# SOLVING FUZZY LINEAR SYSTEMS BY USING THE SCHUR COMPLEMENT WHEN COEFFICIENT MATRIX IS AN $M$-MATRIX

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Abstract. This paper analyzes a linear system of equations when the right-hand side is a fuzzy vector and the coefficient matrix is a crisp $M$-matrix. The fuzzy linear system (FLS) is converted to the equivalent crisp system with coefficient matrix of dimension $2n \times 2n$. However, solving this crisp system is difficult for large $n$ because of dimensionality problems . It is shown that this difficulty may be avoided by computing the inverse of an $n \times n$ matrix instead of $Z^{-1}$.

## 1. Introduction

$n \times n$ fuzzy linear systems have been studied by many authors [1, 3, 5, 6, 7, 9, 10, 11, 12]. In [7] Friedman et al. proposed a general model for solving such systems by the embedding approach and stated conditions for the existence of a unique fuzzy solution to $n \times n$ linear systems. A minimum norm solution of the fuzzy under determined systems has been obtained in [10].

In this paper we investigate the case where the coefficient matrix in a fuzzy linear system is an $M$-matrix. We first study the properties of this system and then propose a solution using the Schur complement.

Section 2 provides preliminaries for fuzzy numbers and fuzzy linear systems. Several $M$-matrix properties and the Schur complement formula are also stated. The relationship between an $M$-matrix and its crisp matrix and the existence and expression of the solution to the fuzzy linear system using the Schur complement are discussed in section 3. Numerical examples to illustrate previous sections are given in Section 4.

## 2. Preliminaries

2.1. **Fuzzy Numbers and Fuzzy Linear Systems.** In this section we recall the basic notion of fuzzy numbers arithmetic and fuzzy linear system.

**Definition 2.1.** [7] A fuzzy number in parametric form is an ordered pair of functions $(\underline{u}(r), \overline{u}(r))$,
$0 \leq r \leq 1$ which satisfy the following conditions:

1. $\underline{u}(r)$ is a bounded left continuous nondecreasing function on [0,1].

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2. $\overline{u}(r)$ is a bounded left continuous nonincreasing function on [0,1].
3. $\underline{u}(r) \leq \overline{u}(r), \quad 0 \leq r \leq 1$.

If $\underline{u}(r) = \overline{u}(r) = \psi, \quad 0 \leq r \leq 1$ then $\psi$ is a crisp number. A popular representation for fuzzy number is the trapezoidal representation $u = (x_0, y_0, \alpha, \beta)$ with defuzzifier interval $[x_0, y_0]$, left fuzziness $\alpha$ and right fuzziness $\beta$. The membership function of this trapezoidal number is as follows:

$$u(x) = \begin{cases} \frac{1}{\alpha}(x - x_0 + \alpha), & x_0 - \alpha \leq x \leq x_0 \\ 1, & x \in [x_0, y_0] \\ \frac{1}{\beta}(y_0 - x + \beta), & y_0 \leq x \leq y_0 + \beta \\ 0, & otherwise. \end{cases}$$

The parametric form of the number is

$$\underline{u}(r) = x_0 - \alpha + \alpha r, \qquad \overline{u}(r) = y_0 + \beta - \beta r.$$

The operations of addition and scalar multiplication for arbitrary fuzzy numbers, $u = (\underline{u}, \overline{u})$ and $v = (\underline{v}, \overline{v})$ and $\lambda \in \Re^1$ are defined by

$$\begin{cases} (\underline{u+v})(r) = \underline{u}(r) + \underline{v}(r), \\ (\overline{u+v})(r) = \overline{u}(r) + \overline{v}(r), \\ (\underline{\lambda u})(r) = \lambda \underline{u}(r), \quad (\overline{\lambda u})(r) = \lambda \overline{u}(r), \quad \lambda \geq 0, \\ (\underline{\lambda u})(r) = \lambda \overline{u}(r), \quad (\overline{\lambda u})(r) = \lambda \underline{u}(r), \quad \lambda \leq 0. \end{cases} \tag{1}$$

A general fuzzy linear system is as follows:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= y_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= y_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= y_n, \end{aligned} \tag{2}$$

where the coefficient matrix $A = (a_{ij}), \quad 1 \leq i, j \leq n$ is a crisp $n \times n$ matrix and $y = (y_i)$, $1 \leq i \leq n$ is a fuzzy vector. This system is called a fuzzy linear system (FLS). By (2.1), the system (2.1) is equivalent to the following parametric system:

$$\begin{aligned} \underline{\textstyle\sum_{j=1}^{n} a_{ij}x_j} = \textstyle\sum_{j=1}^{n} \underline{a_{ij}x_j} = \underline{y_i} \\ \overline{\textstyle\sum_{j=1}^{n} a_{ij}x_j} = \textstyle\sum_{j=1}^{n} \overline{a_{ij}x_j} = \overline{y_i}. \end{aligned} \tag{3}$$

If for a particular $i: \quad a_{ij} > 0, \quad 1 \leq j \leq n$, then we obviously get

$$\sum_{j=1}^{n} a_{ij}\underline{x_j} = \underline{y_i} \quad , \qquad \sum_{j=1}^{n} a_{ij}\overline{x_j} = \overline{y_i}.$$

In general, an arbitrary equation for either $\underline{y_i}$ or $\overline{y_i}$ may include a linear combination of $\underline{x_j}$ 's and $\overline{x_j}$ 's. Consequently, in order to solve the system given by Eq. (**??**) one must solve a $(2n) \times (2n)$ crisp linear system, where the column on the right is the vector $(\underline{y_1}, \underline{y_2}, \dots, \underline{y_n}, \overline{y_1}, \overline{y_2}, \dots, \overline{y_n})^T$.

Let us now rearrange the linear system of Eqs.(**??**) so that the unknowns are $(\underline{x_i}, -\overline{x_i})$, $1 \le i \le n$ and the column on the right is

$$Y = (\underline{y_1}, \underline{y_2}, \dots, \underline{y_n}, -\overline{y_1}, -\overline{y_2}, \dots, -\overline{y_n})^T.$$

Define a $(2n) \times (2n)$ matrix $Z = (z_{ij})$ as follows:

$$\begin{array}{lll} a_{ij} \ge 0 & \Longrightarrow & z_{ij} = a_{ij}, \quad z_{i+n,j+n} = a_{ij}, \\ a_{ij} < 0 & \Longrightarrow & z_{i,j+n} = -a_{ij}, \quad z_{i+n,j} = -a_{ij}. \end{array} \tag{4}$$

Any $z_{ij}$ which is not determined by (2.1) is zero. Then the system (**??**) may be written in the following crisp block form:

$$ZX = Y \longrightarrow \begin{pmatrix} B \ge 0 & C \ge 0 \\ C \ge 0 & B \ge 0 \end{pmatrix} \begin{pmatrix} \underline{X} \\ -\overline{X} \end{pmatrix} = \begin{pmatrix} \underline{Y} \\ -\overline{Y} \end{pmatrix} \tag{5}$$

where $B, C \in \Re^{n,n}$ and

$$\underline{X} = \begin{pmatrix} \underline{x_1}(r) \\ \vdots \\ \underline{x_n}(r) \end{pmatrix}, \quad \overline{X} = \begin{pmatrix} \overline{x_1}(r) \\ \vdots \\ \overline{x_n}(r) \end{pmatrix}, \quad \underline{Y} = \begin{pmatrix} \underline{y_1}(r) \\ \vdots \\ \underline{y_n}(r) \end{pmatrix}, \quad \overline{Y} = \begin{pmatrix} \overline{y_1}(r) \\ \vdots \\ \overline{y_n}(r) \end{pmatrix}.$$

The definition of $Z = (z_{ij})$ implies that $B$ contains the positive entries of $A$, $C$ contains the absolute values of the negative entries of $A$ and $A = B - C$.

### 2.2. $M$-matrices and Some of their Properties.

**Definition 2.2.** [2] A real square matrix $A = (a_{ij})$ is called an $M$-matrix if $a_{ij} \le 0$; $i \ne j$ and $A^{-1} \ge 0$. For any real or complex $n \times n$ matrix $A = (a_{ij})$, the comparison matrix, denoted by $\Xi(A) = (\xi_{ij}(A))$, and defined by

$$\xi_{ij}(A) = \begin{cases} |a_{ij}|, & i = j \\ -|a_{ij}|, & i \ne j \end{cases} \tag{6}$$

is called an $H-$matrix if $\Xi(A)$ is an $M$-matrix.

**Definition 2.3.** [2] A square matrix $A$ is said to be generalized diagonally dominant if

$$|a_{ii}| x_i \ge \sum_{j \ne i} |a_{ij}| x_j, \quad i = 1, 2, \dots, n \tag{7}$$

for some positive vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$, generalized strictly diagonally dominant if (2.3) is valid with strict inequality and (strictly) diagonally dominant if (2.3) is valid for $\mathbf{x}=(1,1,\ldots,1)^T$.

**Lemma 2.4.** [2] *Let $A = (a_{ij})$ be an $n\times n$ matrix, with $a_{ij} \le 0,\ \ i \ne j$ and $a_{ii} > 0$. If $A$ is strictly diagonally dominant, then $A$ is an $M$-matrix.*

**Lemma 2.5.** [2] **a)** *A matrix $A = (a_{ij})$, with $a_{ij} \le 0,\ \ i \ne j$, is an $M$-matrix if and only if there exists a positive vector $\mathbf{x}$, such that $A\mathbf{x}$ is positive.*
**b)** *A matrix $A = (a_{ij})$ is an $H$-matrix if and only if $A$ is generalized strictly diagonally dominant.*

**Theorem 2.6.** [2] *Let $A = (a_{ij})$ be a matrix of order $n$, with $a_{ij} \le 0,\ \ i \ne j$. Then the following are equivalent:*
**a)** *$A$ is generalized strictly diagonally dominant and $a_{ii} \ge 0$.*
**b)** *$A$ is an $M$-matrix.*

2.3. **Schur Complements.** Let the matrix $A$ be partitioned into the two-by-two block form

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \tag{8}$$

where $A_{ii},\ \ i = 1,2$ are square matrices.

**Definition 2.7.** [2] If $A_{11}$ is nonsingular, we define

$$S \equiv A/A_{11} \equiv A_{22} - A_{21}A_{11}^{-1}A_{12}, \tag{9}$$

$S$ is called the Schur complement of $A$ with respect to $A_{11}$.

Note that $S$ is in the position $A_{22}$ of $A$ after we use Gaussian-elimination to convert $A_{21}$ to zero in $A$. In fact, the block matrix triangular factorization of $A$ is readily found to be

$$A = \begin{pmatrix} I & 0 \\ A_{21}A_{11}^{-1} & I \end{pmatrix} \begin{pmatrix} A_{11} & 0 \\ 0 & S \end{pmatrix} \begin{pmatrix} I & A_{11}^{-1}A_{12} \\ 0 & I \end{pmatrix}. \tag{10}$$

Hence we may write the inverse of $A$ in one of the two forms (11) and (12) as follows:

$$A^{-1} = \begin{pmatrix} I & -A_{11}^{-1}A_{12} \\ 0 & I \end{pmatrix} \begin{pmatrix} A_{11}^{-1} & 0 \\ 0 & S^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ -A_{21}A_{11}^{-1} & I \end{pmatrix} \tag{11}$$

$$A^{-1} = \begin{pmatrix} A_{11}^{-1} + A_{11}^{-1}A_{12}S^{-1}A_{21}A_{11}^{-1} & -A_{11}^{-1}A_{12}S^{-1} \\ -S^{-1}A_{21}A_{11}^{-1} & S^{-1} \end{pmatrix} \tag{12}$$

If both $A_{11}^{-1}$ and $A_{22}^{-1}$ exist, an alternative expression for $A^{-1}$ is

$$A^{-1} = \begin{pmatrix} S_1^{-1} & -A_{11}^{-1}A_{12}S_2^{-1} \\ -S_2^{-1}A_{21}A_{11}^{-1} & S_2^{-1} \end{pmatrix} \tag{13}$$

where

$$S_i = A/A_{jj}, \quad i \neq j, \quad i,j = 1,2.$$

**Theorem 2.8.** [2] *Let $A$ be an $M$-matrix that is partitioned in a two-by-two block form. Then the Schur complement*

$$S = A/A_{11} = A_{22} - A_{21}A_{11}^{-1}A_{12}$$

*exists and is also an $M$-matrix.*

## 3. Solving a Fuzzy Linear System when $A$ Is an $M$-matrix

3.1. **Solution to a Fuzzy Linear System.** To study the solution to a fuzzy linear system, it is first necessary to discuss the generalized inverses of the matrix $Z$ in a special structure.

**Definition 3.1.** [5] Let $X = \{\ (\underline{x_i}(r), \overline{x_i}(r));\ \ 1 \le i \le n\ \}$

denote a solution of $ZX = Y$. The fuzzy number vector

$$U = \{(\underline{u_i}(r), \overline{u_i}(r));\ \ 1 \le i \le n\}$$

defined by

$$\begin{aligned} \underline{u_i}(r) &= min\{\ \underline{x_i}(r), \overline{x_i}(r), \underline{x_i}(1), \overline{x_i}(1)\ \} \\ \overline{u_i}(r) &= max\{\ \underline{x_i}(r), \overline{x_i}(r), \underline{x_i}(1), \overline{x_i}(1)\ \} \end{aligned} \tag{14}$$

is called a fuzzy solution of $ZX = Y$. If $(\underline{x_i}(r), \overline{x_i}(r));\ \ 1 \le i \le n$, are all fuzzy numbers and $\underline{u_i}(r) = \underline{x_i}(r),\ \ \overline{u_i}(r) = \overline{x_i}(r),\ \ 1 \le i \le n$, then $U$ is called a strong fuzzy solution. Otherwise, $U$ is a weak fuzzy solution.

From (2.1) we have the following assertion:

**Theorem 3.2.** [12] *Let matrix $Z$ be in the form (2.1). Then the matrix*

$$Z^{-} = \frac{1}{2}\begin{pmatrix} (B+C)^{-} + (B-C)^{-} & (B+C)^{-} - (B-C)^{-} \\ (B+C)^{-} - (B-C)^{-} & (B+C)^{-} + (B-C)^{-} \end{pmatrix} \tag{15}$$

is a $g$-inverse of the matrix $Z$, where $(B+C)^{-}$ and $(B-C)^{-}$ are $g$-inverses of the matrices $B+C$ and $B-C$, respectively. In particular, the Moore-Penrose inverse of the matrix $Z$ is

$$Z^{\dagger} = \frac{1}{2}\begin{pmatrix} (B+C)^{\dagger} + (B-C)^{\dagger} & (B+C)^{\dagger} - (B-C)^{\dagger} \\ (B+C)^{\dagger} - (B-C)^{\dagger} & (B+C)^{\dagger} + (B-C)^{\dagger} \end{pmatrix} \tag{16}$$

In this paper, we consider only the case when $m = n$; thus $Z$ is nonsingular if and only if the matrices $A = B - C$ and $B + C$ are both nonsingular [7, Theorem 1]. Hence, as a direct consequence of Theorem 3.2, we have the following corollary.

**Corollary 3.3.** [7] *If $Z^{-1}$ exists, it must have the same structure as $Z$, i.e.*

$$Z^{-1} = \begin{pmatrix} T_1 & T_2 \\ T_2 & T_1 \end{pmatrix}$$

*where*

$$T_1 = \tfrac{1}{2}[(B+C)^{-1} + (B-C)^{-1}],$$
$$T_2 = \tfrac{1}{2}[(B+C)^{-1} - (B-C)^{-1}].$$

The next result provides a sufficient condition for the solution vector to be a fuzzy vector.

**Theorem 3.4.** [12] *For the consistent system (2.1) and any g-inverse $Z^-$ of the coefficient matrix $Z$, $X = Z^-Y$ is a solution to the system and therefore it admits a weak or strong fuzzy solution. In particular, if $Z^-$ is nonnegative with the special structure (3.2), then $X = Z^-Y$ admits a strong fuzzy solution for arbitrary fuzzy vector $Y$.*

**Corollary 3.5.** [12] *For the $n \times n$ fuzzy linear system, if $Z^{-1}$ exists, then for an arbitrary fuzzy vector $Y$, the unique solution $X = Z^{-1}Y$ is a fuzzy vector if $Z^{-1}$ is nonnegative.*

3.2. **Properties of the Equivalent Crisp Matrix $Z$.** We consider the solution of FLS, $Ax = \widetilde{y}$ when $A$ is an $M$-matrix and $\widetilde{y}$ is a fuzzy vector. Then (2.1), implies that $z_{ij} \geq 0, \quad 1 \leq i, j \leq n$ and $Z = \begin{pmatrix} B & C \\ C & B \end{pmatrix}$ where $B$ is the diagonal matrix containing the diagonal entries of $A$ and $C$ contains the absolute value of the off-diagonal entries of $A$. In other words, $B = diag(a_{11}, a_{22}, \ldots, a_{nn})$, and so $(B^{-1} = diag(\frac{1}{a_{11}}, \frac{1}{a_{22}}, \ldots, \frac{1}{a_{nn}}))$, and $C = (c_{ij})_{n\times n}$ where

$$\begin{cases} c_{ii} = 0, & 1 \leq i \leq n \\ c_{ij} = -a_{ij}, & 1 \leq i, j \leq n, \quad i \neq j. \end{cases}$$

**Theorem 3.6.** *If $A$ is an $M$-matrix then $Z$, as defined by (2.1), is an $H$-matrix.*

*Proof.* From theorem 2.6, it is clear that $A$ is generalized strictly diagonally dominant and $a_{ii} \geq 0$. Thus from lemma 2.5 for some $x > 0, \quad x \in \Re^n$ we have

$$|a_{ii}|x_i > \sum_{j\neq i} |a_{ij}|x_j, \qquad i = 1, 2, \ldots, n. \tag{17}$$

To complete the proof, it is sufficient to show that $Z$ is generalized strictly diagonally dominant. Let $y = [x^T, x^T]$, where $x$ is the vector corresponding to the matrix $A$. Then it may easily be checked that the relation

$$|z_{ii}|y_i > \sum_{j=1}^{2n} |z_{ij}|y_j, \quad j \neq i, \quad 1 \leq i \leq 2n. \tag{18}$$

holds for the matrix $Z$ defined in (2.1). □

3.3. **Solution Using the Schur Complement.** In this section we employ the Schur complement for computing $Z^{-1}$ where $Z$ is defined by (2.1) with block form $Z = \begin{pmatrix} B & C \\ C & B \end{pmatrix}$ where $B$ is a diagonal matrix and $C$ is a nondiagonal matrix whose entries are absolute values of the off-diagonal entries of $A$ if $A$ is an $M$-matrix. It is clear from (2.7), (2.3) and (2.1) that $S = S_1 = S_2 = B - CB^{-1}C$ and so

$$Z^{-1} = \begin{pmatrix} S^{-1} & -B^{-1}CS^{-1} \\ -S^{-1}CB^{-1} & S^{-1} \end{pmatrix} \tag{19}$$

**Theorem 3.7.** *Let $A$ be an $M$-matrix, in FLS, $Z$ be the extended crisp matrix of $A$, and $S$ be its Schur complement. Then we have*

$$S^{-1}CB^{-1} = B^{-1}CS^{-1}$$

*Proof.* Let $P = S^{-1}$, $Q_1 = -B^{-1}CS^{-1}$, $Q_2 = -S^{-1}CB^{-1}$. We must prove $Q_1 = Q_2$. From (3.3) and (2.1) we have

$$ZZ^{-1} = I, \quad or \quad \begin{pmatrix} B & C \\ C & B \end{pmatrix} \begin{pmatrix} P & Q_1 \\ Q_2 & P \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$

thus

$$\begin{cases} BQ_1 + CP = 0 \\ CP + BQ_2 = 0 \end{cases} \Longrightarrow \begin{Bmatrix} Q_1 = -B^{-1}CP \\ Q_2 = -B^{-1}CP \end{Bmatrix}$$

which completes the proof. □

From theorem 3.7 we have

$$Z^{-1} = \begin{pmatrix} P & Q \\ Q & P \end{pmatrix}, \tag{20}$$

where $P = S^{-1}$, $Q = -PCB^{-1}$. Hence (2.1) and (3.3) imply that

$$\begin{pmatrix} \underline{X} \\ -\overline{X} \end{pmatrix} = \begin{pmatrix} P & Q \\ Q & P \end{pmatrix} \begin{pmatrix} \underline{Y} \\ -\overline{Y} \end{pmatrix}$$

or

$$\begin{cases} \underline{X} = P\underline{Y} - Q\overline{Y} \\ \overline{X} = P\overline{Y} - Q\underline{Y}. \end{cases} \tag{21}$$

3.4. **Some Properties of $Z^{-1}$.** To investigate properties of $Z^{-1}$, we shall need the following definitions and lemmas:

**Definition 3.8.** [4] Let $A = (a_{ij})$ be an $m \times n$ matrix. We define a matrix norm $\|A\|$ as follows:

$$\|A\| = max_{1 \le i \le m} \sum_{j=1}^{n} |a_{ij}|.$$

**Lemma 3.9.** [4] *For any matrices, $A$ and $B$, we have*

$$\|AB\| \le \|A\|\|B\|,$$

*and consequently $\|A^2\| \le \|A\|^2$.*

**Lemma 3.10.** [4] *The matrix series*

$$I + A + A^2 + \cdots + A^k + \cdots$$

*converges to* $(I - A)^{-1}$ *if* $\|A\| < 1$.

**Theorem 3.11.** *Let A be a strictly diagonally dominant matrix with the extended crisp matrix defined by (2.1) and the inverse defined by by (3.3). Then*

$$P \geq 0, \qquad Q \leq 0.$$

*Proof.* We know that $P = S^{-1} = (B - CB^{-1}C)^{-1}$, thus

$$B^{-1}S = I - B^{-1}CB^{-1}C.$$

Let $E = B^{-1}CB^{-1}C$. We prove $\|E\| < 1$. From (2.1) we have

$$B^{-1}C = \begin{pmatrix} \frac{1}{a_{11}} & 0 & \cdots & 0 \\ 0 & \frac{1}{a_{22}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{a_{nn}} \end{pmatrix} \begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} \\ a_{21} & 0 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \frac{a_{12}}{a_{11}} & \cdots & \frac{a_{1n}}{a_{11}} \\ \frac{a_{21}}{a_{22}} & 0 & \cdots & \frac{a_{2n}}{a_{22}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{a_{n1}}{a_{nn}} & \frac{a_{n2}}{a_{nn}} & \cdots & 0 \end{pmatrix}.$$

Thus by the Lemma 3.9 and inequalities (3.2) we obtain

$$\|B^{-1}C\| = max_{1\leq i\leq n} \sum_{j=1}^{n} \frac{|a_{ij}|}{|a_{ii}|} < 1.$$

Therefore

$$\|E\| = \|(B^{-1}C)^2\| \leq \|B^{-1}C\|^2 < 1,$$

and from lemma 3.10

$$(I - E)^{-1} = I + E + E^2 + \cdots \geq 0.$$

Hence $S^{-1}B = (I - E)^{-1} \geq 0$ and $P = S^{-1} = S^{-1}BB^{-1} \geq 0$. From $Q = -PCB^{-1}$ and $P \geq 0,\ \ C \geq 0,\ \ B^{-1} \geq 0$, it is clear that $Q \leq 0$ and the proof is complete. □

**Theorem 3.12.** *Let* $A = (a_{ij})$ *be an* $n \times n$ *matrix with the following properties:*
**a)** $a_{ij} \leq 0$ *for* $i \neq j$ *and there exists at least one negative off-diagonal entry.*
**b)** $a_{ii} > 0, \quad i = 1, \cdots, n$.
**c)** *A is strictly diagonally dominant.*

*Then there exists a fuzzy vector Y such that* $X = \begin{pmatrix} \underline{X} \\ -\overline{X} \end{pmatrix} = Z^{-1}Y$ *is not a fuzzy vector i.e.*

$$\begin{cases} x_1(r) = (\underline{x_1}(r), \overline{x_1}(r)) \\ \vdots \\ x_n(r) = (\underline{x_n}(r), \overline{x_n}(r)) \end{cases}$$

*is not a strong fuzzy solution for system (2.1).*

(**Note:** If the assumption (a) fails then C = 0 and there is a fuzzy solution by corollary 3.5)

*Proof.* Suppose

$$P = \begin{pmatrix} p_{11} & \cdots & p_{1n} \\ \vdots & \ddots & \vdots \\ p_{n1} & \cdots & p_{nn} \end{pmatrix}, \quad Q = \begin{pmatrix} q_{11} & \cdots & q_{1n} \\ \vdots & \ddots & \vdots \\ q_{n1} & \cdots & q_{nn} \end{pmatrix}.$$

From the assumptions, Theorem 3.11 holds and from second part of assumption (a),$Q \neq 0$. Since $P$ is non-singular and non-negative, hence there exist $i_0$ and $j_1$ such that $q_{i_0,j_1} < 0$ and $p_{i_0,j_0} > 0, \quad 1 \leq i_0, j_0, j_1 \leq n$. Now we construct a fuzzy vector $Y$ such that $X = Z^{-1}Y$ is a non-fuzzy vector. Let us write

$$X = Z^{-1}Y = \begin{pmatrix} \begin{pmatrix} p_{11} & \cdots & p_{1n} \\ \vdots & \ddots & \vdots \\ p_{n1} & \cdots & p_{nn} \end{pmatrix} & \begin{pmatrix} q_{11} & \cdots & q_{1n} \\ \vdots & \ddots & \vdots \\ q_{n1} & \cdots & q_{nn} \end{pmatrix} \\ \begin{pmatrix} q_{11} & \cdots & q_{1n} \\ \vdots & \ddots & \vdots \\ q_{n1} & \cdots & q_{nn} \end{pmatrix} & \begin{pmatrix} p_{11} & \cdots & p_{1n} \\ \vdots & \ddots & \vdots \\ p_{n1} & \cdots & p_{nn} \end{pmatrix} \end{pmatrix} \begin{pmatrix} \underline{y_1}(r) \\ \vdots \\ \underline{y_n}(r) \\ -\overline{y_1}(r) \\ \vdots \\ -\overline{y_n}(r) \end{pmatrix}$$

$$= \begin{pmatrix} \underline{x_1}(r) \\ \vdots \\ \underline{x_n}(r) \\ -\overline{x_1}(r) \\ \vdots \\ -\overline{x_n}(r) \end{pmatrix}. \qquad (22)$$

It is sufficient to construct $Y$ such that $\underline{x_{i_0}}(r)$ is a non-fuzzy number. From (3.4) we have

$$\begin{aligned} \underline{x_{i_0}}(r) = p_{i_0 1}\underline{y_1}(r) + \cdots + p_{i_0 j_0}\underline{y_{j_0}}(r) + \cdots \\ +p_{i_0 n}\underline{y_n}(r) - q_{i_0 1}\overline{y_1}(r) - \cdots - q_{i_0 j_1}\overline{y_{j_1}}(r) - \cdots - q_{i_0 n}\overline{y_n}(r). \end{aligned} \qquad (23)$$

We have the following two cases:

**Case1.** $j_0 = j_1$. In this case, for the real numbers $a_i, \quad 1 \leq i \leq n$, $c_{j_0}$ and $d_{j_0}$ we define

$$\begin{cases} \underline{y_i}(r) = \overline{y_i}(r) = a_i, \quad (1 \leq i \leq n, \quad i \neq j_0) \\ \underline{y_{j_0}}(r) = a_{j_0} r, \quad \overline{y_{j_0}}(r) = c_{j_0} r + d_{j_0} \end{cases} \qquad (24)$$

in which $a_{j_0} > 0$, $c_{j_0} < 0$ and $c_{j_0} \neq -a_{j_0}$. Then from (3.4) we will have

$$\underline{x_{i_0}}(r) = \sum_{j=1_{j \neq j_0}}^{n} (P_{i_0 j} - q_{i_0 j})a_j + p_{i_0 j_0} a_{j_0} r - q_{i_0 j_0} c_{j_0} r - q_{i_0 j_0} d_{j_0}$$

$$\Rightarrow \underline{x_{i_0}}(r) = (p_{i_0j_0}a_{j_0} - q_{i_0j_0}c_{j_0})r + \sum_{j=1_{j\neq j_0}}^{n} (P_{i_0j} - q_{i_0j})a_j - q_{i_0j_0}d_{j_0}.$$

Now it is sufficient to define the coefficient of the parameter $r$ so that the function $\underline{x_{i_0}}(r)$ is a decreasing function i.e. $\frac{d\underline{x_{i_0}}(r)}{dr} < 0$. Hence the coefficients $c_{j_0}$ and $a_{j_0}$ are determined so that $p_{i_0j_0}a_{j_0} - q_{i_0j_0}c_{j_0} < 0$. Also, to preserve the fuzziness of $Y$, it is sufficient to have $\underline{y_{j_0}} \leq \overline{y_{j_0}}$, or, equivalently, $a_{j_0}r \leq c_{j_0}r + d_{j_0}$; i.e. $(a_{j_0} - c_{j_0})r \leq d_{j_0}$ or $r \leq \frac{d_{j_0}}{a_{j_0} - c_{j_0}}$. If we set $d_{j_0} = a_{j_0} - c_{j_0}$, then $0 \leq r \leq 1$.

**Case2.** $j_0 \neq j_1$. In this case, for the real numbers $a_i$, $1 \leq i \leq n$, $c_{j_0}$ and $d_{j_0}$ we define

$$\begin{cases} \underline{y_i}(r) = \overline{y_i}(r) = a_i, \quad (1 \leq i \leq n, \quad i \neq j_0) \\ \underline{y_{j_0}}(r) = a_{j_0}r, \quad \overline{y_{j_0}}(r) = c_{j_0}r + d_{j_0} \\ \underline{y_{j_1}}(r) = a_{j_1}r, \quad \overline{y_{j_1}}(r) = c_{j_1}r + d_{j_1} \end{cases} \tag{25}$$

where $a_{j_0} > 0$, $c_{j_0} < 0$, $c_{j_0} \neq -a_{j_0}$, $a_{j_1} > 0$, $c_{j_1} < 0$ and $c_{j_1} \neq -a_{j_1}$. From (3.4), we have

$$\underline{x_{i_0}}(r) = \sum_{j=1_{j\neq j_0,j_1}}^{n} (P_{i_0j} - q_{i_0j})a_j + p_{i_0j_0}a_{j_0}r - q_{i_0j_0}c_{j_0}r - q_{i_0j_0}d_{j_0}$$

$$+p_{i_0j_1}a_{j_1}r - q_{i_0j_1}c_{j_1}r - q_{i_0j_1}d_{j_1}$$

or

$$\underline{x_{i_0}}(r) = (p_{i_0j_0}a_{j_0} + p_{i_0j_1}a_{j_1} - q_{i_0j_0}c_{j_0} - q_{i_0j_1}c_{j_1})r$$

$$+ \sum_{j=1_{j\neq j_0,j_1}}^{n} (P_{i_0j} - q_{i_0j})a_j - q_{i_0j_0}d_{j_0} - q_{i_0j_1}d_{j_1}.$$

Choosing $a_{j_0}, a_{j_1}, c_{j_0}$ and $c_{j_1}$ to satisfy

$$p_{i_0j_0}a_{j_0} + p_{i_0j_1}a_{j_1} < q_{i_0j_0}c_{j_0} + q_{i_0j_1}c_{j_1}$$

guarantees $\frac{d\underline{x_{i_0}}(r)}{dr} < 0$. Thus if

$$d_{j_0} = a_{j_0} - c_{j_0}, \quad d_{j_1} = a_{j_1} - c_{j_1},$$

the vector $Y$ will be fuzzy, but the number $\underline{x}_{i_0}(r)$ will not be fuzzy. □

## 4. Numerical Results

**Example 1:** Consider the $2 \times 2$ fuzzy linear system

$$\begin{cases} 2x_1 - x_2 = (r, 3 - r) \\ -3x_1 + 4x_2 = (r - 1, 5 - r) \end{cases}$$

where $A = \begin{pmatrix} 2 & -1 \\ -3 & 4 \end{pmatrix}$ is obviously an $M$-matrix. The extended $4 \times 4$ matrix is

$$Z = \begin{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 3 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 \\ 3 & 0 \end{pmatrix} & \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} \end{pmatrix}$$

where

$$B = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} \quad , \quad C = \begin{pmatrix} 0 & 1 \\ 3 & 0 \end{pmatrix}.$$

From (2.2) we have

$$\Xi(Z) = \begin{pmatrix} 2 & 0 & 0 & -1 \\ 0 & 4 & -3 & 0 \\ 0 & -1 & 2 & 0 \\ -3 & 0 & 0 & 4 \end{pmatrix}.$$

Clearly $\Xi(Z)$, is an $M$-matrix, thus $Z$ is an $H$-matrix. By (3.3)

$$P = S^{-1} = (B - CB^{-1}C)^{-1} = \begin{pmatrix} 1.25 & 0 \\ 0 & 2.5 \end{pmatrix}^{-1} = \begin{pmatrix} 0.8 & 0 \\ 0 & 0.4 \end{pmatrix}$$

$$Q = -PCB^{-1} = \begin{pmatrix} 0 & -0.2 \\ -0.6 & 0 \end{pmatrix}$$

and thus

$$Z^{-1} = \begin{pmatrix} P & Q \\ Q & P \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 0.8 & 0 \\ 0 & 0.4 \end{pmatrix} & \begin{pmatrix} 0 & -0.2 \\ -0.6 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & -0.2 \\ -0.6 & 0 \end{pmatrix} & \begin{pmatrix} 0.8 & 0 \\ 0 & 0.4 \end{pmatrix} \end{pmatrix}.$$

Let

$$\underline{X} = \begin{pmatrix} \underline{x_1}(r) \\ \underline{x_2}(r) \end{pmatrix}, \quad \overline{X} = \begin{pmatrix} \overline{x_1}(r) \\ \overline{x_2}(r) \end{pmatrix},$$

so we have

$$\underline{Y} = \begin{pmatrix} r \\ r-1 \end{pmatrix}, \quad \overline{Y} = \begin{pmatrix} 3-r \\ 5-r \end{pmatrix}$$

and from (3.3),

$$\begin{aligned} \underline{X} = P\underline{Y} - Q\overline{Y} &= \begin{pmatrix} 0.8 & 0 \\ 0 & 0.4 \end{pmatrix} \begin{pmatrix} r \\ r-1 \end{pmatrix} - \begin{pmatrix} 0 & -0.2 \\ -0.6 & 0 \end{pmatrix} \begin{pmatrix} 3-r \\ 5-r \end{pmatrix} \\ &= \begin{pmatrix} 0.6r+1 \\ -0.2r+1.4 \end{pmatrix} \\ \overline{X} = P\overline{Y} - Q\underline{Y} &= \begin{pmatrix} 0.8 & 0 \\ 0 & 0.4 \end{pmatrix} \begin{pmatrix} 3-r \\ 5-r \end{pmatrix} - \begin{pmatrix} 0 & -0.2 \\ -0.6 & 0 \end{pmatrix} \begin{pmatrix} r \\ r-1 \end{pmatrix} \\ &= \begin{pmatrix} -0.6r+2.2 \\ 0.2r+2 \end{pmatrix} \end{aligned}$$

and thus

$$x_1(r) = (\underline{x_1}(r), \overline{x_1}(r)) = (0.6r + 1, -0.6r + 2.2),$$
$$x_2(r) = (\underline{x_2}(r), \overline{x_2}(r)) = (-0.2r + 1.4, 0.2r + 2).$$

Obviously, $x_1$, $x_2$ are not fuzzy numbers. Therefore the corresponding fuzzy solution is a weak fuzzy solution given by

$$\begin{cases} u_1(r) = (0.6r + 1, -0.6r + 2.2) \\ u_2(r) = (1.2, 2.2). \end{cases}$$

**Example 2:** Now consider a system where the coefficient matrix is the same as in the previous example, but where the fuzzy vector on the right hand side leads to a strongly fuzzy solution.

$$\begin{cases} 2x_1 - x_2 = (2r, -3r + 5) \\ -3x_1 + 4x_2 = (5r + 1, -4r + 10) \end{cases}$$

where $A = \begin{pmatrix} 2 & -1 \\ -3 & 4 \end{pmatrix}$ and

$$Z = \begin{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 3 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 \\ 3 & 0 \end{pmatrix} & \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} \end{pmatrix}.$$

From the previous example we have

$$P = \begin{pmatrix} 0.8 & 0 \\ 0 & 0.4 \end{pmatrix} \quad , \quad Q = \begin{pmatrix} 0 & -0.2 \\ -0.6 & 0 \end{pmatrix},$$

$$Z^{-1} = \begin{pmatrix} \begin{pmatrix} 0.8 & 0 \\ 0 & 0.4 \end{pmatrix} & \begin{pmatrix} 0 & -0.2 \\ -0.6 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & -0.2 \\ -0.6 & 0 \end{pmatrix} & \begin{pmatrix} 0.8 & 0 \\ 0 & 0.4 \end{pmatrix} \end{pmatrix}.$$

Also

$$\underline{Y} = \begin{pmatrix} 2r \\ 5r + 1 \end{pmatrix} \quad , \quad \overline{Y} = \begin{pmatrix} -3r + 5 \\ -4r + 10 \end{pmatrix}$$

and thus from (3.9) it follows that

$$\underline{X} = P\underline{Y} - Q\overline{Y} = \begin{pmatrix} 0.8 & 0 \\ 0 & 0.4 \end{pmatrix} \begin{pmatrix} 2r \\ 5r + 1 \end{pmatrix} - \begin{pmatrix} 0 & -0.2 \\ -0.6 & 0 \end{pmatrix} \begin{pmatrix} -3r + 5 \\ -4r + 10 \end{pmatrix}$$
$$= \begin{pmatrix} 0.8r + 2 \\ 0.2r + 3.4 \end{pmatrix}$$
$$\overline{X} = P\overline{Y} - Q\underline{Y} = \begin{pmatrix} 0.8 & 0 \\ 0 & 0.4 \end{pmatrix} \begin{pmatrix} -3r + 5 \\ -4r + 10 \end{pmatrix} - \begin{pmatrix} 0 & -0.2 \\ -0.6 & 0 \end{pmatrix} \begin{pmatrix} 2r \\ 5r + 1 \end{pmatrix}$$
$$= \begin{pmatrix} -1.4r + 4.2 \\ -0.4r + 4 \end{pmatrix}$$

so that

$$x_1(r) = (\underline{x_1}(r), \overline{x_1}(r)) = (0.8r + 2, -1.4r + 4.2),$$

$$x_2(r) = (\underline{x_2}(r), \overline{x_2}(r)) = (0.2r + 3.4, -0.4r + 4).$$

Obviously $x_1(r)$, $x_2(r)$ are fuzzy numbers for $0 \le r \le 1$, and

$$2 \le \underline{x_1}(r) \le 2.8, \quad 2.8 \le \overline{x_1}(r) \le 4.2, \quad 3.4 \le \underline{x_2}(r) \le 3.6, \quad 3.6 \le \overline{x_2}(r) \le 4$$

$$\underline{x_1}(1) = 2.8, \quad \overline{x_1}(1) = 2.8, \quad \underline{x_2}(1) = 3.6, \quad \overline{x_2}(1) = 3.6.$$

Therefore

$$\begin{aligned}
\underline{x_1}(r) &= min\{\underline{x_1}(r), \overline{x_1}(r), \underline{x_1}(1), \overline{x_1}(1)\},\\
\overline{x_1}(r) &= max\{\underline{x_1}(r), \overline{x_1}(r), \underline{x_1}(1), \overline{x_1}(1)\},\\
\underline{x_2}(r) &= min\{\underline{x_2}(r), \overline{x_2}(r), \underline{x_2}(1), \overline{x_2}(1)\},\\
\overline{x_2}(r) &= max\{\underline{x_2}(r), \overline{x_2}(r), \underline{x_2}(1), \overline{x_2}(1)\},
\end{aligned}$$

so $X$ is a strong fuzzy solution.

**Example 3:** Consider the $3 \times 3$ fuzzy linear system

$$\begin{cases} 6x_1 - 4x_2 - x_3 = (r, 1-r) \\ -x_1 + 5x_2 - 2x_3 = (r+1, 3-r) \\ -x_1 + \quad +4x_3 = (r-2, -r). \end{cases}$$

The coefficient matrix is

$$A = \begin{pmatrix} 6 & -4 & -1 \\ -1 & 5 & -2 \\ -1 & 0 & 4 \end{pmatrix}$$

which is an $M$-matrix. Also,

$$\underline{Y} = \begin{pmatrix} r \\ r+1 \\ r-2 \end{pmatrix}, \quad \overline{Y} = \begin{pmatrix} 1-r \\ 3-r \\ -r \end{pmatrix}.$$

The extended $6 \times 6$ crisp matrix is as follows:

$$Z = \begin{pmatrix} \begin{pmatrix} 6 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{pmatrix} & \begin{pmatrix} 0 & 4 & 1 \\ 1 & 0 & 2 \\ 1 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 4 & 1 \\ 1 & 0 & 2 \\ 1 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 6 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{pmatrix} \end{pmatrix}$$

where

$$B = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 4 & 1 \\ 1 & 0 & 2 \\ 1 & 0 & 0 \end{pmatrix}$$

and

$$\Xi(Z) = \begin{pmatrix} 6 & 0 & 0 & 0 & -4 & -1 \\ 0 & 5 & 0 & -1 & 0 & -2 \\ 0 & 0 & 4 & -1 & 0 & 0 \\ 0 & -4 & -1 & 6 & 0 & 0 \\ -1 & 0 & -2 & 0 & 5 & 0 \\ -1 & 0 & 0 & 0 & 0 & 4 \end{pmatrix}.$$

Since $\Xi(Z)$ is an $M$-matrix, $Z$ is an $H$-matrix. Moreover,

$$P = (B - CB^{-1}C)^{-1} = \begin{pmatrix} \frac{413}{2031} & \frac{128}{9737} & \frac{64}{749} \\ \frac{230}{9737} & \frac{181}{774} & \frac{15}{749} \\ \frac{40}{9737} & \frac{396}{9737} & \frac{198}{749} \end{pmatrix},$$

$$Q = -PCB^{-1} = \begin{pmatrix} -\frac{160}{9737} & -\frac{809}{4973} & -\frac{43}{749} \\ -\frac{221}{5223} & -\frac{184}{9737} & -\frac{92}{749} \\ -\frac{495}{9737} & -\frac{32}{9737} & -\frac{16}{749} \end{pmatrix},$$

thus

$$Z^{-1} = \begin{pmatrix} \begin{pmatrix} \frac{413}{2031} & \frac{128}{9737} & \frac{64}{749} \\ \frac{230}{9737} & \frac{181}{774} & \frac{15}{749} \\ \frac{40}{9737} & \frac{396}{9737} & \frac{198}{749} \end{pmatrix} & \begin{pmatrix} -\frac{160}{9737} & -\frac{809}{4973} & -\frac{43}{749} \\ -\frac{221}{5223} & -\frac{184}{9737} & -\frac{92}{749} \\ -\frac{495}{9737} & -\frac{32}{9737} & -\frac{16}{749} \end{pmatrix} \\ \begin{pmatrix} -\frac{160}{9737} & -\frac{809}{4973} & -\frac{43}{749} \\ -\frac{221}{5223} & -\frac{184}{9737} & -\frac{92}{749} \\ -\frac{495}{9737} & -\frac{32}{9737} & -\frac{16}{749} \end{pmatrix} & \begin{pmatrix} \frac{413}{2031} & \frac{128}{9737} & \frac{64}{749} \\ \frac{230}{9737} & \frac{181}{774} & \frac{15}{749} \\ \frac{40}{9737} & \frac{396}{9737} & \frac{198}{749} \end{pmatrix} \end{pmatrix},$$

and from (3.3) we have:

$$\begin{cases} \underline{X} = P\underline{Y} - Q\overline{Y} = \begin{pmatrix} \frac{7}{107}r + \frac{3376}{9737} \\ \frac{10}{107}r + \frac{2851}{9737} \\ \frac{25}{107}r - \frac{4161}{9737} \end{pmatrix} \\ \overline{X} = P\overline{Y} - Q\underline{Y} = \begin{pmatrix} -\frac{7}{107}r + \frac{2830}{9737} \\ -\frac{10}{107}r + \frac{4853}{9737} \\ -\frac{25}{107}r + \frac{844}{9737} \end{pmatrix} \end{cases};$$

i.e.

$$\begin{cases} x_1(r) = (\underline{x_1}(r), \overline{x_1}(r)) = (\frac{7}{107}r + \frac{3376}{9737}, -\frac{7}{107}r + \frac{2830}{9737}) \\ x_2(r) = (\underline{x_2}(r), \overline{x_2}(r)) = (\frac{10}{107}r + \frac{2851}{9737}, -\frac{10}{107}r + \frac{4853}{9737}) \\ x_3(r) = (\underline{x_3}(r), \overline{x_3}(r)) = (\frac{25}{107}r - \frac{4161}{9737}, -\frac{25}{107}r + \frac{844}{9737}) \end{cases}$$

which $x_1$, $x_2$, $x_3$ are fuzzy numbers, but $\underline{u_1}(r) = \overline{x_1}(r)$, $\overline{u_1}(r) = \underline{x_1}(r)$, thus the corresponding fuzzy solution is a weak fuzzy solution given by

$$\begin{cases} u_1(r) = (\underline{u_1}(r) \ , \ \overline{u_1}(r)) = (\overline{x_1}(1) \ , \ \underline{x_1}(1)) = (\frac{2193}{9737} \ , \ \frac{4013}{9737}) \\ u_2(r) = (\underline{u_2}(r), \overline{u_2}(r)) = (\underline{x_2}(r), \overline{x_2}(r)) = (\frac{10}{107}r + \frac{2851}{9737}, -\frac{10}{107}r + \frac{4853}{9737}) \\ u_3(r) = (\underline{u_3}(r), \overline{u_3}(r)) = (\underline{x_3}(r), \overline{x_3}(r)) = (\frac{25}{107}r - \frac{4161}{9737}, -\frac{25}{107}r + \frac{844}{9737}). \end{cases}$$

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