L-FUZZY CONVEXITY INDUCED BY L-CONVEX FUZZY SUBLATTICE DEGREE

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Abstract. In this paper, the notion of L-convex fuzzy sublattices is introduced and their characterizations are given. Furthermore, the notion of the degree to which an L-subset is an L-convex fuzzy sublattice is proposed and its some characterizations are given. Besides, the L-convex fuzzy sublattice degrees of the homomorphic image and pre-image of an L-subset are studied. Finally, we obtain an L-fuzzy convexity, which is induced by the L-convex fuzzy sublattice degrees, in the sense of Shi and Xiu.

1. Introduction

Convexity theory has been accepted to be of increasing importance in recent years in the study of many different problems. The notion of convexities was mainly defined and studied in $\mathbb{R}^n$ in the pioneering works of Newton, Minkowski, and others as described in [2]. In fact, there exist convexities in many other mathematical structures such as vector spaces, posets, semilattices, lattices, metric spaces, graphs, and median algebras. In the literature, many authors generalized usual convexities from different points of view. In general, there are two different kinds of generalizations for convexities according to [11]. On the one hand, there are those that are motivated by concrete problems (e.g. [12, 17]), in which case these convexities first handle non-fuzzy sets and then fuzzy sets and intuitionistic fuzzy sets [6, 16, 29, 32]. On the other hand, there are those that are stated from an axiomatic point of view, where the notion of convexities is based on properties of a family of sets [4, 25].

In [25], by axiomatizing the properties of convex sets in $\mathbb{R}^n$, van de Vel introduced the notion of convexities on a nonempty set X. A convexity, denoted by $\mathcal{C}$, is a subfamily of 2^X satisfying three axioms. With the development of fuzzy mathematics, the notion of convexities has already been extended to fuzzy set theory. In 1994, Rosa [18, 19] presented the notion of fuzzy convexities as a subfamily of $[0,1]^X$. In [13, 14, 15], an L-convexity was presented as a subfamily of L^X. In [21], an L-fuzzifying convexity was introduced as a mapping $\mathcal{C} : 2^X \to L$, that is, $\mathcal{C}$ is an L-subset of 2^X. More recently, the notion of convexities was further extended to a more general fuzzy setting in [22, 28], which is called (L, M)-fuzzy convexities. An (L, M)-fuzzy convexity is an M-subset of L^X, in this case, each L-subset of X can be regarded as convex to some degree.

Received: April 2016; Revised: November 2016; Accepted: February 2017

Key words and phrases: L-convex fuzzy sublattice, Implication operator, L-convex fuzzy sublattice degree, L-fuzzy convexity.

One of the structures that are most extensively used and discussed in the crisp mathematical theory is the lattice structure. As is well known, the lattice structure can be considered as an order structure or an algebra structure [3, 5]. This provides sufficient motivations to generalize various notions and results in lattice structure to fuzzy setting.

Based on the notion of $[0, 1]$-subsets in the sense of Zadeh [31], Yuan and Wu gave the notion of fuzzy sublattices in [30]. Ajmal and Thomas developed the theory of (convex) fuzzy sublattices in [1]. Xin and Fu gave some results of convex fuzzy sublattices in [27]. Afterwards, when L is a complete lattice, Tepavčević and Trajkovski [24] introduced the notion of L-fuzzy sublattices based on L-subsets in the sense of Goguen [8], and gave characterizations of this notion in terms of one kind of cuts of L-subsets.

In this paper, we try to construct a natural L-fuzzy convexity on a lattice. In details, we introduce the notion of L-convex fuzzy sublattices when L is a completely distributive lattice and give their characterizations in terms of four kinds of cut sets of L-subsets. Inspired by the work of Shi and Xin in [23], we further introduce the notion of the degree to which an L-subset is an L-convex fuzzy sublattice, which generalizes the notion of L-convex fuzzy sublattices. As it is expected, we obtain an L-fuzzy convexity induced by L-convex fuzzy sublattice degrees.

2. **Preliminaries**

We first recall (L, M)-fuzzy convexities in [28, 22], which are more general fuzzy convexities.

Let L and M be two completely distributive lattices. $\top_M$ denotes the largest element in M and L^X denotes the set of all L-subsets of X. An (L, M)-fuzzy convexity on X is defined as follows:

A mapping $\mathcal{C} : L^X \to M$ is called an (L, M)-fuzzy convexity if it satisfies the following conditions:

(**LMC1**) $\mathcal{C}(\chi_\emptyset) = \mathcal{C}(\chi_X) = \top_M$;

(**LMC2**) if $\{\mu_i : i \in \Omega\} \subseteq L^X$ is nonempty , then $\mathcal{C}(\bigwedge_{i\in\Omega} \mu_i) \geqslant \bigwedge_{i\in\Omega} \mathcal{C}(\mu_i)$;

(**LMC3**) if $\{\mu_i : i \in \Omega\} \subseteq L^X$ is nonempty and totally ordered by the order in L, then $\mathcal{C}(\bigvee_{i\in\Omega} \mu_i) \geqslant \bigwedge_{i\in\Omega} \mathcal{C}(\mu_i)$.

The pair $(X, \mathcal{C})$ is called an (L, M)-fuzzy convex space. An (L, L)-fuzzy convexity is briefly called an L-fuzzy convexity.

An $(L, 2)$-fuzzy convexity is an L-convexity in [13, 14]. An $(I, 2)$-fuzzy convexity is a fuzzy convexity in [18, 19], where $I = [0, 1]$. A $(2, M)$-fuzzy convexity is an M-fuzzifying convexity in [21]. A $(2, 2)$-fuzzy convexity is a convexity in [25].

The smallest element and the largest element in L are denoted by $\bot$ and $\top$, respectively. An element a in L is called co-prime if $a \leqslant b \vee c$ implies $a \leqslant c$ or $a \leqslant b$. The set of nonzero co-prime elements in L is denoted by $J(L)$. An element a in L is called prime if $a \geqslant b \wedge c$ implies $a \geqslant c$ or $a \geqslant b$. The set of non-unit prime elements in L is denoted by $P(L)$. From [7], we know that each element of L is the sup of co-prime elements and the inf of prime elements.

For $a, b \in L$, we say that a is well below b in L, in symbols $a \prec b$, if for every subset $D \subseteq L$, $b \leqslant \bigvee D$ implies $a \leqslant d$ for some $d \in D$. A complete lattice L is completely distributive if and only if $b = \bigvee\{a \in L | a \prec b\}$ for each $b \in L$. The set $\{a \in L : a \prec b\}$, denoted by $\beta(b)$, is called the greatest minimal family of b in the sense of [26]. Let $\beta^*(b) = \beta(b) \cap J(L)$. Moreover, define a binary relation $\prec^{op}$ as follows: for $a, b \in L$, $b \prec^{op} a$ if and only if for every subset $D \subseteq L$, $\bigwedge D \leqslant b$ implies $d \leqslant a$ for some $d \in D$. The set $\{a \in M : b \prec^{op} a\}$, denoted by $\alpha(b)$, is the greatest maximal family of b in the sense of [26]. Let $\alpha^*(b) = \alpha(b) \cap P(L)$. We know that α is an $\bigwedge$-$\bigcup$ mapping, i.e., $\alpha(\bigwedge\limits_{i\in\Omega} a_i) = \bigcup\limits_{i\in\Omega} \alpha(a_i)$ for all $\{a_i\}_{i\in I} \subseteq L$, β is a union-preserving mapping, i.e., $\beta(\bigvee\limits_{i\in\Omega} a_i) = \bigcup\limits_{i\in\Omega} \beta(a_i)$ for all $\{a_i\}_{i\in I} \subseteq L$ and $b = \bigvee \beta(b) = \bigvee \beta^*(b) = \bigwedge \alpha(b) = \bigwedge \alpha^*(b)$ for each $b \in L$ (see [26]). Also, we have $\alpha(\top) = \emptyset$.

We define a residual implication operator $\to: L \times L \to L$ as the right adjoint for the operation of binary meets $\bigwedge$ by

$$a \to b = \bigvee\{c \in L : a \wedge c \leqslant b\}.$$

Some basic properties of this operation are listed in the following [9].

(1) $a \to b = \top \Leftrightarrow a \leqslant b$;
(2) $a \leqslant b \to c \Leftrightarrow a \wedge b \leqslant c$;
(3) $a \leqslant b \Rightarrow c \to a \leqslant c \to b$
and $a \leqslant b \Rightarrow a \to c \geqslant b \to c$;
(4) $a \to \bigwedge_i b_i = \bigwedge_i (a \to b_i)$;
(5) $\bigvee_i a_i \to b = \bigwedge_i (a_i \to b)$;
(6) $(a \to c) \wedge (c \to b) \leqslant a \to b$;
(7) $(a \to b) \wedge (c \to d) \leqslant a \wedge c \to b \wedge d$.

Lemma 2.1. *Let L be a completely distributive lattice and $a, b \in L$. Then the following conditions are equivalent:*

(1) $a \leqslant b$.
(2) $\forall p \in L$, $p \leqslant a \Rightarrow p \leqslant b$.
(3) $\forall p \in J(L)$, $p \leqslant a \Rightarrow p \leqslant b$.
(4) $\forall q \in P(L)$, $a \nleqslant q \Rightarrow b \nleqslant q$.
(5) $\forall p \in \alpha^*(\bot)$, $p \notin \alpha^*(a) \Rightarrow p \notin \alpha^*(b)$.

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3), (1) $\Rightarrow$ (4) and (1) $\Rightarrow$ (5) are obvious. Next, we prove (4) $\Rightarrow$ (1) and (5) $\Rightarrow$ (1).

(4) $\Rightarrow$ (1) Assume that for all $q \in P(L)$, $a \nleqslant q$ implies $b \nleqslant q$. Then $\{q \in P(L) \mid a \leqslant q\} \supseteq \{q \in P(L) \mid b \leqslant q\}$. From the fact that each element of L is the inf of some prime elements, we know $a = \bigwedge\{q \in P(L) \mid a \leqslant q\} \leqslant \bigwedge\{q \in P(L) \mid b \leqslant q\} = b$.

(5) $\Rightarrow$ (1) Assume that for all $p \in \alpha^*(\bot)$, $p \notin \alpha^*(a)$ implies $p \notin \alpha^*(b)$. By $\alpha^*(a) \subseteq \alpha^*(\bot)$ and $\alpha^*(b) \subseteq \alpha^*(\bot)$, we know $\alpha^*(a) \supseteq \alpha^*(b)$. Therefore $a = \bigwedge \alpha^*(a) \leqslant \bigwedge \alpha^*(b) = b$. □

Definition 2.2. [20] For a nonempty set X. Let $\lambda \in L^X$ and $a \in L$, we define
$\lambda_{[a]} = \{x \in X : \lambda(x) \geqslant a\}$,

$\lambda^{(a)} = \{x \in X : \lambda(x) \nleqslant a\}$,
$\lambda_{(a)} = \{x \in X : a \in \beta(\lambda(x))\}$,
$\lambda^{[a]} = \{x \in X : a \notin \alpha(\lambda(x))\}$.

Let $f : X \to Y$ be a mapping. Define $f_L^{\to}(\lambda)(y) = \bigvee_{f(x)=y} \lambda(x)$ for $\lambda \in L^X, y \in Y$ and $f_L^{\leftarrow}(\mu) = \mu \circ f$ for $\mu \in L^Y$.

Definition 2.3. [3, 5] Let M be a lattice, $X \subseteq M$ and $X \neq \emptyset$. If for all $a, b \in X$, we have $a \wedge b \in X$ and $a \vee b \in X$, then X is called a sublattice of M.

Definition 2.4. [3, 5] Let X be a sublattice of M. For all $a, b \in X$, if $a \leqslant c \leqslant b$ implies $c \in X$, then X is called a convex sublattice of M.

3. *L*-fuzzy Sublattice

In this section, we give characterizations of L-fuzzy sublattices in terms of four kinds of cut sets of L-subsets and consider homomorphic image and preimage of an L-fuzzy sublattice.

In [24], when L is a complete lattice, the following definition is introduced.

Definition 3.1. Let $(M, \vee, \wedge)$ be a lattice and $\lambda \in L^M$. λ is called an L-fuzzy sublattice if for all $x, y \in M$,

$$\lambda(x) \wedge \lambda(y) \leqslant \lambda(x \wedge y) \wedge \lambda(x \vee y).$$

Remark 3.2. From the above definition, it is obvious that λ is an L-fuzzy sublattice if and only if for all $x, y \in M$,

$$\lambda(x) \wedge \lambda(y) \leqslant \lambda(x \wedge y)$$

and

$$\lambda(x) \wedge \lambda(y) \leqslant \lambda(x \vee y).$$

In [24], a characterization of an L-fuzzy sublattice was considered:

λ is an L-fuzzy sublattice if and only if $\forall a \in L$, $\lambda_{[a]}$ is a sublattice of M.

Excepting this, we give some new characterizations of an L-fuzzy sublattice in the following two theorems.

Theorem 3.3. *Let $(M, \vee, \wedge)$ be a lattice and $\lambda \in L^M$. Then the following conditions are equivalent:*

(1) *λ is an L-fuzzy sublattice.*
(2) *$\forall a \in L$, $\lambda_{[a]}$ is a sublattice of M.*
(3) *$\forall a \in J(L)$, $\lambda_{[a]}$ is a sublattice of M.*
(4) *$\forall a \in L$, $\lambda^{[a]}$ is a sublattice of M.*
(5) *$\forall a \in P(L)$, $\lambda^{[a]}$ is a sublattice of M.*
(6) *$\forall a \in P(L)$, $\lambda^{(a)}$ is a sublattice of M.*

Proof. (1) $\Rightarrow$ (2) has been proved in [24].

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1) Assume that $\lambda_{[a]}$ is a sublattice of M for all $a \in J(L)$. For any $x, y \in M$, let $a \in J(L)$ with $a \leqslant \lambda(x) \wedge \lambda(y)$, then $a \leqslant \lambda(x)$ and $a \leqslant \lambda(y)$, i.e.,

$x, y \in \lambda_{[a]}$. Since $\lambda_{[a]}$ is a sublattice of M, we have $x \vee y \in \lambda_{[a]}$ and $x \wedge y \in \lambda_{[a]}$, i.e., $a \leqslant \lambda(x \wedge y)$ and $a \leqslant \lambda(x \vee y)$. Thus $a \leqslant \lambda(x \wedge y) \wedge \lambda(x \vee y)$. From Lemma 2.1, we know $\lambda(x) \wedge \lambda(y) \leqslant \lambda(x \wedge y) \wedge \lambda(x \vee y)$. Therefore λ is an L-fuzzy sublattice.

(1) $\Rightarrow$ (4) Assume that λ is an L-fuzzy sublattice. For any $a \in L$, let $x, y \in \lambda^{[a]}$, i.e., $a \notin \alpha(\lambda(x))$ and $a \notin \alpha(\lambda(y))$, then $a \notin \alpha(\lambda(x)) \bigcup \alpha(\lambda(y))$. Since α is an $\bigwedge$-$\bigcup$ mapping, we have $a \notin \alpha(\lambda(x) \wedge \lambda(y))$. By $\lambda(x) \wedge \lambda(y) \leqslant \lambda(x \wedge y)$ and $\lambda(x) \wedge \lambda(y) \leqslant \lambda(x \vee y)$, we have $a \notin \alpha(\lambda(x \wedge y))$ and $a \notin \alpha(\lambda(x \vee y))$, i.e., $x \wedge y \in \lambda^{[a]}$ and $x \vee y \in \lambda^{[a]}$. Therefore $\lambda^{[a]}$ is a sublattice.

(4) $\Rightarrow$ (5) is obvious.

(5) $\Rightarrow$ (1) Assume that $\lambda^{[a]}$ is a sublattice of M for all $a \in P(L)$. For any $x, y \in M$, let $a \in P(L)$ with $a \notin \alpha^*(\lambda(x) \wedge \lambda(y))$, then $a \notin \alpha(\lambda(x) \wedge \lambda(y))$. By $\alpha(\lambda(x) \wedge \lambda(y)) = \alpha(\lambda(x)) \bigcup \alpha(\lambda(y))$, we have $a \notin \alpha(\lambda(x))$ and $a \notin \alpha(\lambda(y))$, i.e., $x, y \in \lambda^{[a]}$. Since $\lambda^{[a]}$ is a sublattice of M, we have $x \wedge y \in \lambda^{[a]}$ and $x \vee y \in \lambda^{[a]}$, i.e., $a \notin \alpha(\lambda(x \wedge y))$ and $a \notin \alpha(\lambda(x \vee y))$. Further, we obtain $a \notin \alpha^*(\lambda(x \wedge y))$ and $a \notin \alpha^*(\lambda(x \vee y))$. This implies $\alpha^*(\lambda(x) \wedge \lambda(y)) \supseteq \alpha^*(\lambda(x \wedge y))$ and $\alpha^*(\lambda(x) \wedge \lambda(y)) \supseteq \alpha^*(\lambda(x \vee y))$. Hence $\lambda(x) \wedge \lambda(y) = \bigwedge \alpha^*(\lambda(x) \wedge \lambda(y)) \leqslant \bigwedge \alpha^*(\lambda(x \wedge y)) = \lambda(x \wedge y)$ and $\lambda(x) \wedge \lambda(y) = \bigwedge \alpha^*(\lambda(x) \wedge \lambda(y)) \leqslant \bigwedge \alpha^*(\lambda(x \vee y)) = \lambda(x \vee y)$. Therefore λ is an L-fuzzy sublattice.

(1) $\Rightarrow$ (6) Assume that λ is an L-fuzzy sublattice. For any $a \in P(L)$, let $x, y \in \lambda^{(a)}$, i.e., $\lambda(x) \nleqslant a$, $\lambda(y) \nleqslant a$, then $\lambda(x) \wedge \lambda(y) \nleqslant a$. Since λ is an L-fuzzy sublattice, we have $\lambda(x) \wedge \lambda(y) \leqslant \lambda(x \wedge y) \wedge \lambda(x \vee y)$. This implies $\lambda(x \wedge y) \nleqslant a$ and $\lambda(x \vee y) \nleqslant a$, i.e., $x \vee y \in \lambda^{(a)}$, $x \wedge y \in \lambda^{(a)}$. Therefore $\lambda^{(a)}$ is a sublattice.

(6) $\Rightarrow$ (1) Assume that $\lambda^{(a)}$ is a sublattice for all $a \in P(L)$. For any $x, y \in M$, let $a \in P(L)$ with $\lambda(x) \wedge \lambda(y) \nleqslant a$, then $\lambda(x) \nleqslant a$ and $\lambda(y) \nleqslant a$, i.e., $x, y \in \lambda^{(a)}$. Since $\lambda^{(a)}$ is a sublattice, we have $x \vee y \in \lambda^{(a)}$ and $x \wedge y \in \lambda^{(a)}$, i.e., $\lambda(x \vee y) \nleqslant a$ and $\lambda(x \wedge y) \nleqslant a$. Thus $\lambda(x \wedge y) \wedge \lambda(x \vee y) \nleqslant a$. From Lemma 2.1, we know that $\lambda(x) \wedge \lambda(y) \leqslant \lambda(x \wedge y) \wedge \lambda(x \vee y)$. Therefore λ is an L-fuzzy sublattice. □

Theorem 3.4. *Let $(M, \vee, \wedge)$ be a lattice and $\lambda \in L^M$. If $\beta(a \wedge b) = \beta(a) \bigcap \beta(b)$ for all $a, b \in L$, then the following conditions are equivalent:*

(1) *λ is an L-fuzzy sublattice.*

(2) *$\forall a \in L$, $\lambda_{(a)}$ is a sublattice of M.*

(3) *$\forall a \in J(L)$, $\lambda_{(a)}$ is a sublattice of M.*

Proof. (1) $\Rightarrow$ (2) Assume that λ is an L-fuzzy sublattice. For any $a \in L$, let $x, y \in \lambda_{(a)}$, i.e., $a \in \beta(\lambda(x))$ and $a \in \beta(\lambda(y))$. Then $a \in \beta(\lambda(x)) \bigcap \beta(\lambda(y))$. By $\beta(\lambda(x)) \bigcap \beta(\lambda(y)) = \beta(\lambda(x) \wedge \lambda(y))$, we have $a \in \beta(\lambda(x) \wedge \lambda(y))$. Since λ is an L-fuzzy sublattice, we have $\lambda(x) \wedge \lambda(y) \leqslant \lambda(x \wedge y)$ and $\lambda(x) \wedge \lambda(y) \leqslant \lambda(x \vee y)$. This implies $a \in \beta(\lambda(x \vee y))$ and $a \in \beta(\lambda(x \wedge y))$, i.e., $x \vee y \in \lambda_{(a)}$ and $x \wedge y \in \lambda_{(a)}$. Therefore $\lambda_{(a)}$ is a sublattice of M.

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1) Assume that $\lambda_{(a)}$ is a sublattice of M for all $a \in J(L)$. For any $x, y \in M$, let $a \in J(L)$ with $a \in \beta(\lambda(x) \wedge \lambda(y))$. By $\beta(\lambda(x) \wedge \lambda(y)) = \beta(\lambda(x)) \bigcap \beta(\lambda(y))$, then $a \in \beta(\lambda(x))$ and $a \in \beta(\lambda(y))$, i.e., $x, y \in \lambda_{(a)}$. Since $\lambda_{(a)}$ is a sublattice of M , we have $x \vee y \in \lambda_{(a)}$ and $x \wedge y \in \lambda_{(a)}$, i.e., $a \in \beta(\lambda(x \vee y))$ and $a \in$

$\beta(\lambda(x \wedge y))$. This implies $\beta(\lambda(x) \wedge \lambda(y)) \subseteq \beta(\lambda(x \vee y))$ and $\beta(\lambda(x) \wedge \lambda(y)) \subseteq \beta(\lambda(x \wedge y))$. So $\lambda(x) \wedge \lambda(y) = \bigvee \beta(\lambda(x) \wedge \lambda(y)) \leqslant \bigvee \beta(\lambda(x \vee y)) = \lambda(x \vee y)$ and $\lambda(x) \wedge \lambda(y) = \bigvee \beta(\lambda(x) \wedge \lambda(y)) \leqslant \bigvee \beta(\lambda(x \wedge y)) = \lambda(x \wedge y)$. Therefore λ is an L-fuzzy sublattice. □

Theorem 3.5. *Let $f : M \to N$ be a lattice homomorphism and $\mu \in L^M$, $\lambda \in L^N$.*
(1) *If λ is an L-fuzzy sublattice of N, then $f_L^{\leftarrow}(\lambda)$ is an L-fuzzy sublattice of M.*
(2) *If μ is an L-fuzzy sublattice of M, then $f_L^{\rightarrow}(\mu)$ is an L-fuzzy sublattice of N.*

Proof. (1) and (2) are straightforward. □

Let M and N be two lattices, $\lambda \in L^M$ and $\mu \in L^N$. Define $\lambda \times \mu : M \times N \longrightarrow L$ as follows:

$$(\lambda \times \mu)(x, y) = \lambda(x) \wedge \mu(y)$$

for all $(x, y) \in M \times N$.

Theorem 3.6. *Let $\lambda \in L^M$ and $\mu \in L^N$ be L-fuzzy sublattices of M and N, respectively. Then $\lambda \times \mu$ is an L-fuzzy sublattice of $M \times N$.*

Proof. It is straightforward. □

4. *L*-convex Fuzzy Sublattice

In this section, we introduce the notion of L-convex fuzzy sublattices and give their characterizations in terms of four kinds of cut sets of L-subsets. Furthermore, we consider homomorphic image and preimage of an L-convex fuzzy sublattice.

The notion of convex fuzzy sublattices in [10] is generalized to L-fuzzy setting as follows:

Definition 4.1. Let $(M, \vee, \wedge)$ be a lattice and λ be an L-fuzzy sublattice. λ is called an L-convex fuzzy sublattice if for all $x, y, z \in M$, $x \leqslant z \leqslant y$ implies $\lambda(z) \geqslant \lambda(x) \wedge \lambda(y)$.

Since for all $x, y, z \in M$, $x \leqslant z \leqslant y$ implies $\lambda(z) \geqslant \lambda(x) \wedge \lambda(y)$ if and only if for all $x, y, z \in M$, $x \wedge y \leqslant z \leqslant x \vee y$ implies $\lambda(z) \geqslant \lambda(x \wedge y) \wedge \lambda(x \vee y)$, we have the following lemma.

Lemma 4.2. *Let $(M, \vee, \wedge)$ be a lattice and $\lambda \in L^M$ be an L-fuzzy sublattice. Then λ is an L-convex fuzzy sublattice if and only if for all $x, y, z \in M$, $x \wedge y \leqslant z \leqslant x \vee y$ implies $\lambda(z) \geqslant \lambda(x \wedge y) \wedge \lambda(x \vee y)$.*

Analogous to the characterizations of L-fuzzy sublattices, we can obtain the following characterizations of L-convex fuzzy sublattices.

Theorem 4.3. *Let $(M, \vee, \wedge)$ be a lattice and $\lambda \in L^M$. Then the following conditions are equivalent:*
(1) *λ is an L-convex fuzzy sublattice.*
(2) *$\forall a \in L$, $\lambda_{[a]}$ is a convex sublattice of M.*
(3) *$\forall a \in J(L)$, $\lambda_{[a]}$ is a convex sublattice of M.*
(4) *$\forall a \in L$, $\lambda^{[a]}$ is a convex sublattice of M.*

(5) $\forall a \in P(L)$, $\lambda^{[a]}$ *is a convex sublattice of* M.
(6) $\forall a \in P(L)$, $\lambda^{(a)}$ *is a convex sublattice of* M.

Proof. (1) $\Rightarrow$ (2) Assume that λ is an L-convex fuzzy sublattice. By Theorem 3.3, we know that $\lambda_{[a]}$ is a sublattice of M for all $a \in L$. Now we prove $\lambda_{[a]}$ is convex. Let $x, y \in \lambda_{[a]}$ with $x \leqslant z \leqslant y$, then $\lambda(x) \geqslant a$ and $\lambda(y) \geqslant a$. Since λ is L-convex, we have $\lambda(z) \geqslant \lambda(x) \wedge \lambda(y) \geqslant a$, i.e., $z \in \lambda_{[a]}$. Therefore $\lambda_{[a]}$ is a convex sublattice of M.

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1) Assume that $\lambda_{[a]}$ is a convex sublattice of M for all $a \in J(L)$. By Theorem 3.3, we knowthat λ is an L-fuzzy sublattice. Now we prove λ is L-convex. For any $x, y, z \in M$ with $x \leqslant z \leqslant y$, let $a \in J(L)$ with $a \leqslant \lambda(x) \wedge \lambda(y)$, then $a \leqslant \lambda(x)$ and $a \leqslant \lambda(y)$, i.e., $x, y \in \lambda_{[a]}$. Since $\lambda_{[a]}$ is convex, we have $z \in \lambda_{[a]}$, i.e., $a \leqslant \lambda(z)$. From Lemma 2.1, it holds that $\lambda(z) \geqslant \lambda(x) \wedge \lambda(y)$. Therefore λ is an L-convex fuzzy sublattice.

(1) $\Rightarrow$ (4) Assume that λ is an L-convex fuzzy sublattice. By Theorem 3.3, we know that $\lambda^{[a]}$ is a sublattice for all $a \in L$. Now we prove $\lambda^{[a]}$ is convex. Let $x, y \in \lambda^{[a]}$ with $x \leqslant z \leqslant y$. Then $a \notin \alpha(\lambda(x))$ and $a \notin \alpha(\lambda(y))$. Thus $a \notin \alpha(\lambda(x)) \bigcup \alpha(\lambda(y))$. Since $\alpha(\lambda(x) \wedge \lambda(y)) = \alpha(\lambda(x)) \bigcup \alpha(\lambda(y))$, we have $a \notin \alpha(\lambda(x) \wedge \lambda(y))$. By $\lambda(z) \geqslant \lambda(x) \wedge \lambda(y)$, we have $a \notin \alpha(\lambda(z)))$, i.e., $z \in \lambda^{[a]}$. Therefore $\lambda^{[a]}$ is a convex sublattice.

(4) $\Rightarrow$ (5) is obvious.

(5) $\Rightarrow$ (1) Assume that $\lambda^{[a]}$ is a convex sublattice of M for all $a \in P(L)$. By Theorem 3.3, we know that λ is an L-fuzzy sublattice. Now we prove λ is L-convex. For any $x, y, z \in M$ with $x \leqslant z \leqslant y$, let $a \in P(L)$ with $a \notin \alpha^*(\lambda(x) \wedge \lambda(y))$, then $a \notin \alpha(\lambda(x) \wedge \lambda(y))$. By $\alpha(\lambda(x) \wedge \lambda(y)) = \alpha(\lambda(x)) \bigcup \alpha(\lambda(y))$, we have $a \notin \alpha(\lambda(x)) \bigcup \alpha(\lambda(y))$, i.e., $a \notin \alpha(\lambda(x))$ and $a \notin \alpha(\lambda(y))$, i.e., $x, y \in \lambda^{[a]}$. Since $\lambda^{[a]}$ is convex, we have $z \in \lambda^{[a]}$, i.e., $a \notin \alpha(\lambda(z))$, thus $a \notin \alpha^*(\lambda(z))$. This implies $\alpha^*(\lambda(x) \wedge \lambda(y)) \supseteq \alpha^*(\lambda(z))$. So $\lambda(x) \wedge \lambda(y) = \bigwedge \alpha^*(\lambda(x) \wedge \lambda(y)) \leqslant \bigwedge \alpha^*(\lambda(z)) = \lambda(z)$. Therefore λ is an L-convex fuzzy sublattice.

(1) $\Rightarrow$ (6) Assume that λ is an L-convex fuzzy sublattice. By Theorem 3.3, we know $\lambda^{(a)}$ is a sublattice for all $a \in P(L)$. Now we prove $\lambda^{(a)}$ is convex. Let $x, y \in \lambda^{(a)}$ with $x \leqslant z \leqslant y$, then $\lambda(x) \nleqslant a$, $\lambda(y) \nleqslant a$. This implies $\lambda(x) \wedge \lambda(y) \nleqslant a$. Since λ is L-convex, we have $\lambda(z) \geqslant \lambda(x) \wedge \lambda(y)$. This means that $\lambda(z) \nleqslant a$, i.e., $z \in \lambda^{[a]}$. Therefore $\lambda^{[a]}$ is a convex sublattice.

(6) $\Rightarrow$ (1) Assume that $\lambda^{(a)}$ is a convex sublattice for all $a \in P(L)$. By Theorem 3.3, λ is an L-fuzzy sublattice. Now we prove λ is L-convex. For any $x, y, z \in M$ with $x \leqslant z \leqslant y$, let $a \in P(L)$ with $\lambda(x) \wedge \lambda(y) \nleqslant a$. Then $\lambda(x) \nleqslant a$ and $\lambda(y) \nleqslant a$, i.e., $x, y \in \lambda^{(a)}$. Since $\lambda^{(a)}$ is a convex sublattice, we have $z \in \lambda^{(a)}$, i.e., $\lambda(z) \nleqslant a$. From Lemma 2.1, it holds that $\lambda(x) \wedge \lambda(y) \leqslant \lambda(z)$. Therefore λ is an L-convex fuzzy sublattice. □

Theorem 4.4. *Let* $(M, \vee, \wedge)$ *be a lattice and* $\lambda \in L^M$. *If* $\beta(a \wedge b) = \beta(a) \bigcap \beta(b)$ *for all* $a, b \in L$, *then the following conditions are equivalent:*

(1) λ *is an* L*-convex fuzzy sublattice.*

(2) $\forall a \in L$, $\lambda_{(a)}$ *is a convex sublattice of* M.

(3) $\forall a \in J(L)$, $\lambda_{(a)}$ *is a convex sublattice of* M.

Proof. (1) $\Rightarrow$ (2) Assume that λ is an L-convex fuzzy sublattice. By Theorem 3.4, $\lambda_{(a)}$ is a sublattice for all $a \in L$. Now we prove $\lambda_{(a)}$ is convex. Let $x, y \in \lambda_{(a)}$ with $x \leqslant z \leqslant y$, then $a \in \beta(\lambda(x))$ and $a \in \beta(\lambda(y))$. By $\beta(\lambda(x)) \bigcap \beta(\lambda(y)) = \beta(\lambda(x) \wedge \lambda(y))$, we have $a \in \beta(\lambda(x) \wedge \lambda(y))$. Since λ is an L-convex fuzzy sublattice, we have $\lambda(x) \wedge \lambda(y) \leqslant \lambda(z)$. Thus $a \in \beta(\lambda(z))$, i.e., $z \in \lambda_{(a)}$. Therefore $\lambda_{(a)}$ is a convex sublattice of M.

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1) Assume that $\lambda_{(a)}$ is a convex sublattice of M for all $a \in J(L)$. By Theorem 3.4, λ is an L-fuzzy sublattice of M. Now we prove λ is L-convex. For any $x, y, z \in M$ with $x \leqslant z \leqslant y$, let $a \in J(L)$ with $a \in \beta(\lambda(x) \wedge \lambda(y))$. By $\beta(\lambda(x) \wedge \lambda(y)) = \beta(\lambda(x)) \bigcap \beta(\lambda(y))$, we have $a \in \beta(\lambda(x)) \bigcap \beta(\lambda(y))$, thus $a \in \beta(\lambda(x))$ and $a \in \beta(\lambda(y))$, i.e., $x, y \in \lambda_{(a)}$. Since $\lambda_{(a)}$ is a convex sublattice of M , we have $z \in \lambda_{(a)}$, i.e., $a \in \beta(\lambda(z))$. This implies $\beta(\lambda(x) \wedge \lambda(y)) \subseteq \beta(\lambda(z))$. So $\lambda(x) \wedge \lambda(y) = \bigvee \beta(\lambda(x) \wedge \lambda(y)) \leqslant \bigvee \beta(\lambda(z)) = \lambda(z)$. Therefore λ is an L-convex fuzzy sublattice. □

Based on the notion of f-invariant fuzzy subset in [1], we give the following definition.

Definition 4.5. Let $f : M \to N$ be a lattice mapping and $\mu \in L^M$. Then μ is called f-invariant if $f(x) = f(y)$ implies $\mu(x) = \mu(y)$ for all $x, y \in M$.

Lemma 4.6. *Let* $f : M \to N$ *be a lattice homomorphism and* μ *be an* f*-invariant* L*-subset of* M. *Then* $f(x) \leqslant f(y)$ *implies* $\mu(x) = \mu(x \wedge y)$ *and* $\mu(y) = \mu(x \vee y)$ *for all* $x, y \in M$.

The following Theorem 4.7 and Theorem 4.8 are special cases of Theorem 6.10 and Theorem 6.11 in Section 6, respectively. Here the proof is omitted.

Theorem 4.7. *Let* $f : M \to N$ *be a lattice homomorphism and* $\mu \in L^M$, $\lambda \in L^N$.

(1) *If* λ *is an* L*-convex fuzzy sublattice of* N, *then* $f_L^{\leftarrow}(\lambda)$ *is an* L*-convex fuzzy sublattice of* M.

(2) *If* μ *is an* f*-invariant* L*-convex fuzzy sublattice of* M *and* f *is surjective, then* $f_L^{\rightarrow}(\mu)$ *is an* L*-convex fuzzy sublattice of* N.

Theorem 4.8. *Let* $\lambda \in L^M$ *and* $\mu \in L^N$ *be* L*-convex fuzzy sublattices of* M *and* N, *respectively. Then* $\lambda \times \mu$ *is an* L*-convex fuzzy sublattice of* $M \times N$.

5. L-fuzzy Sublattice Degree

In this section, we generalize the notion of L-fuzzy sublattices and introduce the notion of the degree to which an L-subset is an L-fuzzy sublattice by means of the implication operator of L.

Definition 5.1. Let $(M, \vee, \wedge)$ be a lattice and $\lambda \in L^M$. The L-fuzzy sublattice degree $LFL(\lambda)$ of λ is defined as

$$LFL(\lambda) = \bigwedge_{x,y \in M} (\lambda(x) \wedge \lambda(y) \to \lambda(x \wedge y) \wedge \lambda(x \vee y)).$$

Remark 5.2. (1) It is obvious that λ is an L-fuzzy sublattice if and only if $LFL(\lambda) = \top$.

(2) $LFL(\lambda) = \bigwedge_{x,y\in M} ((\lambda(x) \wedge \lambda(y) \to \lambda(x \wedge y)) \wedge (\lambda(x) \wedge \lambda(y) \to \lambda(x \vee y))$.

From properties (3) of the implication operator, the following lemma is obvious.

Lemma 5.3. *Let $(M, \vee, \wedge)$ be a lattice and $\lambda \in L^M$. Then $LFL(\lambda) \geqslant a$ if and only if for all $x, y \in M$,*

$$\lambda(x) \wedge \lambda(y) \wedge a \leqslant \lambda(x \wedge y) \wedge \lambda(x \vee y),$$

or equivalently,

$$\lambda(x) \wedge \lambda(y) \wedge a \leqslant \lambda(x \wedge y)$$

and

$$\lambda(x) \wedge \lambda(y) \wedge a \leqslant \lambda(x \vee y).$$

By Lemma 5.3, we can easily obtain the following theorem.

Theorem 5.4. *Let $(M, \vee, \wedge)$ be a lattice and $\lambda \in L^M$. Then*
$LFL(\lambda) = \bigvee\{a \in L : \lambda(x)\wedge\lambda(y)\wedge a \leqslant \lambda(x\wedge y), \lambda(x)\wedge\lambda(y)\wedge a \leqslant \lambda(x\vee y), x, y \in M\}$.

The following four theorems give characterizations of L-fuzzy sublattice degree of an L-subset by means of four kinds of levels.

Theorem 5.5. *Let $(M, \vee, \wedge)$ be a lattice and $\lambda \in L^M$. Then*
$LFL(\lambda) = \bigvee\{a \in L : \forall b \leqslant a, \lambda_{[b]}$ *is a sublattice of* $M\}$.

Proof. Assume that $a \in L$ with the property of $\lambda(x) \wedge \lambda(y) \wedge a \leqslant \lambda(x \wedge y)$ and $\lambda(x) \wedge \lambda(y) \wedge a \leqslant \lambda(x \vee y)$ for all $x, y \in M$. For any $b \leqslant a$, let $x, y \in \lambda_{[b]}$, then $\lambda(x\wedge y) \geqslant \lambda(x)\wedge\lambda(y)\wedge a \geqslant b\wedge a = b$ and $\lambda(x\vee y) \geqslant \lambda(x)\wedge\lambda(y)\wedge a \geqslant b\wedge a = b$, i.e., $x \wedge y, x \vee y \in \lambda_{[b]}$. Hence $\lambda_{[b]}$ is a sublattice of M. This shows $LFL(\lambda) \leqslant \bigvee\{a \in L : \forall b \leqslant a, \lambda_{[b]}$ is a sublattice of $M\}$.

Conversely, assume $a \in L$ and $\lambda_{[b]}$ is a sublattice of M for all $b \leqslant a$. For any $x, y \in M$, let $b = \lambda(x)\wedge\lambda(y)\wedge a$, then $b \leqslant a$ and $x, y \in \lambda_{[b]}$. Since $\lambda_{[b]}$ is a sublattice of M, we have $x \vee y \in \lambda_{[b]}$ and $x \wedge y \in \lambda_{[b]}$, i.e., $\lambda(x) \wedge \lambda(y) \wedge a \leqslant \lambda(x \wedge y)$ and $\lambda(x) \wedge \lambda(y) \wedge a \leqslant \lambda(x \vee y)$. This shows $LFL(\lambda) \geqslant \bigvee\{a \in L : \forall b \leqslant a, \lambda_{[b]}$ is a sublattice of $M\}$. This completes the proof. □

Theorem 5.6. *Let $(M, \vee, \wedge)$ be a lattice and $\lambda \in L^M$. Then*
$LFL(\lambda) = \bigvee\{a \in L : \forall b \in P(L), b \not\geqslant a, \lambda^{(b)}$ *is a sublattice of* $M\}$.

Proof. Assume that $a \in L$ with the property of $\lambda(x) \wedge \lambda(y) \wedge a \leqslant \lambda(x \wedge y)$ and $\lambda(x) \wedge \lambda(y) \wedge a \leqslant \lambda(x \vee y)$ for all $x, y \in M$. For any $b \in P(L)$ with $b \not\geqslant a$, let $x, y \in \lambda^{(b)}$, i.e., $\lambda(x) \not\leqslant b$ and $\lambda(y) \not\leqslant b$, then $\lambda(x)\wedge\lambda(y)\wedge a \not\leqslant b$. By $\lambda(x)\wedge\lambda(y)\wedge a \leqslant \lambda(x\wedge y)$ and $\lambda(x) \wedge \lambda(y) \wedge a \leqslant \lambda(x \vee y)$, it follows that $\lambda(x \wedge y) \not\leqslant b$ and $\lambda(x \vee y) \not\leqslant b$, i.e., $x \wedge y \in \lambda^{(b)}$ and $x \vee y \in \lambda^{(b)}$. Hence $\lambda^{(b)}$ is a sublattice of M. This shows that $LFL(\lambda) \leqslant \bigvee\{a \in L : \forall b \in P(L), b \not\geqslant a, \lambda^{(b)}$ is a sublattice of $M\}$.

Conversely, assume that $a \in L$ and $\lambda^{(b)}$ is a sublattice of M for all $b \in P(L)$ with $b \not\geqslant a$. For any $x, y \in M$, let $b \in P(L)$ and $\lambda(x) \wedge \lambda(y) \wedge a \not\leqslant b$, then

$x, y \in \lambda^{(b)}$ and $a \nleqslant b$. Since $\lambda^{(b)}$ is a sublattice of M, it holds that $x \wedge y \in \lambda^{(b)}$ and $x \vee y \in \lambda^{(b)}$, i.e., $\lambda(x \wedge y) \nleqslant b$ and $\lambda(x \vee y) \nleqslant b$. Hence $\lambda(x) \wedge \lambda(y) \wedge a \leqslant \lambda(x \wedge y)$ and $\lambda(x) \wedge \lambda(y) \wedge a \leqslant \lambda(x \vee y)$. This shows $LFL(\lambda) \geqslant \bigvee\{a \in L : \forall b \in P(L), b \ngeqslant a, \lambda^{(b)}$ is a sublattice of $M\}$. This completes the proof. □

Theorem 5.7. *Let $(M, \vee, \wedge)$ be a lattice and $\lambda \in L^M$. Then $LFL(\lambda) = \bigvee\{a \in L : \forall b \notin \alpha(a), \lambda^{[b]}$ is a sublattice of $M\}$.*

Proof. Assume that $a \in L$ with the property of $\lambda(x) \wedge \lambda(y) \wedge a \leqslant \lambda(x \wedge y)$ and $\lambda(x) \wedge \lambda(y) \wedge a \leqslant \lambda(x \vee y)$ for all $x, y \in M$. For any $b \notin \alpha(a)$, let $x, y \in \lambda^{[b]}$, i.e., $b \notin \alpha(\lambda(x))$ and $b \notin \alpha(\lambda(y))$, then by $b \notin \alpha(a)$, we have $b \notin \alpha(\lambda(x)) \bigcup \alpha(\lambda(y)) \bigcup \alpha(a)$. From $\alpha(\lambda(x)) \bigcup \alpha(\lambda(y)) \bigcup \alpha(a) = \alpha(\lambda(x) \wedge \lambda(y) \wedge a)$, we know $b \notin \alpha(\lambda(x) \wedge \lambda(y) \wedge a)$. By $\lambda(x) \wedge \lambda(y) \wedge a \leqslant \lambda(x \wedge y)$ and $\lambda(x) \wedge \lambda(y) \wedge a \leqslant \lambda(x \vee y)$, we have $b \notin \alpha(\lambda(x \vee y))$ and $b \notin \alpha(\lambda(x \wedge y))$, i.e., $x \vee y \in \lambda^{[b]}$, $x \wedge y \in \lambda^{[b]}$. Hence $\lambda^{[b]}$ is a sublattice of M. This shows $LFL(\lambda) \leqslant \bigvee\{a \in L : \forall b \notin \alpha(a), \lambda^{[b]}$ is a sublattice of $M\}$.

Conversely, assume that $a \in L$ and $\lambda^{[b]}$ is a sublattice of M for all $b \notin \alpha(a)$. Now we prove $\lambda(x) \wedge \lambda(y) \wedge a \leqslant \lambda(x \wedge y)$ and $\lambda(x) \wedge \lambda(y) \wedge a \leqslant \lambda(x \vee y)$ for any $x, y \in M$. Let $b \in L$ and $b \notin \alpha(\lambda(x) \wedge \lambda(y) \wedge a)$, then $b \notin \alpha(\lambda(x)) \bigcup \alpha(\lambda(y)) \bigcup \alpha(a)$. Thus $x, y \in \lambda^{[b]}$ and $b \notin \alpha(a)$. Since $\lambda^{[b]}$ is a sublattice of M, it holds that $x \vee y \in \lambda^{[b]}$ and $x \wedge y \in \lambda^{[b]}$, i.e., $b \notin \alpha(\lambda(x \wedge y))$ and $b \notin \alpha(\lambda(x \wedge y))$. This means $\alpha(\lambda(x) \wedge \lambda(y) \wedge a) \supseteq \alpha(\lambda(x \wedge y))$ and $\alpha(\lambda(x) \wedge \lambda(y) \wedge a) \supseteq \alpha(\lambda(x \vee y))$. Hence $\lambda(x) \wedge \lambda(y) \wedge a = \bigwedge \alpha(\lambda(x) \wedge \lambda(y) \wedge a) \leqslant \bigwedge \alpha(\lambda(x \wedge y)) = \lambda(x \wedge y)$ and $\lambda(x) \wedge \lambda(y) \wedge a = \bigwedge \alpha(\lambda(x) \wedge \lambda(y) \wedge a) \leqslant \bigwedge \alpha(\lambda(x \vee y)) = \lambda(x \vee y)$. This shows $LFL(\lambda) \geqslant \bigvee\{a \in L : \forall b \notin \alpha(a), \lambda^{[b]}$ is a sublattice of $M\}$. This completes the proof. □

Theorem 5.8. *Let $(M, \vee, \wedge)$ be a lattice and $\lambda \in L^M$. If $\beta(a \wedge b) = \beta(a) \bigcap \beta(b)$, then $LFL(\lambda) = \bigvee\{a \in L : \forall b \in \beta(a), \lambda_{(b)}$ is a sublattice of $M\}$.*

Proof. Assume that $a \in L$ with the property of $\lambda(x) \wedge \lambda(y) \wedge a \leqslant \lambda(x \wedge y)$ and $\lambda(x) \wedge \lambda(y) \wedge a \leqslant \lambda(x \vee y)$ for all $x, y \in M$. For any $b \in \beta(a)$, let $x, y \in \lambda_{(b)}$, then $b \in \beta(\lambda(x))$ and $b \in \beta(\lambda(y))$. From $b \in \beta(a)$, we have $b \in \beta(\lambda(x)) \bigcap \beta(\lambda(y)) \bigcap \beta(a)$. Thus $b \in \beta(\lambda(x) \wedge \lambda(y) \wedge a)$. By $\lambda(x) \wedge \lambda(y) \wedge a \leqslant \lambda(x \wedge y)$ and $\lambda(x) \wedge \lambda(y) \wedge a \leqslant \lambda(x \vee y)$, we have $b \in \beta(\lambda(x \wedge y))$ and $b \in \beta(\lambda(x \vee y))$, i.e., $x \wedge y \in \lambda_{(b)}$ and $x \vee y \in \lambda_{(b)}$. Therefore $\lambda_{(b)}$ is a sublattice of M. This shows that $LFL(\lambda) \leqslant \bigvee\{a \in L : \forall b \in \beta(a), \lambda_{(b)}$ is a sublattice of $M\}$.

Conversely, assume that $a \in L$ and $\lambda_{(b)}$ is a sublattice of M for all $b \in \beta(a)$. Now we prove $\lambda(x) \wedge \lambda(y) \wedge a \leqslant \lambda(x \wedge y)$ and $\lambda(x) \wedge \lambda(y) \wedge a \leqslant \lambda(x \vee y)$ for any $x, y \in M$. Let $b \in L$ and $b \in \beta(\lambda(x) \wedge \lambda(y) \wedge a)$, then $b \in \beta(\lambda(x)) \bigcap \beta(\lambda(y)) \bigcap \beta(a)$. Thus $x, y \in \lambda_{(b)}$ and $b \in \beta(a)$. Since $\lambda_{(b)}$ is a sublattice of M, it holds that $x \wedge y \in \lambda_{(b)}$ and $x \vee y \in \lambda_{(b)}$, i.e., $b \in \beta(\lambda(x \wedge y)$ and $b \in \beta(\lambda(x \vee y)$. This implies $\beta(\lambda(x) \wedge \lambda(y) \wedge a) \subseteq \beta(\lambda(x \wedge y)$ and $\beta(\lambda(x) \wedge \lambda(y) \wedge a) \subseteq \beta(\lambda(x \vee y)$. So $\lambda(x) \wedge \lambda(y) \wedge a = \bigvee \beta(\lambda(x) \wedge \lambda(y) \wedge a) \leqslant \bigvee \beta(\lambda(x \wedge y) = \lambda(x \wedge y)$ and $\lambda(x) \wedge \lambda(y) \wedge a = \bigvee \beta(\lambda(x) \wedge \lambda(y) \wedge a) \leqslant \bigvee \beta(\lambda(x \vee y) = \lambda(x \vee y)$. This shows that $LFL(\lambda) \geqslant \bigvee\{a \in L : \forall b \in \beta(a), \lambda_{(b)}$ is a sublattice of $M\}$. This completes the proof. □

Remark 5.9. It is not difficult to see that Theorem 5.5, Theorem 5.6 and Theorem 5.7 are generalizations of (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3), (1) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) and (1) $\Leftrightarrow$ (6) in Theorem 3.3, respectively. That is, Theorem 5.5, Theorem 5.6 and Theorem 5.7 reduce to Theorem 3.3 when $LFL(\lambda) = \top$. Similarly, Theorem 5.8 is a generalization of Theorem 3.4.

Theorem 5.10. *Let $f : M \to N$ be a lattice homomorphism and $\mu \in L^M$, $\lambda \in L^N$.*

(1) *$LFL_M(f_L^{\leftarrow}(\lambda)) \geqslant LFL_N(\lambda)$, and if f is surjective, then $LFL_M(f_L^{\leftarrow}(\lambda)) = LFL_N(\lambda)$.*

(2) *$LFL_M(\mu) \leqslant LFL_N(f_L^{\rightarrow}(\mu))$, and if f is injective, then $LFL_M(\mu) = LFL_N(f_L^{\rightarrow}(\mu))$.*

Proof. (1) Let $a \in L$ and $\lambda(x) \wedge \lambda(y) \wedge a \leqslant \lambda(x \wedge y)$ and $\lambda(x) \wedge \lambda(y) \wedge a \leqslant \lambda(x \vee y)$ for all $x, y \in N$. Then for any $s, t \in M$,

$$\begin{aligned} & f_L^{\leftarrow}(\lambda)(s) \wedge f_L^{\leftarrow}(\lambda)(t) \wedge a \\ & = \lambda(f(s)) \wedge \lambda(f(t)) \wedge a \leqslant \lambda(f(s) \wedge f(t)) \\ & = \lambda(f(s \wedge t) = f_L^{\leftarrow}(\lambda)(s \wedge t). \end{aligned}$$

Similarly, $f_L^{\leftarrow}(\lambda)(s) \wedge f_L^{\leftarrow}(\lambda)(t) \wedge a \leqslant f_L^{\leftarrow}(\lambda)(s \vee t)$. This shows that $LFL_M(f_L^{\leftarrow}(\lambda)) \geqslant LFL_N(\lambda)$. If f is surjective, then it is easily checked that $LFL_M(f_L^{\leftarrow}(\lambda)) \leqslant LFL_N(\lambda)$. Therefore $LFL_M(f_L^{\leftarrow}(\lambda)) = LFL_N(\lambda)$.

(2) Let $a \in L$, $\mu(s) \wedge \mu(t) \wedge a \leqslant \mu(s \wedge t)$ and $\mu(s) \wedge \mu(t) \wedge a \leqslant \mu(s \vee t)$ for all $s, t \in M$. Then for any $x, y \in N$,

$$\begin{aligned} & f_L^{\rightarrow}(\mu)(x) \wedge f_L^{\rightarrow}(\mu)(y) \wedge a \\ & = \bigvee_{f(s)=x} \mu(s) \wedge \bigvee_{f(t)=y} \mu(t) \wedge a \\ & = \bigvee \{\mu(s) \wedge \mu(t) \wedge a : f(s) = x, f(t) = y\} \\ & \leqslant \bigvee \{\mu(s \wedge t) : f(s) = x, f(t) = y\} \\ & \leqslant \bigvee \{\mu(s \wedge t) : f(s \wedge t) = x \wedge y\} \\ & \leqslant \bigvee \{\mu(w) : f(w) = x \wedge y\} \\ & = f_L^{\rightarrow}(\mu)(x \wedge y). \end{aligned}$$

Similarly, we can prove that $f_L^{\rightarrow}(\mu)(x) \wedge f_L^{\rightarrow}(\mu)(y) \wedge a \leqslant f_L^{\rightarrow}(\mu)(x \vee y)$. This shows that $LFL_M(\mu) \leqslant LFL_N(f_L^{\rightarrow}(\mu))$.

If f is injective, in order to prove $LFL_M(\mu) = LFL_N(f_L^{\rightarrow}(\mu))$, we only need to prove $LFL_N(f_L^{\rightarrow}(\mu)) \leqslant LFL_M(\mu)$. Let $a \in L$, $f_L^{\rightarrow}(\mu)(x) \wedge f_L^{\rightarrow}(\mu)(y) \wedge a \leqslant f_L^{\rightarrow}(\mu)(x \wedge y)$ and $f_L^{\rightarrow}(\mu)(x) \wedge f_L^{\rightarrow}(\mu)(y) \wedge a \leqslant f_L^{\rightarrow}(\mu)(x \vee y)$ for all $x, y \in N$. Then for any $s, t \in M$, let $f(s) = x$ and $f(t) = y$. Since f is injective and f is a homomorphism, we have $f_L^{\rightarrow}(\mu)(x) = \mu(s)$, $f_L^{\rightarrow}(\mu)(y) = \mu(t)$ and $f_L^{\rightarrow}(\mu)(x \wedge y) = \mu(s \wedge t)$. Thus

$$\begin{aligned} & \mu(s) \wedge \mu(t) \wedge a \\ & = f_L^{\rightarrow}(\mu)(x) \wedge f_L^{\rightarrow}(\mu)(y) \wedge a \\ & \leqslant f_L^{\rightarrow}(\mu)(x \wedge y) \\ & = \mu(s \wedge t). \end{aligned}$$

Similarly, we can prove that $\mu(s)\wedge\mu(t)\wedge a \leqslant \mu(s\vee t)$. This shows that $LFL_N(f_L^{\rightarrow}(\mu)) \leqslant LFL_M(\mu)$. □

6. L-convex Fuzzy Sublattice Degree

In this section, analogously, we generalize the notion of L-convex fuzzy sublattices and introduce the notion of the degree to which an L-subset is an L-convex fuzzy sublattice.

Definition 6.1. Let $(M,\vee,\wedge)$ be a lattice and $\lambda \in L^M$. The L-convex fuzzy sublattice degree $LCFL(\lambda)$ of λ is defined as
$LCFL(\lambda) = LFL(\lambda)\wedge\bigwedge\{\lambda(x\wedge y)\wedge\lambda(x\vee y) \to \lambda(z) : x\wedge y \leqslant z \leqslant x\vee y, x, y, z \in M\}$.

Remark 6.2. It is obvious that λ is an L-fuzzy convex sublattice if and only if $LCFL(\lambda) = \top$, and λ always is an L-fuzzy convex sublattice to the degree $LCFL(\lambda)$.

Example 6.3. Let $M = \{\bot_M, a, b, c, \top_M\}$ be a lattice and $L = [0,1]$, where a, b, c are incomparable.

(1) Define $\lambda \in L^M$ by $\lambda(\bot_M) = 0.64$, $\lambda(a) = 0.82$, $\lambda(b) = 0.91$, $\lambda(c) = 0.45$ and $\lambda(\top_M) = 0.85$. Then $LCFL(\lambda) = (0.82 \to 0.85) \wedge (0.82 \to 0.64) \wedge (0.45 \to 0.85) \wedge (0.45 \to 0.64) \wedge (0.64 \to 0.82) \wedge (0.64 \to 0.91) \wedge (0.64 \to 0.45) = 0.64 \wedge 0.45 = 0.45$.

(2) Define $\mu \in L^X$ by $\mu(\bot_M) = 0.35$, $\mu(a) = 0.32$, $\mu(b) = 0.32$, $\mu(c) = 0.48$ and $\mu(\top_M) = 0.32$. Then it is easily checked that $LCFL(\lambda) = \top$, and it is easy to see that μ is an L-convex fuzzy sublattice.

Analogous to Lemma 5.3, we have the following lemma.

Lemma 6.4. *Let $(M,\vee,\wedge)$ be a lattice and $\lambda \in L^M$. Then $LCFL(\lambda) \geqslant a$ if and only if $LFL(\lambda) \geqslant a$ and $\lambda(x \wedge y) \wedge \lambda(x \vee y) \wedge a \leqslant \lambda(z)$ for all $x, y, z \in M$ with $x \wedge y \leqslant z \leqslant x \vee y$.*

By Lemma 5.3 and Lemma 6.4, we can easily obtain the following lemma.

Lemma 6.5. *Let $(M,\vee,\wedge)$ be a lattice and $\lambda \in L^M$. Then*
$LCFL(\lambda) = \bigvee\{a \in L : \lambda(x) \wedge \lambda(y) \wedge a \leqslant \lambda(x \wedge y), \lambda(x) \wedge \lambda(y) \wedge a \leqslant \lambda(x \vee y), \lambda(x \wedge y) \wedge \lambda(x \vee y) \wedge a \leqslant \lambda(z), x \wedge y \leqslant z \leqslant x \vee y, x, y, z \in M\}$.

Theorem 6.6. *Let $(M,\vee,\wedge)$ be a lattice and $\lambda \in L^M$. Then*
$LCFL(\lambda) = \bigvee\{a \in L : \forall b \leqslant a, \lambda_{[b]}$ *is a convex sublattice of* $M\}$.

Proof. Let $a \in L$, $LFL(\lambda) \geqslant a$ and let $\lambda(x\wedge y)\wedge\lambda(x\vee y)\wedge a \leqslant \lambda(z)$ for $x, y, z \in M$ with $x \wedge y \leqslant z \leqslant x \vee y$. By the proof of Theorem 5.5, we know that $\lambda_{[b]}$ is a sublattice of M for all $b \leqslant a$. Now we prove that $\lambda_{[b]}$ is convex. Let $x, y \in \lambda_{[b]}$ and $x \wedge y \leqslant z \leqslant x \vee y$, then $\lambda(x \wedge y) \geqslant b$, $\lambda(x \vee y) \geqslant b$. By $a \geqslant b$, we have $\lambda(z) \geqslant \lambda(x \wedge y) \wedge \lambda(x \vee y) \wedge a \geqslant b$, i.e., $z \in \lambda_{[b]}$. Hence $\lambda_{[b]}$ is a convex sublattice of M. This shows $LCFL(\lambda) \leqslant \bigvee\{a \in L : \forall b \leqslant a, \lambda_{[b]}$ is a convex sublattice of $M\}$.

Conversely, assume that $a \in L$ and $\lambda_{[b]}$ is a convex sublattice of M for all $b \leqslant a$. Then by Theorem 5.5, we know $LFL(\lambda) \geqslant a$. For any $x, y, z \in M$ with $x \wedge y \leqslant z \leqslant x \vee y$, in order to prove $\lambda(x \wedge y) \wedge \lambda(x \vee y) \wedge a \leqslant \lambda(z)$, let $\lambda(x \wedge y) \wedge \lambda(x \vee y) \wedge a = b$, then $x \vee y \in \lambda_{[b]}$ and $x \wedge y \in \lambda_{[b]}$ and $a \geqslant b$. Since $\lambda_{[b]}$ is a convex sublattice of M, it holds that $z \in \lambda_{[b]}$, i.e., $\lambda(z) \geqslant b$. Hence $\lambda(x \wedge y) \wedge \lambda(x \vee y) \wedge a \leqslant \lambda(z)$. This shows $LCFL(\lambda) \geqslant \bigvee\{a \in L : \forall b \leqslant a, \lambda_{[b]}$ is a convex sublattice of $M\}$. This completes the proof. □

Theorem 6.7. *Let* $(M, \vee, \wedge)$ *be a lattice and* $\lambda \in L^M$. *Then*
$LCFL(\lambda) = \bigvee\{a \in L : \forall b \in P(L), b \ngeqslant a, \lambda^{(b)}$ *is a convex sublattice of* $M\}$.

Proof. Assume that $a \in L$, $LFL(\lambda) \geqslant a$ and $\lambda(x \wedge y) \wedge \lambda(x \vee y) \wedge a \leqslant \lambda(z)$ for $x, y, z \in M$ with $x \wedge y \leqslant z \leqslant x \vee y$. By the proof of Theorem 5.6, we know that $\lambda^{(b)}$ is a sublattice of M for all $b \in P(L)$ with $b \ngeqslant a$. Now we prove that $\lambda^{(b)}$ is convex, let $x, y \in \lambda^{(b)}$ and $x \wedge y \leqslant z \leqslant x \vee y$. Then $\lambda(x \wedge y) \nleqslant b$ and $\lambda(x \vee y) \nleqslant b$. By $a \nleqslant b$, we have $\lambda(x \wedge y) \wedge \lambda(x \vee y) \wedge a \nleqslant b$. By $\lambda(x \wedge y) \wedge \lambda(x \vee y) \wedge a \leqslant \lambda(z)$, we have $\lambda(z) \nleqslant b$, i.e., $z \in \lambda^{(b)}$. Hence $\lambda^{(b)}$ is a convex sublattice of M. This shows that $LCFL(\lambda) \leqslant \bigvee\{a \in L : \forall b \in P(L), b \ngeqslant a, \lambda^{(b)}$ is a convex sublattice of $M\}$.

Conversely, assume that $a \in L$ and $\lambda^{(b)}$ is a convex sublattice of M for all $b \in P(L)$ with $b \ngeqslant a$. Then by Theorem 5.6, we know that $LFL(\lambda) \geqslant a$. Let $x, y, z \in M$ and $x \wedge y \leqslant z \leqslant x \vee y$. Now we prove $\lambda(x \wedge y) \wedge \lambda(x \vee y) \wedge a \leqslant \lambda(z)$. Let $\lambda(x \wedge y) \wedge \lambda(x \vee y) \wedge a \nleqslant b$, then $x \vee y \in \lambda^{(b)}$, $x \wedge y \in \lambda^{(b)}$ and $a \nleqslant b$. Since $\lambda^{(b)}$ is a convex sublattice of M, we have $z \in \lambda^{(b)}$, i.e., $\lambda(z) \nleqslant b$. Hence $\lambda(x \wedge y) \wedge \lambda(x \vee y) \wedge a \leqslant \lambda(z)$. This shows $LCFL(\lambda) \geqslant \bigvee\{a \in L : \forall b \in P(L), b \ngeqslant a, \lambda^{(b)}$ is a convex sublattice of $M\}$. This completes the proof. □

Theorem 6.8. *Let* $(M, \vee, \wedge)$ *be a lattice and* $\lambda \in L^M$. *Then*
$LCFL(\lambda) = \bigvee\{a \in L : \forall b \notin \alpha(a), \lambda^{[b]}$ *is a convex sublattice of* $M\}$.

Proof. Assume $a \in L$, $LFL(\lambda) \geqslant a$ and $\lambda(x \wedge y) \wedge \lambda(x \vee y) \wedge a \leqslant \lambda(z)$ for all $x, y, z \in M$ with $x \wedge y \leqslant z \leqslant x \vee y$. By the proof of Theorem 5.7, we know that $\lambda^{[b]}$ is a sublattice of M for all $b \notin \alpha(a)$. Now we prove that $\lambda^{[b]}$ is convex. Let $x, y \in \lambda^{[b]}$, $z \in M$ and $x \wedge y \leqslant z \leqslant x \vee y$. Then $b \notin \alpha(\lambda(x \wedge y))$ and $b \notin \alpha(\lambda(x \vee y))$. By $b \notin \alpha(a)$, we have $b \notin \alpha(\lambda(x \wedge y)) \bigcup \alpha(\lambda(x \vee y)) \bigcup \alpha(a) = \alpha(\lambda(x \wedge y) \wedge \lambda(x \vee y) \wedge a)$. By $\lambda(x \wedge y) \wedge \lambda(x \vee y) \wedge a \leqslant \lambda(z)$, we know $b \notin \alpha(\lambda(z))$, i.e., $z \in \lambda^{[b]}$. Hence $\lambda^{[b]}$ is a convex sublattice of M. This shows that $LCFL(\lambda) \leqslant \bigvee\{a \in L : \forall b \notin \alpha(a), \lambda^{[b]}$ is a convex sublattice of $M\}$.

Conversely, assume that $a \in L$ and $\lambda^{[b]}$ is a convex sublattice of M for all $b \notin \alpha(a)$. Then by Theorem 5.7, we know that$LFL(\lambda) \geqslant a$. Let $x, y, z \in M$ and $x \wedge y \leqslant z \leqslant x \vee y$. Now we prove $\lambda(x \wedge y) \wedge \lambda(x \vee y) \wedge a \leqslant \lambda(z)$. Let $b \notin \alpha(\lambda(x \wedge y) \wedge \lambda(x \vee y) \wedge a) = \alpha(\lambda(x \wedge y)) \bigcup \alpha(\lambda(x \vee y)) \bigcup \alpha(a)$, then $x \wedge y$, $x \vee y \in \lambda^{[b]}$ and $b \notin \alpha(a)$. Since $\lambda^{[b]}$ is a convex sublattice of M, it holds that $z \in \lambda^{[b]}$, i.e., $b \notin \alpha(\lambda(z))$. This implies $\alpha(\lambda(x \wedge y) \wedge \lambda(x \vee y) \wedge a) \supseteq \alpha(\lambda(z))$. Hence $\lambda(x \wedge y) \wedge \lambda(x \vee y) \wedge a = \bigwedge \alpha(\lambda(x \wedge y) \wedge \lambda(x \vee y) \wedge a) \leqslant \bigwedge \alpha(\lambda(z)) = \lambda(z)$. This shows $LCFL(\lambda) \geqslant \bigvee\{a \in L : \forall b \notin \alpha(a), \lambda^{[b]}$ is a convex sublattice of $M\}$. This completes the proof. □

Theorem 6.9. *Let* $(M, \vee, \wedge)$ *be a lattice and* $\lambda \in L^M$. *If* $\beta(a \wedge b) = \beta(a) \bigcap \beta(b)$, *then* $LCFL(\lambda) = \bigvee\{a \in L : \forall b \in \beta(a), \lambda_{(b)}$ *is a convex sublattice of* $M\}$.

Proof. Let that $a \in L$, $LFL(\lambda) \geqslant a$ and let $\lambda(x \wedge y) \wedge \lambda(x \vee y) \wedge a \leqslant \lambda(z)$ for all $x, y, z \in M$ with $x \wedge y \leqslant z \leqslant x \vee y$. By the proof of Theorem 5.8, we know $\lambda_{(b)}$ is a sublattice of M for all $b \in \beta(a)$. Now we prove that $\lambda_{(b)}$ is convex. Let $x, y \in \lambda_{(b)}$, $z \in M$ and $x \wedge y \leqslant z \leqslant x \vee y$, then $b \in \beta(\lambda(x \wedge y))$ and $b \in \beta(\lambda(x \vee y))$, and by $b \in \beta(a)$, we have $b \in \beta(\lambda(x \wedge y)) \bigcap \beta(\lambda(x \vee y)) \bigcap \beta(a)$. Thus $b \in \beta(\lambda(x \wedge y) \wedge \lambda(x \vee y) \wedge a)$. By $\lambda(x \wedge y) \wedge \lambda(x \vee y) \wedge a \leqslant \lambda(z)$, we have $b \in \beta(\lambda(z))$, i.e., $z \in \lambda_{(b)}$. Hence $\lambda_{(b)}$ is a convex sublattice of M. This shows $LCFL(\lambda) \leqslant \bigvee\{a \in L : \forall b \in \beta(a), \lambda_{(b)}$ is a convex sublattice of $M\}$.

Conversely, suppose that $a \in L$ and $\lambda_{(b)}$ is a convex sublattice of M for all $b \in \beta(a)$. Then by Theorem 5.8, we know $LFL(\lambda) \geqslant a$. For any $x, y, z \in M$ with $x \wedge y \leqslant z \leqslant x \vee y$, in order to prove $\lambda(x \wedge y) \wedge \lambda(x \vee y) \wedge a \leqslant \lambda(z)$, let $b \in \beta(\lambda(x \wedge y) \wedge \lambda(x \vee y) \wedge a)$, then $b \in \beta(\lambda(x \wedge y)) \bigcap \beta(\lambda(x \vee y)) \bigcap \beta(a)$. Thus $x \vee y, x \wedge y \in \lambda_{(b)}$ and $b \in \beta(a)$. Since $\lambda_{(b)}$ is a convex sublattice of M and $x \wedge y \leqslant z \leqslant x \vee y$, we have $z \in \lambda_{(b)}$, i.e., $b \in \beta(\lambda(z))$. This implies $\beta(\lambda(x \wedge y) \wedge \lambda(x \vee y) \wedge a) \subseteq \beta(\lambda(z))$. So $\lambda(x \wedge y) \wedge \lambda(x \vee y) \wedge a = \bigvee \beta(\lambda(x \wedge y) \wedge \lambda(x \vee y) \wedge a) \leqslant \bigvee \beta(\lambda(z)) = \lambda(z)$. This shows that $LCFL(\lambda) \geqslant \bigvee\{a \in L : \forall b \in \beta(a), \lambda_{(b)}$ is a convex sublattice of $M\}$. This completes the proof. □

Theorem 6.10. *Let $f : M \to N$ be a lattice homomorphism, $\mu \in L^M$ and $\lambda \in L^N$.*

(1) *$LCFL_M(f_L^{\leftarrow}(\lambda)) \geqslant LCFL_N(\lambda)$, and if f is surjective, then $LCFL_M(f_L^{\leftarrow}(\lambda)) = LCFL_N(\lambda)$.*

(2) *If f is surjective and μ is f-invariant, then $LCFL_M(\mu) \leqslant LCFL_N(f_L^{\rightarrow}(\mu))$.*

Proof. (1) By Theorem 5.10, we know $LFL_M(f_L^{\leftarrow}(\lambda)) \geqslant LFL_N(\lambda)$. Let $a \in L$, $LFL_M(f_L^{\leftarrow}(\lambda)) \geqslant a$ and $\lambda(x \wedge y) \wedge \lambda(x \vee y) \wedge a \leqslant \lambda(z)$ for all $x, y, z \in N$ with $x \wedge y \leqslant z \leqslant x \vee y$. Then $LFL_M(f_L^{\leftarrow}(\lambda)) \geqslant a$. Let $s, t, w \in M$ and $s \wedge t \leqslant w \leqslant s \vee t$. Since f is order-preserving, we have $f(s \wedge t) \leqslant f(w) \leqslant f(s \vee t)$. Hence

$$\begin{aligned}
&f_L^{\leftarrow}(\lambda)(s \wedge t) \wedge f_L^{\leftarrow}(\lambda)(s \vee t) \wedge a \\
&= \lambda(f(s \wedge t)) \wedge \lambda(f(s \vee t)) \wedge a \\
&\leqslant \lambda(f(w)) \\
&= f^{\leftarrow}(\lambda)(w).
\end{aligned}$$

This shows $LCFL_M(f_L^{\leftarrow}(\lambda)) \geqslant LCFL_N(\lambda)$. If f is surjective, then it is easily checked that $LCFL_M(f_L^{\leftarrow}(\lambda)) \leqslant LCFL_N(\lambda)$. Therefore $LCFL_M(f_L^{\leftarrow}(\lambda)) = LCFL_N(\lambda)$.

(2) Let $a \in L$, $a \leqslant LFL_M(\mu)$ and $\mu(s \wedge t) \wedge \mu(s \vee t) \wedge a \leqslant \mu(w)$ for all $s, t, w \in M$ with $s \wedge t \leqslant w \leqslant s \vee t$. By Theorem 5.10, we have $a \leqslant LFL_N(f_L^{\rightarrow}(\mu))$. Let $x, y, z \in N$ and $x \wedge y \leqslant z \leqslant x \vee y$. Then there exist $s, t, w \in M$ such that $f(s) = x \wedge y$, $f(t) = x \vee y$ and $f(w) = z$. Thus $f(s) \leqslant f(w) \leqslant f(t)$. Since f is f-invariant, by Lemma 4.6, we know $\mu(s) = \mu(s \wedge w)$ and $\mu(t) = \mu(w \vee t)$. Since $s \wedge w \leqslant w \vee t$, we have $\mu(s \wedge w) \wedge \mu(w \vee t) \wedge a \leqslant \mu(w)$, i.e., $\mu(s) \wedge \mu(t) \wedge a \leqslant \mu(w)$. Hence

$$\begin{aligned}
&f_L^{\rightarrow}(\mu)(x) \wedge f_L^{\rightarrow}(\mu)(y) \wedge a \\
&= \bigvee_{f(s)=x\wedge y} \mu(s) \wedge \bigvee_{f(t)=x\vee y} \mu(t) \wedge a \\
&= \bigvee\{\mu(s) \wedge \mu(t) \wedge a : f(s) = x \wedge y, f(t) = x \vee y\} \\
&\leqslant \bigvee\{\mu(w) : f(w) = z\} \\
&= f_L^{\rightarrow}(\mu)(z).
\end{aligned}$$

This shows $LCFL_M(\mu) \leqslant LCFL_N(f_L^{\rightarrow}(\mu))$. □

Theorem 6.11. *Let M and N be two lattices. Then $LCFL(\mu \times \lambda) \geqslant LCFL(\mu) \wedge LCFL(\lambda)$ for $\mu \in L^M$ and $\lambda \in L^N$.*

Proof. It is straightforward. □

7. L-fuzzy Convexity Induced by L-convex Fuzzy Sublattice Degree

In this section, we obtain an L-fuzzy convexity on a lattice by means of L-convex fuzzy sublattice degrees. Moreover, we discuss some properties of this kind of L-fuzzy convexity.

For a lattice M, let $\lambda \in L^M$. $LCFL(\lambda)$ can be considered as the degree to which λ is an L-convex fuzzy sublattice. Thus $LCFL$ can be naturally considered as a mapping from L^M to L by $\lambda \mapsto LCFL(\lambda)$. By the following theorem, we know that $LCFL$ is an L-fuzzy convexity on the lattice M, which is called the L-fuzzy convexity induced by L-convex fuzzy sublattice degrees.

Theorem 7.1. *Let $(M, \vee, \wedge)$ be a lattice and $\lambda \in L^M$. Then $LCFL$ is an L-fuzzy convexity on M.*

Proof. (**LMC1**) It is obvious that $LCFL(\chi_{\emptyset}) = LCFL(\chi_M) = \top$.

(**LMC2**) Let $\{\lambda_i : i \in \Omega\} \subseteq L^M$ is nonempty. In order to prove $LCFL(\bigwedge_{i\in\Omega} \lambda_i) \geqslant \bigwedge_{i\in\Omega} LCFL(\lambda_i)$, let $a \in L$ with $a \leqslant \bigwedge_{i\in\Omega} LCFL(\lambda_i)$. Then for any $i \in \Omega$, $\lambda_i(x)\wedge\lambda_i(y)\wedge a \leqslant \lambda_i(x \wedge y)$, $\lambda_i(x) \wedge \lambda_i(y) \wedge a \leqslant \lambda_i(x \vee y)$ and $\lambda_i(x \wedge y) \wedge \lambda_i(x \vee y) \wedge a \leqslant \lambda_i(z)$ for all $x, y, z \in M$ with $x \wedge y \leqslant z \leqslant x \vee y$. This implies

$$\bigwedge_{i\in\Omega} \lambda_i(x) \wedge \bigwedge_{i\in\Omega} \lambda_i(y) \wedge a \leqslant \bigwedge_{i\in\Omega} \lambda_i(x \wedge y),$$

$$\bigwedge_{i\in\Omega} \lambda_i(x) \wedge \bigwedge_{i\in\Omega} \lambda_i(y) \wedge a \leqslant \bigwedge_{i\in\Omega} \lambda_i(x \vee y)$$

and

$$\bigwedge_{i\in\Omega} \lambda_i(x \wedge y) \wedge \bigwedge_{i\in\Omega} \lambda_i(x \vee y) \wedge a \leqslant \bigwedge_{i\in\Omega} \lambda_i(z)$$

for all $x, y, z \in M$ with $x\wedge y \leqslant z \leqslant x\vee y$. This shows $a \leqslant LCFL(\bigwedge_{i\in\Omega} \lambda_i)$. Therefore $LCFL(\bigwedge_{i\in\Omega} \lambda_i) \geqslant \bigwedge_{i\in\Omega} LCFL(\lambda_i)$

(**LMC3**) Let $\{\lambda_i : i \in \Omega\} \subseteq L^M$ be nonempty and totally ordered by the order in L. In order to prove $LCFL(\bigvee_{i\in\Omega} \lambda_i) \geqslant \bigwedge_{i\in\Omega} LCFL(\lambda_i)$, let $a \in L$ with $a \leqslant \bigwedge_{i\in\Omega} LCFL(\lambda_i)$. Then for any $i \in \Omega$, $x, y, z \in M$ with $x \wedge y \leqslant z \leqslant x \vee y$, $\lambda_i(x)\wedge\lambda_i(y)\wedge a \leqslant \lambda_i(x\wedge y)$, $\lambda_i(x)\wedge\lambda_i(y)\wedge a \leqslant \lambda_i(x\vee y)$ and $\lambda_i(x\wedge y)\wedge\lambda_i(x\vee y)\wedge a \leqslant \lambda_i(z)$. Now we prove $\bigvee_{i\in\Omega} \lambda_i(x) \wedge \bigvee_{i\in\Omega} \lambda_i(y) \wedge a \leqslant \bigvee_{i\in\Omega} \lambda_i(x \wedge y)$. Let $b \in J(L)$ with $b \prec \bigvee_{i\in\Omega} \lambda_i(x)\wedge \bigvee_{i\in\Omega} \lambda_i(y)\wedge a$, then there exists $i, j \in \Omega$ such that $b \leqslant \lambda_i(x)$, $b \leqslant \lambda_j(y)$ and $b \leqslant a$. Since $\{\lambda_i : i \in \Omega\}$ is totally ordered, we assume that $\lambda_j \leqslant \lambda_i$. Thus

$b \leqslant \lambda_i(x) \wedge \lambda_i(y) \wedge a$. By $\lambda_i(x) \wedge \lambda_i(y) \wedge a \leqslant \lambda_i(x \wedge y)$, we have $b \leqslant \lambda_i(x \wedge y)$. Hence $b \leqslant \bigvee_{i\in\Omega} \lambda_i(x \wedge y)$. This implies

$$\bigvee_{i\in\Omega} \lambda_i(x) \wedge \bigvee_{i\in\Omega} \lambda_i(y) \wedge a \leqslant \bigvee_{i\in\Omega} \lambda_i(x \wedge y).$$

Similarly, we can show that

$$\bigvee_{i\in\Omega} \lambda_i(x) \wedge \bigvee_{i\in\Omega} \lambda_i(y) \wedge a \leqslant \bigvee_{i\in\Omega} \lambda_i(x \vee y)$$

and

$$\bigvee_{i\in\Omega} \lambda_i(x \wedge y) \wedge \bigvee_{i\in\Omega} \lambda_i(x \vee y) \wedge a \leqslant \bigvee_{i\in\Omega} \lambda_i(z).$$

This shows $a \leqslant LCFL(\bigvee_{i\in\Omega} \lambda_i)$. Therefore $LCFL(\bigvee_{i\in\Omega} \lambda_i) \geqslant \bigwedge_{i\in\Omega} LCFL(\lambda_i)$. This completes the proof. □

In the following, some properties of the L-fuzzy convexity $LCFL$ are discussed.

From papers [28, 22], we know $LCFL_{[a]}$ is an L-convexity for all $a \in L\setminus\{\bot\}$ and $LCFL^{[a]}$ is an L-convexity for all $a \in \alpha(\bot)$ on the lattice M. Then the following theorem is obvious.

Theorem 7.2. *Let $(M, \vee, \wedge)$ be a lattice and $LCFL$ be the L-fuzzy convexity induced by L-convex fuzzy sublattice degrees. Then the following hold:*

(1) $\forall a \in J(L)$, $LCFL_{[a]}$ is an L-convexity on M.

(2) $\forall a \in \alpha^*(\bot)$, $LCFL^{[a]}$ is an L-convexity on M.

Definition 7.3. [28, 22] Let $(X, \mathcal{C})$ and $(Y, \mathcal{D})$ be two (L, M)-fuzzy convex spaces. A mapping $f : X \to Y$ is called an (L, M)-fuzzy convexity preserving mapping if $\mathcal{C}(f_L^{\leftarrow}(B)) \geqslant \mathcal{D}(B)$ for all $B \in L^Y$.

An (L, L)-fuzzy convexity preserving mapping is briefly called an L-fuzzy convexity preserving mapping.

An $(L, 2)$-fuzzy convexity preserving mapping is an L-convexity preserving mapping in [13].

By Theorem 6.10, we have the following theorem.

Theorem 7.4. *Let $f : M \to M'$ be a lattice homomorphism. Then $f : (M, LCFL_M) \to (M', LCFL_{M'})$ is an L-fuzzy convexity preserving mapping.*

From Theorem 7.4, we obtain a concrete functor from the category **L** of lattices and lattice homomorphisms and the category **LCS** of L-fuzzy convex spaces and L-fuzzy convexity preserving mappings.

Theorem 7.5. *Let M and N be two lattices and $f : M \to N$ be a mapping. Then the following conditions are equivalent:*

(1) $f : (M, LCFL_M) \to (N, LCFL_N)$ is an L-fuzzy convexity preserving mapping.

(2) $f : (M, LCFL_{M[a]}) \to (N, LCFL_{N[a]})$ is an L-convexity preserving mapping for each $a \in L \setminus \{\bot\}$.

(3) $f : (M, LCFL_{M[a]}) \to (N, LCFL_{N[a]})$ *is an* L*-convexity preserving mapping for each* $a \in J(L)$.

(4) $f : (M, LCFL_M^{[a]}) \to (N, LCFL_N^{[a]})$ *is an* L*-convexity preserving mapping for each* $a \in \alpha(\perp)$.

(5) $f : (M, LCFL_M^{[a]}) \to (N, LCFL_N^{[a]})$ *is an* L*-convexity preserving mapping for each* $a \in \alpha^*(\perp)$.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1) are obvious.

(1) $\Rightarrow$ (4) For each $a \in \alpha(\perp)$, let $A \in LCFL_N^{[a]}$, i.e., $a \notin \alpha(LCFL_N(A))$. Since $LCFL_N(A) \leqslant LCFL_M(f_L^{\leftarrow}(A))$, we have $\alpha(LCFL_N(A)) \supseteq \alpha(LCFL_M(f_L^{\leftarrow}(A)))$. This implies $a \notin \alpha(LCFL_M(f_L^{\leftarrow}(A)))$, i.e., $f_L^{\leftarrow}(A) \in LCFL_M^{[a]}$.

(4) $\Rightarrow$ (5) is obvious.

(5) $\Rightarrow$ (1) For any $A \in L^N$, let $a \in \alpha^*(\perp)$ and $a \notin \alpha^*(LCFL_N(A))$. By $a \in P(L)$, we know $a \notin \alpha(LCFL_N(A))$. Then $A \in LCFL_N^{[a]}$. Since $f : (M, LCFL_M^{[a]}) \to (N, LCFL_N^{[a]})$ is an L-convex preserving mapping, we have $f_L^{\leftarrow}(A)) \in LCFL_M^{[a]}$, i.e., $a \notin \alpha(LCFL_M(f_L^{\leftarrow}(A)))$. Thus $a \notin \alpha^*(LCFL_M(f_L^{\leftarrow}(A)))$. By Lemma 2.1, we know $LCFL_N(A) \leqslant LCFL_M(f_L^{\leftarrow}(A))$. This completes the proof. □

Definition 7.6. [28, 22] Let $(X, \mathcal{C})$ and $(Y, \mathcal{D})$ be two (L, M)-fuzzy convex spaces. A mapping $f : X \to Y$ is called an (L, M)-fuzzy convex-to-convex mapping if $\mathcal{D}(f_L^{\rightarrow}(A)) \geqslant \mathcal{C}(A)$ for all $A \in L^X$.

An (L, L)-fuzzy convex-to-convex mapping is briefly called an L-fuzzy convex-to-convex mapping.

An $(L, 2)$-fuzzy convex-to-convex mapping is an L-convex-to-convex mapping in [13].

Theorem 7.7. *Let* M *and* N *be two lattices and* $f : M \to N$ *be a mapping. Then the following conditions are equivalent:*

(1) $f : (M, LCFL_M) \to (N, LCFL_N)$ *is an* L*-fuzzy convex-to-convex mapping.*

(2) $f : (M, LCFL_{M[a]}) \to (N, LCFL_{N[a]})$ *is an* L*-convex-to-convex mapping for each* $a \in L \setminus \{\perp\}$.

(3) $f : (M, LCFL_{M[a]}) \to (N, LCFL_{N[a]})$ *is an* L*-convex-to-convex mapping for each* $a \in J(L)$.

(4) $f : (M, LCFL_M^{[a]}) \to (N, LCFL_N^{[a]})$ *is an* L*-convex-to-convex mapping for each* $a \in \alpha(\perp)$.

(5) $f : (M, LCFL_M^{[a]}) \to (N, LCFL_N^{[a]})$ *is an* L*-convex-to-convex mapping for each* $a \in \alpha^*(\perp)$.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1) are obvious.

(1) $\Rightarrow$ (4) For each $a \in \alpha(\perp)$, let $A \in LCFL_M^{[a]}$, i.e., $a \notin \alpha(LCFL_M(A))$. Since $LCFL_M(A) \leqslant LCFL_N(f_L^{\rightarrow}(A))$, we have $\alpha(LCFL_M(A)) \supseteq \alpha(LCFL_N(f_L^{\rightarrow}(A)))$. This implies $a \notin \alpha(LCFL_N(f_L^{\rightarrow}(A)))$, i.e., $f_L^{\rightarrow}(A) \in LCFL_N^{[a]}$.

(4) $\Rightarrow$ (5) is obvious.

(5) $\Rightarrow$ (1) For any $A \in L^M$, let $a \in \alpha^*(\perp)$ and $a \notin \alpha^*(LCFL_M(A))$. Thus $a \notin \alpha(LCFL_M(A))$ i.e., $A \in LCFL_M^{[a]}$. Since $f : (M, LCFL_M^{[a]}) \to (N, LCFL_N^{[a]})$

is an L-convex-to-convex preserving mapping, we have $f_L^{\rightarrow}(A)) \in LCFL_N^{[a]}$, i.e., $a \notin \alpha(LCFL_N(f_L^{\rightarrow}(A)))$. Thus $a \notin \alpha^*(LCFL_N(f_L^{\rightarrow}(A)))$. By Lemma 2.1, we know $LCFL_M(A) \leqslant LCFL_N(f_L^{\rightarrow}(A))$. This completes the proof. □

Definition 7.8. [28, 22] Let $(X, \mathcal{C})$ and $(Y, \mathcal{D})$ be two (L, M)-fuzzy convex spaces. $f : X \to Y$ is called an (L, M)-fuzzy isomorphism if f is bijective, (L, M)-fuzzy convexity preserving and (L, M)-fuzzy convex-to-convex.

An (L, L)-fuzzy isomorphism is briefly called an L-fuzzy isomorphism.

An $(L, 2)$-fuzzy isomorphism is an L-isomorphism in [13].

Obviously, if f is bijective, then f is L-fuzzy convex-to-convex if and only if f^{-1} is L-fuzzy convexity preserving. Thus from Theorem 7.5 and Theorem 7.7, we have the following theorem.

Theorem 7.9. *Let M and N be two lattices and $f : M \to N$ be a mapping. Then the following conditions are equivalent:*

(1) *$f : (M, LCFL_M) \to (N, LCFL_N)$ is an L-fuzzy isomorphism.*

(2) *$f : (M, LCFL_{M[a]}) \to (N, LCFL_{N[a]})$ is an L-isomorphism for each $a \in L \setminus \{\perp\}$.*

(3) *$f : (M, LCFL_{M[a]}) \to (N, LCFL_{N[a]})$ is an L-isomorphism for each $a \in J(L)$.*

(4) *$f : (M, LCFL_M^{[a]}) \to (N, LCFL_N^{[a]})$ is an L-isomorphism for each $a \in \alpha(\perp)$.*

(5) *$f : (M, LCFL_M^{[a]}) \to (N, LCFL_N^{[a]})$ is an L-isomorphism for each $a \in \alpha^*(\perp)$.*

Theorem 7.10. [28, 22] *(***Quotient structure***)*
Let $(X, \mathcal{C})$ be an (L, M)-fuzzy convex space and $f : X \to Y$ be surjective. Define a mapping $\mathcal{C}_f : L^Y \to M$ by $\forall B \in L^Y$, $\mathcal{C}_f(B) = \mathcal{C}(f_L^{\leftarrow}(B))$.

Then $(Y, \mathcal{C}_f)$ is an (L, M)-fuzzy convex space, and we call $\mathcal{C}_f$ a quotient (L, M)-fuzzy convexity of X with respect to f and $\mathcal{C}$.

By Theorem 6.10, we have the following theorem.

Theorem 7.11. *(***Quotient structure of lattice***)*
Let $f : M \to M'$ be a surjective lattice homomorphism. Then $(M', LCFL_{M'})$ is a quotient L-fuzzy convex space.

Theorem 7.12. [28, 22] *(***Substructure***)*
Let $(X, \mathcal{C})$ be an (L, M)-fuzzy convex space and $Y \subseteq X$. Then $\mathcal{C}\mid_Y$ is an (L, M)-fuzzy convexity on Y, where $\mathcal{C}\mid_Y (A) = \bigvee\{\mathcal{C}(B) : B \in L^X, B\mid_Y = A\}$, $A \in L^Y$. In this case, $(Y, \mathcal{C}\mid_Y)$ is called a subspace of $(X, \mathcal{C})$.

Theorem 7.13. *(***Substructure of lattice***)*
Let $(M, LCFL_M)$ be an L-fuzzy convex space and $N \subseteq M$ be a sublattice. Then $(N, LCFL_N)$ is a subspace of $(M, LCFL_M)$.

Proof. By Theorem 7.12, we need to prove that

$$LCFL_N(\lambda) = LCFL_M\mid_N(\lambda)$$

for all $\lambda \in L^N$.

Let $\lambda \in L^N$. Then λ induces an L-subset $\bar{\lambda}$ of M defined by $\bar{\lambda}(x) = \lambda(x)$ for $x \in N$ and $\bar{\lambda}(x) = \perp$ for $x \notin N$. Obviously, it holds that $\bar{\lambda} |_N = \lambda$ and $LCFL_N(\lambda) = LCFL_M(\bar{\lambda})$. This implies $LCFL_N(\lambda) \leqslant LCFL_M |_N (\lambda) = \bigvee\{LCFL_M(\mu) : \mu \in L^M, \mu |_N = \lambda\}$.

Conversely, we prove that $LCFL_M |_N (\lambda) \leqslant LCFL_N(\lambda)$. First, it is easy to see that $LCFL_M(\mu) \leqslant LCFL_N(\lambda)$ for $\lambda \in L^N$ and $\mu \in L^M$ with $\mu |_N = \lambda$. Let $p \in J(L)$ with $p \prec LCFL_M |_N (\lambda)$, then there exists $\mu \in L^M$, $\mu |_N = \lambda$ such that $p \leqslant LCFL_M(\mu)$. By $LCFL_M(\mu) \leqslant LCFL_N(\lambda)$, we have $p \leqslant LCFL_N(\lambda)$. By Lemma 2.1, we know that $LCFL_M |_N (\lambda) \leqslant LCFL_N(\lambda)$. This completes the proof. □

8. **Conclusion**

In this paper, we introduced the notion of L-convex fuzzy sublattices based on a completely distributive lattice L. We gave their characterizations by means of four kinds of cut sets of L-subsets. We also introduced the notion of the degree to which an L-subset is an L-convex fuzzy sublattice. An L-fuzzy convexity on a lattice was naturally constructed and its some properties were studied.

It is noted that, by means of the same idea, we can construct L-fuzzy convexities in many other mathematical structures such as vector spaces, posets, semilattices, metric spaces, order groups. Thus many examples are provided for L-fuzzy convexities.

Acknowledgements. The authors appreciate the referee and Editor-in-Chief's carefully reading and suggestions on this paper. This work is supported by the National Natural Science Foundation of China (11371002) and Specialized Research Fund for the Doctoral Program Higher Education (20131101110048).

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