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K-FLAT PROJECTIVE FUZZY QUANTALES

J. LU, K. WANG AND B. ZHAO

Abstract. In this paper, we introduce the notion of **K**-flat projective fuzzy quantales, and give an elementary characterization in terms of a fuzzy binary relation on the fuzzy quantale. Moreover, we prove that **K**-flat projective fuzzy quantales are precisely the coalgebras for a certain comonad on the category of fuzzy quantales. Finally, we present two special cases of **K** as examples.

1. **Introduction**

Quantales were introduced by Mulvey in order to provide a lattice-theoretic setting for studying non-commutative C^*-algebra, as well as constructive foundations for quantum mechanics [13, 14]. The study that analyzed partially ordered algebraic structure goes back to a series of papers by Ward and Dilworth [4, 20, 22] in the 1930s. Since the theory of quantales provides a powerful tool in studying non-commutative structures, following Mulvey, various types and aspects of quantales have been considered by many researchers [9, 16].

Since Zadeh introduced fuzzy sets to model the uncertainty associated with the concept of imprecision [27], several extensions of fuzzy sets have been introduced. In order to establish a fuzzy counterpart of the Isbell-adjunction between topological spaces and locales [8], based on a frame L, Yao [25, 26] introduced an L-frame as an L-ordered set equipped with some further conditions and proved that the categories of Zhang-Liu-L-frames, Yao-L-frames and L-algebras are isomorphic. Based on the work of Yao, for a unital commutative quantale Q, Wang and Zhao [18] defined a Q-quantale as a Q-ordered semigroup equipped with some further conditions, and they also showed that the category of Q-quantales is isomorphic to the category of Q-algebras [17, 21]. The study of injectivity and projectivity in quantales was initiated by Li, Zhou and Li [11]. Banaschewski [2] established both the internal and the external characterization of "projective" objects in the category of frames. Moreover, Paseka [15] proposed a general view of projective quantales in the spirit of Banaschewski. Following this viewpoint, we shall present a general view with respect to the projectivity notion in the category of Q-quantales.

The content of the paper is organized as follows. Section 2 lists some preliminary notions and results about fuzzy posets. In Section 3, we discuss the relations between **Q-Quant**-morphisms and **K**-morphisms. In Section 4, we give an elementary characterization of a **K**-flat projective Q-quantale in terms of a Q-binary

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relation on the Q-quantale. We also prove that the **K**-flat projective Q-quantales are precisely the coalgebras for a certain comonad on the category of Q-quantales. In Section 5, we consider two special cases of **K**.

2. Preliminaries

We refer to [16] for quantale theory, to [1, 7] for category theory, to [6, 8] for lattice theory and to [3, 5, 10, 23, 24, 28] for fuzzy orders.

Definition 2.1. [16] A quantale is a complete lattice Q with an associative binary operation & satisfying

$$a\&\Big(\bigvee_{i\in I} b_i\Big) = \bigvee_{i\in I}(a\&b_i) \quad and \quad \Big(\bigvee_{i\in I} b_i\Big)\&a = \bigvee_{i\in I}(b_i\&a)$$

for all $a \in Q$ and $\{b_i\}_{i\in I} \subseteq Q$.

A quantale Q is said to be unital provided that there exists an element $1 \in Q$ such that $a\&1 = 1\&a = a$ for all $a \in Q$. Q is said to be commutative provided that $a\&b = b\&a$ for all $a, b \in Q$. From now on, unless otherwise stated, Q always denotes a unital commutative quantale. Since the map $a\&\_$ preserves arbitrary sups, it has right adjoint, which we shall denote by $a \to \_$. Thus, $a\&c \leq b$ iff $c \leq a \to b$ for all $a, b, c \in Q$.

Definition 2.2. [3, 5] Let X be a set. A map $e : X \times X \longrightarrow Q$ is called a fuzzy order (or, Q-order) on X if for all $x, y, z \in X$,

(E1) $e(x, x) \geq 1$ (reflexivity);

(E2) $e(x, y)\&e(y, z) \leq e(x, z)$ (transitivity);

(E3) $e(x, y) \geq 1$, $e(y, x) \geq 1$ imply $x = y$ (antisymmetry).

The pair (X, e) is called a Q-ordered set.

Let $(X, \leq)$ be a classical poset. Then $(X, e_{\leq})$ is a Q-ordered set, where $e_{\leq}$ is defined as follows:

$$e_{\leq}(x, y) = \begin{cases} 1, & x \leq y, \\ 0, & \text{otherwise.} \end{cases}$$

For a Q-ordered set (X, e), $\leq_e = \{(x, y) \mid e(x, y) \geq 1\}$ is a crisp partial order on X. Unless otherwise stated, throughout the paper, whenever a partial order is mentioned in the context of a Q-ordered set (X, e), it is to be interpreted with respect to the crisp partial order on X. We often denote the crisp partial order on (X, e) by $\leq$ if there would be no confusion.

Let X be a set. Q^X denotes the set of all Q-subsets of X. For all $A, B \in Q^X$, the subsethood degree of A in B is defined by $sub_X(A, B) = \bigwedge_{x\in X}(A(x) \to B(x))$. Then (Q^X, sub_X) is a Q-ordered set.

A map $f : (X, e_X) \longrightarrow (Y, e_Y)$ between Q-ordered sets is called Q-order-preserving if $e_X(x_1, x_2) \leq e_Y(f(x_1), f(x_2))$ for all $x_1, x_2 \in X$.

Definition 2.3. [28] Let (X, e_X) be a Q-ordered set, $x \in X$ and $A \in Q^X$. The element x_0 is called a join (resp., meet) of A, in symbols $x_0 = \sqcup A$ (resp., $x_0 = \sqcap A$), if

(1) For all $x \in X$, $A(x) \leq e_X(x, x_0)$ (resp., $A(x) \leq e_X(x_0, x)$);

(2) For all $y \in X$, $\bigwedge_{x\in X}(A(x) \to e_X(x, y)) \leq e_X(x_0, y)$ (resp., $\bigwedge_{x\in X}(A(x) \to e_X(y, x)) \leq e_X(y, x_0)$).

Proposition 2.4. [23] *Let (X, e_X) be a Q-ordered set, $x_0, x_1 \in X$ and $A \in Q^X$.*
(1) $x_0 = \sqcup A$ iff for all $y \in X$, $e_X(x_0, y) = \bigwedge_{x\in X}(A(x) \to e_X(x, y))$;
(2) $x_1 = \sqcap A$ iff for all $y \in X$, $e_X(y, x_1) = \bigwedge_{x\in X}(A(x) \to e_X(y, x))$.

Definition 2.5. [28] Let (X, e_X) be a Q-ordered set, and $A \in Q^X$. A is called a lower Q-subset of X if $e_X(x, y)\&A(y) \leq A(x)$ for all $x, y \in X$. Let $D(X)$ denote the collection of all lower Q-subsets of X.

Definition 2.6. [28] A Q-ordered set (X, e_X) is called a complete Q-lattice if $\sqcup A$ and $\sqcap A$ exist for all $A \in Q^X$.

Remark 2.7. Let (X, e_X) be a complete Q-lattice. Then $(X, \leq_{e_X})$ is a complete lattice.

Proposition 2.8. [23] *Let (X, e_X) be a Q-ordered set. The following statements are equivalent:*
(1) (X, e_X) is a complete Q-lattice;
(2) For all $A \in Q^X$, $\sqcup A$ exists;
(3) For all $A \in Q^X$, $\sqcap A$ exists.

Definition 2.9. [23] Let (X, e_X), (Y, e_Y) be two Q-ordered sets and $f : X \longrightarrow Y$, $g : Y \longrightarrow X$ two Q-order-preserving maps. The pair (f, g) is called a Q-adjunction between X and Y if for all $x \in X$, $y \in Y$, $e_Y(f(x), y) = e_X(x, g(y))$. In this case, f is called the left adjoint and dually g is called the right adjoint.

Lemma 2.10. *Let (X, e_X), (Y, e_Y) be two Q-ordered sets and $f : X \longrightarrow Y$, $g : Y \longrightarrow X$ two Q-order-preserving maps. If $f \circ g = id_Y$ and $id_X \leq g \circ f$, then (f, g) is a Q-adjunction.*

Proof. For all $x \in X$ and $y \in Y$, we have that $e_Y(f(x), y) \leq e_X(g(f(x)), g(y))\&1 \leq e_X(g(f(x)), g(y))\&e_X(x, g(f(x))) \leq e_X(x, g(y)) \leq e_Y(f(x), f(g(y))) = e_Y(f(x), y)$. Then $e_Y(f(x), y) = e_X(x, g(y))$, and thus (f, g) is a Q-adjunction. □

Let X, Y be sets and $f : X \longrightarrow Y$ be a map. Then the Zadeh forward power set operator $f_Q^{\rightarrow} : Q^X \longrightarrow Q^Y$ and the Zadeh backward power set operator $f_Q^{\leftarrow} : Q^Y \longrightarrow Q^X$ are defined, respectively, by $f_Q^{\rightarrow}(A)(y) = \bigvee_{f(x)=y} A(x)$, $f_Q^{\leftarrow}(B) = B \circ f$ for all $A \in Q^X$, $y \in Y$, $B \in Q^Y$. It can be easily seen (see [23]) that $(f_Q^{\rightarrow}, f_Q^{\leftarrow})$ is a Q-adjunction between (Q^X, sub_X) and (Q^Y, sub_Y).

Definition 2.11. [18, 19] A complete Q-lattice (X, e) with an associative binary operation $\otimes$ is called a fuzzy quantale (or simply, Q-quantale) if for all $a \in X$, $a\otimes\_ : X \longrightarrow X$ and $\_ \otimes a : X \longrightarrow X$ preserve joins of every Q-subset of X, that is, for all $A \in Q^X$, $a \otimes (\sqcup A) = \sqcup(a\otimes\_)_Q^{\rightarrow}(A)$, $(\sqcup A) \otimes a = \sqcup(\_\otimes a)_Q^{\rightarrow}(A)$.

Let X be a Q-quantale. $S \subseteq X$ is called a sub-Q-quantale of X provided that S is closed under joins of every Q-subset of S and $\otimes$. Let $\mathcal{S}$ denote the set of all sub-Q-quantales of X. For $A \subseteq X$, $\bigcap\{S \in \mathcal{S} | A \subseteq S\}$ is a sub-Q-quantale of X, called the sub-Q-quantale generated by A, denoted by $\langle A \rangle$.

Since $a \otimes \_$ and $\_ \otimes a$ preserve joins of every Q-subset of X, they have right adjoint (see [23]), which we shall denote by $a \rightarrow_r -$, $a \rightarrow_l -$ respectively. A map $f : (X, \otimes_X, e_X) \longrightarrow (Y, \otimes_Y, e_Y)$ between two Q-quantales is called a Q-quantale homomorphism if $f(a \otimes_X b) = f(a) \otimes_Y f(b)$ and $f(\sqcup A) = \sqcup f_Q^{\rightarrow}(A)$ for all $a, b \in X$, $A \in Q^X$. Let **Q-Quant** denote the category of Q-quantales with Q-quantale homomorphisms.

Remark 2.12. Any Q-quantale (resp., sub-Q-quantale) is a quantale (resp., sub-quantale) with respect to the crisp partial order. Similarly, any Q-quantale homomorphism is a quantale homomorphism.

Let (X, e) be a Q-quantale. If $S \subseteq X$ is a sub-Q-quantale of X, and $i : S \longrightarrow X$ is defined as follows:

$$\forall\, x \in S,\ i(x) = x,$$

then i is a Q-quantale homomorphism. We call i an identical sub-Q-quantale embedding.

Definition 2.13. [10, 24] Let (X, e) be a Q-ordered set and $D \in Q^X$. D is called a Q-directed subset of X if

(1) $\bigvee_{x \in X} D(x) \geq 1$;

(2) $D(x) \& D(y) \leq \bigvee_{z \in X} (D(z) \& e(x, z) \& e(y, z))$ for all $x, y \in X$.

Let $\mathcal{D}(X)$ denote the collection of all Q-directed subsets of X.

Definition 2.14. [24] A Q-ordered set (X, e) is called a fuzzy dcpo if $\sqcup A$ exists for all $A \in \mathcal{D}(X)$.

3. The Relations Between Q-Quant-morphisms and K-morphisms

A Q-ordered set (X, e_X) with an associative binary operation $\star$ is called a Q-ordered semigroup if $e_X(a, b) \leq e_X(a \star c, b \star c)$ and $e_X(a, b) \leq e_X(c \star a, c \star b)$ for all $a, b, c \in X$. A Q-order-preserving map $f : (X, e_X, \star_X) \longrightarrow (Y, e_Y, \star_Y)$ between Q-ordered semigroups is called a Q-ordered semigroup homomorphism if $f(a \star_X b) = f(a) \star_Y f(b)$ for all $a, b \in X$. Let **Q-OSgr** denote the category of Q-ordered semigroups with Q-ordered semigroup homomorphisms. Clearly, **Q-Quant** is a subcategory of **Q-OSgr** (see [19]). Now we consider the category **K**, which is a subcategory of **Q-OSgr**. Moreover, **K** contains the category **Q-Quant** reflectively, subject to the following condition:

(C) For any $\phi : A \longrightarrow L$ in **K** where L is a Q-quantale and A arbitrary, the corestriction of ϕ to any sub-Q-quantale of L containing the image of ϕ also belongs to **K**.

Remark 3.1. For any object A in **K**, we have a universal map $\eta_A : A \longrightarrow FA$ in **K**. In particular, for any Q-quantale L, there exists a unique Q-quantale homomorphism $\varepsilon_L : FL \longrightarrow L$, such that $\varepsilon_L \circ \eta_L = id_L$.

Proposition 3.2. *Let A be an object in* **K**. *Then FA is generated by the image $Im\eta_A$ of A.*

Proof. Let $M \subseteq FA$ be the sub-Q-quantale generated by $Im\eta_A$, $\phi : A \longrightarrow M$ be the corestriction of $\eta_A : A \longrightarrow FA$, and $i : M \longrightarrow FA$ be the identical sub-Q-quantale embedding. Then by (C) we have a unique Q-quantale homomorphism $h : FA \longrightarrow M$ such that $h \circ \eta_A = \phi$. Hence $i \circ h \circ \eta_A = \eta_A$. By the universal property of η_A we get i is onto. Thus $M = FA$. □

Corollary 3.3. *Let A be an object in* **K** *and $b \in FA$. Define a map $k_b : FA \longrightarrow Q$ as follows:*

$$\forall\ y \in FA,\ k_b(y) = \begin{cases} e_{FA}(y, b), & y \in Im\eta_A, \\ 0, & \textit{otherwise.} \end{cases}$$

Then $b = \sqcup k_b$.

Proof. Let $\mathcal{A} = \{\sqcup B \mid B \in Q^{Im\eta_A}\}$. For all $B_1,\ B_2 \in Q^{Im\eta_A}$, we define a map $B_1 \oplus B_2 : Im\eta_A \longrightarrow Q$ as follows:

$$\forall\ y \in Im\eta_A,\ (B_1 \oplus B_2)(y) = \bigvee_{a \otimes_{FL} b = y, a, b \in Im\eta_A} (B_1(a) \& B_2(b)).$$

We can check that $(\sqcup B_1) \otimes_{FL} (\sqcup B_1) = \sqcup(B_1 \oplus B_2)$. For any $\mathcal{B} \in Q^{\mathcal{A}}$, $z \in FA$, we have that

$$\begin{aligned}
\bigwedge_{x \in FA} \big(\mathcal{B}(x) \to e_{FA}(x, z)\big) &= \bigwedge_{x \in \mathcal{A}} (\mathcal{B}(x) \to e_{FA}(x, z)) \\
&= \bigwedge_{\sqcup B \in \mathcal{A}} (\mathcal{B}(\sqcup B) \to e_{FA}(\sqcup B, z)) \\
&= \bigwedge_{\sqcup B \in \mathcal{A}} \Big(\mathcal{B}(\sqcup B) \to \Big(\bigwedge_{a \in Im\eta_A} (B(a) \to e_{FA}(a, z))\Big)\Big) \\
&= \bigwedge_{\sqcup B \in \mathcal{A}} \bigwedge_{a \in Im\eta_A} ((\mathcal{B}(\sqcup B) \& B(a)) \to e_{FA}(a, z)) \\
&= \bigwedge_{a \in Im\eta_A} \Big(\Big(\bigvee_{B \in Q^{Im\eta_A}} \mathcal{B}(\sqcup B) \& B(a)\Big) \to e_{FA}(a, z)\Big) \\
&= e_{FA}(\sqcup(\bigvee_{B \in Q^{Im\eta_A}} \mathcal{B}(\sqcup B) \& B), z).
\end{aligned}$$

Then $\sqcup \mathcal{B} = \sqcup(\bigvee_{B \in Q^{Im\eta_A}} \mathcal{B}(\sqcup B) \& B)$, and thus $\mathcal{A}$ is a sub-Q-quantale. For any $y \in Im\eta_A$, we define a map $\chi_{\{y\}} : Im\eta_A \longrightarrow Q$ as follows:

$$\forall\ x \in Im\eta_A,\ \chi_{\{y\}}(x) = \begin{cases} 1, & x = y, \\ 0, & \text{otherwise.} \end{cases}$$

It follows that $\sqcup \chi_{\{y\}} = y$. This shows that $Im\eta_A \subseteq \mathcal{A}$. Let Y be a sub-Q-quantale of FA with $Im\eta_A \subseteq Y$, and $B \in Q^{Im\eta_A}$. Define a map $C : Y \longrightarrow Q$ as follows:

$$\forall\ y \in Y,\ C(y) = \begin{cases} B(y), & y \in Im\eta_A, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\sqcup B = \sqcup C$, and thus $\mathcal{A} \subseteq Y$. This means $\langle Im\eta_A \rangle = \mathcal{A}$. By Proposition 3.2, we have that $FA = \mathcal{A}$. For every $b \in FA$, there exists $B \in Q^{Im\eta_A}$ such that $b = \sqcup B$. For all $y \in Im\eta_A$, $B(y) \leq k_b(y)$. Then $\sqcup B \leq \sqcup k_b \leq b$, and thus $\sqcup k_b = b$. □

Proposition 3.4. *Let L be a Q-quantale. Then $id_{FL} \leq \eta_L \circ \varepsilon_L$.*

Proof. Assume that $b \in FL$. By Corollary 3.3, $b = \sqcup k_b$. Now, for all $y \in Im\eta_L$, there exists $a \in L$ such that $y = \eta_L(a)$. Thus $(\eta_L \circ \varepsilon_L)(y) = (\eta_L \circ \varepsilon_L)(\eta_L(a)) = \eta_L(\varepsilon_L(\eta_L(a))) = \eta_L(a) = y$. Since $\eta_L \circ \varepsilon_L$ is Q-order-preserving, we have

$$
\begin{aligned}
&e_{FL}(b, (\eta_L \circ \varepsilon_L)(b)) \\
&= e_{FL}(\sqcup k_b, (\eta_L \circ \varepsilon_L)(b)) \\
&= \bigwedge_{y \in FL} (k_b(y) \rightarrow e_{FL}(y, (\eta_L \circ \varepsilon_L)(b))) \\
&= \bigwedge_{y \in Im\eta_L} (e_{FL}(y, b) \rightarrow e_{FL}(y, (\eta_L \circ \varepsilon_L)(b))) \\
&\geq \bigwedge_{y \in Im\eta_L} (e_{FL}((\eta_L \circ \varepsilon_L)(y), (\eta_L \circ \varepsilon_L)(b)) \rightarrow e_{FL}(y, (\eta_L \circ \varepsilon_L)(b))) \\
&= \bigwedge_{y \in Im\eta_L} (e_{FL}(y, (\eta_L \circ \varepsilon_L)(b)) \rightarrow e_{FL}(y, (\eta_L \circ \varepsilon_L)(b))) \\
&\geq 1,
\end{aligned}
$$

consequently, $id_{FL} \leq \eta_L \circ \varepsilon_L$. □

Corollary 3.5. *Let L be a Q-quantale. Then (ε_L, η_L) is a Q-adjunction between FL and L.*

Proposition 3.6. *Let L be a Q-quantale, A an object in* **K**, *and $f,\ g : FA \longrightarrow L$ Q-quantale homomorphisms. Then $f \circ \eta_A \leq g \circ \eta_A$ implies $f \leq g$.*

Proof. Since $f \circ \eta_A \leq g \circ \eta_A$, $e_L((f \circ \eta_A)(d), (g \circ \eta_A)(d)) \geq 1$ for all $d \in A$. For all $b \in FA$, we have $b = \sqcup k_b$, $f(b) = f(\sqcup k_b) = \sqcup f_Q^{\rightarrow}(k_b)$, $g(b) = g(\sqcup k_b) = \sqcup g_Q^{\rightarrow}(k_b)$. Thus

$$
\begin{aligned}
e_L(f(b), g(b)) &= e_L(\sqcup f_Q^{\rightarrow}(k_b), g(b)) \\
&= \bigwedge_{x \in L} \Big(\Big(\bigvee_{f(a)=x} k_b(a) \Big) \rightarrow e_L(x, g(b)) \Big) \\
&= \bigwedge_{a \in FA} (k_b(a) \rightarrow e_L(f(a), g(b))) \\
&= \bigwedge_{a \in Im\eta_A} (e_{FA}(a, b) \rightarrow e_L(f(a), g(b))) \\
&\geq \bigwedge_{a \in Im\eta_A} ((e_L(g(a), g(b)) \& e_L(f(a), g(a))) \rightarrow e_L(f(a), g(b))) \\
&= \bigwedge_{a \in Im\eta_A} (e_L(g(a), g(b)) \rightarrow (e_L(f(a), g(a)) \rightarrow e_L(f(a), g(b)))) \\
&\geq \bigwedge_{a \in Im\eta_A} (e_L(g(a), g(b)) \rightarrow e_L(g(a), g(b))) \\
&\geq 1.
\end{aligned}
$$

That is, $f \leq g$. □

Proposition 3.7. *Let L be a Q-quantale and $h : L \longrightarrow FL$ be a right inverse of $\varepsilon_L : FL \longrightarrow L$. Then $h \circ \varepsilon_L \leq id_{FL}$.*

Proof. For all $x \in L$, since $e_{FL}((h\circ\varepsilon_L)(\eta_L(x)), \eta_L(x)) = e_{FL}((h\circ(\varepsilon_L\circ\eta_L))(x), \eta_L(x))$ $= e_{FL}(h(x), \eta_L(x)) = e_{FL}(h(x), (\eta_L \circ (\varepsilon_L \circ h))(x)) = e_{FL}(h(x), (\eta_L \circ \varepsilon_L)(h(x))) \geq 1$, we have that $e_{FL}((h \circ \varepsilon_L)(a), a) \geq 1$ for all $a \in Im\eta_L$. For all $b \in FL$, $e_{FL}((h \circ \varepsilon_L)(b), b)$

$$\begin{aligned}
&= e_{FL}((h \circ \varepsilon_L)(\sqcup k_b), b) \\
&= e_{FL}(\sqcup (h \circ \varepsilon_L)^{\rightarrow}_Q(k_b), b) \\
&= \bigwedge_{y\in FL} ((h \circ \varepsilon_L)^{\rightarrow}_Q(k_b)(y) \rightarrow e_{FL}(y, b)) \\
&= \bigwedge_{y\in FL} \Big(\Big(\bigvee_{(h\circ\varepsilon_L)(a)=y} k_b(a)\Big) \rightarrow e_{FL}(y, b)\Big) \\
&= \bigwedge_{a\in Im\eta_A} (e_{FL}(a, b) \rightarrow e_{FL}((h \circ \varepsilon_L)(a), b)) \\
&\geq \bigwedge_{a\in Im\eta_A} ((e_{FL}(a, b)\& e_{FL}((h \circ \varepsilon_L)(a), a)) \rightarrow e_{FL}((h \circ \varepsilon_L)(a), b)) \\
&= \bigwedge_{a\in Im\eta_A} (e_{FL}(a, b) \rightarrow (e_{FL}((h \circ \varepsilon_L)(a), a) \rightarrow e_{FL}((h \circ \varepsilon_L)(a), b))) \\
&\geq \bigwedge_{a\in Im\eta_A} (e_{FL}(a, b) \rightarrow e_{FL}(a, b)) \\
&\geq 1.
\end{aligned}$$

Thus $h \circ \varepsilon_L \leq id_{FL}$. □

Proposition 3.8. *Let A, B be objects in **K** and $g : A \longrightarrow B$ be a **K**-morphism. Suppose L, P are Q-quantales and $f : L \longrightarrow P$ is a Q-quantale homomorphism. Then the following statements hold:*

(1) $Fg \circ \eta_A = \eta_B \circ g$;

(2) $f \circ \varepsilon_L = \varepsilon_P \circ Ff$;

*(3) If $\phi, \varphi : A \longrightarrow B$ are **K**-morphisms and $\phi \leq \varphi$, then $F\phi \leq F\varphi$;*

(4) $(F\eta_A, \varepsilon_{FA})$ is a Q-adjunction between FA and FFA.

Proof. (1) The statement is straightforward since **K** contains the category **Q-Quant** reflectively.

(2) Since $\eta_P \circ f = Ff \circ \eta_L$, $\varepsilon_P \circ \eta_P \circ f = \varepsilon_P \circ Ff \circ \eta_L$, $f = \varepsilon_P \circ Ff \circ \eta_L$, $f \circ \varepsilon_L \circ \eta_L = \varepsilon_P \circ Ff \circ \eta_L$, therefore $f \circ \varepsilon_L = \varepsilon_P \circ Ff$ by the universal property of η_L.

(3) Let $\phi \leq \varphi$. For $a \in A$. By (1), we have that $e_{FB}((F\phi\circ\eta_A)(a), (\eta_B\circ\phi)(a)) \geq 1$, and $e_{FB}((\eta_B \circ \varphi)(a), (F\varphi \circ \eta_A)(a)) \geq 1$. Thus, by transitivity of e_{FB} and the fact that $\eta_B(\phi(a)) \leq \eta_B(\varphi(a))$, $e_{FB}(F\phi(\eta_A(a)), F\varphi(\eta_A(a))) \geq 1$. By Proposition 3.6, we have that $F\phi \leq F\varphi$.

(4) Since $id_{FA} \circ \eta_A = \varepsilon_{FA} \circ \eta_{FA} \circ \eta_A = \varepsilon_{FA} \circ F\eta_A \circ \eta_A$, we have that $id_{FA} = \varepsilon_{FA} \circ F\eta_A$. By Proposition 3.7, we have that $(F\eta_A, \varepsilon_{FA})$ is a Q-adjunction between FA and FFA. □

4. K-flat Projective Q-quantales

Definition 4.1. A Q-quantale L is said to be projective if for any Q-quantale homomorphism $f : L \longrightarrow M$ and an epimorphism $g : N \longrightarrow M$ in **Q-Quant**, there exists a Q-quantale homomorphism $h : L \longrightarrow N$ such that $f = g \circ h$.

Definition 4.2. A Q-quantale L is called **K**-flat projective if L is projective in **Q-Quant** relative to the onto Q-quantale homomorphism $h : N \longrightarrow M$ for which the right adjoint $h_* : M \longrightarrow N$ belongs to **K**.

Remark 4.3. A Q-quantale L is a **K**-flat projective Q-quantale if L is a projective Q-quantale.

Definition 4.4. Let L be a Q-quantale and $a \in L$. Define a map $\Downarrow a : L \longrightarrow Q$ as follows:

$$\forall\, x \in L, \Downarrow a(x) = \bigwedge_{b \in FL} (e_L(a, \varepsilon_L(b)) \to e_{FL}(\eta_L(x), b)).$$

We call $\Downarrow : L \times L \longrightarrow Q$ a Q-binary relation on the Q-quantale L.

Lemma 4.5. *Let L, P be Q-quantales. Then for all $a, x, y, u, v \in L$,*

(1) $\Downarrow a \leq \downarrow a$;

(2) $e_L(x, y) \& \Downarrow u(y) \& e_L(u, v) \leq \Downarrow v(x)$.

Proof. (1) For all $m \in L$, we have that

$$\begin{aligned} \Downarrow a(m) &= \bigwedge_{b \in FL} (e_L(a, \varepsilon_L(b)) \to e_{FL}(\eta_L(m), b)) \\ &\leq e_L(a, \varepsilon_L(\eta_L(a))) \to e_{FL}(\eta_L(m), \eta_L(a)) \\ &= e_L(a, a) \to e_{FL}(\eta_L(m), \eta_L(a)) \\ &\leq e_L(\varepsilon_L(\eta_L(m)), \varepsilon_L(\eta_L(a))) \\ &= \downarrow a(m). \end{aligned}$$

Thus $\Downarrow a \leq \downarrow a$.

(2) For all $b \in FL$, we have that $e_L(x, y) \& e_L(u, v) \& (e_L(u, \varepsilon_L(b)) \to e_{FL}(\eta_L(y), b))$ $\& e_L(v, \varepsilon_L(b)) \leq e_L(x, y) \& e_{FL}(\eta_L(y), b) \leq e_{FL}(\eta_L(x), \eta_L(y)) \& e_{FL}(\eta_L(y), b) \leq e_{FL}(\eta_L(x), b)$. Then $e_L(x, y) \& e_L(u, v) \& (e_L(u, \varepsilon_L(b)) \to e_{FL}(\eta_L(y), b)) \leq e_L(v, \varepsilon_L(b)) \to e_{FL}(\eta_L(x), b)$. So we can conclude that $e_L(x, y) \& e_L(u, v) \& \bigwedge_{b \in FL} (e_L(u, \varepsilon_L(b)) \to e_{FL}(\eta_L(y), b)) \leq \bigwedge_{b \in FL} (e_L(v, \varepsilon_L(b)) \to e_{FL}(\eta_L(x), b)) = \Downarrow v(x)$. □

Theorem 4.6. *Let L be a Q-quantale. Then the following statements are equivalent:*

*(1) L is **K**-flat projective;*

(2) ε_L has a right inverse;

*(3) There exists an object A in **K** such that L is a retraction of FA;*

(4) $a = \sqcup \Downarrow a$ and $\Downarrow a(x) \& \Downarrow b(y) \leq \Downarrow (a \otimes_L b)(x \otimes_L y)$ for all $a, b, x, y \in L$.

Proof. (1)⇒(2) Since $\varepsilon_L \circ \eta_L = id_L$, we have that ε_L is an onto Q-quantale homomorphism. By Corollary 3.5, we have that ε_L has a right adjoint η_L, which belongs to **K**. Since L is **K**-flat projective, there exists a Q-quantale homomorphism $h : L \longrightarrow FL$ such that $\varepsilon_L \circ h = id_L$.

(2)⇒(3) Let $A = L$. Then A is an object in **K**. By (2), we have that L is a retract of FL.

(3)⇒(1) By (3), there exist two Q-quantale homomorphisms $n : FA \longrightarrow L$, $j : L \longrightarrow FA$ such that $n \circ j = id_L$. Firstly, let P, T be Q-quantales and the onto $Q-$quantale homomorphism $h : P \longrightarrow T$ for which the right adjoint $h_* : T \longrightarrow P$ belongs to **K**, $f : FA \longrightarrow T$ be a Q-quantale homomorphism. Then $h_* \circ f \circ \eta_A$ belongs to **K**, there exists a unique Q-quantale homomorphism g such that $h_* \circ f \circ \eta_A = g \circ \eta_A$, $h \circ h_* \circ f \circ \eta_A = h \circ g \circ \eta_A$. By the universal of η_A, we have $f = h \circ g$. Thus FA is **K**-flat projective. Moreover, let $m : L \longrightarrow T$ be a Q-quantale homomorphism, then $m \circ n : FA \longrightarrow T$ is a Q-quantale homomorphism, so there exists a Q-quantale homomorphism $p : FA \longrightarrow P$ such that $h \circ p = m \circ n$, $(h \circ p) \circ j = (m \circ n) \circ j$, $h \circ (p \circ j) = m \circ (n \circ j) = m \circ id_L = m$. Thus L is **K**-flat projective.

(2)⇒(4) Let h be a right inverse of ε_L. For all $a, x \in L$, we have

$$\begin{aligned}
\Downarrow a(x) &= \bigwedge_{b \in FL} (e_L(a, \varepsilon_L(b)) \to e_{FL}(\eta_L(x), b)) \\
&\leq e_L(a, \varepsilon_L(h(a))) \to e_{FL}(\eta_L(x), h(a)) \\
&= e_L(a, a) \to e_{FL}(\eta_L(x), h(a)) \\
&\leq 1 \to e_{FL}(\eta_L(x), h(a)) \\
&\leq e_{FL}(\eta_L(x), h(a)).
\end{aligned}$$

For all $b \in FL$, by Proposition 3.7, we can conclude that

$$\begin{aligned}
e_{FL}(\eta_L(x), h(a)) \& e_L(a, \varepsilon_L(b)) &\leq e_{FL}(\eta_L(x), h(a)) \& e_{FL}(h(a), h(\varepsilon_L(b))) \\
&\leq e_{FL}(\eta_L(x), h(\varepsilon_L(b))) \& e_{FL}(h(\varepsilon_L(b)), b) \\
&\leq e_{FL}(\eta_L(x), b).
\end{aligned}$$

Then $e_{FL}(\eta_L(x), h(a)) \leq \bigwedge_{b \in FL} (e_L(a, \varepsilon_L(b)) \to e_{FL}(\eta_L(x), b)) = \Downarrow a(x)$, and thus $\Downarrow a(x) = e_{FL}(\eta_L(x), h(a))$. Since $h(a) = \sqcup k_{h(a)}$ and $\varepsilon_L \circ h = id_L$, we have that $a = (\varepsilon_L \circ h)(a) = \varepsilon_L(\sqcup k_{h(a)}) = \sqcup (\varepsilon_L)^{\rightarrow}_Q(k_{h(a)})$. For all $y \in L$,

$$\begin{aligned}
e_L(a, y) &= \bigwedge_{d \in L} ((\varepsilon_L)^{\rightarrow}_Q(k_{h(a)})(d) \to e_L(d, y)) \\
&= \bigwedge_{d \in L} \bigwedge_{\varepsilon_L(p) = d} (k_{h(a)}(p) \to e_L(d, y)) \\
&= \bigwedge_{p \in FL} (k_{h(a)}(p) \to e_L(\varepsilon_L(p), y)) \\
&= \bigwedge_{p \in Im \eta_L} (e_{FL}(p, h(a)) \to e_L(\varepsilon_L(p), y)) \\
&= \bigwedge_{d \in L} (e_{FL}(\eta_L(d), h(a)) \to e_L(d, y)) \\
&= \bigwedge_{d \in L} (\Downarrow a(d) \to e_L(d, y)).
\end{aligned}$$

This means that $a = \sqcup \Downarrow a$.

$$\begin{aligned}\Downarrow a(x) \& \Downarrow b(y) &= e_{FL}(\eta_L(x), h(a)) \& e_{FL}(\eta_L(y), h(b)) \\ &\leq e_{FL}(\eta_L(x) \otimes_{FL} \eta_L(y), h(a) \otimes_{FL} h(b)) \\ &= e(\eta_L(x \otimes_L y), h(a \otimes_L b)) \\ &= \Downarrow (a \otimes_L b)(x \otimes_L y).\end{aligned}$$

(4)⇒(2) Define a map $h_L : L \longrightarrow FL$ as follows:

$$\forall\ a \in L,\ h_L(a) = \sqcup A_a,$$

where $A_a : FL \longrightarrow Q$ is defined by

$$\forall\ b \in FL,\ A_a(b) = \bigvee_{x \in L} (\Downarrow a(x) \& e_{FL}(b, \eta_L(x))).$$

We shall prove that h_L is a right inverse of ε_L. Firstly, for all $a \in L$, we have that $(\varepsilon_L \circ h_L)(a) = \varepsilon_L(\sqcup A_a) = \sqcup (\varepsilon_L)^{\rightarrow}_Q(A_a)$. For all $t \in L$, we have

$$\begin{aligned}\bigwedge_{x \in L} ((\varepsilon_L)^{\rightarrow}_Q(A_a)(x) \rightarrow e_L(x,t)) &= \bigwedge_{x \in L} \bigwedge_{\varepsilon_L(b)=x} (A_a(b) \rightarrow e_L(x,t)) \\ &= \bigwedge_{b \in FL} (A_a(b) \rightarrow e_L(\varepsilon_L(b), t)) \\ &= \bigwedge_{b \in FL} \bigwedge_{z \in L} ((\Downarrow a(z) \& e_{FL}(b, \eta_L(z))) \rightarrow e_L(\varepsilon_L(b), t) \\ &= \bigwedge_{b \in FL} \bigwedge_{z \in L} (\Downarrow a(z) \rightarrow (e_{FL}(b, \eta_L(z)) \rightarrow e_L(\varepsilon_L(b), t))) \\ &= \bigwedge_{z \in L} (\Downarrow a(z) \rightarrow e_L(z,t)) \\ &= e_L(\sqcup \Downarrow a, t) \\ &= e_L(a,t).\end{aligned}$$

This means that $(\varepsilon_L \circ h_L)(a) = \sqcup (\varepsilon_L)^{\rightarrow}_Q(A_a) = a$. So $\varepsilon_L \circ h_L = id_L$.

Moreover, for all $b \in FL$, $(h_L \circ \varepsilon_L)(b) = \sqcup A_{\varepsilon_L(b)}$,

$$\begin{aligned}e_{FL}((h_L \circ \varepsilon_L)(b), b) &= e_{FL}(\sqcup A_{\varepsilon_L(b)}, b) \\ &= \bigwedge_{c \in FL} (A_{\varepsilon_L(b)}(c) \rightarrow e_{FL}(c,b)) \\ &= \bigwedge_{c \in FL} \Big(\Big(\bigvee_{z \in L} \Downarrow \varepsilon_L(b)(z) \& e_{FL}(c, \eta_L(z))\Big) \rightarrow e_{FL}(c,b)\Big) \\ &\geq \bigwedge_{c \in FL} \bigwedge_{z \in L} (e_{FL}(c,b) \rightarrow e_{FL}(c,b)) \\ &\geq 1.\end{aligned}$$

Then $h_L \circ \varepsilon_L \leq id_{FL}$, so (h_L, ε_L) is a Q-adjunction between L and FL, hence h_L preserves joins. Since

$$\begin{aligned}
e_{FL}(h_L(a)\otimes_{FL} h_L(b), h_L(a\otimes_L b)) &= e_{FL}(\sqcup A_a\otimes_{FL}\sqcup A_b, \sqcup A_{a\otimes_L b})\\
&= e_{FL}(\sqcup(\sqcup A_a\otimes_{FL}-)^{\rightarrow}_Q(A_b), \sqcup A_{a\otimes_L b})\\
&= \bigwedge_{c\in FL}\Big(\Big(\bigvee_{\sqcup A_a\otimes_{FL} d=c} A_b(d)\Big)\to e_{FL}(c,\sqcup A_{a\otimes_L b})\Big)\\
&= \bigwedge_{d\in FL}(A_b(d)\to e_{FL}(\sqcup A_a\otimes_{FL} d, \sqcup A_{a\otimes_L b}))\\
&= \bigwedge_{d\in FL}\bigwedge_{z\in FL}(A_b(d)\&(-\otimes_{FL} d)^{\rightarrow}_Q(A_a)(z)\to e_{FL}(z,\sqcup A_{a\otimes_L b}))\\
&= \bigwedge_{d\in FL}\bigwedge_{z\in FL}\bigwedge_{l\otimes_{FL} d=z}(A_b(d)\& A_a(l)\to e_{FL}(z,\sqcup A_{a\otimes_L b}))\\
&= \bigwedge_{d\in FL}\bigwedge_{l\in FL}(A_b(d)\& A_a(l)\to e_{FL}(l\otimes_{FL} d,\sqcup A_{a\otimes_L b}))\\
&\geq 1,
\end{aligned}$$

we have that $h_L(a)\otimes_{FL} h_L(b)\leq h_L(a\otimes_L b)$. Moreover, one can conclude that

$$\begin{aligned}
h_L(a\otimes_L b) &= h_L((\varepsilon_L\circ h_L)(a)\otimes_{FL}(\varepsilon_L\circ h_L)(b))\\
&= (h_L\circ\varepsilon_L)(h_L(a)\otimes_{FL} h_L(b))\\
&\leq h_L(a)\otimes_{FL} h_L(b).
\end{aligned}$$

Then $h_L(a\otimes_L b) = h_L(a)\otimes_{FL} h_L(b)$, and thus h_L is a right inverse of ε_L. □

The comonad determined by F (viewed as an endofunctor of **Q-Quant**) is $(F,\varepsilon,F\eta)$, and its coalgebras are pairs (L,g_L), where the structure map $g_L : L\longrightarrow FL$ satisfies the conditions as follows:

$$(U)\quad \varepsilon_L\circ g_L = id_L;\ (A)\quad (Fg_L)\circ g_L = (F\eta_L)\circ g_L.$$

Proposition 4.7. *Let L be a Q-quantale. Then L is **K**-flat projective iff it has a coalgebra structure for the $(F,\varepsilon,F\eta)$.*

Proof. We only have to show necessity. For all $b\in FL$,

$$\begin{aligned}
e_{FFL}(F\eta_L(b),\eta_{FL}(b)) &= e_{FFL}(F\eta_L(\sqcup k_b),\eta_{FL}(b))\\
&= e_{FFL}(\sqcup(F\eta_L)^{\rightarrow}_Q(k_b),\eta_{FL}(b))\\
&= \bigwedge_{c\in FFL}((F\eta_L)^{\rightarrow}_Q(k_b)(c)\to e_{FFL}(c,\eta_{FL}(b)))\\
&= \bigwedge_{c\in FFL}\bigwedge_{F\eta_L(y)=c}(k_b(y)\to e_{FFL}(c,\eta_{FL}(b)))\\
&= \bigwedge_{y\in FL}(k_b(y)\to e_{FFL}(F\eta_L(y),\eta_{FL}(b)))\\
&= \bigwedge_{y\in Im\eta_L}(e_{FL}(y,b)\to e_{FFL}(F\eta_L(y),\eta_{FL}(b)))\\
&= \bigwedge_{a\in L}(e_{FL}(\eta_L(a),b)\to e_{FFL}(F\eta_L(\eta_L(a)),\eta_{FL}(b)))\\
&= \bigwedge_{a\in L}(e_{FL}(\eta_L(a),b)\to e_{FFL}(\eta_{FL}(\eta_L(a)),\eta_{FL}(b)))\\
&\geq 1.
\end{aligned}$$

Then $e_{FFL}(F\eta_L(b),\eta_{FL}(b))\geq 1$. By Theorem 4.6, there exists a Q-quantale homomorphism $h_L : L\longrightarrow FL$ such that $\varepsilon_L\circ h_L = id_L$. Next, we shall prove that

$Fh_L \circ h_L = F\eta_L \circ h_L$. For all $b \in FL$, $e_{FL}(b, (\eta_L \circ \varepsilon_L)(b)) \geq 1$. Then for all $a \in L$, $e_{FL}(h_L(a), \eta_L(a)) = e_{FL}(h_L(a), (\eta_L \circ \varepsilon_L)(h_L(a))) \geq 1$, By Proposition 3.8(3), we have that $e_{FFL}(Fh_L(h_L(a)), F\eta_L(h_L(a))) \geq 1$. Moreover, since $a = \sqcup \Downarrow a$, we have that $e_{FFL}(F\eta_L(h_L(a)), Fh_L(h_L(a)))$

$$\begin{aligned}
&= e_{FFL}(F\eta_L(\sqcup(h_L)^{\rightarrow}_{Q}(\Downarrow a)), Fh_L(h_L(a))) \\
&= e_{FFL}(\sqcup(F\eta_L \circ h_L)^{\rightarrow}_{Q}(\Downarrow a), Fh_L(h_L(a))) \\
&= \bigwedge_{y \in FFL} ((F\eta_L \circ h_L)^{\rightarrow}_{Q}(\Downarrow a)(y) \rightarrow e_{FFL}(y, Fh_L(h_L(a)))) \\
&= \bigwedge_{y \in FFL} \bigwedge_{F\eta_L \circ h_L(x) = y} (\Downarrow a(x) \rightarrow e_{FFL}(y, Fh_L(h_L(a)))) \\
&\geq \bigwedge_{x \in L} (e_{FL}(\eta_L(x), h_L(a)) \rightarrow e_{FFL}(F\eta_L(h_L(x)), Fh_L(h_L(a)))) \\
&\geq \bigwedge_{x \in L} (e_{FL}(\eta_L(x), h_L(a)) \rightarrow e_{FFL}(\eta_{FL}(h_L(x)), Fh_L(h_L(a)))) \\
&= \bigwedge_{x \in L} (e_{FL}(\eta_L(x), h_L(a)) \rightarrow e_{FFL}(Fh_L(\eta_L(x)), Fh_L(h_L(a)))) \\
&\geq 1.
\end{aligned}$$

Therefore $Fh_L \circ h_L = F\eta_L \circ h_L$. □

5. **Examples**

Example 5.1. It is proved in [19] that **Q-Quant** is a reflective subcategory of **Q-OSgr**. When **K**=**Q-OSgr**, we can prove that a **Q-OSgr**-flat projective fuzzy quantale L is exactly the fuzzy weakly $\otimes$-stable completely distributive lattice (see [12]).

Definition 5.2. A fuzzy dcpo (A, e_A) with an associative binary operator $\otimes$ is called a pre-Q-quantale if for all $a \in A$, $a \otimes \_ : A \longrightarrow A$ and $\_ \otimes a : A \longrightarrow A$ preserve joins of every Q-directed subset of A.

Remark 5.3. Clearly, for $Q = 2$, a pre-Q-quantale is just a pre-quantale [15].

A map $f : (X, \otimes_X, e_X) \longrightarrow (Y, \otimes_Y, e_Y)$ between two pre-Q-quantales is called a pre-Q-quantale homomorphism if $f(a \otimes_X b) = f(a) \otimes_Y f(b)$ and $f(\sqcup S) = \sqcup f^{\rightarrow}_{Q}(S)$ for all $a, b \in X$, $S \in \mathcal{D}(X)$. Let **PQ-Quant** denote the category of pre-Q-quantales with pre-Q-quantale homomorphisms. Clearly, **PQ-Quant** is a subcategory of **Q-OSgr**.

Definition 5.4. Let $(L, \otimes, e_L)$ be a Q-quantale. A Q-order-preserving map $j : L \longrightarrow L$ is called a pre-Q-nucleus if it satisfies the following conditions:
(1) $e_L(x, j(x)) \geq 1$ for all $x \in L$;
(2) $e_L(a \otimes j(b), j(a \otimes b)) \geq 1$, $e_L(j(a) \otimes b, j(a \otimes b)) \geq 1$ for all $a, b \in L$.

Definition 5.5. [19] Let $(L, \otimes, e_L)$ be a Q-quantale. A Q-order-preserving map $j : L \longrightarrow L$ is called a Q-nucleus if it satisfies the following conditions:

(1) $e_L(x, j(x)) \geq 1$ for all $x \in L$;
(1) $e_L(j(j(x)), j(x)) \geq 1$ for all $x \in L$;
(2) $e_L(j(a) \otimes j(b), j(a \otimes b)) \geq 1$ for all $a, b \in L$.

Definition 5.6. [19] Let $(L, \otimes, e_L)$ be a Q-quantale. A subset $S \subseteq L$ is called a quotient Q-quantale of L if there exists a Q-nucleus j on L such that $Imj = S$.

Lemma 5.7. [19] *Let $(L, \otimes, e_L)$ be a Q-quantale, $S \subseteq L$. Then S is closed under Q-inf and for all $a \in L, s \in S, a \to_r s, a \to_l s \in S$ iff S is a quotient Q-quantale of L.*

Proposition 5.8. *Let $(L, \otimes, e_L)$ be a Q-quantale and j be a pre-Q-nucleus. Then the set of fixed points $Fix(j)$ of j is a quotient Q-quantale of L.*

Proof. Let $i : Fix(j) \longrightarrow L$ be the inclusion map. For all $A \in Q^{Fix(j)}$, since

$$\begin{aligned}
e_L(j(\sqcap i_Q^{\rightarrow}(A)), \sqcap i_Q^{\rightarrow}(A)) &= \bigwedge_{a \in L} (i_Q^{\rightarrow}(A)(a) \to e_L(j(\sqcap i_Q^{\rightarrow}(A)), a)) \\
&= \bigwedge_{a \in L} \bigwedge_{i(x)=a} (A(x) \to e_L(j(\sqcap i_Q^{\rightarrow}(A)), a)) \\
&= \bigwedge_{x \in Fix(j)} (A(x) \to e_L(j(\sqcap i_Q^{\rightarrow}(A)), i(x))) \\
&\geq \bigwedge_{x \in Fix(j)} (A(x) \to e_L(\sqcap i_Q^{\rightarrow}(A), x)) \\
&\geq \bigwedge_{x \in Fix(j)} (A(x) \to i_Q^{\rightarrow}(A)(x)) \\
&= \bigwedge_{x \in Fix(j)} (A(x) \to A(x)) \\
&\geq 1,
\end{aligned}$$

and $e_L(\sqcap i_Q^{\rightarrow}(A), j(\sqcap i_Q^{\rightarrow}(A))) \geq 1$, $j(\sqcap i_Q^{\rightarrow}(A)) = \sqcap i_Q^{\rightarrow}(A)$.

Moreover, for all $a \in L$, $s \in Fix(j)$, since

$$\begin{aligned}
e_L(j(a \to_r s), a \to_r s) &= e_L(j(a \to_r s), a \to_r j(s)) \\
&= e_L(a \otimes j(a \to_r s), j(s)) \\
&\geq e_L(a \otimes (a \to_r s), s) \\
&= e_L(a \to_r s, a \to_r s) \\
&\geq 1,
\end{aligned}$$

and $e_L(a \to_r s, j(a \to_r s)) \geq 1$. Therefore, $a \to_r s = j(a \to_r s)$. Similarly, we can prove $a \to_l s = j(a \to_l s)$. Hence, $Fix(j)$ is a quotient Q-quantale of L. □

Theorem 5.9. Q-Quant *is a reflective subcategory of* **PQ-Quant**.

Proof. Let $(A, \cdot, e_A)$ be a pre-Q-quantale and $\Upsilon(A) = \{U \in D(A) \mid \text{for all } S \in \mathcal{D}(A),$ $sub_A(S, U) \leq U(\sqcup S)\}$.

(1) We define a map $j : D(A) \longrightarrow D(A)$ as follows:

$$\forall\ U \in D(A),\ j(U) = k_U,$$

where $k_U : A \longrightarrow Q$ is defined by

$$\forall\ x \in A,\ k_U(x) = U(x) \vee \Big(\bigvee_{S\in\mathcal{D}(A)} sub_A(S, U)\&e_A(x, \sqcup S)\Big).$$

Then j is a pre-Q-nucleus and $\Upsilon(A) = Fix(j)$. Thus $\Upsilon(A)$ is a Q-quantale.

(2) Now, we define a map $\delta_A :\ A \longrightarrow \Upsilon(A)$ as follows:

$$\forall\ a \in A,\ \delta_A(a) =\downarrow a.$$

We can easily prove that $\delta_A(x) \otimes_j \delta_A(y) = j(\downarrow (x \cdot y)) =\downarrow (x \cdot y) = \delta_A(x \cdot y)$. It remains to show that $\delta_A(\sqcup X) = \sqcup(\delta_A)^{\rightarrow}_Q(X)$ for all $X \in \mathcal{D}(A)$.

For all $X \in \mathcal{D}(A), U \in \Upsilon(A)$. If $U =\downarrow a$ for some $a \in A$, then $sub_A(\downarrow a, \downarrow (\sqcup X)) = e_A(a, \sqcup X)$. Hence $X(a) \leq sub_A(\downarrow a, \downarrow (\sqcup X))$. Thus $(\delta_A)^{\rightarrow}_Q(X)(U) = \bigvee_{\delta_A(z)=U} X(z) \leq sub_A(U, \downarrow (\sqcup X))$. For all $Y \in \Upsilon(A), y \in A$, $Y(\sqcup X)\&e_A(y, \sqcup X) \leq Y(y)$, so we have $Y(\sqcup X) \leq e_A(y, \sqcup X) \rightarrow Y(y)$, then $Y(\sqcup X) \leq \bigwedge_{y\in A} (e_A(y, \sqcup X) \rightarrow Y(y)) = sub_A(\downarrow \sqcup X, Y)$. Since

$$\begin{aligned}
\bigwedge_{U\in\Upsilon(A)} \Big((\delta_A)^{\rightarrow}_Q(X)(U) \rightarrow sub_A(U, Y)\Big) &= \bigwedge_{U\in\Upsilon(A)} \Big(\Big(\bigvee_{\delta_A(a)=U} X(a)\Big) \rightarrow sub_A(U, Y)\Big)\\
&= \bigwedge_{a\in A} (X(a) \rightarrow sub_A(\downarrow a, Y))\\
&= \bigwedge_{a\in A} \Big(X(a) \rightarrow \Big(\bigwedge_{y\in A} (\downarrow a(y) \rightarrow Y(y))\Big)\Big)\\
&= \bigwedge_{a,y\in A} (X(a)\&e_A(y, a) \rightarrow Y(y))\\
&= \bigwedge_{y\in A} \Big(\Big(\bigvee_{a\in A} X(a)\&e_A(y, a)\Big) \rightarrow Y(y)\Big)\\
&\leq \bigwedge_{y\in A} (X(y) \rightarrow Y(y))\\
&\leq Y(\sqcup X).
\end{aligned}$$

Then $\bigwedge_{U\in\Upsilon(A)} ((\delta_A)^{\rightarrow}_Q(U) \rightarrow sub_A(U, Y)) \leq sub_A(\downarrow \sqcup X, Y)$, and thus $\delta_A(\sqcup X) = \sqcup(\delta_A)^{\rightarrow}_Q(X)$. Thus δ_A is a pre-Q-quantale homomorphism.

(3) Let X be a Q-quantale and $g : A \longrightarrow X$ be a pre-Q-quantale homomorphism. Define a map $h : D(A) \longrightarrow X$ as follows:

$$\forall\ U \in D(A),\ h(U) = \sqcup g^{\rightarrow}_Q(U).$$

It is easily proved that h is a Q-quantale homomorphism and $h \circ j = h$. Define a map $f : \Upsilon(A) \longrightarrow X$ as follows:

$$\forall\ U \in \Upsilon(A), f(U) = h(U).$$

For all $\mathcal{B} \in Q^{\Upsilon(A)}$, we have that $f(\sqcup_{\Upsilon(A)}\mathcal{B}) = h(\sqcup_{\Upsilon(A)}\mathcal{B}) = h(j(\sqcup i_Q^{\rightarrow}(\mathcal{B}))) = h(\sqcup i_Q^{\rightarrow}(\mathcal{B})) = \sqcup h_Q^{\rightarrow}(i_Q^{\rightarrow}(\mathcal{B})) = \sqcup(h \circ i)_Q^{\rightarrow}(\mathcal{B}) = \sqcup h_Q^{\rightarrow}(\mathcal{B}) = \sqcup f_Q^{\rightarrow}(\mathcal{B})$. For all $B,\ C \in \Upsilon(A)$, $f(B \otimes_j C) = h(j(B \otimes C)) = h(B \otimes C) = h(B) \otimes h(C) = f(B) \otimes f(C)$. For all $x \in A$, $y \in X$,

$$\begin{aligned}\bigwedge_{a\in A}(g_Q^{\rightarrow}(\downarrow x)(a) \rightarrow e_X(a,y)) &= \bigwedge_{a\in A}\Big(\Big(\bigvee_{g(z)=a} \downarrow x(z)\Big) \rightarrow e_X(a,y)\Big)\\ &= \bigwedge_{z\in A}(e_A(z,x) \rightarrow e_X(g(z),y))\\ &\leq e_A(x,x) \rightarrow e_X(g(x),y)\\ &= e_X(g(x),y).\end{aligned}$$

Then $e_X(g(x),y)\&e_A(z,x) \leq e_X(g(x),y)\&e_X(g(z),g(x)) \leq e_X(g(z),y)$ for all $z \in A$. Hence $e_X(g(x),y) \leq \bigwedge_{z\in A}(e_A(z,x) \rightarrow e_X(g(z),y)) = \bigwedge_{a\in A}(g_Q^{\rightarrow}(\downarrow x)(a) \rightarrow e_X(a,y))$. Thus $f(\downarrow x) = g(x)$.

(4) Suppose there exists a Q-quantale homomorphism l such that $l \circ \delta_A = g$. For all $X \in \Upsilon(A)$, we have $X = \sqcup(\delta_A)_Q^{\rightarrow}(X)$. Then $l(X) = l(\sqcup(\delta_A)_Q^{\rightarrow}(X)) = \sqcup l_Q^{\rightarrow}((\delta_A)_Q^{\rightarrow})(X)) = \sqcup(l \circ \delta_A)_Q^{\rightarrow}(X) = \sqcup g_Q^{\rightarrow}(X) = f(X)$, $l = f$. □

Remark 5.10. Let $(A,\cdot,e)$ be a pre-Q-quantale. Then $\Upsilon(A)$ is **PQ-Quant**-flat projective. Moreover, by Theorem 5.9, we know that **PQ-Quant** is an special case of **K**. In this case, suppose L is a Q-quantale. Then $\Downarrow a(x) = \bigwedge_{U\in\Upsilon(A)}(e_L(a,\sqcup U) \rightarrow U(x))$ for all $a, x \in L$. Thus L is **PQ-Quant**-flat projective iff $a = \sqcup \Downarrow a$ and $\Downarrow a(x) \& \Downarrow b(y) \leq \Downarrow (a \otimes_L b)(x \otimes_L y)$.

6. Conclusions

In this paper, we obtain some characterizations of the **K**-flat projective fuzzy quantales. Especially, we prove that a Q-quantale L is **K**-flat projective iff it has a coalgebra structure for the $(F, \varepsilon, F\eta)$. Furthermore, we present two examples for special cases of **K**. In further work, we can pursue to characterize projective Q-quantales. That is, we hope to find a satisfactory sufficient and necessary condition for a Q-quantale to be projective.

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Jing Lu, College of Mathematics and Information Science, Shaanxi Normal University, Xi'an 710119, P.R. China
E-mail address: lujing0926@126.com

Kaiyun Wang, College of Mathematics and Information Science, Shaanxi Normal University, Xi'an 710119, P.R. China
E-mail address: wangkaiyun@snnu.edu.cn

Bin Zhao*, College of Mathematics and Information Science, Shaanxi Normal University, Xi'an 710119, P.R. China
E-mail address: zhaobin@snnu.edu.cn

*Corresponding author