BATHTUB HAZARD RATE DISTRIBUTIONS AND FUZZY LIFE TIMES

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Abstract. The development of life time analysis started back in the 20^{th} century and since then comprehensive developments have been made to model life time data efficiently. Recent development in measurements shows that all continuous measurements can not be measured as precise numbers but they are more or less fuzzy. Life time is also a continuous phenomenon, and has already been shown that life time observations are not precise measurements but fuzzy. Therefore, the corresponding analysis techniques employed on the data require to consider fuzziness of the observations to obtain appropriate estimates.

In this study generalized estimators for the parameters and hazard rates are proposed for bathtub failure rate distributions to model fuzzy life time data effectively.

1. Introduction

Life time can generally be defined as, "the time to the occurrence of a specified event". For instance, in biomedical sciences, the specified event may be death or recovery of a patient from an illness, failure of equipment in engineering sciences, learning of skills in art, change of residence in demography etc.

Main emphasis of life time analysis is, to estimate mean survival time, to quantify the prognostic factors related to life time of units, and to compare the survival functions. Models used for survival time are usually termed as "time-to event models" [15].

The analysis techniques of life time data can be traced back centuries but the rapid development started about seven decades ago, especially World War II stimulated interest in the reliability of military equipments [17].

A significant number of books and research papers have already been published for the analysis of life time data, including reliability analysis, e.g. [8], [16], [11], [12], [18], [6], [5].

These publications include many parametric and non-parametric approaches to model life time data. In recent years some probability density functions are developed to model life time data having bathtub shape failure rate.

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A new model for lifetime observations with bathtub shaped failure rate was proposed by [7] with probability density function

$$f(t) = \frac{1+2\alpha}{2t_0\sqrt{\alpha^2+(1+2\alpha)t/t_0}} \quad \text{with} \quad 0 \le t \le t_0 \quad \text{and} \; -1/2 < \alpha < \infty$$

having failure rate

$$h(t) = \frac{1+2\alpha}{(2t_0\sqrt{\alpha^2+(1+2\alpha)t/t_0})(1+\alpha-\sqrt{\alpha^2+(1+2\alpha)t/t_0})} \quad \text{for} \quad 0 \le t \le t_0.$$

The failure rate of the given distribution is bathtub shaped for $-1/3 < \alpha < 1$. For precise life time observations t_1, t_2, ..., t_n and assuming that
$t_{(1)} < t_{(2)} < ... < t_{(n)}$, maximum likelihood estimators of the corresponding parameters are

$$\hat{t}_0 = t_{(n)}, \tag{1}$$

and by solving

$$\frac{2}{1+2\hat{\alpha}} - \frac{1}{n}\sum_{i=1}^{n}\frac{\hat{\alpha}+t_{(i)}/\hat{t}_0}{\hat{\alpha}^2+(1+2\hat{\alpha})t_{(i)}/\hat{t}_0} = 0 \tag{2}$$

$\hat{\alpha}$ can be obtained.

Another new two-parameter lifetime distribution having bathtub hazard rate was proposed by [4] with probability density function

$$f(t) = \lambda\beta e^{\lambda(1-e^{t^\beta})}e^{t^\beta}t^{\beta-1} \qquad \forall t > 0$$

having failure rate

$$h(t) = \lambda\beta e^{t^\beta}t^{\beta-1} \qquad \forall t > 0.$$

For precise life time observations t_1, t_2, ..., t_n and assuming that
$t_{(1)} < t_{(2)} < ... < t_{(r)}$ are the first r order statistics where $r \le n$ then the maximum likelihood estimator $\hat{\beta}$ of the parameter β is the solution of the following equation:

$$\frac{r}{\beta} + \sum_{i=1}^{r}\ln t_{(i)} + \sum_{i=1}^{r}\left(t_{(i)}^{\beta}\ln t_{(i)}\right) - \frac{r\left[\sum_{i=1}^{r}\left(e^{t_{(i)}^{\beta}}t_{(i)}^{\beta}\ln t_{(i)}\right) + (n-r)\left(e^{t_{(r)}^{\beta}}t_{(r)}^{\beta}\ln t_{(r)}\right)\right]}{\sum_{i=1}^{r}e^{t_{(i)}^{\beta}} - n - (n-r)e^{t_{(r)}^{\beta}}} = 0 \tag{3}$$

and

$$\hat{\lambda} = \frac{r}{\sum_{i=1}^{r}e^{t_{(i)}^{\hat{\beta}}} - n - (n-r)e^{t_{(r)}^{\hat{\beta}}}}. \tag{4}$$

Both life time distributions are based on precise life time observations, while in practical applications life time cannot be measured as precise number but more or less fuzzy. About the exact measurements of continuous variables [2] wrote that, in modern technologic world of measurements the words like "exactness" or "equality" need to be banned. In support of his argument he said that even on the measurement obtained from a high quality instrument one can only believe that it is exact, but the characteristic exact cannot be obtained in reality. It can be confirmed by a simple practical of repeating measurement of some continuous phenomenon.

This also leads to the situation that in fact there are two types of uncertainty in the measurements, one is variation among the observations and another is imprecision of the observation called *fuzziness* [23].

For centuries classical statistical tools are developed to model variation among the observations, without considering the imprecision of single observations. Keeping in view the importance of imprecision Zadeh was the first person who wrote on the idea of *fuzzy sets* in [25]. He pointed out that in the physical world many quantities have not precise values but are more or less fuzzy.

Afterward dealing with fuzziness some work has been done like, fuzzy reasoning in information, decision and control systems [19], fuzzy set theory - and its applications [26], first course on fuzzy theory and applications [14], fundamentals of statistics with fuzzy data [10], statistical methods for fuzzy data [22].

Some work has been done with fuzzy life time data like, fuzzy Bayesian estimation on life time data [24], Bayesian reliability analysis for fuzzy lifetime data [9], on reliability estimation based on fuzzy lifetime data [21], reliability estimation in Rayleigh distribution for fuzzy lifetime data [1], Random fuzzy sets: a mathematical tool to develop statistical fuzzy data analysis [3], but still in most of the publications it is ignored.

In [21], it has been shown that life time observations are not precise numbers but fuzzy, therefore life time analysis techniques should be generalized to deal with fuzziness of the observations.

According to [22] some concepts of fuzzy theory are explained below.

Definition 1.1. (Fuzzy Numbers) Let t^* be an element of the system of special fuzzy subsets of $\mathbb{R}$, a so-called *fuzzy number* determined by the so-called *characterizing function* $\xi(\cdot)$, which is a real valued function of one real variable satisfying the following three conditions:

(1) $\xi : \mathbb{R} \to [0\,,\,1]$
(2) $\xi(\cdot)$ has bounded support, i.e.
$supp[\xi(\cdot)] := [t \in \mathbb{R} : \xi\,(t) > 0\,] \subseteq [a\,,\,b]$ with $-\infty < a < b < \infty$
(3) The so-called δ-cut, i.e. $C_\delta(t^*) := \{t \in \mathbb{R} : \xi\,(t) \geq \delta\} \quad \forall \delta \in (0\,,\,1]$ represents a finite union of non-empty and compact intervals, i.e.
$C_\delta(t^*) = \bigcup_{j=1}^{k_\delta} [\underline{t}_{j,\delta}\,,\,\bar{t}_{j,\delta}] \neq \emptyset$

Special case: If all δ-cuts of a fuzzy number are closed bounded intervals, the corresponding fuzzy number is called *fuzzy interval*.

Lemma 1.2. *For any characterizing function of a fuzzy number t^* the following holds true*

$$\xi\,(t) = \max\left\{\delta \cdot \mathbb{1}_{C_\delta(t^*)}(t) : \; \delta \in [0\,,\,1]\right\} \qquad \forall\, t \in \mathbb{R}.$$

For the proof see [22]

Definition 1.3. (Nested Intervals) A family of intervals $\Big(I_\delta;\ \delta \in (0\,,\,1]\Big)$ is called nested if $I_{\delta_1} \subseteq I_{\delta_2}$ for all $\delta_1 > \delta_2$.

Remark 1.4. If $\left(A_\delta\,;\ \delta\ \in (0\,,\,1]\right)$ is a nested family of finite unions of compact intervals it should be noted that not all such families are the δ-cuts of a fuzzy number. But the following construction lemma holds:

Lemma 1.5. *(Construction Lemma) Let* $\left(A_\delta\,;\ \delta\ \in (0\,,\,1]\right)$ *with* $A_\delta = \bigcup_{j=1}^{k_\delta} [a_{j,\delta}\ ,\ b_{j,\delta}]$ *be a nested family of non-empty subsets of* $\mathbb{R}$*. Then the characterizing function of the generated fuzzy number is given by*

$$\xi(t) = \sup\{\delta\cdot \mathbb{1}_{A_\delta}(t) :\ \delta \in [0\,,\,1]\} \qquad \forall\, t \in \mathbb{R}.$$

For the proof see [20].

Definition 1.6. (Extension Principle) This is extension of an arbitrary function G: $M \to N$ for fuzzy argument value c^* in M. Let c^* be a fuzzy element of M with membership function $\nu:\ M \to [0\,,\,1]$, then the fuzzy value $z^*\ =\ G(c^*)$ is the fuzzy element z^* in N, whose membership function $\eta(\cdot)$ is defined by its values

$$\eta(z) := \left\{\begin{matrix} \sup\{\nu(c) : c \in M,\ G(c) = z\} & \text{if}\,\exists c : G(c) = z \\ 0 & \text{if}\,\nexists c : G(c) = z \end{matrix}\right\} \quad \forall\ z \in N$$

compare [13].

Definition 1.7. (Fuzzy Vectors) A so-called *n-dimensional fuzzy vector* $\underline{t}^*$ is determined by a real valued function of n real variables. It is determined by its so-called *vector-characterizing function* $\zeta(.,\ ...,\ .)$, which must obey the following three conditions:

(1) $\zeta:\ \mathbb{R}^n \to [0\,,\,1]$
(2) The support of $\zeta(.,\ ...,\ .)$ is a bounded set
(3) For all $\delta \in (0\,,\,1]$ the so-called δ-cut $C_\delta(\underline{t}^*) :=\ \{\underline{t} \in \mathbb{R}^n : \zeta(\underline{t}) \geq \delta\}$ is non-empty, bounded, and a finite union of simply connected and closed bounded sets

If all the δ-cuts of a n-dimensional fuzzy vector are simply connected compact sets, then the corresponding n-dimensional fuzzy vector is called *n-dimensional fuzzy interval*.

Definition 1.8. (Minimum t-norm) To form fuzzy vectors from fuzzy samples usually the minimum t-norm is applied, which is defined as

$$T(a_1\,,\,a_2) = \min\{a_1\,,\,a_2\} \quad \forall a_1\,,\,a_2\ \in [0\,,\,1].$$

This operation is associative, and therefore it can be generalized to n arguments.

Let T be a stochastic quantity representing life time, having an observation space $M_T \subseteq [0\,,\,\infty)$. For a classical sample of size n of T, i.e. $(t_1,\ t_2,\ ...,\ t_n)$, each t_i, $i = 1(1)n$ is an element of the observation space M_T. While the sample is an element of the so-called *sample space*.

The sample space is the Cartesian product of the n copies of the observation space M_T denoted by M_T^n, i.e. $M_T^n = M_T \times M_T \times ... \times M_T$.

While dealing with fuzzy observations the situation is different, i.e. for a fuzzy sample, i.e. $(t_1^*,\ t_2^*,\ ...,\ t_n^*)$ with corresponding characterizing functions $\xi_i(\cdot)$, $i =$

$1(1)n$, each t_i^*, $i = 1(1)n$ is a fuzzy element of the observation space M_T, but a fuzzy sample is not a fuzzy element of the sample space M_T^n.

To obtain a fuzzy element of the sample space M_X^n from a fuzzy sample t_1^*, t_2^*, ..., t_n^* with corresponding characterizing functions $\xi_1(\cdot)$, $\xi_2(\cdot)$, ..., $\xi_n(\cdot)$ respectively, the minimum t-norm from fuzzy set theory is applied. The fuzziness of observations does not decrease for increasing the number of observations. Therefore the minimum t-norm is used.

From a fuzzy sample, the vector-characterizing function $\zeta(., ..., .)$ of the so-called *combined fuzzy sample* $\underline{t}^*$ is obtained by its values, i.e.

$$\zeta(t_1, t_2, ..., t_n) = \min\{\xi_1(t_1), \xi_2(t_2), ..., \xi_n(t_n)\} \quad \forall (t_1, t_2, ..., t_n) \in \mathbb{R}^n,$$

by this a fuzzy subset of M_{X^n} is obtained.

The corresponding δ-cuts of $\underline{t}^*$ are obtained as the Cartesian products of the δ-cuts, i.e.

$$C_\delta\left[\zeta(., ..., .)\right] = C_\delta(t_1^*) \times C_\delta(t_2^*) \times ... \times C_\delta(t_n^*) \qquad \forall \delta \in (0\,,\,1]$$

see [22].

In [21] it has already been shown that life time observations are not precise numbers but more or less fuzzy, therefore the corresponding estimators and hazard function of the bathtub failure rate distribution need to be generalized for fuzzy data.

2. **Estimation for Fuzzy Life Times**

Let t_1^*, t_2^*, ..., t_n^* are denoting fuzzy life time observations with corresponding δ-cuts $C_\delta(t_i^*) = \left[\underline{t}_{i,\delta}\,,\,\overline{t}_{i,\delta}\right] \quad \forall \delta \in (0\,,\,1]$.

The estimators explained in the introduction need to be generalized for fuzzy life time observations.

Definition 2.1. (Estimation of Haupt and Schabe estimators for fuzzy life times) The estimators explained in equation (1) and (2) from the introduction are generalized in the following way:

The generalized (fuzzy) estimators for the parameters are written as $\hat{t}_0^*$ and $\hat{\alpha}^*$.

The generating families of intervals for the fuzzy estimators $\hat{t}_0^*$ and $\hat{\alpha}^*$ are denoted as $(A_\delta(\hat{t}_0^*);\ \forall \delta \in (0\,,\,1])$ and $(B_\delta(\hat{\alpha}^*);\ \forall \delta \in (0\,,\,1])$, respectively.

Denoting by $\underline{t}_{(n),\delta}$ and $\overline{t}_{(n),\delta}$ the lower and upper ends of the δ-cut of the largest order statistic respectively, the generating family for the fuzzy estimate $\hat{t}_0^*$ is given by the following equations

$$A_\delta(\hat{t}_0^*) = \left[\underline{t}_{(n),\delta}\,,\,\overline{t}_{(n),\delta}\right] \quad \forall \delta \in (0\,,\,1],$$

For the lower and upper ends of the generating family of the fuzzy estimator $\hat{\alpha}^*$, the estimator given in equation (2) is generalized in the following way:

Taking the solutions $\alpha_{1,\delta}$, $\alpha_{2,\delta}$, $\alpha_{3,\delta}$, and $\alpha_{4,\delta}$ of the following equations:

for $\alpha_{1,\delta}$: $\quad \frac{2}{1+2\alpha} - \frac{1}{n}\sum_{i=1}^{n} \frac{\alpha+\underline{t}_{(i),\delta}/\underline{t}_{0,\delta}}{\alpha^2+(1+2\alpha)\overline{t}_{(i),\delta}/\underline{t}_{0,\delta}} = 0$

for $\alpha_{2,\delta}$: $\quad \frac{2}{1+2\alpha} - \frac{1}{n}\sum_{i=1}^{n} \frac{\alpha+\overline{t}_{(i),\delta}/\overline{t}_{0,\delta}}{\alpha^2+(1+2\alpha)\underline{t}_{(i),\delta}/\overline{t}_{0,\delta}} = 0$

for $\alpha_{3,\delta}$: $\frac{2}{1+2\alpha} - \frac{1}{n}\sum_{i=1}^{n} \frac{\alpha + \underline{t}_{(i),\delta}/\overline{t}_{0,\delta}}{\alpha^2+(1+2\alpha)\underline{t}_{(i),\delta}/\overline{t}_{0,\delta}} = 0$

for $\alpha_{4,\delta}$: $\frac{2}{1+2\alpha} - \frac{1}{n}\sum_{i=1}^{n} \frac{\alpha + \overline{t}_{(i),\delta}/\underline{t}_{0,\delta}}{\alpha^2+(1+2\alpha)\overline{t}_{(i),\delta}/\underline{t}_{0,\delta}} = 0$

and defining

$$\underline{\alpha}_\delta = \min\left[\alpha_{1,\delta},\ \alpha_{2,\delta},\ \alpha_{3,\delta},\ \alpha_{4,\delta}\right]$$

$$\overline{\alpha}_\delta = \max\left[\alpha_{1,\delta},\ \alpha_{2,\delta},\ \alpha_{3,\delta},\ \alpha_{4,\delta}\right],$$

$$B_\delta(\hat{\alpha}^*) := [\underline{\alpha}_\delta\, ,\, \overline{\alpha}_\delta] \qquad \forall \delta \in (0\, ,\, 1].$$

The characterizing functions for the estimates of the parameters based on fuzzy data are obtained by the mentioned construction lemma.

The computations for the following examples are realized by Matlab software.

Example 2.2. Based on the fuzzy sample whose characterizing functions are given in figure 1 the characterizing functions of the fuzzy estimates are given in Figure 2 and Figure 3.

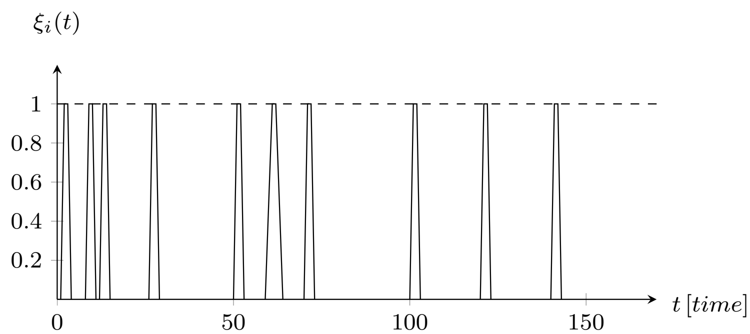

FIGURE 1. Characterizing Functions of a Fuzzy Sample

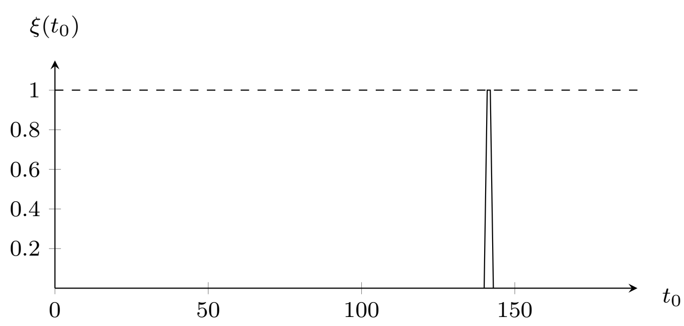

FIGURE 2. Characterizing Function of the Fuzzy Estimator $\hat{t}_0^*$

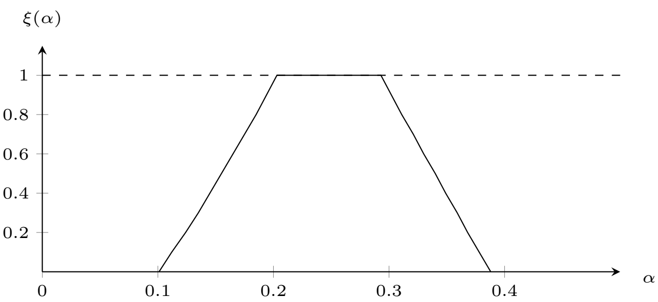

FIGURE 3. Characterizing Function of the Fuzzy Estimator $\hat{\alpha}^*$

In the above figures the characterizing functions of the generalized estimators are depicted. These proposed estimators are based on stochastic variation among the observations in addition with the fuzziness of single life time observations. Therefore, the obtained results are based on all the available information, which makes these more suitable and useful.

Based on the fuzzy estimates of t_0 and α the δ-level curves of the fuzzy estimate of the hazard rate $\hat{h}^*(\cdot)$ are given by the δ-cuts

$$C_\delta(h^*(t)) = [\underline{h}_\delta(t)\,,\,\overline{h}_\delta(t)] \quad \forall\delta \in (0\,,\,1], \quad \forall t \in (0\,,\,\infty).$$

The corresponding δ-level curves are obtained as

$$\underline{h}_\delta(t) = \frac{1 + 2\underline{\alpha}_\delta}{(2\overline{t}_{0,\delta}\sqrt{\overline{\alpha}_\delta^2 + (1+2\overline{\alpha}_\delta)t/\underline{t}_{0,\delta}})(1 + \overline{\alpha}_\delta - \sqrt{\underline{\alpha}_\delta^2 + (1+2\underline{\alpha}_\delta)t/\overline{t}_{0,\delta}})} \quad \forall\delta \in (0\,,\,1]$$

and

$$\overline{h}_\delta(t) = \frac{1 + 2\overline{\alpha}_\delta}{(2\underline{t}_{0,\delta}\sqrt{\underline{\alpha}_\delta^2 + (1+2\underline{\alpha}_\delta)t/\overline{t}_{0,\delta}})(1 + \underline{\alpha}_\delta - \sqrt{\overline{\alpha}_\delta^2 + (1+2\overline{\alpha}_\delta)t/\underline{t}_{0,\delta}})} \quad \forall\delta \in (0\,,\,1].$$

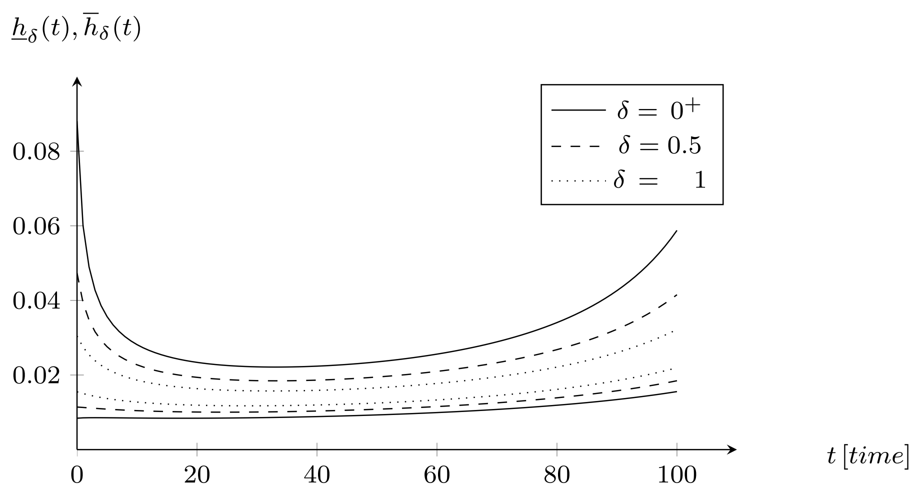

FIGURE 4. Some Lower and Upper δ-level Curves of the Fuzzy Estimate of the Hazard Rate

The curves for $\delta = 0^+$, 0.5, 1 show the boundaries of the supports of the corresponding characterizing functions.

The above δ-level curves of the fuzzy estimate of the hazard rate are based on both uncertainties, i.e. random variation and fuzziness which makes these estimators more realistic. This means that results of such analysis are fuzzy numbers as generalized probabilities of events. These are generalization of imprecise probabilities in fuzzy framework.

Definition 2.3. (Estimation of parameters in the model by Chen for fuzzy life times) For fuzzy life times fuzzy parameter estimators and a fuzzy estimator for the hazard rate of the new two-parameter distribution defined by [4] can be written as:

$$C_\delta(\hat{\beta}^*) = [\underline{\beta}_\delta\,,\,\overline{\beta}_\delta] \quad \forall \delta \in (0\,,\,1]$$
$$C_\delta(\hat{\lambda}^*) = [\underline{\lambda}_\delta\,,\,\overline{\lambda}_\delta] \quad \forall \delta \in (0\,,\,1]$$

and

$$C_\delta(h^*(t)) = [\underline{h}_\delta(t)\,,\,\overline{h}_\delta(t)] \quad \forall \delta \in (0\,,\,1]$$

respectively.

The estimate of β has no explicit solution and is obtained by iterative method, therefore, the lower and upper limits of the generating families of intervals for the fuzzy estimator $\hat{\beta}^*$ are approximated through the following equations:

$$\beta_{1,\delta} = \tfrac{r}{\beta} + \textstyle\sum_{i=1}^{r} \ln \underline{t}_{(i),\delta} + \sum_{i=1}^{r} \left(\underline{t}_{(i),\delta}^{\beta} \ln \underline{t}_{(i),\delta} \right) - \frac{r\left[\sum_{i=1}^{r} (e^{\underline{t}_{(i),\delta}^{\beta}} \underline{t}_{(i),\delta}^{\beta} \ln \underline{t}_{(i),\delta}) + (n-r)(e^{\underline{t}_{(r),\delta}^{\beta}} \underline{t}_{(r),\delta}^{\beta} \ln \underline{t}_{(r),\delta})\right]}{\sum_{i=1}^{r} e^{\underline{t}_{(i),\delta}^{\beta}} - n - (n-r) e^{\underline{t}_{(r),\delta}^{\beta}}} = 0$$

and

$$\beta_{2,\delta} = \tfrac{r}{\beta} + \textstyle\sum_{i=1}^{r} \ln \overline{t}_{(i),\delta} + \sum_{i=1}^{r} \left(\overline{t}_{(i),\delta}^{\beta} \ln \overline{t}_{(i),\delta} \right) - \frac{r\left[\sum_{i=1}^{r} \left(e^{\overline{t}_{(i),\delta}^{\beta}} \overline{t}_{(i),\delta}^{\beta} \ln \overline{t}_{(i),\delta}\right) + (n-r)(e^{\overline{t}_{(r),\delta}^{\beta}} \overline{t}_{(r),\delta}^{\beta} \ln \overline{t}_{(r),\delta})\right]}{\sum_{i=1}^{r} e^{\overline{t}_{(i),\delta}^{\beta}} - n - (n-r) e^{\overline{t}_{(r),\delta}^{\beta}}} = 0$$

where

$$\left(A_\delta(\hat{\beta}^*) = [\{\min(\beta_{1,\delta},\ \beta_{2,\delta})\}\,,\ \{\max(\beta_{1,\delta}\,,\ \beta_{2,\delta})\}] \quad \forall \delta \in (0\,,\,1] \right)$$

The characterizing function of $\hat{\beta}^*$ can be obtained by the mentioned Construction lemma.

Lower and upper limits of the generating family of intervals for the fuzzy estimator $\hat{\lambda}^*$ are obtained through the following equations:

$$A_\delta(\hat{\lambda}^*) = \left[\frac{r}{\sum_{i=1}^{r} e^{\overline{t}_{(i),\delta}^{\overline{\beta}_\delta}} - n - (n-r) e^{\underline{t}_{(r),\delta}^{\underline{\beta}_\delta}}}, \frac{r}{\sum_{i=1}^{r} e^{\underline{t}_{(i),\delta}^{\underline{\beta}_\delta}} - n - (n-r) e^{\overline{t}_{(r),\delta}^{\overline{\beta}_\delta}}} \right] \quad \forall \delta \in (0\,,\,1]$$

$\underline{\lambda}_\delta$ and $\overline{\lambda}_\delta$ represent the lower and upper limits of the generating interval at level δ. The characterizing function of $\hat{\lambda}^*$ can be constructed by the Construction lemma.

Example 2.4. A fuzzy sample is given in figure 5. The corresponding fuzzy estimates for the parameters are depicted in Figure 6 and Figure 7 respectively.

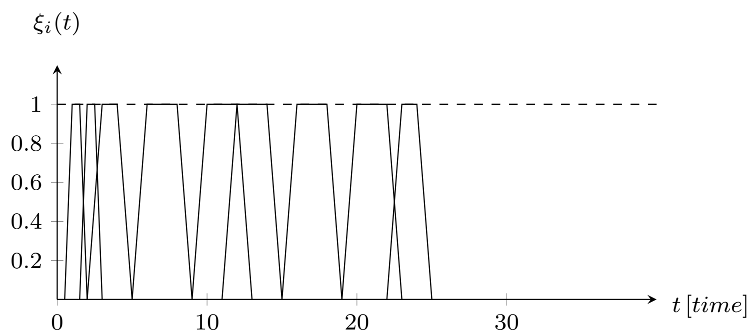

FIGURE 5. Characterizing Functions of a Fuzzy Sample

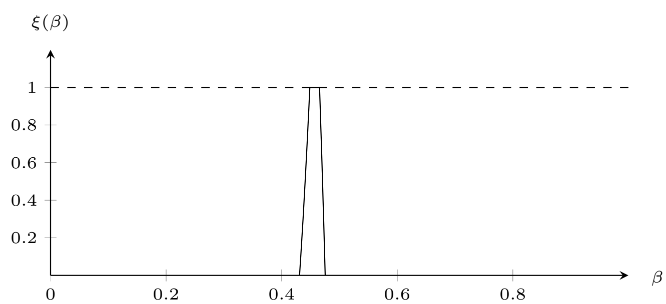

FIGURE 6. Characterizing Function of the Fuzzy Estimator $\hat{\beta}^*$

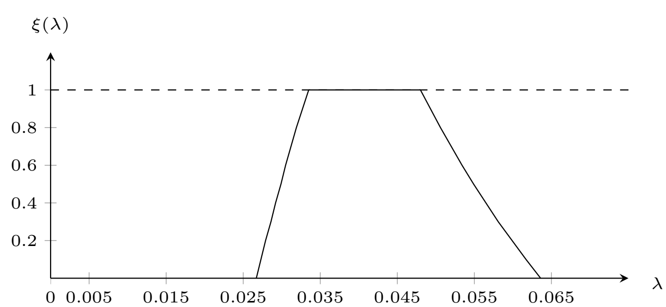

FIGURE 7. Characterizing Function of the Fuzzy Estimator $\hat{\lambda}^*$

The inclusion of fuzziness is essential, otherwise misleading results are obtained which hide the other uncertainty coming from fuzziness.

Lower and upper δ-level curves of the fuzzy hazard rate are given by:

$$\underline{h}_\delta(t) = \underline{\lambda}_\delta \underline{\beta}_\delta e^{t^{\overline{\beta}_\delta}} t^{\overline{\beta}_\delta - 1} \qquad \forall \delta \in (0\,,\,1], \qquad \forall t \in [0\,,\,\infty)$$

$$\overline{h}_\delta(t) = \overline{\lambda}_\delta \overline{\beta}_\delta e^{t^{\underline{\beta}_\delta}} t^{\underline{\beta}_\delta - 1} \qquad \forall \delta \in (0\,,\,1], \qquad \forall t \in [0\,,\,\infty)$$

Some related δ-level functions are graphically displayed in Figure 8.

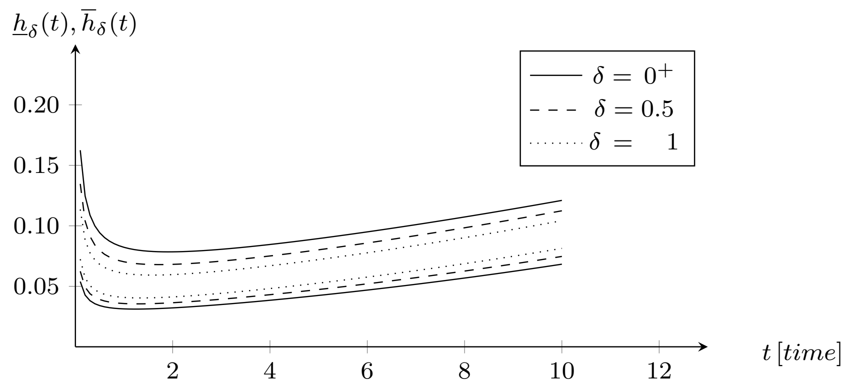

FIGURE 8. Some Lower and Upper δ-level Curves of the Fuzzy Estimate of the Hazard Rate

The above depicted hazard rate is based on both uncertainties in the life time data, which make this more appropriate for realistic life time data.

Instead of reliability estimates as classical probabilities these are generalized fuzzy results yield fuzzy probabilities. Which are obtained from the δ-level curves.

3. **Conclusion**

In survival analysis life time observations are usually regarded as precise measurements; whereas, the modern science of measurements says that the characteristic 'exact' cannot be achieved in reality. The analysis techniques based on precise life time observations without considering fuzziness of the single observations use incomplete information. Estimations based on incomplete information lead to unrealistic results.

The proposed generalized estimators are based on fuzzy life time observations; hence, they are more suitable and realistic to analyze life time data.

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