# DISTINGUISHABILITY AND COMPLETENESS OF CRISP DETERMINISTIC FUZZY AUTOMATA

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Abstract. In this paper, we introduce and study notions like state-distinguishability, input-distinguishability and output completeness of states of a crisp deterministic fuzzy automaton. We show that for each crisp deterministic fuzzy automaton there corresponds a unique (up to isomorphism), equivalent distinguished crisp deterministic fuzzy automaton. Finally, we introduce two axioms related to output completeness of states and discuss the interrelationship between them.

## 1. Introduction

Algebraic study of automata have been done by many authors in many forms (cf., e.g., [1, 5, 7, 10, 26, 27]). Among these studies, in [1], the concepts like subautomaton, separatedness, connectedness and retrievability of an automaton were studied; the study of distinguishability and completeness of abstract machines has been carried out in [5]; in [7], decompositions and several products of automata were studied; while [10] is towards the recent contribution in this area to determine the structure of an automaton. In [1], it has also been pointed out that the study of such concepts of automata naturally contributes better understanding of the structure of automata and their applications; while in [7], it has been stated that such concepts have arisen from a desire to understand the behaviour of a system in an environment and to play a large role in the development of the fundamentals of computer science.

The algebraic study of fuzzy automata has been initiated by Malik, Mordeson and Sen [20] (cf., [21] for details), and after that a number of works were reported in this direction (cf., e.g., [11, 12, 13, 14, 15, 16]). Similar studies for fuzzy automata with membership values in complete residuated lattice were proposed in [23, 24]. But, interestingly, for fuzzy automata with membership values in lattice-ordered monoid, it is shown that the results discussed in [21] depend on the associated monoid structure (cf., [28, 30]). The recent work done in [11] further enriches the algebraic study of fuzzy automata. Also, the studies done in [2, 3, 17, 19, 31, 32] provide the information about some new and different aspects of fuzzy automata theory.

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Chiefly inspired from [5], in this paper, we study the distinguishability and completeness of a crisp deterministic fuzzy automaton. Specifically, in Section 2, we recall some concepts of a lattice-ordered monoid and introduce the concept of a crisp deterministic fuzzy automaton, while Section 3 is towards the study of indistinguishability of states of a crisp deterministic fuzzy automaton. In Section 4, we introduce the concept of input-indistinguishability of a crisp deterministic fuzzy automaton. Finally, Section 5 deals with output completeness of states of a given crisp deterministic fuzzy automaton.

## 2. Preliminaries

The fuzzy sets considered in this paper are in the sense of [6], i.e., a fuzzy set $\lambda$ on a set $X$ is a map $\lambda : X \to L$, where $L$ is a lattice-ordered monoid. We first recall the following concept of a lattice order monoid given in [6].

**Definition 2.1.** An algebra $L = (L, \leq, \wedge, \vee, \bullet, 0, 1)$ is called a **lattice-ordered monoid** if

(1) $L = (L, \leq, \wedge, \vee, 0, 1)$ is a lattice with the least element 0 and the greatest element 1.
(2) $(L, \bullet, 1)$ is a monoid with identity $1 \in L$ such that for all $a, b, c \in L$.
(i) $a \bullet 0 = 0 \bullet a = 0$,
(ii) $a \leq b \Rightarrow \forall x \in L, a \bullet x \leq b \bullet x$ and $x \bullet a \leq x \bullet b$,
(iii) $a \bullet (b \vee c) = (a \bullet b) \vee (a \bullet c)$ and $(b \vee c) \bullet a = (b \bullet a) \vee (c \bullet a)$.

Throughout, we work with a lattice-ordered monoid $L$ so that the monoid $(L, \bullet, 1)$ satisfies the left cancellation law.

The concept of a fuzzy automaton was studied by different authors in different forms, e.g., cf., the works done in [4, 18, 21, 22, 25]. In this paper, we work with fuzzy automata having deterministic transition function and fuzzy output function, and called them crisp deterministic fuzzy automata. Similar concept of fuzzy automata have already been appeared in the literature (cf., [8, 9, 29]). Unlike the works done in [8, 9, 29], it is to be noted here that the crisp deterministic fuzzy automaton introduced here has no initial state and fuzzy set of final states. Even, in case of crisp automaton, for the study of many aspects of an automaton, the initial state and the set of final states play no role. Accordingly, a modified and simplified version of the notion an automaton is also considered and is still called an automaton (cf., [1, 7]).

**Definition 2.2.** A **crisp deterministic fuzzy automaton** is a 5-tuple $S = (Q, X, Y, \delta, \lambda)$, where

(i) $Q$ is a nonempty set called the **state-set**.
(ii) $X$ and $Y$ are monoids with identity $e$, called the **input set** and the **output set**, respectively.
(iii) $\delta : Q \times X \to Q$ is a map called the **transition map** satisfying
  (a) $\delta(q, e) = q, \forall\ q \in Q$, and
  (b) $\delta(q, w_1 w_2) = \delta(\delta(q, w_1), w_2), \forall\ q \in Q$ and $w_1, w_2 \in X$.

(iv) $\lambda : Q \times X \times Y \to L$ is a map called the **output function** such that

$$\lambda(p, w, u) = \begin{cases} 1 & \text{if } w = u = e, \\ 0 & \text{if either } w \neq e \text{ and } u = e \text{ or } w = e \text{ and } u \neq e, \text{ and} \end{cases}$$

$\lambda(p, w_1w_2, u_1u_2) = \lambda(p, w_1, u_1) \bullet \lambda(\delta(p, w_1), w_2, u_2)$, $\forall\, p \in Q$, $w_1, w_2 \in X$ and $u_1, u_2 \in Y$.

**Definition 2.3.** A crisp deterministic fuzzy automaton $S = (Q, X, Y, \delta, \lambda)$ is called **free** if for each $q \in Q$ and for each $w \in X$ there exists $y \in Y$ such that $\lambda(q, w, y) > 0$.

We now turn to the concept of a subautomaton of a crisp deterministic fuzzy automaton.

**Definition 2.4.** A crisp deterministic fuzzy automaton $T = (R, X, Y, \delta_1, \lambda_1)$ is called a **subautomaton** of a crisp deterministic fuzzy automaton $S = (Q, X, Y, \delta, \lambda)$ if $R \subseteq Q$, $\delta_1 = \delta\mid_{R\times X}$ and $\lambda_1 = \lambda\mid_{R\times X\times Y}$.

**Remark 2.5.** We note that for a given crisp deterministic fuzzy automaton $S = (Q, X, Y, \delta, \lambda)$ and any nonempty $A \subseteq Q, (< A >, X, Y, \delta_A, \lambda)$, where $< A >= \{\delta(q, w) : \text{for some } q \in Q \text{ and } w \in X\}$, $\delta_A :< A > \times X \to< A >$, is the restriction of $\delta$ to $< A > \times X$ and $\lambda_A :< A > \times X \times Y \to L$, is the restriction of $\lambda$ to $< A > \times X \times Y$ is always a subautomaton of $S$. We shall refer to it as the subautomaton of $S$ **generated by** $A$. In case $A = \{q\}$ for some $q \in Q$, then $< A >$ is denoted by $< q >$ and called a **singly generated subautomaton** of $A$.

Finally, we introduce the following concept of a homomorphism between two crisp deterministic fuzzy automata.

**Definition 2.6.** A **homomorphism** from a crisp deterministic fuzzy automaton $S = (Q, X, Y, \delta, \lambda)$ to a crisp deterministic fuzzy automaton $T = (Q', X', Y', \delta', \lambda')$ is a triple $(f, g, h)$, where $f : Q \to Q'$ is a map, $g : X \to X'$ and $h : Y \to Y'$ are monoid homomorphisms such that $f(\delta(q, w)) = \delta'(f(q), g(w))$ and $\lambda(q, w, u) \leq \lambda'(f(q), g(w), h(u))$, $\forall\ q \in Q$, $\forall\ w \in X$ and $\forall\ u \in Y$. Further, a homomorphism $(f, g, h) : S \to T$ is called an **isomorphism** if $f, g$ and $h$ are bijective. Also, a homomorphism $(f, g, h) : S \to T$ is called **widely-isomorphism** if $f$ is injective and $g, h$ are bijective.

## 3. Distinguishability of States

In this section, we introduce and study the notion of 'distinguishability' and 'indistinguishability' of states of a crisp deterministic fuzzy automaton. Interestingly, we show that for each crisp deterministic fuzzy automaton there corresponds a unique (up to isomorphism), equivalent distinguished crisp deterministic fuzzy automaton.

**Definition 3.1.** Let $S = (Q, X, Y, \delta, \lambda)$ and $T = (Q', X, Y, \delta', \lambda')$ be crisp deterministic fuzzy automata. A state $q \in Q$ is said to be **indistinguishable** from state $q' \in Q'$ if $\lambda(q, w, u) = \lambda'(q', w, u)$, $\forall\ w \in X$ and $\forall\ u \in Y$. If $q$ and $q'$ are not indistinguishable, they are said to be **distinguishable**. Further, a crisp deterministic fuzzy automaton $S$ is said to be **indistinguished** if each of two distinct states in $S$ are indistinguishable.

**Example 3.2.** Let $X = Y = \{1, 2, 3, ...\}$. Then $X$ and $Y$ form a monoid w.r.t. multiplication. Also, let $Q = \{1, 2, 3, ...\}$, $Q' = \{-1, -2, -3, ...\}$. Now, consider the crisp deterministic fuzzy automata $S = (Q, X, Y, \delta, \lambda)$ and $T = (Q', X, Y, \delta', \lambda')$, where $\delta : Q \times X \to Q$, $\delta' : Q' \times X \to Q'$ are maps such that $\delta(q, w) = qw$ and $\delta'(q', w) = q'w$, $\forall\ q \in Q, q' \in Q'$ and $w \in X$. Also, $\lambda : Q \times X \times Y \to L$ and $\lambda' : Q' \times X \times Y \to L$ are maps such that $\forall\ q \in Q, q' \in Q', w \in X$ and $\forall\ u \in Y$,

$$\lambda(q, w, u) = \lambda'(q', w, u) = \begin{cases} 1 & \text{if } w = u = e, \\ 0 & \text{if } w \neq e \text{ and } u = e \\ 0 & w = e \text{ and } u \neq e \\ \frac{1}{u} & \text{otherwise.} \end{cases}$$

Then all the states in $S$ are indistinguishable from all of the states in $T$.

**Proposition 3.3.** *Let $S = (Q, X, Y, \delta, \lambda)$ and $T = (Q', X, Y, \delta', \lambda')$ be crisp deterministic fuzzy automata. If state $q \in Q$ is indistinguishable from state $q' \in Q'$, then $\delta(q, w)$ is indistinguishable from $\delta'(q', w)$, $\forall\ w \in X$.*

*Proof.* From the indistinguishability of states $q$ and $q'$, $\lambda(q, w, u) = \lambda'(q', w, u)$, $\forall$ $w \in X$ and $\forall\ u \in Y$. Now, for $w' \in X$ and $u' \in Y$,

$$\begin{aligned} \lambda(q, w, u) \bullet \lambda(\delta(q, w), w', u') &= \lambda(q, ww', uu') \\ &= \lambda'(q', ww', uu') \\ &= \lambda'(q', w, u) \bullet \lambda'(\delta'((q', w), w', u'), \end{aligned}$$

or that, $\lambda(\delta(q, w), w', u') = \lambda'(\delta'(q', w), w', u')$, as $L$ satisfies the left cancellation law. Thus $\delta(q, w)$ is indistinguishable from $\delta'(q', w)$. □

**Definition 3.4.** Two crisp deterministic fuzzy automata $S = (Q, X, Y, \delta, \lambda)$ and $T = (Q', X, Y, \delta', \lambda')$ are said to be **equivalent** if for each state $q \in Q$ there exists a state $q' \in Q'$, which is indistinguishable from $q$ and vice-versa.

**Theorem 3.5.** *To each crisp deterministic fuzzy automaton $S = (Q, X, Y, \delta, \lambda)$ there corresponds a unique (up to isomorphism), distinguished crisp deterministic fuzzy automaton which is equivalent to $S$.*

*Proof.* Define a relation $\sim$ on $Q$ by $q \sim p$ iff $\lambda(q, w, u) = \lambda(p, w, u)$, $\forall\ q, p \in Q$, $\forall\ w \in X$, and $\forall u \in Y$. It is clear that $\sim$ is an equivalence relation on $Q$. Now, let $Q' = Q/\sim = \{[q] : q \in Q\}$, where $[q] = \{p \in Q : q \sim p\}$ and define the maps $\delta' : Q' \times X \to Q'$ and $\lambda' : Q' \times X \times Y \to L$ such that $\delta'([q], w) = [\delta(q, w)]$ and $\lambda'([q], w, u) = \lambda(q, w, u)$ $\forall q \in Q, w \in X$ and $\forall u \in Y$. Both the maps $\delta'$ and $\lambda'$ are well-defined, which are shown as under:
Let $q, p \in Q$ such that $[q] = [p]$, or that, $q \sim p$. Then $q \sim p \Rightarrow \lambda(q, w, u) = \lambda(p, w, u), \forall w \in X, u \in Y$, or that, $\lambda(q, ww_1, uu_1) = \lambda(p, ww_1, uu_1)$, $\forall w, w_1 \in X, u, u_1 \in Y$.

Now, for $w, w_1 \in X$ and $u, u_1 \in Y$,

$$\begin{aligned} \lambda(q, ww_1, uu_1) &= \lambda(p, ww_1, uu_1) \\ \Rightarrow \lambda(q, w, u) \bullet \lambda(\delta(q, w), w_1, u_1) &= \lambda(p, w, u) \bullet \lambda(\delta(p, w), w_1, u_1) \\ \Rightarrow \lambda(\delta(q, w), w_1, u_1) &= \lambda(\delta(p, w), w_1, u_1) \\ \Rightarrow \delta(q, w) &\sim \delta(p, w) \\ \Rightarrow [\delta(q, w)] &= [\delta(p, w)] \\ \Rightarrow \delta'([q], w) &= \delta'([p], w), \end{aligned}$$

whereby $\delta'$ is well-defined. Again, let $q, p \in Q$ such that $[q] = [p]$. Then $q \sim p$. Also,

$$\begin{aligned} q \sim p \ \Rightarrow \lambda(q, w, u) &= \lambda(p, w, u), \forall w \in X, u \in Y \\ \Rightarrow \lambda'([q], w, u) &= \lambda'([p], w, u), \end{aligned}$$

whereby $\lambda'$ is well-defined. Thus $T = (Q', X, Y, \delta', \lambda')$ is a crisp deterministic fuzzy automaton, which is equivalent to $S$.
In order to show the uniqueness (up to isomorphism), let $T' = (Q'', X, Y, \delta'', \lambda'')$ and $T$ be two distinguished crisp deterministic fuzzy automata, which are equivalent to $S$. From indistinguishability, $q'' \in Q''$ implies $\lambda(q, w, u) = \lambda''(q'', w, u)$, $\forall\ w \in X$ and $\forall u \in Y$. Now, define a function $f : T' \to T$ such that $f(q'') = [q]$, $\forall\ q'' \in Q''$. It is to be noted here that $[q]$ is indistinguishable from $q''$. Again, let $q'', p'' \in Q''$. Then $q'' = p''$ iff $\lambda''(q'', w, u) = \lambda''(p'', w, u)$, $\forall\ w \in X$ and $\forall u \in Y$ iff $[q] = [p]$. Thus $f$ is well-defined and one-one. From distinguishedness of $T'$ and $T$, it follows that $f$ is onto. Finally, let $w = w_1w_2 \in X$ and $u = u_1u_2 \in Y$. Then

$$\begin{aligned} f(\delta''(q'', w)) &= \delta'([q], w), \\ &= \delta'(f(q''), w), and \\ \lambda''(q'', w, u) &= \lambda''(q'', w_1w_2, u_1u_2) \\ &= \lambda''(q'', w_1, u_1) \bullet \lambda''(\delta''(q'', w_1), w_2, u_2) \\ &\leq \lambda'(f(q''), w_1, u_1) \bullet \lambda'(\delta'(f(q''), w_1), w_2, u_2) \\ &= \lambda'(f(q''), w_1w_2, u_1u_2) \\ &= \lambda'(f(q''), w, u). \end{aligned}$$

Thus $f$ is a homomorphism. Hence the crisp deterministic fuzzy automaton $T$ is unique (up to isomorphism). □

**Theorem 3.6.** *Let $k$ be a positive integer and for each positive integer $j \leq k$, let $Q_j$ be a countable set of states of the crisp deterministic fuzzy automaton $S_j$ and $R_j$ be a stable subset of $S_j$ generated by $Q_j$. Also, assume that for every finite sequence $\{w_i\}$ of inputs, there exists a set of states $\{q_j : q_j \in Q_j, j \leq k\}$ such that $\lambda_j(q_j, w_1....w_i, u) = \lambda_m(q_m, w_1....w_i, u)$, $\forall j,\ m \leq k$, $\forall i$. Then there exists a set $\{p_j : p_j \in R_j, j \leq k\}$ of states which are pairwise indistinguishable.*

*Proof.* From countability of $Q_j$, the set of all $k$-tuples $\{(q_1, q_2, ..., q_k) : q_j \in Q_j\}$ is countable. We assume that the conclusion of the theorem (the states in set $\{p_j : p_j \in R_j, j \leq k\}$ are pairwise indistinguishable) is false. Then there exists an input $w_1$ and integers $s_1$ and $t_1$ such that $\lambda_{s_1}(p_1^{s_1}, w_1, u) \neq \lambda_{t_1}(p_1^{t_1}, w_1, u)$, $\forall\, w_1 \in X$, and for each integer $i \geq 1$ there exists an input $w_i$ and integers $s_i$ and $t_i$ such that $\lambda_{s_i}(\delta_{s_i}(p_i^{s_i}, w_1...w_{i-1}), w_i, u) \neq \lambda_{t_i}(\delta_{t_i}(p_i^{t_i}, w_1....w_{i-1}), w_i, u)$. Also, $R_j$ being the stable subset of $S_j$, $\delta_j(p_i^j, w_1....w_{i-1})$ is in $R_j$. By hypothesis, for the sequence of inputs $\{w_i\}$, there exists an integer $n$ such that

$$\begin{aligned} \lambda_j(p_n^j, w_1...w_i, u) &= \lambda_m(p_n^m, w_1...w_i, u), \ \forall i, \text{and} \\ \lambda_j(p_n^j, w_1...w_n, u) &= \lambda_m(p_n^m, w_1...w_n, u), \ \text{for } j, m \leq k. \end{aligned}$$

Now, if $n > 1$, then $\lambda_j(\delta_j(p_n^j, w_1...w_{n-1}), w_n, u) = \lambda_m(\delta_m(p_n^m, w_1...w_{n-1}), w_n, u)$. This contradicts our assumption. Again, if $n = 1$, then $\lambda_j(p_1^j, w_1, u) = \lambda_m(p_1^m, w_1, u)$, for $j, m \leq k$, which is again a contradiction. Thus the conclusion of the theorem is true. Hence there exists a set $\{p_j : p_j \in R_j, j \leq k\}$ of states which are pairwise indistinguishable. □

## 4. **Input-distinguishability**

This section is towards the study of input-distinguishability of a crisp deterministic fuzzy automaton.

**Definition 4.1.** Let $S = (Q, X, Y, \delta, \lambda)$ be a crisp deterministic fuzzy automaton. Two inputs $w_1, w_2 \in X$ are said to be **input-indistinguishable** if $\lambda(q, w_1, u) = \lambda(q, w_2, u)$ and $\lambda(q, w_1w, u_1u_2) = \lambda(q, w_2w, u_1u_2)$, $\forall\ q \in Q$, $\forall\ w \in X$ and $\forall\ u, u_1, u_2 \in Y$. Otherwise, $w_1$ and $w_2$ are said to be **input-distinguishable**. Further, a crisp deterministic fuzzy automaton is said to be **input-distinguished** if each of two distinct inputs are input-distinguishable.

**Example 4.2.** Consider the crisp deterministic fuzzy automaton $S = (Q, X, Y, \delta, \lambda)$ of Example 3.2. Then each pair of distinct input from $X$ are input-indistinguishable.

**Proposition 4.3.** *Let $S = (Q, X, Y, \delta, \lambda)$ be a crisp deterministic fuzzy automaton and $w_1, w_2 \in X$ be input-indistinguishable. Then for all $q \in Q$, $\delta(q, w_1)$ and $\delta(q, w_2)$ are indistinguishable states.*

*Proof.* From the input-indistinguishability of $w_1, w_2 \in X$, $\lambda(q, w_1, u) = \lambda(q, w_2, u)$ and $\lambda(q, w_1w, u_1u_2) = \lambda(q, w_2w, u_1u_2)$, $\forall\ q \in Q$, $w \in X$ and $\forall\ u, u_1, u_2 \in Y$. Now,

$$\begin{aligned} \lambda(q, w_1, u_1) \bullet \lambda(\delta(q, w_1), w, u_2) &= \lambda(q, w_1w, u_1u_2) \\ &= \lambda(q, w_2w, u_1u_2) \\ &= \lambda(q, w_2, u_1) \bullet \lambda(\delta(q, w_2), w, u_2), \end{aligned}$$

or that, $\lambda(\delta(q, w_1), w, u_2) = \lambda(\delta(q, w_2), w, u_2)$, as $L$ satisfies the left cancellation law. Thus $\delta(q, w_1)$ and $\delta(q, w_2)$ are indistinguishable states. □

**Proposition 4.4.** *Let $S = (Q, X, Y, \delta, \lambda)$ be a crisp deterministic fuzzy automaton. If $w_1, w_2 \in X$ are input-indistinguishable and $q_1, q_2 \in Q$ are indistinguishable states, then $\delta(q_1, w_1)$ and $\delta(q_2, w_2)$ are indistinguishable states.*

*Proof.* From the input-indistinguishability of $w_1, w_2 \in X$, $\lambda(q, w_1, u) = \lambda(q, w_2, u)$ and $\lambda(q, w_1w, u_1u_2) = \lambda(q, w_2w, u_1u_2)$, $\forall q \in Q$, $w \in X$ and $u, u_1, u_2 \in Y$. Also, as the states $q_1$ and $q_2$ are indistinguishable, $\lambda(q_1, w, u) = \lambda(q_2, w, u)$, $\forall q_1, q_2 \in Q$, $\forall$ $w \in X$ and $\forall$ $u \in Y$. Now,

$$\begin{aligned}
\lambda(q_1, w_1, u_1) \bullet \lambda(\delta(q_1, w_1), w, u_2)) &= \lambda(q_1, w_1w, u_1u_2) \\
&= \lambda(q_1, w_2w, u_1u_2) \\
&= \lambda(q_2, w_2w, u_1u_2) \\
&= \lambda(q_2, w_2, u_1) \bullet \lambda(\delta(q_2, w_2), w, u_2) \\
&= \lambda(q_2, w_1, u_1) \bullet \lambda(\delta(q_2, w_2), w, u_2) \\
&= \lambda(q_1, w_1, u_1) \bullet \lambda(\delta(q_2, w_2), w, u_2),
\end{aligned}$$

or that, $\lambda(\delta(q_1, w_1), w, u_2) = \lambda(\delta(q_2, w_2), w, u_2)$, $\forall w \in X$ and $\forall u_2 \in Y$, as $L$ satisfies the left cancellation law. Thus $\delta(q_1, w_1)$ and $\delta(q_2, w_2)$ are indistinguishable states. □

**Proposition 4.5.** *Let $S = (Q, X, Y, \delta, \lambda)$ be a crisp deterministic fuzzy automaton. If $w_1$ and $w_2$, $w_3$ and $w_4$ are input-indistinguishable, then $w_1w_3$ and $w_2w_4$ are input-indistinguishable.*

*Proof.* From the input-indistinguishability of $w_1, w_2 \in X$, $\lambda(q, w_1, u) = \lambda(q, w_2, u)$ and $\lambda(q, w_1w, u_1u_2) = \lambda(q, w_2w, u_1u_2)$, $\forall q \in Q$, $w \in X$ and $u, u_1, u_2 \in Y$. Also, the states $\delta(q, w_1)$ and $\delta(q, w_2)$ are indistinguishable states (cf., Proposition 4.3). Now, from the input-indistinguishable of $w_3$ and $w_4$,

$$\begin{aligned}
\lambda(\delta(q, w_1), w_3, u_2) &= \lambda(\delta(q, w_2), w_3, u_2) \\
&= \lambda(\delta(q, w_2), w_4, u_2).
\end{aligned}$$

Also,

$$\begin{aligned}
\lambda(q, w_1w_3, u_1u_2) &= \lambda(q, w_1, u_1) \bullet \lambda(\delta(q, w_1), w_3, u_2) \\
&= \lambda(q, w_1, u_1) \bullet \lambda(\delta(q, w_2), w_3, u_2) \\
&= \lambda(q, w_2, u_1) \bullet \lambda(\delta(q, w_2), w_4, u_2) \\
&= \lambda(q, w_2w_4, u_1u_2).
\end{aligned}$$

Again, $\delta(q, w_1w_3) = \delta(\delta(q, w_1), w_3)$ and $\delta(\delta(q, w_2), w_4) = \delta(q, w_2w_4)$ are indistinguishable states. Thus

$$\begin{aligned}
\lambda(q, w_1w_3w, u_1u_2u) &= \lambda(q, w_1w_3, u_1u_2) \bullet \lambda(\delta(q, w_1w_3), w, u) \\
&= \lambda(q, w_2w_4, u_1u_2) \bullet \lambda(\delta(q, w_2w_4), w, u) \\
&= \lambda(q, w_2w_4w, u_1u_2u).
\end{aligned}$$

Hence $w_1w_3$ and $w_2w_4$ are input-indistinguishable. □

**Theorem 4.6.** *To each crisp deterministic fuzzy automaton $S = (Q, X, Y, \delta, \lambda)$ there corresponds an input-distinguished crisp deterministic fuzzy automaton $T = (Q', X', Y', \delta', \lambda')$ such that*

(i) *$Q = Q'$ and $Y = Y'$.*

(ii) *There exists a homomorphism $k$ of $X$ onto $X'$ such that $\lambda(q, w, u) = \lambda'(q, k(w), u)$, $\forall$ $w \in X$ and $\forall q \in Q$.*

(iii) *If $S$ is distinguished, then any input-distinguished crisp deterministic fuzzy automaton $T''$ satisfying (i) and (ii) is widely isomorphic to $T$.*

*Proof.* Define a relation $\sim$ on $X$ by $w_1 \sim w_2$ iff $\lambda(q, w_1, u) = \lambda(q, w_2, u)$ and $\lambda(q, w_1 w, u) = \lambda(q, w_2 w, u)$, $\forall q \in Q$, $w \in X$ and $\forall u \in Y$. It is clear that $\sim$ is an equivalence relation on $X$. Now, let $X' = \{[w'] : w \in X\}$, where $[w'] = \{w \in X : w \sim w'\}$ and define the maps $\delta' : Q \times X' \to Q$ and $\lambda' : Q \times X' \times Y \to L$ such that $\delta'(q, [w']) = [\delta(q, w)]\ \forall\ q \in Q, [w'] \in X$, where $w \in [w']$ and $\lambda'(q, [w'], u) = \lambda(q, w, u)$, $q \in Q\ \forall\ [w'] \in X'$, $\forall\ u \in Y$, where $w \in [w']$. It is trivial to show that the inputs $w$ and $[w']$ are input - indistinguishable and both maps $\delta'$ and $\lambda'$ are well-defined. In order to show that $T$ define a crisp deterministic fuzzy automaton, it is enough to show that the conditions $(iii)$ and $(iv)$ of Definition 2.2 hold. Now, $\delta'(q, [w'_1 w'_2]) = [\delta(q, w_1 w_2)]$. Thus $\delta'(\delta'(q, [w'_1]), [w'_2]) = [\delta(\delta(q, w_1), w_2)]$, as $\delta(\delta(q, w_1), w_2) = [\delta(q, w_1 w_2)]$, whereby $\delta'(q, [w'_1][w'_2]) = \delta'(\delta'(q, [w'_1]), [w'_2])$. Also,

$$
\begin{aligned}
\lambda'(q, [w'_1 w'_2], u_1 u_2) &= \lambda'(q, [w'_1 w'_2], u_1 u_2) \\
&= \lambda(q, w_1 w_2, u_1 u_2) \\
&= \lambda(q, w_1, u_1) \bullet \lambda([\delta(q, w_1)], w_2, u_2) \\
&= \lambda'(q, [w'_1], u_1) \bullet \lambda(\delta'(q, [w'_1]), w_2, u_2), \\
&= \lambda'(q, [w'_1], u_1) \bullet \lambda'(\delta'(q, [w'_1]), [w'_2], u_2),
\end{aligned}
$$

which follows from the fact that $[\delta(q, w_1)]$ and $\delta'(q, [w'_1])$ are indistinguishable states. Now, define a function $k : X \to X'$ such that $k(w) = [w']$, $\forall\ w \in X$. Again, let $w_1, w_2 \in X$. Then $w_1 = w_2$ iff $\lambda(q, w_1, u) = \lambda(q, w_2, u)$, $\forall\ q \in Q$ and $\forall u \in Y$ iff $[w'_1] = [w'_2]$. Thus $k$ is well-defined and one-one. From input-distinguishability of $S$ and $T$, it follows that $k$ is onto, whereby

$$
\begin{aligned}
k([\delta(q, w)]) &= \delta'(q, [w']) \\
&= \delta'(q, k(w)), \text{and} \\
\lambda(q, w, u) &= \lambda'(q, [w'], u) \\
&= \lambda'(q, k(w), u).
\end{aligned}
$$

Thus $k$ is a homomorphism such that $\lambda(q, w, u) = \lambda'(q, k(w), u)$, $\forall\ w \in X$ and $q \in Q$. We now assume that $S$ is a distinguished crisp deterministic fuzzy automaton and $T'' = (Q'', X'', Y'', \delta'', \lambda'')$ is an input-distinguished crisp deterministic fuzzy automaton satisfying the given conditions $(i)$ and $(ii)$. For, $w_1, t_1 \in [w'_1]$, let $k''$ be a homomorphism from $X$ onto $X''$. Then

$$
\begin{aligned}
\lambda''(q, k''(w_1), u) &= \lambda(q, w_1, u) \\
&= \lambda(q, t_1, u) \\
&= \lambda''(q, k''(t_1), u), \text{and}
\end{aligned}
$$

$$
\begin{aligned}
\lambda''(q, k''(w_1)k''(w), u_1 u_2) &= \lambda(q, w_1 w, u_1 u_2) \\
&= \lambda(q, t_1 w, u_1 u_2) \\
&= \lambda''(q, k''(t_1)k''(w), u_1 u_2),
\end{aligned}
$$

or that, $k''(w_1) = k''(t_1)$. Now, define $h([w'_1]) = k''(w_1)$. Then $h$ is well-defined function from $X'$ onto $X''$. From input-distinguishability of crisp deterministic fuzzy automaton $S$, $\lambda(q, w_1, u) \neq \lambda(q, w_2, u)$ and $\lambda(q, w_1 w, u_1 u_2) \neq \lambda(q, w_2 w, u_1 u_2)$, or that, $\lambda''(q, h([w'_1]), u) \neq \lambda''(q, h([w'_2]), u)$ and $\lambda''(q, h([w'_1])h([w']), u_1 u_2) \neq \lambda''(q, h([w'_2])h([w']), u_1 u_2)$. Thus $h$ is injective. As $k''$ is a homomorphism of $X$ onto $X''$, $h([w'_1 w'_2]) = k''(w_1 w_2) = k''(w_1)k''(w_2) = h([w'_1])h([w'_2])$, whereby $h$ is an isomorphism of $X'$ onto $X''$. Finally, in order to show that $T$ and $T''$ are widely isomorphic, it is enough to show that $\delta'(q, [w'_1]) = \delta''(q, h([w'_1]))$, $\forall q \in Q$ and $\forall [w'_1] \in X'$. For which, let $q \in Q, [w'] \in X', u_1 \in Y$. Then

$$\begin{aligned}
\lambda'(q, [w'_1], u_1) \bullet \lambda'(\delta'(q, [w'_1]), [w'], u_2) \\
&= \lambda'(q, [w'_1][w'], u_1 u_2) \\
&= \lambda''(q, h([w'_1])h([w']), u_1 u_2) \\
&= \lambda''(q, h([w'_1]), u_1) \bullet \lambda''(\delta''(q, h([w'_1])), h([w']), u_2),
\end{aligned}$$

or that, $\lambda'(\delta'(q, [w'_1]), [w'], u_2) = \lambda''(\delta''(q, h([w'_1])), h([w']), u_2)$, $\forall\, [w'] \in X'$, whereby $\delta'(q, [w'_1]) = \delta''(q, h([w'_1]))$. Hence the crisp deterministic fuzzy automaton $T''$ is widely isomorphic to $T$. □

## 5. **Output Complete States**

In this section, we introduce and study the output completeness of a crisp deterministic fuzzy automaton. We introduce two axioms for output completeness and established some results interrelating them.

**Definition 5.1.** For a crisp deterministic fuzzy automaton $S = (Q, X, Y, \delta, \lambda)$, $q \in Q$ is called **output complete** if for all $u \in Y$ there exists $w \in X$ such that $\lambda(q, w, u) > 0$. Also, if each state of $S$ is output complete, then $S$ is called output complete.

**Definition 5.2.** A crisp deterministic fuzzy automaton $S = (Q, X, Y, \delta, \lambda)$ is said to satisfy

• the $OC_1$-axiom, if $\lambda(q, w, u_1 u_2) > 0$, for $q \in Q, w \in X$ and $u_1, u_2 \in Y$ then there exists $w_1, w_2 \in X$ such that $w = w_1 w_2$, $\lambda(q, w_1, u_1) > 0$ and $\lambda(\delta(q, w_1), w_2, u_2) > 0$. Further, a subautomaton $T = (Q', X', Y', \delta', \lambda')$ of $S$ is said to satisfy

• the $OC_2$-axiom, if for all $q \in Q'$ there exists $u' \in Y'$ such that $\lambda(q, w', u') = 0$, $\forall\, w' \in X'$ and if for $p \in Q$ there exists an input $w \in X'$ such that $\delta(p, w) = q$; then $\lambda(p, w, uu') = 0$, $\forall\, w \in X'$ provided $\lambda(p, w, u) > 0$.

**Proposition 5.3.** *Let $S = (Q, X, Y, \delta, \lambda)$ be a crisp deterministic fuzzy automaton satisfy the $OC_1$-axiom and $p$ be an output complete state. For given $w, w_0 \in X$ if there exists $u \in Y$ such that $\lambda(p, w, u) > 0$ and $\lambda(p, w_0, u) > 0$, then $q = \delta(p, w_0) = \delta(p, w)$. Then $q$ is also output complete.*

*Proof.* Let $p$ be output complete. Then there exists $w \in X$ such that $\lambda(p, w, u_0 u) > 0$ for all $u, u_0 \in Y$. Now, as $S$ satisfies the $OC_1$-axiom, $\exists w_1, w_2 \in X$ such that $w = w_1 w_2$, $\lambda(p, w_1, u_0) > 0$ and $\lambda(\delta(p, w_1), w_2, u) > 0$. Again, from output completeness of $p$, there exists $w_0 \in X$ such that $\lambda(p, w_0, u_0) > 0$. Thus $\lambda(p, w_0, u_0) > 0$ and

$\lambda(p, w_1, u_0) > 0$, whereby $q = \delta(p, w_0) = \delta(p, w_1)$, or that, $\lambda(q, w_2, u) > 0$. Hence $q$ is output complete. □

**Proposition 5.4.** *Let $S = (Q, X, Y, \delta, \lambda)$ be a crisp deterministic fuzzy automaton and $p$ be an output complete state. Also, let $T =< p >$ be the subautomaton satisfies the $OC_2$-axiom. Then $T$ is output complete.*

*Proof.* Let, if possible, $T$ be not output complete. Then there exists $q \in T =< p >$, which is not output complete. Now, $q \in T =< p >$ implies that there exists $w \in X$ such that $\delta(p, w) = q$ and as $q$ is not output complete, there exists $u \in Y$ such that $\lambda(q, w, u) = 0$, $\forall\ w \in X$. Also, from the output completeness of $p$ there exists $w \in X$ such that $\lambda(p, w, v) > 0$, or that, $\lambda(p, w, vu) = 0$, $\forall w \in X$, as $T$ satisfies the $OC_2$-axiom, which contradict the fact that $p$ is output complete. Hence $q$ is output complete, or that, $T$ is output complete. □

**Proposition 5.5.** *Let $S = (Q, X, Y, \delta, \lambda)$ be a crisp deterministic fuzzy automaton satisfy the $OC_1$-axiom. Also, let for all $q \in Q$ and for all $w_1, w_2 \in X$, $\delta(q, w_1) = \delta(q, w_2)$, whenever $\lambda(q, w_1, u) > 0$ and $\lambda(q, w_2, u) > 0$ for some $u \in Y$. Then $S$ satisfies the $OC_2$-axiom.*

*Proof.* Let, if possible, $S$ does not satisfy the $OC_2$-axiom. Then for all $q \in Q$ there exists $u' \in Y$ such that $\lambda(q, w', u') = 0$, $\forall\ w' \in X$ and if for $p \in Q$ there exists an input $w \in X$ such that $\delta(p, w) = q$; then $\lambda(p, w, uu') > 0$, $\forall\ w \in X$ provided $\lambda(p, w, u) > 0$. Now, as $S$ satisfies the $OC_1$- axiom, there exists $w_1, w_2 \in X$ such that $w = w_1w_2$, $\lambda(p, w_1, u) > 0$ and $\lambda(\delta(p, w_1), w_2, u') > 0$. As $\lambda(p, w_1, u) > 0$ and $\lambda(p, w, u) > 0$, $\delta(p, w_1) = \delta(p, w) = q$. Thus from Proposition 5.3, $q$ is output complete, which is a contradiction. Hence $S$ satisfies the $OC_2$-axiom. □

**Theorem 5.6.** *Let $S = (Q, X, Y, \delta, \lambda)$ be a free crisp deterministic fuzzy automaton and $p$ be a state which satisfies the following properties:*

(i) *For each finite sequence $\{u_r\}$ of outputs of length $n$ or less, there exists a finite sequence $\{w_r\}$ of inputs such that $\lambda(p, w_1...w_r, u_1...u_r) > 0$.*
(ii) *The subautomaton $T =< p >$ has $n$ states and satisfies the $OC_2$-axiom.*

*Then $T$ is output complete.*

*Proof.* We will prove the result by induction on the set of states. For $n = 1$, the result follows from Proposition 5.4. Now, let the $OC_2$-axiom be true $\forall\ j \leq k$, where $n \leq k$ and $u_1, ..., u_{k+1} \in Y$ such that $\delta(p, w_1...w_k) = q'$, $w_1...w_k \in X$ provided $\lambda(p, w_1...w_k, u_1...u_k) > 0$. Also, assume that $\lambda(q', w, u_{k+1}) = 0$, $\forall w \in X$. Then if $q' = p$, $\lambda(q', w, u_1) > 0$, $\forall\ w \in X$. Thus $q' \neq p$. Further, as $S$ is a free crisp deterministic fuzzy automaton having $n$ states, there exists $w \in X$ such that $\delta(p, w) = q'$ and $\exists\ w'_1...w'_r : r \leq (n-1)$ such that $\delta(p, w'_1...w'_r) = q'$ and $\lambda(p, w'_1...w'_r, u'_1...u'_r) > 0$. Thus from $(ii)$, $\lambda(p, w, u'_1...u'_k u_{k+1}) = 0$, but it contradicts $(i)$ as $u'_1...u'_k u_{k+1}$ of length $\leq n$, whereby there exists $w_{k+1}$ such that $\lambda(q', w_{k+1}, u_{k+1}) > 0$. Now, $\lambda(p, w_1...w_k w_{k+1}, u_1...u_k u_{k+1}) = \lambda(p, w_1...w_k, u_1...u_k) \bullet \lambda(\delta(p, w_1...w_k), w_{k+1}, u_{k+1})$ $= \lambda(p, w_1...w_k, u_1...u_k) \bullet \lambda(q', w_{k+1}, u_{k+1})$. Also, as $\lambda(p, w_1...w_k, u_1...u_k) > 0$ and $\lambda(q', w_{k+1}, u_{k+1}) > 0$, $\lambda(p, w_1...w_k w_{k+1}, u_1...u_k u_{k+1}) > 0$, whereby $p$ is output complete, or that, $T$ is output complete. □

## 6. Conclusion

This paper is an algebraic study of a crisp deterministic fuzzy automaton. We have tried to study the concept of distinguishability and completeness of a crisp deterministic fuzzy automaton to enrich the algebraic study. It will be interesting to study such concepts for a fuzzy automaton, which we will try in near future.

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