Abstract. In this paper, we introduce the notion of $K$-flat projective fuzzy quantales, and give an elementary characterization in terms of a fuzzy binary relation on the fuzzy quantale. Moreover, we prove that $K$-flat projective fuzzy quantales are precisely the coalgebras for a certain comonad on the category of fuzzy quantales. Finally, we present two special cases of $K$ as examples.

1. Introduction

Quantales were introduced by Mulvey in order to provide a lattice-theoretic setting for studying non-commutative $C^*$-algebra, as well as constructive foundations for quantum mechanics [13, 14]. The study that analyzed partially ordered algebraic structure goes back to a series of papers by Ward and Dilworth [4, 20, 22] in the 1930s. Since the theory of quantales provides a powerful tool in studying non-commutative structures, following Mulvey, various types and aspects of quantales have been considered by many researchers [9, 16].

Since Zadeh introduced fuzzy sets to model the uncertainty associated with the concept of imprecision [27], several extensions of fuzzy sets have been introduced. In order to establish a fuzzy counterpart of the Isbell-adjunction between topological spaces and locales [8], based on a frame $L$, Yao [25, 26] introduced an $L$-frame as an $L$-ordered set equipped with some further conditions and proved that the categories of Zhang-Liu-$L$-frames, Yao-$L$-frames and $L$-algebras are isomorphic. Based on the work of Yao, for a unital commutative quantale $Q$, Wang and Zhao [18] defined a $Q$-quantale as a $Q$-ordered semigroup equipped with some further conditions, and they also showed that the category of $Q$-quantales is isomorphic to the category of $Q$-algebras [17, 21]. The study of injectivity and projectivity in quantales was initiated by Li, Zhou and Li [11]. Banaschewski [2] established both the internal and the external characterization of “projective” objects in the category of frames. Moreover, Paseka [15] proposed a general view of projective quantales in the spirit of Banaschewski. Following this viewpoint, we shall present a general view with respect to the projectivity notion in the category of $Q$-quantales.

The content of the paper is organized as follows. Section 2 lists some preliminary notions and results about fuzzy posets. In Section 3, we discuss the relations between $Q$-Quant-morphisms and $K$-morphisms. In Section 4, we give an elementary characterization of a $K$-flat projective $Q$-quantale in terms of a $Q$-binary relation.

Received: January 2016; Revised: October 2016; Accepted: January 2017

Key words and phrases: Fuzzy quantale, Fuzzy binary relation, $K$-flat projective fuzzy quantale, Comonad.
relation on the \(Q\)-quantale. We also prove that the \(K\)-flat projective \(Q\)-quantales are precisely the coalgebras for a certain comonad on the category of \(Q\)-quantales. In Section 5, we consider two special cases of \(K\).

2. Preliminaries

We refer to [16] for quantale theory, to [1, 7] for category theory, to [6, 8] for lattice theory and to [3, 5, 10, 23, 24, 28] for fuzzy orders.

**Definition 2.1.** [16] A quantale is a complete lattice \(Q\) with an associative binary operation \& satisfying

\[
a \& \left( \bigvee_{i \in I} b_i \right) = \bigvee_{i \in I} (a \& b_i) \quad \text{and} \quad \left( \bigvee_{i \in I} b_i \right) \& a = \bigvee_{i \in I} (b_i \& a)
\]

for all \(a \in Q\) and \(\{b_i\}_{i \in I} \subseteq Q\).

A quantale \(Q\) is said to be unital provided that there exists an element \(1 \in Q\) such that \(a \& 1 = 1 \& a = a\) for all \(a \in Q\). \(Q\) is said to be commutative provided that \(a \& b = b \& a\) for all \(a, b \in Q\). From now on, unless otherwise stated, \(Q\) always denotes a unital commutative quantale. Since the map \(a \&\) preserves arbitrary sups, it has right adjoint, which we shall denote by \(a \rightarrow -\). Thus, \(a \& c \leq b\) iff \(c \leq a \rightarrow b\) for all \(a, b, c \in Q\).

**Definition 2.2.** [3, 5] Let \(X\) be a set. A map \(e : X \times X \rightarrow Q\) is called a fuzzy order (or, \(Q\)-order) on \(X\) if for all \(x, y, z \in X\),

- \((E1)\) \(e(x, x) \geq 1\) (reflexivity);
- \((E2)\) \(e(x, y) \& e(y, z) \leq e(x, z)\) (transitivity);
- \((E3)\) \(e(x, y) \geq 1\), \(e(y, x) \geq 1\) imply \(x = y\) (antisymmetry).

The pair \((X, e)\) is called a \(Q\)-ordered set.

Let \((X, \leq)\) be a classical poset. Then \((X, e_{\leq})\) is a \(Q\)-ordered set, where \(e_{\leq}\) is defined as follows:

\[
e_{\leq}(x, y) = \begin{cases} 1, & x \leq y, \\ 0, & \text{otherwise}. \end{cases}
\]

For a \(Q\)-ordered set \((X, e)\), \(\leq_{e} = \{(x, y) \mid e(x, y) \geq 1\}\) is a crisp partial order on \(X\). Unless otherwise stated, throughout the paper, whenever a partial order is mentioned in the context of a \(Q\)-ordered set \((X, e)\), it is to be interpreted with respect to the crisp partial order on \(X\). We often denote the crisp partial order on \((X, e)\) by \(\leq\) if there would be no confusion.

Let \(X\) be a set. \(Q^X\) denotes the set of all \(Q\)-subsets of \(X\). For all \(A, B \subseteq Q^X\), the subsethood degree of \(A\) in \(B\) is defined by \(\text{sub}_X(A, B) = \bigwedge_{x \in X} (A(x) \rightarrow B(x))\). Then \((Q^X, \text{sub}_X)\) is a \(Q\)-ordered set.

A map \(f : (X, e_X) \rightarrow (Y, \epsilon_Y)\) between \(Q\)-ordered sets is called \(Q\)-order-preserving if \(\epsilon_X(x_1, x_2) \leq \epsilon_Y(f(x_1), f(x_2))\) for all \(x_1, x_2 \in X\).

**Definition 2.3.** [28] Let \((X, e_X)\) be a \(Q\)-ordered set, \(x \in X\) and \(A \subseteq Q^X\). The element \(x_0\) is called a join (resp., meet) of \(A\), in symbols \(x_0 = \bigvee A\) (resp., \(x_0 = \bigwedge A\), if
Proposition 2.4. [23] Let $(X,e_X)$ be a $Q$-ordered set, $x_0, x_1 \in X$ and $A \in Q^X$.
(1) $x_0 = \sqcup A$ if for all $y \in X$, $e_X(x_0, y) = \bigwedge_{x \in X} (A(x) \rightarrow e_X(x, y))$;
(2) $x_1 = \sqcap A$ if for all $y \in X$, $e_X(y, x_1) = \bigvee_{x \in X} (A(x) \rightarrow e_X(y, x))$.

Definition 2.5. [28] Let $(X,e_X)$ be a $Q$-ordered set, and $A \in Q^X$. $A$ is called a lower $Q$-subset of $X$ if $e_X(x, y) \& A(y) \leq A(x)$ for all $x, y \in X$. Let $D(X)$ denote the collection of all lower $Q$-subsets of $X$.

Definition 2.6. [28] A $Q$-ordered set $(X,e_X)$ is called a complete $Q$-lattice if $\sqcup A$ and $\sqcap A$ exist for all $A \in Q^X$.

Remark 2.7. Let $(X,e_X)$ be a complete $Q$-lattice. Then $(X, \leq_{e_X})$ is a complete lattice.

Proposition 2.8. [23] Let $(X,e_X)$ be a $Q$-ordered set. The following statements are equivalent:
(1) $(X,e_X)$ is a complete $Q$-lattice;
(2) For all $A \in Q^X$, $\sqcup A$ exists;
(3) For all $A \in Q^X$, $\sqcap A$ exists.

Definition 2.9. [23] Let $(X,e_X), (Y,e_Y)$ be two $Q$-ordered sets and $f : X \rightarrow Y$, $g : Y \rightarrow X$ two $Q$-order-preserving maps. The pair $(f, g)$ is called a $Q$-adjunction between $X$ and $Y$ if for all $x \in X$, $y \in Y$, $e_Y(f(x), y) = e_X(x, g(y))$. In this case, $f$ is called the left adjoint and dually $g$ is called the right adjoint.

Lemma 2.10. Let $(X,e_X), (Y,e_Y)$ be two $Q$-ordered sets and $f : X \rightarrow Y$, $g : Y \rightarrow X$ two $Q$-order-preserving maps. If $f \circ g = id_Y$ and $id_X \leq g \circ f$, then $(f, g)$ is a $Q$-adjunction.

Proof. For all $x \in X$ and $y \in Y$, we have that $e_Y(f(x), y) = e_X(g(f(x)), g(y)) \& 1 \leq e_X(x, g(f(x)), g(y)) \& e_X(x, g(f(x))) \leq e_X(x, g(y)) \leq e_Y(f(x), g(y)) = e_Y(f(x), y)$. Then $e_Y(f(x), y) = e_X(x, g(y))$, and thus $(f, g)$ is a $Q$-adjunction.

Let $X, Y$ be sets and $f : X \rightarrow Y$ be a map. Then the Zadeh forward power set operator $f^+ : Q^Y \rightarrow Q^X$ and the Zadeh backward power set operator $f^− : Q^X \rightarrow Q^Y$ are defined, respectively, by $f^+(A)(y) = \bigvee_{f(x)=y} A(x)$, $f^−(B) = B \circ f$ for all $A \in Q^X$, $y \in Y$, $B \in Q^Y$. It can be easily seen (see [23]) that $(f^+, f^−)$ is a $Q$-adjunction between $(Q^X, sub_X)$ and $(Q^Y, sub_Y)$.

Definition 2.11. [18, 19] A complete $Q$-lattice $(X, e)$ with an associative binary operation $\otimes$ is called a fuzzy quantale (or simply, $Q$-quantale) if for all $a \in X$, $a \otimes x : X \rightarrow X$ and $a \otimes y : X \rightarrow X$ preserve joins of every $Q$-subset of $X$, that is, for all $A \in Q^X$, $a \otimes (\sqcup A) = \sqcup (a \otimes) A$ and $(\sqcup A) \otimes a = \sqcup (\sqcup a \otimes) A$.

Let $X$ be a $Q$-quantale. $S \subseteq X$ is called a sub-$Q$-quantale of $X$ provided that $S$ is closed under joins of every $Q$-subset of $S$ and $\otimes$. Let $S$ denote the set of all
Remark 3.1. For any object \( m : \mathcal{M} \rightarrow \mathcal{N} \) in \( \mathcal{C} \), the corestriction of \( m \) is defined as \( \phi \circ m \) for any \( \phi : \mathcal{N} \rightarrow \mathcal{M} \), subject to the following condition: 

A subcategory of \( \mathcal{Q}-\mathcal{OSgr} \) is a subcategory of \( \mathcal{Q} \)-ordered semigroups if \( a \circ b \leq c \circ d \) for all \( a, b, c, d \in \mathcal{Q} \). 

Definition 2.14. A \( \mathcal{Q} \)-ordered set \( (X, e) \) is called a \( \mathcal{Q} \)-ordered semigroup if \( a \circ b \leq c \circ d \) for all \( a, b, c, d \in \mathcal{Q} \). 

Remark 2.12. Any \( \mathcal{Q} \)-quantale (resp., sub-\( \mathcal{Q} \)-quantale) is a quantale (resp., sub-quantale) with respect to the crisp partial order. Similarly, any \( \mathcal{Q} \)-quantale homomorphism is a quantale homomorphism.

Let \( (X, e) \) be a \( \mathcal{Q} \)-quantale. If \( S \subseteq X \) is a sub-\( \mathcal{Q} \)-quantale of \( X \), and \( i : S \rightarrow X \) is defined as follows:

\[ \forall x \in S, i(x) = x, \]

then \( i \) is a \( \mathcal{Q} \)-quantale homomorphism. We call \( i \) an identical sub-\( \mathcal{Q} \)-quantale embedding.

Definition 2.13. [10, 24] Let \( (X, e) \) be a \( \mathcal{Q} \)-ordered set and \( D \in \mathcal{Q} X \). \( D \) is called a \( \mathcal{Q} \)-directed subset of \( X \) if 

1. \( \bigvee_{x \in X} D(x) \geq 1 \); 
2. \( D(x) \circ D(y) \leq \bigvee_{z \in X} (D(z) \circ e(x, z) \circ e(y, z)) \) for all \( x, y \in X \).

Let \( \mathcal{D}(X) \) denote the collection of all \( \mathcal{Q} \)-directed subsets of \( X \).

Definition 2.14. [24] A \( \mathcal{Q} \)-ordered set \( (X, e) \) is called a fuzzy dcpo if \( \bigvee A \) exists for all \( A \in \mathcal{D}(X) \).

3. The Relations Between \( \mathcal{Q} \)-Quant-morphisms and \( \mathcal{K} \)-morphisms

A \( \mathcal{Q} \)-ordered set \( (X, e_X) \) with an associative binary operation \( \ast \) is called a \( \mathcal{Q} \)-ordered semigroup if \( e_X(a, b) \leq e_X(a \ast c, b \ast c) \) and \( e_X(a, b) \leq e_X(c \ast a, c \ast b) \) for all \( a, b, c \in \mathcal{Q} \). A \( \mathcal{Q} \)-order-preserving map \( f : (X, e_X, \ast) \rightarrow (Y, e_Y, \ast_Y) \) between \( \mathcal{Q} \)-ordered semigroups is called a \( \mathcal{Q} \)-ordered semigroup homomorphism if \( f(a \ast_X b) = f(a) \ast_Y f(b) \) for all \( a, b \in \mathcal{X} \). Let \( \mathcal{Q} \)-\( \mathcal{OSgr} \) denote the category of \( \mathcal{Q} \)-ordered semigroups with \( \mathcal{Q} \)-ordered semigroup homomorphisms. Clearly, \( \mathcal{Q} \)-\( \mathcal{Quant} \) is a subcategory of \( \mathcal{Q} \)-\( \mathcal{OSgr} \) (see [19]). Now we consider the category \( \mathcal{K} \), which is a subcategory of \( \mathcal{Q} \)-\( \mathcal{OSgr} \). Moreover, \( \mathcal{K} \) contains the category \( \mathcal{Q} \)-\( \mathcal{Quant} \) reflectively, subject to the following condition:

(C) For any \( \phi : A \rightarrow L \) in \( \mathcal{K} \) where \( L \) is a \( \mathcal{Q} \)-quantale and \( A \) arbitrary, the corestriction of \( \phi \) to any sub-\( \mathcal{Q} \)-quantale of \( L \) containing the image of \( \phi \) also belongs to \( \mathcal{K} \).

Remark 3.1. For any object \( A \) in \( \mathcal{K} \), we have a universal map \( \eta_A : A \rightarrow FA \) in \( \mathcal{K} \). In particular, for any \( \mathcal{Q} \)-quantale \( L \), there exists a unique \( \mathcal{Q} \)-quantale homomorphism \( \varepsilon_L : FL \rightarrow L \), such that \( \varepsilon_L \circ \eta_L = id_L \).
Proposition 3.2. Let $A$ be an object in $\mathbf{K}$. Then $FA$ is generated by the image $\text{Im} \eta_A$ of $A$.

Proof. Let $M \subseteq FA$ be the sub-$Q$-quantale generated by $\text{Im} \eta_A$, $\phi : A \to M$ be the corestriction of $\eta_A : A \to FA$, and $i : M \to FA$ be the identical sub-$Q$-quantale embedding. Then by (C) we have a unique $Q$-quantale homomorphism $h : FA \to M$ such that $h \circ \eta_A = \phi$. Hence $i \circ h \circ \eta_A = \eta_A$. By the universal property of $\eta_A$ we get $i$ is onto. Thus $M = FA$. □

Corollary 3.3. Let $A$ be an object in $\mathbf{K}$ and $b \in FA$. Define a map $k_b : FA \to Q$ as follows:

$$\forall y \in FA, \quad k_b(y) = \begin{cases} e_{FA}(y, b), & y \in \text{Im} \eta_A, \\ 0, & \text{otherwise.} \end{cases}$$

Then $b = \sqcup k_b$.

Proof. Let $A = \{ \sqcup B \mid B \in Q^{\text{Im} \eta_A} \}$. For all $B_1, B_2 \in Q^{\text{Im} \eta_A}$, we define a map $B_1 \oplus B_2 : \text{Im} \eta_A \to Q$ as follows:

$$\forall y \in \text{Im} \eta_A, \quad (B_1 \oplus B_2)(y) = \bigvee_{a \in FL, b = y, a \in \text{Im} \eta_A} (B_1(a) \& B_2(b)).$$

We can check that $(\sqcup B_1) \otimes_{FL} (\sqcup B_1) = \sqcup (B_1 \oplus B_2)$. For any $B \in Q^A$, $z \in FA$, we have that

$$\bigwedge_{x \in FA} (B(x) \to e_{FA}(x, z)) = \bigwedge_{a \in A} (B(a) \to e_{FA}(a, z))$$

Then $\sqcup B = \sqcup (\bigvee_{B \in Q^{\text{Im} \eta_A}} B(\sqcup B) \& B)$, and thus $A$ is a sub-$Q$-quantale. For any $y \in \text{Im} \eta_A$, we define a map $\chi_{(y)} : \text{Im} \eta_A \to Q$ as follows:

$$\forall x \in \text{Im} \eta_A, \quad \chi_{(y)}(x) = \begin{cases} 1, & x = y, \\ 0, & \text{otherwise.} \end{cases}$$

It follows that $\sqcup \chi_{(y)} = y$. This shows that $\text{Im} \eta_A \subseteq A$. Let $Y$ be a sub-$Q$-quantale of $FA$ with $\text{Im} \eta_A \subseteq Y$, and $B \in Q^{\text{Im} \eta_A}$. Define a map $C : Y \to Q$ as follows:

$$\forall y \in Y, \quad C(y) = \begin{cases} B(y), & y \in \text{Im} \eta_A, \\ 0, & \text{otherwise.} \end{cases}$$
Then $\sqcup B = \sqcup C$, and thus $A \subseteq Y$. This means $\langle \text{Im} \eta_A \rangle = A$. By Proposition 3.2, we have that $FA = A$. For every $b \in FA$, there exists $B \in Q^{\text{Im} \eta_A}$ such that $b = \sqcup B$. For all $y \in \text{Im} \eta_A$, $B(y) \leq k_b(y)$. Then $\sqcup B \leq \sqcup k_b \leq b$, and thus $\sqcup k_b = b$. □

Proposition 3.4. Let $L$ be a $Q$-quantale. Then $\text{id}_{FL} \leq \eta_L \circ \varepsilon_L$.

Proof. Assume that $b \in FL$. By Corollary 3.3, $b = \sqcup k_b$. Now, for all $y \in \text{Im} \eta_L$, there exists $a \in L$ such that $y = \eta_L(a)$. Thus $(\eta_L \circ \varepsilon_L)(y) = (\eta_L \circ \varepsilon_L)(\eta_L(a)) = 

\eta_L(\varepsilon_L(\eta_L(a))) = \eta_L(a) = y$. Since $\eta_L \circ \varepsilon_L$ is $Q$-order-preserving, we have $e_{FL}(b, (\eta_L \circ \varepsilon_L)(b))$

$= e_{FL}(\sqcup k_b, (\eta_L \circ \varepsilon_L)(b))$

$= \bigwedge_{y \in FL} (k_b(y) \rightarrow e_{FL}(y, (\eta_L \circ \varepsilon_L)(b)))$

$= \bigwedge_{y \in \text{Im} \eta_L} (e_{FL}(y, b) \rightarrow e_{FL}(y, (\eta_L \circ \varepsilon_L)(b)))$

$\geq \bigwedge_{y \in \text{Im} \eta_L} (e_{FL}(y, (\eta_L \circ \varepsilon_L)(y), (\eta_L \circ \varepsilon_L)(b)) \rightarrow e_{FL}(y, (\eta_L \circ \varepsilon_L)(b)))$

$= \bigwedge_{y \in \text{Im} \eta_L} (e_{FL}(y, (\eta_L \circ \varepsilon_L)(b)) \rightarrow e_{FL}(y, (\eta_L \circ \varepsilon_L)(b)))$

$\geq 1$, consequently, $\text{id}_{FL} \leq \eta_L \circ \varepsilon_L$. □

Corollary 3.5. Let $L$ be a $Q$-quantale. Then $(\varepsilon_L, \eta_L)$ is a $Q$-adjunction between $FL$ and $L$.

Proposition 3.6. Let $L$ be a $Q$-quantale, $A$ an object in $K$, and $f, g : FA \rightarrow L$ $Q$-quantale homomorphisms. Then $f \circ \eta_A \leq g \circ \eta_A$ implies $f \leq g$.

Proof. Since $f \circ \eta_A \leq g \circ \eta_A$, $e_L((f \circ \eta_A)(d), (g \circ \eta_A)(d)) \geq 1$ for all $d \in A$. For all $b \in FA$, we have $b = \sqcup k_b$, $f(b) = f(\sqcup k_b) = \sqcup f_{Q^{-}}(k_b)$, $g(b) = g(\sqcup k_b) = \sqcup g_{Q^{-}}(k_b)$. Thus

$e_L(f(b), g(b)) = e_L(\sqcup f_{Q^{-}}(k_b), g(b))$

$= \bigwedge_{x \in L} \left( \bigvee_{f(a) = x} k_b(a) \rightarrow e_L(x, g(b)) \right)$

$= \bigwedge_{a \in FA} (k_b(a) \rightarrow e_L(f(a), g(b)))$

$= \bigwedge_{a \in \text{Im} \eta_A} (e_{FA}(a, b) \rightarrow e_L(f(a), g(b)))$

$\geq \bigwedge_{a \in \text{Im} \eta_A} ((e_L(g(a), g(b))) \& e_L(f(a), g(a))) \rightarrow e_L(f(a), g(b)))$

$= \bigwedge_{a \in \text{Im} \eta_A} (e_L(g(a), g(b)) \rightarrow (e_L(f(a), g(a)) \rightarrow e_L(f(a), g(b))))$

$\geq \bigwedge_{a \in \text{Im} \eta_A} (e_L(g(a), g(b)) \rightarrow e_L(g(a), g(b)))$

$\geq 1$.

That is, $f \leq g$. □
Proposition 3.7. Let $L$ be a $Q$-quantale and $h : L \rightarrow FL$ be a right inverse of $\varepsilon_L : FL \rightarrow L$. Then $h \circ \varepsilon_L \leq id_{FL}$.

Proof. For all $x \in L$, since $e_{FL}((h \circ \varepsilon_L)(\eta_L(x)), \eta_L(x)) = e_{FL}((h \circ \varepsilon_L)(\eta_L(x)), \eta_L(x)) = e_{FL}(h(x), \eta_L(x)) = e_{FL}(h(x), (\eta_L \circ (\varepsilon_L \circ h))(x)) = e_{FL}(h(x), (\eta_L \circ \varepsilon_L)(h(x))) \geq 1$, we have that $e_{FL}((h \circ \varepsilon_L)(a), a) \geq 1$ for all $a \in Im\eta_L$. For all $b \in FL$, $e_{FL}((h \circ \varepsilon_L)(b), b) = e_{FL}((h \circ \varepsilon_L)((\sqcup k_b), b) = e_{FL}((h \circ \varepsilon_L)_{Q}(k_b), b) = \bigwedge_{y \in FL} \left( (h \circ \varepsilon_L)_{Q}(k_b)(y) \rightarrow e_{FL}(y, b) \right) = \bigwedge_{a \in Im\eta_A} \left( (e_{FL}(a, b) \rightarrow e_{FL}((h \circ \varepsilon_L)(a), b)) \right) \geq \bigwedge_{a \in Im\eta_A} \left( (e_{FL}(a, b) \rightarrow (e_{FL}((h \circ \varepsilon_L)(a), a) \rightarrow e_{FL}((h \circ \varepsilon_L)(a), b)) \right) \geq 1.

Thus $h \circ \varepsilon_L \leq id_{FL}$. \hfill \Box

Proposition 3.8. Let $A, B$ be objects in $K$ and $g : A \rightarrow B$ be a $K$-morphism. Suppose $L, P$ are $Q$-quantales and $f : L \rightarrow P$ is a $Q$-quantale homomorphism. Then the following statements hold:

1. $Fg \circ \eta_A = \eta_B \circ g$;
2. $f \circ \varepsilon_L = \varepsilon_P \circ Ff$;
3. If $\phi, \varphi : A \rightarrow B$ are $K$-morphisms and $\phi \leq \varphi$, then $F\phi \leq F\varphi$;
4. $(F\eta_A, F\varepsilon_A)$ is a $Q$-adjunction between $FA$ and $FFA$.

Proof. (1) The statement is straightforward since $K$ contains the category $Q$-Quant reflectively.

(2) Since $\eta_P \circ f = Ff \circ \eta_L$, $\varepsilon_P \circ \eta_P \circ f = \varepsilon_P \circ Ff \circ \eta_L$, $f \circ \varepsilon_L \circ \eta_L = \varepsilon_P \circ Ff \circ \eta_L$, therefore $f \circ \varepsilon_L = \varepsilon_P \circ Ff$ by the universal property of $\eta_L$.

(3) Let $\phi \leq \varphi$. For $a \in A$. By (1), we have that $e_{FB}((F\phi \circ \eta_A)(a), (\eta_B \circ \phi)(a)) \geq 1$, and $e_{FB}((\eta_B \circ \varphi)(a), (F\varphi \circ \eta_A)(a)) \geq 1$. Thus, by transitivity of $e_{FB}$ and the fact that $\eta_B(\phi(a)) \leq \eta_B(\varphi(a))$, $e_{FB}(F\phi(\eta_A(a)), F\varphi(\eta_A(a))) \geq 1$. By Proposition 3.6, we have that $F\phi \leq F\varphi$. 
(4) Since $id_{FA} \circ \eta_A = \varepsilon_{FA} \circ \eta_{FA} \circ \eta_A = \varepsilon_{FA} \circ F\eta_A \circ \eta_A$, we have that $id_{FA} = \varepsilon_{FA} \circ F\eta_A$. By Proposition 3.7, we have that $(F\eta_A, \varepsilon_{FA})$ is a $Q$-adjunction between $FA$ and $FFA$.

4. K-flat Projective Q-quantales

Definition 4.1. A $Q$-quantale $L$ is said to be projective if for any $Q$-quantale homomorphism $f : L \rightarrow M$ and an epimorphism $g : N \rightarrow M$ in $Q$-Quant, there exists a $Q$-quantale homomorphism $h : L \rightarrow N$ such that $f = g \circ h$.

Definition 4.2. A $Q$-quantale $L$ is called $K$-flat projective if $L$ is projective in $Q$-Quant relative to the onto $Q$-quantale homomorphism $h : N \rightarrow M$ for which the right adjoint $h_* : M \rightarrow N$ belongs to $K$.

Remark 4.3. A $Q$-quantale $L$ is a $K$-flat projective $Q$-quantale if $L$ is a projective $Q$-quantale.

Definition 4.4. Let $L$ be a $Q$-quantale and $a \in L$. Define a map $\downarrow a : L \rightarrow Q$ as follows:

$$\forall x \in L, \downarrow a(x) = \bigwedge_{b \in FL} (e_L(a, \varepsilon_L(b)) \rightarrow e_{FL}(\eta_L(x), b)).$$

We call $\downarrow : L \times L \rightarrow Q$ a $Q$-binary relation on the $Q$-quantale $L$.

Lemma 4.5. Let $L, P$ be $Q$-quantales. Then for all $a, x, y, u, v \in L$,

1. $\downarrow a \leq \downarrow a$;
2. $e_L(x, y) \& \downarrow u(y) \& e_L(u, v) \leq \downarrow v(x)$.

Proof. (1) For all $m \in L$, we have that

$$\downarrow a(m) = \bigwedge_{b \in FL} (e_L(a, \varepsilon_L(b)) \rightarrow e_{FL}(\eta_L(m), b))$$

$$\leq e_L(a, \varepsilon_L(\eta_L(a))) \rightarrow e_{FL}(\eta_L(m), \eta_L(a))$$

$$= e_L(a, a) \rightarrow e_{FL}(\eta_L(m), \eta_L(a))$$

$$\leq e_L(\varepsilon_L(\eta_L(m)), \varepsilon_L(\eta_L(a)))$$

$$= \downarrow a(m).$$

Thus $\downarrow a \leq \downarrow a$.

(2) For all $b \in FL$, we have that $e_L(x, y) \& e_L(u, v) \& e_L(u, \varepsilon_L(b)) \rightarrow e_{FL}(\eta_L(y), b))$ $\& e_L(v, \varepsilon_L(b)) \leq e_L(x, y) \& e_{FL}(\eta_L(y), b) \leq e_{FL}(\eta_L(x), \eta_L(y)) \& e_L(v, \varepsilon_L(b)) \leq e_{FL}(\eta_L(x), b)$. Then $e_L(x, y) \& e_L(u, v) \& e_L(u, \varepsilon_L(b)) \rightarrow e_{FL}(\eta_L(y), b) \leq e_L(v, \varepsilon_L(b)) \rightarrow e_{FL}(\eta_L(x), b)$. So we can conclude that $e_L(x, y) \& e_L(u, v) \& \bigwedge_{b \in FL} (e_L(u, \varepsilon_L(b)) \rightarrow e_{FL}(\eta_L(x), b)) = \downarrow v(x).$
Proof. (1)⇒(2) Since $\varepsilon_L \circ \eta_L = id_L$, we have that $\varepsilon_L$ is an onto $Q$-quantale homomorphism. By Corollary 3.5, we have that $\varepsilon_L$ has a right adjoint $\eta_L$, which belongs to $K$. Since $L$ is $K$-flat projective, there exists a $Q$-quantale homomorphism $h : L \rightarrow FL$ such that $\varepsilon_L \circ h = id_L$.

(2)⇒(3) Let $A = L$. Then $A$ is an object in $K$. By (2), we have that $L$ is a retract of $FL$.

(3)⇒(1) By (3), there exist two $Q$-quantale homomorphisms $n : FA \rightarrow L$, $j : L \rightarrow FA$ such that $n \circ j = id_L$. Firstly, let $P, T$ be $Q$-quantales and the onto $Q$-quantale homomorphism $h : P \rightarrow T$ for which the right adjoint $h_\ast : T \rightarrow P$ belongs to $K$, $f : FA \rightarrow T$ be a $Q$-quantale homomorphism. Then $h_\ast \circ f \circ \eta_A$ belongs to $K$, there exists a unique $Q$-quantale homomorphism $g$ such that $h_\ast \circ f \circ \eta_A = g \circ \eta_A$, $h \circ h_\ast \circ f \circ \eta_A = h \circ g \circ \eta_A$. By the universal of $\eta_A$, we have $f = h \circ g$. Thus $FA$ is $K$-flat projective. Moreover, let $m : L \rightarrow T$ be a $Q$-quantale homomorphism, then $m \circ n : FA \rightarrow T$ is a $Q$-quantale homomorphism, so there exists a $Q$-quantale homomorphism $p : FA \rightarrow P$ such that $h \circ p = m \circ n$, $(h \circ p) \circ j = (m \circ n) \circ j$, $h \circ (p \circ j) = m \circ (n \circ j) = m \circ id_L = m$. Thus $L$ is $K$-flat projective.

(2)⇒(4) Let $h$ be a right inverse of $\varepsilon_L$. For all $a, x \in L$, we have

\[ \vDash a(x) = \bigwedge_{b \in FL} (e_L(a, x) \rightarrow e_{FL}(\eta_L(x), b)) \]

\[ \leq e_L(a, \varepsilon_L(h(a))) \rightarrow e_{FL}(\eta_L(x), h(a)) \]

\[ = e_L(a, a) \rightarrow e_{FL}(\eta_L(x), h(a)) \]

\[ \leq 1 \rightarrow e_{FL}(\eta_L(x), h(a)) \]

\[ \leq e_{FL}(\eta_L(x), h(a)). \]

For all $b \in FL$, by Proposition 3.7, we can conclude that

\[ e_{FL}(\eta_L(x), h(a)) \& e_L(a, \varepsilon_L(b)) \leq e_{FL}(\eta_L(x), h(a)) \& e_{FL}(h(a), \varepsilon_L(b)) \]

\[ \leq e_{FL}(\eta_L(x), h(\varepsilon_L(b)), b) \].

Then $\vDash a(x) = e_{FL}(\eta_L(x), h(a))$. Since $h(a) = \bigcup k_{h(a)}$ and $\varepsilon_L \circ h = id_L$, we have that $a = (\varepsilon_L \circ h)(a) = \varepsilon_L(\bigcup k_{h(a)}) = \bigcup (\varepsilon_L)^{-1}(k_{h(a)})$. For all $y \in L$,

\[ e_L(a, y) = \bigwedge_{d \in L} (e_L(\varepsilon_L^{-1}(k_{h(a)})(d) \rightarrow e_L(d, y)) \]

\[ = \bigwedge_{d \in L} \bigwedge_{\varepsilon_L(p) = d} (k_{h(a)}(p) \rightarrow e_L(d, y)) \]

\[ = \bigwedge_{p \in FL} (k_{h(a)}(p) \rightarrow e_L(\varepsilon_L(p), y)) \]

\[ = \bigwedge_{p \in FL} e_{FL}(p, h(a)) \rightarrow e_L(\varepsilon_L(p), y) \]

\[ = \bigwedge_{d \in L} (e_{FL}(\eta_L(d), h(a)) \rightarrow e_L(d, y)) \]

\[ = \bigwedge_{d \in L} (\vDash d \rightarrow e_L(d, y)). \]
This means that \( a = \sqcup \downarrow a \).

\[
\downarrow a(x) \& \downarrow b(y) = e_{FL}(\eta_L(x), h(a)) \& e_{FL}(\eta_L(y), h(b)) \\
\leq e_{FL}(\eta_L(x) \otimes_{FL} \eta_L(y), h(a) \otimes_{FL} h(b)) \\
= e(\eta_L(x \otimes_L y), h(a \otimes_L b)) \\
= \downarrow (a \otimes_L b)(x \otimes_L y).
\]

\( (4) \Rightarrow (2) \) Define a map \( h_L : L \rightarrow FL \) as follows:

\[
\forall a \in L, \ h_L(a) = \sqcup A_a,
\]

where \( A_a : FL \rightarrow Q \) is defined by

\[
\forall b \in FL, \ A_a(b) = \bigvee_{x \in L} (\downarrow a(x) \& e_{FL}(b, \eta_L(x))).
\]

We shall prove that \( h_L \) is a right inverse of \( \varepsilon_L \). Firstly, for all \( a \in L \), we have that

\[
\varepsilon_L \circ h_L(a) = \varepsilon_L(\sqcup A_a) = \sqcup(\varepsilon_L)_Q^\ast(A_a).
\]

For all \( t \in L \), we have

\[
\bigwedge_{x \in L} ((\varepsilon_L)_Q^\ast(A_a)(x) \rightarrow e_L(x, t)) = \bigwedge_{x \in L \in \varepsilon_L(b) = x} (A_a(b) \rightarrow e_L(x, t)) \\
= \bigwedge_{b \in FL} (A_a(b) \rightarrow e_L(e_L(b), t)) \\
= \bigwedge_{b \in FL} \bigwedge_{z \in L} ((\downarrow a(z) \& e_{FL}(b, \eta_L(z))) \rightarrow e_L(e_L(b), t)) \\
= \bigwedge_{b \in FL} \bigwedge_{z \in L} ((\downarrow a(z) \rightarrow (e_{FL}(b, \eta_L(z)) \rightarrow e_L(e_L(b), t))) \\
= \bigwedge_{z \in L} (\downarrow a(z) \rightarrow e_L(z, t)) \\
= e_L(\sqcup \downarrow a, t) \\
= e_L(a, t).
\]

This means that \( \varepsilon_L \circ h_L(a) = \sqcup(\varepsilon_L)_Q^\ast(A_a) = a \). So \( \varepsilon_L \circ h_L = \text{id}_L \).

Moreover, for all \( b \in FL \), \( (h_L \circ \varepsilon_L)(b) = \sqcup A_{\varepsilon_L(b)}, \)

\[
e_{FL}(h_L \circ \varepsilon_L)(b, b) = e_{FL}(\sqcup A_{\varepsilon_L(b)}, b) \\
= \bigwedge_{c \in FL} (A_{\varepsilon_L(b)}(c) \rightarrow e_{FL}(c, b)) \\
= \bigwedge_{c \in FL} \left( \left( \bigvee_{z \in L} (\downarrow \varepsilon_L(b)(z) \& e_{FL}(c, \eta_L(z))) \rightarrow e_{FL}(c, b) \right) \right) \\
\geq \bigwedge_{c \in FL} \bigwedge_{z \in L} (e_{FL}(c, b) \rightarrow e_{FL}(c, b)) \\
\geq 1.
\]

Then \( h_L \circ \varepsilon_L \leq \text{id}_{FL} \), so \( (h_L, \varepsilon_L) \) is a \( Q \)-adjunction between \( L \) and \( FL \), hence \( h_L \) preserves joins. Since
$e_{FL}(h_L(a) \otimes_{FL} h_L(b), h_L(a \otimes_L b)) = e_{FL}(\sqcup A_b \otimes_{FL} \sqcup A_b, \sqcup A_b \otimes_L b)$
\[= e_{FL}(\sqcup(\sqcup A_b \otimes_{FL} -), (\sqcup A_b, \sqcup A_b \otimes_L b))
\[= \bigwedge_{c \in FL} \left( \bigvee_{d \in c} A_b(d) \to e_{FL}(c, \sqcup A_b \otimes_L b) \right) \bigwedge_{d \in FL} (A_b(d) \to e_{FL}(\sqcup A_b \otimes_{FL} d, \sqcup A_b \otimes_L b))
\[= \bigwedge_{d \in FL} (A_b(d) \to e_{FL}(\sqcup A_b \otimes_{FL} d, \sqcup A_b \otimes_L b))
\[= \bigwedge_{d \in FL} (A_b(d) \otimes_{FL} A_L(l) \to e_{FL}(l, \sqcup A_b \otimes_L b))
\[= \bigwedge_{d \in FL} (A_b(d) \otimes_{FL} A_L(l) \to e_{FL}(l \otimes_{FL} d, \sqcup A_b \otimes_L b))
\[\geq 1.
\]
we have that $h_L(a) \otimes_{FL} h_L(b) \leq h_L(a \otimes_L b)$. Moreover, one can conclude that
\[h_L(a \otimes_L b) = h_L((\varepsilon_L \circ h_L)(a) \otimes_{FL} (\varepsilon_L \circ h_L)(b))
\[= (h_L \circ \varepsilon_L)(h_L(a) \otimes_{FL} h_L(b))
\[\leq h_L(a) \otimes_{FL} h_L(b).$

Then $h_L(a \otimes_L b) = h_L(a) \otimes_{FL} h_L(b)$, and thus $h_L$ is a right inverse of $\varepsilon_L$. □

The comonad determined by $F$ (viewed as an endofunctor of $\mathbf{Q-Quant}$) is $(F, \varepsilon, F\eta)$, and its coalgebras are pairs $(L, g_L)$, where the structure map $g_L : L \to FL$ satisfies the conditions as follows:

$$(U) \quad \varepsilon_L \circ g_L = id_L; \quad (A) \quad (Fg_L) \circ g_L = (F\eta_L) \circ g_L.$$

**Proposition 4.7.** Let $L$ be a $Q$-quantale. Then $L$ is $K$-flat projective iff it has a coalgebra structure for the $(F, \varepsilon, F\eta)$.

**Proof.** We only have to show necessity. For all $b \in FL$,
\[e_{FFL}(F\eta_L(b), \eta_{FL}(b)) = e_{FFL}(F\eta_L(\sqcup b), \eta_{FL}(b))
\[= \bigwedge_{c \in FFL} ((F\eta_L)_c(k_b)(c) \to e_{FFL}(c, \eta_{FL}(b)))
\[= \bigwedge_{c \in FFL} \bigwedge_{y \in c \sqcap FFL} (k_b(y) \to e_{FFL}(c, \eta_{FL}(b)))
\[= \bigwedge_{y \in FL} (k_b(y) \to e_{FFL}(F\eta_L(y), \eta_{FL}(b)))
\[= \bigwedge_{y \in FL} (e_{FL}(y, b) \to e_{FFL}(F\eta_L(y), \eta_{FL}(b)))
\[= \bigwedge_{a \in L} (e_{FL}(\eta_L(a), b) \to e_{FFL}(F\eta_L(\eta_L(a)), \eta_{FL}(b)))
\[= \bigwedge_{a \in L} (e_{FL}(\eta_L(a), b) \to e_{FFL}(\eta_{FL}(\eta_L(a)), \eta_{FL}(b)))
\[\geq 1.
\]
Then $e_{FFL}(F\eta_L(b), \eta_{FL}(b)) \geq 1$. By Theorem 4.6, there exists a $Q$-quantale homomorphism $h_L : L \to FL$ such that $\varepsilon_L \circ h_L = id_L$. Next, we shall prove that
For all \( b \in FL \), \( e_{FL}(b, (\eta_L \circ \varepsilon_L)(b)) \geq 1 \). Then for all \( a \in L \),
\[
e_{FL}(h_L(a), \eta_L(a)) = e_{FL}(h_L(a), (\eta_L \circ \varepsilon_L)(h_L(a))) \geq 1.
\]
By Proposition 3.8(3), we have that \( e_{FFL}(Fh_L(h_L(a)), F\eta_L(h_L(a))) \geq 1 \).
Moreover, since \( a = \bigcup \underline{\varepsilon} a \), we have that \( e_{FFL}(F\eta_L(h_L(a)), Fh_L(h_L(a))) \geq 1 \).
Therefore \( Fh_L \circ h_L = F\eta_L \circ h_L \).

5. Examples

Example 5.1. It is proved in [19] that \( \textbf{Q-Quant} \) is a reflective subcategory of \( \textbf{Q-OSgr} \). When \( K=\textbf{Q-OSgr} \), we can prove that a \( \textbf{Q-OSgr} \)-flat projective fuzzy quantale \( L \) is exactly the fuzzy weakly \( \otimes \)-stable completely distributive lattice (see [12]).

Definition 5.2. A fuzzy depo \((A, e_A)\) with an associative binary operator \( \otimes \) is called a pre-\( Q \)-quantale if for all \( a \in A \), \( a \otimes \_ \) : \( A \rightarrow A \) and \( \_ \otimes a : A \rightarrow A \) preserve joins of every \( Q \)-directed subset of \( A \).

Remark 5.3. Clearly, for \( Q = 2 \), a pre-\( Q \)-quantale is just a pre-quantale [15].

A map \( f : (X, \otimes_X, e_X) \rightarrow (Y, \otimes_Y, e_Y) \) between two pre-\( Q \)-quantales is called a pre-\( Q \)-quantale homomorphism if \( f(a \otimes_X b) = f(a) \otimes_Y f(b) \) and \( f(\sqcup S) = \sqcup f_X^-(S) \) for all \( a, b \in X, S \in D(X) \). Let \( \textbf{PQ-Quant} \) denote the category of pre-\( Q \)-quantales with pre-\( Q \)-quantale homomorphisms. Clearly, \( \textbf{PQ-Quant} \) is a subcategory of \( \textbf{Q-OSgr} \).

Definition 5.4. Let \((L, \otimes, e_L)\) be a \( Q \)-quantale. A \( Q \)-order-preserving map \( j : L \rightarrow L \) is called a pre-\( Q \)-nucleus if it satisfies the following conditions:

1. \( e_L(x, j(x)) \geq 1 \) for all \( x \in L \);
2. \( e_L(a \otimes j(b), j(a \otimes b)) \geq 1, e_L(j(a) \otimes b, j(a \otimes b)) \geq 1 \) for all \( a, b \in L \).

Definition 5.5. [19] Let \((L, \otimes, e_L)\) be a \( Q \)-quantale. A \( Q \)-order-preserving map \( j : L \rightarrow L \) is called a \( Q \)-nucleus if it satisfies the following conditions:
Proof. Let \( \text{Fix} \) be the set of fixed points \( \text{Fix} \).

Proposition 5.8. Let \((L, \otimes, e_L)\) be a \( Q \)-quantale. A subset \( S \subseteq L \) is called a quotient \( Q \)-quantale of \( L \) if there exists a \( Q \)-nucleus \( j \) on \( L \) such that \( \text{Im} j = S \).

Lemma 5.7. Let \((L, \otimes, e_L)\) be a \( Q \)-quantale, \( S \subseteq L \). Then \( S \) is closed under \( Q \)-inf and for all \( a \in L, s \in S, a \rightarrow_r s, a \rightarrow_l s \in S \) iff \( S \) is a quotient \( Q \)-quantale of \( L \).

Proposition 5.8. Let \((L, \otimes, e_L)\) be a \( Q \)-quantale and \( j \) be a pre-\( Q \)-nucleus. Then the set of fixed points \( \text{Fix}(j) \) of \( j \) is a quotient \( Q \)-quantale of \( L \).

Proof. Let \( i : \text{Fix}(j) \rightarrow L \) be the inclusion map. For all \( A \in Q^{\text{Fix}(j)}, \) since

\[
e_L(j(\sqcap Q^A(A)), \sqcap Q^A(A)) = \bigwedge_{a \in L} (i_Q^A(A) \rightarrow e_L(j(\sqcap Q^A(A)), a))
\]

\[
= \bigwedge_{a \in L} \bigwedge_{i(x) = a} (A(x) \rightarrow e_L(j(\sqcap Q^A(A)), a))
\]

\[
= \bigwedge_{x \in \text{Fix}(j)} (A(x) \rightarrow e_L(j(\sqcap Q^A(A)), i(x)))
\]

\[
\geq \bigwedge_{x \in \text{Fix}(j)} (A(x) \rightarrow e_L(\sqcap Q^A(A), x))
\]

\[
\geq \bigwedge_{x \in \text{Fix}(j)} (A(x) \rightarrow i_Q^A(A(x))
\]

\[
= \bigwedge_{x \in \text{Fix}(j)} (A(x) \rightarrow A(x))
\]

\[
\geq 1,
\]

and \( e_L(\sqcap Q^A(A), j(\sqcap Q^A(A))) \geq 1, j(\sqcap Q^A(A)) = \sqcap Q^A(A) \).

Moreover, for all \( a \in L, s \in \text{Fix}(j), \) since

\[
e_L(j(a \rightarrow_r s), a \rightarrow_r s) = e_L(j(a \rightarrow_r s), a \rightarrow_r s)
\]

\[
= e_L(a \rightarrow_r s, j(s))
\]

\[
\geq e_L(a \rightarrow_r s, s)
\]

\[
e_L(a \rightarrow_r s, a \rightarrow_r s)
\]

\[
\geq 1,
\]

and \( e_L(a \rightarrow_r s, j(a \rightarrow_r s)) \geq 1 \). Therefore, \( a \rightarrow_r s = j(a \rightarrow_r s) \). Similarly, we can prove \( a \rightarrow_l s = j(a \rightarrow_l s) \). Hence, \( \text{Fix}(j) \) is a quotient \( Q \)-quantale of \( L \). \( \square \)

Theorem 5.9. \( Q \)-Quant is a reflective subcategory of \( \text{PQ-Quant} \).

Proof. Let \((A, \cdot, e_A)\) be a pre-\( Q \)-quantale and \( \Upsilon(A) = \{ U \in D(A) \mid \text{for all } S \in D(A), \text{sub}_A(S, U) \leq U(\sqcup S) \} \).
(1) We define a map \( j : D(A) \rightarrow D(A) \) as follows:
\[
\forall U \in D(A), \ j(U) = k_U,
\]
where \( k_U : A \rightarrow Q \) is defined by
\[
\forall x \in A, \ k_U(x) = U(x) \lor \left( \bigvee_{S \in D(A)} \text{sub}_A(S, U) \land e_A(x, \sqcap S) \right).
\]
Then \( j \) is a pre-\( Q \)-nucleus and \( \Upsilon(A) = \text{Fix}(j) \). Thus \( \Upsilon(A) \) is a \( Q \)-quantale.

(2) Now, we define a map \( \delta_A : A \rightarrow \Upsilon(A) \) as follows:
\[
\forall a \in A, \ \delta_A(a) = \downarrow a.
\]
We can easily prove that \( \delta_A(x) \otimes \delta_A(y) = j(\downarrow (x \cdot y)) = \downarrow (x \cdot y) = \delta_A(x \cdot y) \). It remains to show that \( \delta_A(\sqcap X) = \sqcap (\delta_A)^{-}Q(X) \) for all \( X \in D(A) \).

For all \( X \in D(A), U \in \Upsilon(A) \). If \( U = \downarrow a \) for some \( a \in A \), then \( \text{sub}_A(\downarrow a, \downarrow (\sqcap X)) = e_A(a, \sqcap X) \). Hence \( X(a) \leq \text{sub}_A(\downarrow a, \sqcap (\sqcap X)) \). Thus \( (\delta_A)^{-}Q(X)(U) = \bigvee_{\delta_A(z) = U} X(z) \leq \text{sub}_A(U, \downarrow (\sqcap X)) \). For all \( Y \in \Upsilon(A), y \in A, Y(\sqcap X) \land e_A(y, \sqcap X) \leq Y(y) \), so we have \( Y(\sqcap X) \leq e_A(y, \sqcap X) \rightarrow Y(y) \), then \( Y(\sqcap X) \leq \bigwedge_{y \in A} (e_A(y, \sqcap X) \rightarrow Y(y)) = \text{sub}_A(\downarrow \sqcap X, Y) \). Since
\[
\bigwedge_{U \in \Upsilon(A)} \left( (\delta_A)^{-}Q(U) \rightarrow \text{sub}_A(U, Y) \right) = \bigwedge_{U \in \Upsilon(A)} \left( \left( \bigvee_{\delta_A(a) = U} X(a) \right) \rightarrow \text{sub}_A(U, Y) \right)
\]
\[
= \bigwedge_{a \in A} \left( X(a) \rightarrow \text{sub}_A(\downarrow a, Y) \right)
\]
\[
= \bigwedge_{a \in A} \left( X(a) \rightarrow \left( \bigwedge_{y \in A} (\downarrow a(y) \rightarrow Y(y)) \right) \right)
\]
\[
= \bigwedge_{a, y \in A} \left( X(a) \land e_A(y, a) \rightarrow Y(y) \right)
\]
\[
= \bigwedge_{y \in A} \left( \left( \bigvee_{a \in A} X(a) \land e_A(y, a) \right) \rightarrow Y(y) \right)
\]
\[
\leq \bigwedge_{y \in A} (X(y) \rightarrow Y(y))
\]
\[
\leq Y(\sqcap X).
\]
Then
\[
\bigwedge_{U \in \Upsilon(A)} ((\delta_A)^{-}Q(U) \rightarrow \text{sub}_A(U, Y)) \leq \text{sub}_A(\downarrow \sqcap X, Y), \text{ and thus } \delta_A(\sqcap X) = \sqcap (\delta_A)^{-}Q(X). \text{ Thus } \delta_A \text{ is a pre-}Q\text{-quantale homomorphism.}
\]

(3) Let \( X \) be a \( Q \)-quantale and \( g : A \rightarrow X \) be a pre-\( Q \)-quantale homomorphism. Define a map \( h : D(A) \rightarrow X \) as follows:
\[
\forall U \in D(A), \ h(U) = \sqcap g_Q^{-}(U).
\]
It is easily proved that \( h \) is a \( Q \)-quantale homomorphism and \( h \circ j = h \). Define a map \( f : \Upsilon(A) \rightarrow X \) as follows:
\[
\forall U \in \Upsilon(A), f(U) = h(U).
\]
For all $B \in Q^Y(A)$, we have that $f(\sqcup_{Y(A)} B) = h(\sqcup_{Y(A)} B) = h(j(\sqcup_{Y(A)} i_Q(B))) = h(\sqcup_{Y(A)} j_Q(B)) = \sqcup_{Y(A)} h_Q(B) = \sqcup h_Q(B) = \sqcup f_Q(B)$. For all $B, C \in Y(A)$, $f(B \otimes C) = h(j(B \otimes C)) = h(B \otimes h(C)) = h(B) \otimes h(C) = f(B) \otimes f(C)$. For all $x \in A$, $y \in X$,

$$\bigwedge_{a \in A} (g_Q(\downarrow x)(a) \rightarrow e_X(a, y)) = \bigwedge_{a \in A} \left( \left( \bigvee_{g(z) = a} \downarrow x(z) \right) \rightarrow e_X(a, y) \right)$$

$$= \bigwedge_{z \in A} (e_A(z, x) \rightarrow e_X(g(z), y))$$

$$\leq e_A(x, x) \rightarrow e_X(g(x), y)$$

$$= e_X(g(x), y).$$

Then $e_X(g(x), y) \& e_A(z, x) \leq e_X(g(x), y) \& e_X(g(z), g(x)) \leq e_X(g(z), y)$ for all $z \in A$. Hence $e_X(g(x), y) \leq \bigwedge_{z \in A} (e_A(z, x) \rightarrow e_X(g(z), y)) = \bigwedge_{a \in A} (g_Q(\downarrow x)(a) \rightarrow e_X(a, y))$.

Thus $f(\downarrow x) = g(x)$.

(4) Suppose there exists a $Q$-quantale homomorphism $l$ such that $l \circ \delta_A = g$. For all $X \in Y(A)$, we have $X = \sqcup (\delta_A)\_Q(X)$). Then $l(X) = l(\sqcup (\delta_A)\_Q(X)) = \sqcup l(\delta_A)\_Q(X) = \sqcup l(g_Q(X)) = f(X)$, $l = f$. □

**Remark 5.10.** Let $(A, \cdot, e)$ be a pre-$Q$-quantale. Then $Y(A)$ is $\text{PQ-Quant}$-flat projective. Moreover, by Theorem 5.9, we know that $\text{PQ-Quant}$ is an special case of $K$. In this case, suppose $L$ is a $Q$-quantale. Then $\downarrow a(x) = \bigwedge_{U \in Y(A)} (e_L(a, \sqcup U) \rightarrow U(x))$ for all $a, x \in L$. Thus $L$ is $\text{PQ-Quant}$-flat projective iff $a = \sqcup \downarrow a$ and $\downarrow a(x) \& \downarrow b(y) \leq \downarrow (a \otimes_L b)(x \otimes_L y)$.

## 6. Conclusions

In this paper, we obtain some characterizations of the $K$-flat projective fuzzy quantales. Especially, we prove that a $Q$-quantale $L$ is $K$-flat projective iff it has a coalgebra structure for the $(F, \varepsilon, F\eta)$. Furthermore, we present two examples for special cases of $K$. In further work, we can pursue to characterize projective $Q$-quantales. That is, we hope to find a satisfactory sufficient and necessary condition for a $Q$-quantale to be projective.

**Acknowledgements.** The authors would like to thank the anonymous referees for their valuable comments and suggestions. This work is supported by the National Natural Science Foundation of China (Grant nos. 11531009, 11301316) and the Fundamental Research Funds for the Central Universities (Grant no. GK201501001).

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