometimes simply by $(X, f^{\to})$ if our emphasis is only X and $f^{\to}$, where (X,δ) is an R_0-type three-valued Stone space and $f^{\to}$ is a Zadeh type idempotent continuous endofunction.

Proposition 4.4. *Let (M,σ) be a state R_0-algebra and $f = \sigma^{-1} : Max(M) \to Max(M)$ defined by $f(F) = \sigma^{-1}(F) = \{x \in M \mid \sigma(x) \in F\}$. Then the Zadeh type extension $f^{\to}$ of f, called the dual of σ and denoted by $f^{\to} = \sigma^d$, is idempotent and continuous on the space $(Max(M),\delta)$ with δ the three-valued topology generated by (5).*

Proof. Since σ is a homomorphism, $f = \sigma^{-1}$ is well-defined on $Max(M)$, and so is $f^{\to}$. Since $f \circ f = \sigma^{-1} \circ \sigma^{-1} = \sigma^{-1} = f$ by the idempotency of σ, we then have $f^{\to} \circ f^{\to} = (f \circ f)^{\to} = f^{\to}$ [23, Theorem 2.1.23]. Hence, $f^{\to}$ is idempotent. Now, take any clopen set $s(x)$ (see (5)), then, $\forall F \in Max(M)$,

$$\begin{aligned} f^{\leftarrow}(s(x))(F) &= s(x)(f(F)) = s(x)(\sigma^{-1}(F)) \\ &= \xi_{\sigma^{-1}(F)}(x) \\ &= \xi_F(\sigma(x)) \\ &= s(\sigma(x))(F). \end{aligned}$$

This shows that $f^{\leftarrow}(s(x)) = s(\sigma(x))$ is clopen, and thus, $f^{\to}$ is continuous. □

Proposition 4.5. *Let $((X,\delta), f^{\to})$ be an R_0-type three-valued Stone state space. Then $(Clop(X), \sigma_f)$, with $\sigma_f = (f^{\to})^d = f^{\leftarrow}$, is a state R_0-algebra.*

Proof. By Theorem 3.3, $M = \mathrm{Clop}(X)$, under the pointwise R_0-operations $\neg$, $\vee$ and $\to$, is an R_0-algebra. Now, it is sufficient to show that σ_f is an internal state on M. $\sigma_f(0_M) = f^{\leftarrow}(0_M) = 0_M \circ f = 0_M$ and similarly, $\sigma_f(1_M) = 1_M$. Take any $\mu, \lambda \in M$ and any $x \in X$, then

$$\begin{aligned} [\sigma_f(\mu \to \lambda) \to \sigma_f(\lambda)](x) &= \sigma_f(\mu \to \lambda)(x) \to \sigma_f(\lambda)(x) \\ &= (\mu \to \lambda)(f(x)) \to \lambda(f(x)) \\ &= [\mu(f(x)) \to \lambda(f(x))] \to \lambda(f(x)) \\ &= [\lambda(f(x)) \to \mu(f(x))] \to \mu(f(x)) \\ &= [\sigma_f(\lambda \to \mu) \to \sigma_f(\mu)](x). \end{aligned}$$

This shows Definition 4.1 (i). Moreover,

$$\begin{aligned} \sigma_f(\sigma_f(\mu) \to \sigma_f(\lambda)) &= (\mu \circ f \to \lambda \circ f) \circ f \\ &= (\mu \to \lambda) \circ (f \circ f) \\ &= (\mu \to \lambda) \circ f \\ &= \mu \circ f \to \lambda \circ f \\ &= \sigma_f(\mu) \to \sigma_f(\lambda). \end{aligned}$$

This is Definition 4.1 (ii). Therefore, σ_f is an internal state on M. □

Example 4.6. (i) Let $M = F(\omega)$ be the free R_0-algebra with countably many generators, and ϕ defined on M through (2), then $(F(\omega), \phi)$ is a state R_0-algebra. Let $f = \phi^{-1} : Max(M) \to Max(M)$, then it follows from Proposition 4.4 that $f^{\to}$ is an idempotent Zadeh type continuous endofunction on the three valued Stone space $(Max(M), \delta)$ with δ generated by (5).

(ii) Let $X = \{a, b\}$ with $a \neq b$, and $\delta = \left\{0, \frac{1}{2}, 1\right\}^X$, then (X, δ) is a (discrete) three-valued Stone space. Define $f : X \to X$ by $f(a) = a$ and $f(b) = a$, then $f^{\to}$ is an idempotent Zadeh type continuous endofunction on (X, δ). Denote by $s(x, y)$ the member of δ such that $s(x, y)(a) = x$ and $s(x, y)(b) = y$. Then $\mathrm{Clop}(X) = \delta = \left\{s(x, y) \mid x, y \in \left\{0, \frac{1}{2}, 1\right\}\right\}$ is an R_0-algebra under the pointwise operations, which is indeed isomorphic to the direct product $W_3 \times W_3$ of the three-valued R_0-chain W_3. It follows from Proposition 4.5 that $\sigma_f = f^{\leftarrow}$ is an internal state on $\mathrm{Clop}(X)$. In addition, one can check that $\sigma_f(s(x, y)) = f^{\leftarrow}(s(x, y)) = s(x, y) \circ f = s(x, x)$ for all $x, y \in \left\{0, \frac{1}{2}, 1\right\}$.

Let $\mathcal{SMV}_3(\mathcal{SR}_0)$ be the class of all state MV-skeletons of state R_0-algebras. For $(MV_3(M_1), \sigma_1), (MV_3(M_2), \sigma_2) \in \mathcal{SMV}_3(\mathcal{SR}_0)$, a morphism from $(MV_3(M_1), \sigma_1)$ into $(MV_3(M_2), \sigma_2)$ is an MV-homomorphism h from M_1 into M_2 such that

$$h \circ \sigma_1 = \sigma_2 \circ h.$$

Then it is straightforward to verify that the class $\mathcal{SMV}_3(\mathcal{SR}_0)$ with the set of all morphisms forms a category.

Let $\mathcal{R}_0\mathcal{TSSF}$ be the class of all R_0-type three-valued Stone state spaces $((X, \delta), f^{\to})$ with Zadeh type continuous functions $f^{\to} : X \to X$ possessing the property $f^{\to} \circ f^{\to} = f^{\to}$. By a morphism from $((X_1, \delta_1), f_1^{\to})$ into $((X_2, \delta_2), f_2^{\to})$ we mean a Zadeh type continuous function $g^{\to} : X_1 \to X_2$ such that

$$g^{\to} \circ f_1^{\to} = f_2^{\to} \circ g^{\to}.$$

The class $\mathcal{R}_0\mathcal{TSSF}$ with such morphisms forms a category. In the sequel we show a duality between the categories $\mathcal{SMV}_3(\mathcal{SR}_0)$ and $\mathcal{R}_0\mathcal{TSSF}$.

Proposition 4.7. *The function $\varphi : \mathcal{SMV}_3(\mathcal{SR}_0) \to \mathcal{R}_0\mathcal{TSSF}$ is a contravariant functor, which is defined by*

$$\varphi(MV_3(M), \sigma) = ((Max(M), \delta), f^{\to})$$

with $f^{\to} = \sigma^d = (\sigma^{-1})^{\to}$ for $(MV_3(M), \sigma) \in \mathcal{SMV}_3(\mathcal{SR}_0)$, where δ is the three-valued topology on $Max(M)$ generated through (5), and

$$\varphi(h) = h^d = (h^{-1})^{\to}$$

for each morphism $h : (MV_3(M_2), \sigma_2) \to (MV_3(M_1), \sigma_1)$.

Proof. By Corollary 3.6, φ is well-defined and $\varphi(h) : Max(M_1) \to Max(M_2)$ is a Zadeh type continuous function, and by Proposition 4.4, $f^{\to}$ is an idempotent Zadeh type continuous endofunction. It is now left to show that

$$\varphi(h) \circ f_1^{\to} = f_2^{\to} \circ \varphi(h)$$

where $f_1^{\rightarrow} = \sigma_1^d = (\sigma_1^{-1})^{\rightarrow}$ and $f_2^{\rightarrow} = \sigma_2^d = (\sigma_2^{-1})^{\rightarrow}$.

Since $h \circ \sigma_2 = \sigma_1 \circ h$, we have $\sigma_2^{-1} \circ h^{-1} = h^{-1} \circ \sigma_1^{-1}$, i.e., $f_2 \circ h^{-1} = h^{-1} \circ f_1$. Take any $\mu_1 \in \{0, \frac{1}{2}, 1\}^{Max(M_1)}$ and $F_2 \in Max(M_2)$, then

$$\begin{aligned}(\varphi(h) \circ f_1^{\rightarrow})(\mu_1)(F_2) &= [\varphi(h)(f_1^{\rightarrow}(\mu_1))](F_2) \\ &= \vee\{f_1^{\rightarrow}(\mu_1)(F_1) \mid h^{-1}(F_1) = F_2\} \\ &= \vee\{\mu_1(F_1^{'}) \mid f_1(F_1^{'}) = F_1, h^{-1}(F_1) = F_2\} \\ &= \vee\{\mu_1(F_1^{'}) \mid (h^{-1} \circ f_1)(F_1^{'}) = F_2\} \\ &= \vee\{\mu_1(F_1^{'}) \mid (f_2 \circ h^{-1})(F_1^{'}) = F_2\} \\ &= \vee\{\mu_1(F_1^{'}) \mid h^{-1}(F_1^{'}) = F_2^{'}, f_2(F_2^{'}) = F_2\} \\ &= \vee\{\varphi(h)(\mu_1)(F_2^{'}) \mid f_2(F_2^{'}) = F_2\} \\ &= f_2^{\rightarrow}(\varphi(h)(\mu_1))(F_2) \\ &= (f_2^{\rightarrow} \circ \varphi(h))(\mu_1)(F_2).\end{aligned}$$

This shows $\varphi(h) \circ f_1^{\rightarrow} = f_2^{\rightarrow} \circ \varphi(h)$. □

Example 4.8. Let $M_1 = F(\omega)$ be the free R_0-algebra with countably many generators, and $M_2 = W_3 \times W_3$ the direct product of the three-valued R_0-chain W_3.

(i) Let $\sigma_1 = \phi$ be defined on M_1 by (2) and $f_1 = \phi^{-1} : Max(M_1) \to Max(M_1)$, then $\varphi(MV_3(M_1), \sigma_1) = ((Max(M_1), \delta_1), f_1^{\rightarrow})$ is an R_0-type three-valued Stone state space, see also Example 4.6 (i).

(ii) Let σ_2 be defined on M_2 by $\sigma_2(x, y) = (x, x)$ for all $(x, y) \in M_2$, then, by Example 4.3 (iv), (M_2, σ_2) is a state R_0-algebra. Note that $MV_3(M_2) = M_2$. Let $f_2 = \sigma_2^{-1} : Max(M_2) \to Max(M_2)$ and δ_2 the three-valued Stone topology on $Max(M_2)$ generated through (3) and (5), then it follows from Proposition 4.7 that $\varphi(MV_3(M_2), \sigma_2) = ((Max(M_2), \delta_2), f_2^{\rightarrow})$ is an R_0-type three-valued Stone state space.

(iii) Fix a homomorphism $v : M_1 \to W_3$, and define $h : M_1 \to M_2$ by $h(\alpha) = (v(\alpha), v(\alpha)) = (\overline{\alpha}(v), \overline{\alpha}(v))$ for all $\alpha \in M_1$. Then h is an MV-homomorphism from M_1 into M_2 and satisfies also $h \circ \sigma_1 = \sigma_2 \circ h$, and thus h is a morphism from $(MV_3(M_1), \sigma_1)$ into $(MV_3(M_2), \sigma_2)$ in the category $\mathcal{SMV}_3(\mathcal{SR}_0)$. We have now that $\varphi(h) = (h^{-1})^{\rightarrow} : (Max(M_2), \delta_2) \to (Max(M_1), \delta_1)$ is a morphism in the category $\mathcal{R}_0\mathcal{TSSF}$, where

$$\begin{aligned}\varphi(h)(s(x, y))(F_1) &= (h^{-1})^{\rightarrow}(s(x, y))(F_1) \\ &= \vee\{s(x, y)(F_2) \mid h^{-1}(F_2) = F_1, F_2 \in Max(M_2)\} \\ &= \begin{cases} \max\{x, y\}, & F_1 = v^{-1}(1), \\ 0, & \text{otherwise}, \end{cases}\end{aligned}$$

for all $s(x, y) \in \{0, \frac{1}{2}, 1\}^{Max(M_2)}$ and for all $F_1 \in Max(M_1)$. Here we adopt a similar notation $s(x, y)$ as in Example 4.6 (ii) for the function with the property that $s(x, y)(F_{21}) = x, s(x, y)(F_{22}) = y$, where $F_{21} = \{(1, 1), (1, \frac{1}{2}), (1, 0)\}$, $F_{22} =$

$\{(1,1),\left(\frac{1}{2},1\right),(0,1)\}$ and $Max(M_2)=\{F_{21},F_{22}\}$, and $v^{-1}(1)=\{\alpha\in F(\omega)\mid v(\alpha)=1\}$ is a maximal filter of $F(\omega)$, see [41, Theorem 3.16].

Proposition 4.9. *The function* $\psi:\mathcal{R}_0\mathcal{TSSF}\to\mathcal{SMV}_3(\mathcal{SR}_0)$ *is a contravariant functor, which is defined by*

$$\psi((X,\delta),f^{\to})=(Clop(X),\sigma_f)$$

where $\sigma_f=f^{\leftarrow}$, *for every* $((X,\delta),f^{\to})\in\mathcal{R}_0\mathcal{TSSF}$, *and*

$$\psi(g^{\to})=(g^{\to})^d=g^{\leftarrow}$$

for each morphism $g^{\to}:((X_2,\delta_2),f_2^{\to})\to((X_1,\delta_1),f_1^{\to})$.

Proof. By Corollary 3.6, it is sufficient to show that $\sigma_f\circ\sigma_f=\sigma_f$ and $\psi(g^{\to})\circ\sigma_{f_1}=\sigma_{f_2}\circ\psi(g^{\to})$.

Firstly, $\sigma_f\circ\sigma_f=f^{\leftarrow}\circ f^{\leftarrow}=(f\circ f)^{\leftarrow}=f^{\leftarrow}=\sigma_f$, by the idempotency of f. Secondly, by $(f_1\circ g)^{\to}=f_1^{\to}\circ g^{\to}=g^{\to}\circ f_2^{\to}=(g\circ f_2)^{\to}$ we have $f_1\circ g=g\circ f_2$. Take now any $\mu_1\in\mathrm{Clop}(X_1)$ and $x_2\in X_2$, then

$$\begin{aligned}(\psi(g^{\to})\circ\sigma_{f_1})(\mu_1)(x_2)&=\psi(g^{\to})(\sigma_{f_1}(\mu_1))(x_2)=g^{\leftarrow}(f_1^{\leftarrow}(\mu_1))(x_2)\\&=(\mu_1\circ f_1\circ g)(x_2)=\mu_1[(f_1\circ g)(x_2)]\\&=\mu_1[(g\circ f_2)(x_2)]=[(\mu_1\circ g)\circ f_2](x_2)\\&=f_2^{\leftarrow}(g^{\leftarrow}(\mu_1))(x_2)=\sigma_{f_2}(\psi(g^{\to})(\mu_1))(x_2)\\&=(\sigma_{f_2}\circ\psi(g^{\to}))(\mu_1)(x_2).\end{aligned}$$

This shows $\psi(g^{\to})\circ\sigma_{f_1}=\sigma_{f_2}\circ\psi(g^{\to})$. □

Example 4.10. (i) Let $X_1=\{a,b\}$ with $a\neq b$, $\delta_1=\{0,\frac{1}{2},1\}^{X_1}$, define $f_1:X_1\to X_1$ by $f_1(a)=a$ and $f_1(b)=a$, and $\sigma_1=\sigma_{f_1}=f_1^{\leftarrow}$. Then, by Example 4.6 (ii), $((X_1,\delta_1),f_1^{\to})\in\mathcal{R}_0\mathcal{TSSF}$ and $\psi((X_1,\delta_1),f_1^{\to})=(\mathrm{Clop}(X_1),\sigma_1)\in\mathcal{SMV}_3(\mathcal{SR}_0)$.

(ii) Let $X_2=\{c\}$, $\delta_2=\{0,\frac{1}{2},1\}^{X_2}$, $f_2=\mathrm{id}_{X_2}$, and $\sigma_2=\sigma_{f_2}=f_2^{\to}$. Trivially $((X_2,\delta_2),f_2^{\to})\in\mathcal{R}_0\mathcal{TSSF}$ and $\psi((X_2,\delta_2),f_2^{\to})=(\mathrm{Clop}(X_2),\sigma_2)\in\mathcal{SMV}_3(\mathcal{SR}_0)$.

(iii) Define $g:X_1\to X_2$ by $g(a)=g(b)=c$, then $g^{\to}$ is a Zadeh type continuous function from (X_1,δ_1) into (X_2,δ_2) such that $g^{\to}\circ f_1^{\to}=(g\circ f_1)^{\to}=(f_2\circ g)^{\to}=f_2^{\to}\circ g^{\to}$, and so it is a morphism from $((X_1,\delta_1),f_1^{\to})$ into $((X_2,\delta_2),f_2^{\to})$ in the category $\mathcal{R}_0\mathcal{TSSF}$. It follows from Proposition 4.9 that $\psi(g^{\to})=g^{\leftarrow}$ is a morphism from $(\mathrm{Clop}(X_2),\sigma_2)$ into $(\mathrm{Clop}(X_1),\sigma_1)$ in the category $\mathcal{SMV}_3(\mathcal{SR}_0)$.

Proposition 4.11. *Let* $(MV_3(M),\sigma)\in\mathcal{SMV}_3(\mathcal{SR}_0)$ *and*

$$(Clop(Max(M)),\sigma_f)=(\psi\circ\varphi)(MV_3(M),\sigma),$$

where φ *and* ψ *are defined as in Propositions 4.7 and 4.9, respectively. Then* $\sigma_f(s(x))=s(\sigma(x))$ *for* $s(x)\in Clop(Max(M)),x\in M$.

Proof. By Propositions 4.7 and 4.9, $\sigma_f=f^{\leftarrow}$ and $f^{\to}=(\sigma^{-1})^{\to}$, and so $\sigma_f=(\sigma^{-1})^{\leftarrow}$. Take $s(x)\in\mathrm{Clop}(Max(M))$ and $F\in Max(M)$, then $\sigma_f(s(x))(F)=(\sigma^{-1})^{\leftarrow}(s(x))(F)=s(x)(\sigma^{-1}(F))=\xi_{\sigma^{-1}(F)}(x)=\xi_F(\sigma(x))=s(\sigma(x))(F)$. This shows $\sigma_f(s(x))=s(\sigma(x))$. □

Theorem 4.12. *The categories $\mathcal{SMV}_3(\mathcal{SR}_0)$ and $\mathcal{R}_0\mathcal{TSSF}$ are dually equivalent.*

Proof. By Propositions 4.7 and 4.9, it suffices to show that $s : I_{\mathcal{SMV}_3(\mathcal{SR}_0)} \cong \psi \circ \varphi$ and $t^{\rightarrow} : I_{\mathcal{R}_0\mathcal{TSSF}} \cong \varphi \circ \psi$ are natural equivalences. By Corollary 3.6, it remains to show

$$\sigma_f \circ s = s \circ \sigma \tag{18}$$

and

$$t^{\rightarrow} \circ f^{\rightarrow} = g^{\rightarrow} \circ t^{\rightarrow} \tag{19}$$

where $(\varphi \circ \psi)((X, \delta), f^{\rightarrow}) = ((Max(\text{Clop}(X)), \delta'), g^{\rightarrow}) \in \mathcal{R}_0\mathcal{TSSF}$.

(18) is true by Proposition 4.11. Let us limit ourselves to (19). By definition, $g^{\rightarrow} = \sigma_f^d = (\sigma_f^{-1})^{\rightarrow}$ and $\sigma_f = f^{\leftarrow}$. Similar to (11), we have $g \circ t = (\sigma_f^{-1}) \circ t = t \circ f$, and hence, $t^{\rightarrow} \circ f^{\rightarrow} = (t \circ f)^{\rightarrow} = (g \circ t)^{\rightarrow} = g^{\rightarrow} \circ t^{\rightarrow}$. □

Let (M, σ) be a state R_0-algebra, and $B(M)$ the Boolean skeleton of M. Then the restriction of σ on $B(M)$ is a Boolean homomorphism. Let $\mathcal{SB}(\mathcal{SR}_0)$ denote the category whose objects are state Boolean skeletons $(B(M), \sigma)$ of state R_0-algebras (M, σ) and morphisms are (restrictions of) Boolean homomorphisms $h : (M_1, \sigma_1) \to (M_2, \sigma_2)$ with the property $h \circ \sigma_1 = \sigma_2 \circ h$. Let $\mathcal{CSSF}$ denote the category whose objects are pairs $((X, \mathcal{T}), f)$ of (crisp) Stone spaces with an idempotent continuous function f, and morphisms are continuous functions $g : ((X_1, \mathcal{T}_1), f_1) \to ((X_2, \mathcal{T}_2), f_2)$ with the property $g \circ f_1 = f_2 \circ g$.

Analogous to Theorem 4.12, one can show the dual equivalence between the categories $\mathcal{SB}(\mathcal{SR}_0)$ and $\mathcal{CSSF}$.

Theorem 4.13. *The categories $\mathcal{SB}(\mathcal{SR}_0)$ and $\mathcal{CSSF}$ are dually equivalent.*

5. Concluding Remarks

In this paper, we studied categorical dualities between two kinds of subalgebras of R_0-algebras with internal states and R_0-type three-valued Stone spaces and Zadeh type continuous functions. After each duality, we constructed several examples for its being intuitively understood and specific applications. Such dualities provide us a topological viewpoint for the study of algebraic structures of R_0-algebras. This work is in the line of research in [13], where the authors obtained the categorical dualities for weakly divisible σ-complete state-morphism MV-algebras. One of the main differences from [13] is that the topological dualities of R_0-algebras are three-valued, not crisp.

Following the idea of [10, 42], one can introduce internal states of type I (in the sense of Bosbach) as well as Riečan states on R_0-algebras, which are not necessarily algebraic homomorphisms. How to find their topological counterparts deserves our attention in further studies. The same questions hold for other logical algebras such as MV-algebras, BL-algebras and MTL-algebras too.

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