GENERAL FUZZY AUTOMATA BASED ON COMPLETE RESIDUATED LATTICE-VALUED

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Abstract. The present paper has been an attempt to investigate the general fuzzy automata on the basis of complete residuated lattice-valued (L-GFAs). The study has been chiefly inspired from the work by Mockor [14, 15, 16]. Regarding this, the categorical issue of L-GFAs has been studied in more details. The main issues addressed in this research include: (1) investigating the relationship between the category of L-GFAs and the category of non-deterministic automata (NDAs); as well as the relationship between the category of generalized L-GFAs and the category of NDAs; (2) demonstrating the existence of isomorphism between the category of L-GFAs and the subcategory of generalized L-GFAs and between the category of L-GFAs and the category of sets of NDAs; (3) and further scrutinizing some specific relationship between the output L-valued subsets of generalized L-GFAs and the output L-valued of NDAs.

1. Introduction

In 1965, L. A. Zadeh introduced the notion of fuzzy set as a method for representing uncertainty [35]. His ideas have been applied to a wide range of scientific areas. Automata theory was first introduced by W.G. Wee in [31]. Automata have a long history both in theory and application [2, 7, 8, 13]. Automata are the prime examples of general computational systems over discrete spaces. Fuzzy automata and their corresponding fuzzy grammars, combine the capabilities of automata and language theory with fuzzy logic in an elegant way [17]. Notably, residuated lattice-valued logic was dealt with by Pavelka in 1979. Pavelka used residuated lattices as a tool to cope with inexact reasoning and a basis of fuzzy logic. Some relations between residuated lattices and other algebraic structures can also be referred to [19]. Recently, Qiu [20, 21] established a fundamental framework of automata theory, based on complete residuated lattice-valued logic, called L-valued automata (L-VAs), which to a certain extent, generalizes the previous fuzzy finite automata systematically studied by Mordeson and Malik et al. [17]. In addition, L-valued pumping lemma in this setting was established by Qiu [22]. As an important application, fuzzy automata have been used to simulate fuzzy discrete event systems.

Received: November 2015; Revised: October 2016; Accepted: February 2017

Key words and phrases: General fuzzy automata, Active state set, Residuated-lattice, Isomorphism of category, Functor.

This paper is the extended version of presented paper in "4th Iranian Joint Congress of Fuzzy and Intelligent Systems (CFIS)" which was held during 8-11 September 2015 at University of Sistan and Baluchestan, Zahedan, Iran.

Recently, Mockor [15, 16] used the categorical method and filter theory in a lattice to give a uniform treatment of varied definitions of fuzzy automata. Also, Mockor [15, 16] proved that the categories of classical types of automata are equivalent to some subcategories of a category of fuzzy automata. For more details see the recent literature as [25, 26, 27, 32, 33, 34].

The paper is organized as: Section 2 serves to demonstrate some related concepts and theorems concerning general fuzzy automata and complete residuated lattice-valued logic. In Section 3, the researchers mainly discuss the relationship between the category of L-GFAs and the category of NDAs. This captures the idea that these two categories are the same from the categorical point of view. Also, we further investigate some specific relationship between the output L-valued subsets of generalized L-GFAs and the output L-valued subsets of NDAs. In Section 4, some concluding remarks are offered. Finally, the paper concludes that instead of a general fuzzy automaton we can deal equivalently with a chain of non-deterministic automata.

2. Preliminaries

The various definitions of residuated lattice-valued logic and related theorems were provided in this part. For more details, we may refer to [20, 21, 22, 19, 23, 24].

Definition 2.1. A complete residuated lattice is a 5-tuple $\ell = \langle L, +, ., \otimes, \rho \rangle$, where:
(i) $\langle L, +, . \rangle$ is a complete lattice whose least and greatest elements are 0 and 1, respectively,
(ii) $\otimes$ and ρ are two binary operations on L such that $\langle L, \otimes, 1 \rangle$ is a commutative monoid;
(iii) for all $a, b, c \in L$, $a \otimes b \leq c$ if and only if $a \leq b\rho c$.

From the above, it follows that $\otimes$ is isotone, and ρ is antitone, the former being in the first and the latter in the second variable; that is, for any $a_1, a_2, b \in L$, if $a_1 \leq a_2$, then, $a_1 \otimes b \leq a_2 \otimes b$, $b \otimes a_1 \leq b \otimes a_2$, $a_2 \rho b \leq a_1 \rho b$, and $b \rho a_1 \leq b \rho a_2$.

Throughout this paper, L-valued logic refers to the complete residuated lattice-valued logic; that is, the set of truth values is L, which possesses nullary connectives a $(a \in L)$ and a binary connective as well as usual connectives $\vee$, $\wedge$ and implication $\longrightarrow$. Moreover, in L-valued logic, the only designated truth value is 1; in other words, a formula ϕ is valid, written as $=^{\ell} \phi$, if and only if $[\phi] = 1$ for any interpretation, where $[\phi]$ indicates the truth value of ϕ. Now, we present the truth valuation rules of predicate logic and set theoretical formula as follows:

(1) $[a] = a (a \in L), [\phi \vee \varphi] = [\phi] + [\varphi], [\phi \wedge \varphi] = [\phi].[\varphi], [\phi \& \varphi] = [\phi] \otimes [\varphi], [\phi \rightarrow \varphi] = [\phi] \rho [\varphi]$;

(2) If X is the universe, $A \in L^X$ (as usual, L^X denotes the class of L-valued subsets over X, and L-valued subset over X is a mapping from X to L), then;

$$[(\exists x)\phi(x)] = \sum_{x \in X} [\phi(x)], [(\forall x)\phi(x)] = \prod_{x \in X} [\phi(x)], [x \in A] = A(x).$$

In addition, we will use some derived formulas including:
$(a) \longmapsto \phi \ \underline{\underline{def}} \phi \rightarrow 0, \phi \leftrightarrow \varphi \ \underline{\underline{def}} (\phi \rightarrow \varphi) \wedge (\varphi \rightarrow \phi)$;

$(b) A \subseteq B \underline{\underline{def}} (\forall x)((x \in A) \to (x \in B)), A \equiv B \underline{\underline{def}} (A \subseteq B) \wedge (B \subseteq A)$;
and some properties of residuated lattices: for any $a, x_i, y_i \in L$,
(p1) $(\sum_i x_i) \otimes a = \sum_i (x_i \otimes a)$;
(p2) $x_i \otimes y_i = y_i \otimes x_i$;
(p3) $1 \otimes a = a$.
In fuzzy set or fuzzy automata theory, as in the classical ones, there are several definitions and several categories with possible different properties. For example, in Goguen [10], L is said to be closg (i. e., complete lattice ordered semigroup) if it is a complete lattice ($\vee$ denotes supremum, and $\wedge$ infimum in L) and a monoid (with respect to an operation (.)) which is sup-distributive, i. e., $(\vee_i a_i).(\vee_j b_j) = \vee_{i,j}(a_i.b_j)$. If L is a closg, then, L-fuzzy relation from a set A into a set B is a mapping $\mu : A \times B \longrightarrow L$. The value $\mu(a, b)$ is interpreted as the grade that $a \in A$ is related to $b \in B$. Let us indicate $A \stackrel{\mu}{\longrightarrow} B$ and L-fuzzy relation from A to B. If $A \stackrel{\mu_1}{\longrightarrow} B$, $B \stackrel{\mu_2}{\longrightarrow} C$ are L-fuzzy relations, their composition $A \stackrel{\mu_1 \otimes \mu_2}{\longrightarrow} C$ is defined by:

$$\mu_1 \otimes \mu_2(a, c) = \bigvee_{b \in B} \mu_1(a, b) \otimes \mu_2(b, c).$$

All sets and all their L-fuzzy relations are from a category.

For $\ell = \langle L, +, ., \otimes, \rho \rangle$, a category ς comprises a collection of objects, called the set of objects of ς, denoted by object (ς), together with, for each pair $A, B \in object(\varsigma)$, which is included in the whole set of morphisms $Mor(\varsigma)$ in ς, called the set of L-valued morphism from A to B subject to the conditions (i) and (ii). We may write $\alpha : A \times B \longrightarrow L$ or simply $\alpha : A \longrightarrow B$, or $A \stackrel{\alpha}{\longrightarrow} B$ to indicate that $\alpha \in Mor_L(A, B) \subseteq Mor(\varsigma)$:
(i) For any three (not necessarily distinct) objects $A, B, C \in object(\varsigma)$, there is a map defined by

$$Mor(A, B) \times Mor(B, C) \to Mor(A, C) : (A \stackrel{\alpha}{\longrightarrow} B, B \stackrel{\beta}{\longrightarrow} C) \to A \stackrel{\alpha o \beta}{\longrightarrow} C$$

called composition which satisfies the associativity axiom, that is, for all objects $A, B, C, D \in object(\varsigma)$, and morphism $\alpha \in Mor_L(A, B), \beta \in Mor_L(B, C)$ and $\gamma \in Mor_L(C, D)$, if we have $\alpha o(\beta o \gamma) = (\alpha o \beta) o \gamma : A \longrightarrow D$, where for two morphisms $\alpha \in Mor_L(A, B), \beta \in Mor_L(B, C)$, the composition $\alpha o \beta \in Mor_L(A, C)$ is defined as:

$$\alpha o \beta(a, c) = \sum_{b \in B} \alpha(a, b) \otimes \beta(b, c) \text{ for any } a \in A,\ c \in C.$$

(ii) For every object A of ς, the set $Mor_L(A, A)$ contains a morphism id_A, called the identity of A, with the property that, for every object $B \in object(\varsigma)$ and for all $\alpha \in Mor_L(A, B)$ and $\beta \in Mor_L(B, A)$, we have $id_A o \alpha = \alpha$, $\beta o id_A = \beta$.

This concept is in accordance with the principle of fuzzification.

Definition 2.2. Let $\ell = \langle L, +, ., \otimes, \rho \rangle$ be a complete residuated lattice. By ψ_L we denote a category with objects (X, F), where X is a set and $F \subseteq L^X$ is an L-valued subset in X. A morphism from an object (X, F) to another (Y, G) is a pair (α, ω) such that:

(i) $\alpha : X \to Y$ is a map;
(ii) $\omega \subseteq F \times G$ is a relation and $dom(\omega) = F$;
(iii) $\exists l \in L/0, \forall (f, g) \in \omega, x \in X, f(x) \to g(\alpha(x)) \geq l$
(or equivalently $f(x) \otimes l \leq g(\alpha(x))$), where $L/0$ denotes the set consisting of elements in L except 0, i.e., $L/0 = \{l | l \in L, l > 0\}$.

Definition 2.3. Recall that a system (Q, δ) is called an NDA over a monoid $(M, *)$, if Q is a set of states and $\delta : Q \times M \longrightarrow 2^Q$ is a non-deterministic transition, where 2^Q stands for the set of all subsets of Q, such that:
(1) $\forall q \in Q, \delta(q, 1_M) = \{q\}$ and
(2) $\forall m, n \in M, q \in Q, \delta(q, m * n) = \bigcup_{p \in \delta(q,m)} \delta(p, n)$.
An NDA over a monoid $(M, *)$ with initial and terminal states is a system (Q, δ, I, T) such that (Q, δ) is an NDA over M and $I, T \subseteq Q$.

By NA_M, we denote the category of NDAs

A morphism in NA_M is defined as $\alpha : (Q_1, \delta) \longrightarrow (Q_2, \delta')$, i.e., $\alpha : Q_1 \longrightarrow Q_2$ is a map such that

$$\alpha\left(\bigcup_{q' \in Q_1, \alpha(q') = \alpha(q)} \delta(q', m)\right) = \delta'(\alpha(q), m) \cap \alpha(Q_1), \forall q \in Q_1, \forall m \in M.$$

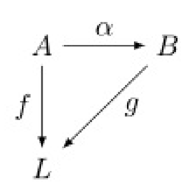
By NA^0_M, we denote the category of NDAs with initial and terminal states.

A morphism $\alpha : (Q_1, \delta, I, T) \longrightarrow (Q_2, \delta', J, S)$ is defined by:
(1) $\alpha \in Mor((Q_1, \delta), (Q_2, \delta'))$;
(2) $\alpha(I) = J \cap \alpha(Q_1), \alpha(T) = S \cap \alpha(Q_1)$.

Any NDA, $A^0 = (Q, \delta, I, T) \in NA^0_M$ defines the output set, denote as $Out(A^0)$. Obviously, $Out(A^0) \subseteq M$, satisfying:

$$m \in Out(A^0) \Longleftrightarrow \exists q \in I, \delta(q, m) \cap T \neq \phi.$$

Definition 2.4. For $\ell = \langle L, +, ., \otimes, \rho\rangle$, L-valued subsets f, g of A, B, respectively (i.e., $f \in L^A$, $g \in L^B$), and $\alpha \in Mor_L(A, B)$, we call a diagram



L-commute if $g(b) = \sum_{a \in A} f(a) \otimes \alpha(a, b), \forall b \in B$.

Definition 2.5. A functor $E : C \longrightarrow D$ is a structure-preserving map between categories, in the same way as a homomorphism being a structure-preserving map, and it is defined by the two functions $A \longrightarrow E(A)$ on objects, and $(A \xrightarrow{f} B) \longrightarrow (E(A) \xrightarrow{E(f)} E(B))$ on maps which satisfy these axioms:

$$E(fg) = E(f)E(g), \forall f, g \in Mor(C)$$

and

$$E(I_A) = I_{E(A)}, \forall A \in object(C).$$

Lemma 2.6. *Let X be a set and $H \in L^X$. We denotd $H_l = \{y \in X | H(y) \geq l\}$ for any $l \in L$. For $x \in X$ and $\omega = \sum_{x \in H_l} l$, we have $x \in H_\omega$ and $H(x) = \omega$.*

A fuzzy finite-state automaton (FFA) is a six-tuple denoted by $\tilde{F} = (Q, \Sigma, R, Z, \delta, \omega)$, where Q is a finite set of states, Σ is a finite set of input symbols, R is the start state of $\tilde{F}$, Z is a finite set of output symbols, $\delta : Q \times \Sigma \times Q \longrightarrow (0, 1]$ is the fuzzy transition function used to map a state (current state) into another state (next state) upon an input symbol, attributing a value in the interval $(0, 1]$ and $\omega : Q \longrightarrow Z$ is the output function. In an FFA, as can be seen, associated with each fuzzy transition, there is a membership value in $(0, 1]$. We call this membership value the weight of the transition.

The transition from state q_i (current state) to state q_j (next state) upon input a_k is denoted by $\delta(q_i, a_k, q_j)$. We use this notation to refer both to a transition and its weight. Whenever $\delta(q_i, a_k, q_j)$ is used as a value, it refers to the weight of the transition. Otherwise, it specifies the transition itself. Also, Δ shows a set of all transitions of $\tilde{F}$. The above definition is generally accepted as a formal definition for FFA [12, 17, 18].

We believe that the current literature and background for fuzzy automata is not well established to characterize the operation of the automaton and, thus, fulfill the implementation requirements in a well-defined manner. There is an important problem which should be clarified in the definition of FFA. It is the assignment of membership values to the next states. There are two issues within state membership assignment. The first one is how to assign a membership value to a next state, upon the completion of a transition. Secondly, how should we deal with the cases where a state is forced to take several membership values simultaneously, via overlapping transition?

During the past few years, several researchers studied the equivalence and isomorphism of fuzzy automata with other types of automata. However, we believe that the role of fuzzy automata goes beyond equivalence. In fact, they can represent not only other types of automata, but also several other computational paradigms. To make the computational generality of fuzzy automata and its generalization capability in more systematic and application-friendly way, we need a more general definition. In 2004, M. Doostfatemeh and S.C. Kremer extended the notion of fuzzy automata and gave the notion of general fuzzy automata [9].

Definition 2.7. [9] A general fuzzy automaton (GFA) $\tilde{F}$ is an eight-tuple machine denoted by $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2)$, where

(i) Q is a finite set of states, $Q = \{q_1, q_2, \ldots, q_n\}$,
(ii) Σ is a finite set of input symbols, $\Sigma = \{a_1, a_2, \ldots, a_m\}$,
(iii) $\tilde{R}$ is the set of fuzzy start states,
(iv) Z is a finite set of output symbols, $Z = \{b_1, b_2, \ldots, b_k\}$,
(v) $\omega : Q \longrightarrow Z$ is the output function,

(vi) $\tilde{\delta} : (Q \times [0,1]) \times \Sigma \times Q \longrightarrow [0,1]$ is the augmented transition function,

(vii) $F_1 : [0,1] \times [0,1] \longrightarrow [0,1]$ is called membership assignment function. Function $F_1(\mu, \delta)$, as seen, is motivated by two parameters μ and δ, where μ is the membership value of a predecessor and δ is the weight of a transition. According to this definition, the process that takes place upon the transition from state q_i to q_j on input a_k, is represented as:

$$\mu^{t+1}(q_j) = \tilde{\delta}((q_i, \mu^t(q_i)), a_k, q_j) = F_1(\mu^t(q_i), \delta(q_i, a_k, q_j))$$

which indicates that membership value (mv) of the state q_j at time $t+1$, is computed by function F_1 using both the membership value of q_i at time t and the weight of the transition.

There are many options which can be used for the function $F_1(\mu, \delta)$. It can be, for example, $\max\{\mu, \delta\}$, $\min\{\mu, \delta\}$, $\dfrac{\mu+\delta}{2}$ or any other applicable mathematical function.

(viii) $F_2 : [0,1]^* \longrightarrow [0,1]$ is called multi- membership resolution function.

The multi-membership resolution function resolves the multi-membership active states and assigns a single membership value to them.

3. General Fuzzy Automata Based on Complete Residuated Lattice-valued Logic

In this section, we consider the general fuzzy automata based on complete residuated lattice-valued (L-GFAs) and study the categorical issue of L-GFAs (ς_G). Finally, we show that the category ς_G is isomorphic to a subcategory of generalized L-GFAs (ψ_G).

Definition 3.1. Let $\ell = \langle L, +, ., \otimes, \rho \rangle$ be a complete residuated lattice and (Σ^*, o) a monoid. Also, let $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2)$ be a general fuzzy automaton. A general fuzzy automaton based on ℓ (for simplicity L-GFA), is an eight-tuple machine denoted by $\tilde{F}_\ell = (Q, \Sigma, \tilde{R}_\ell, Z, \tilde{\delta}_\ell, \omega_\ell, F_1, F_2)$, where

(i) $Q = \{q_1, q_2, \ldots, q_n\}$ is a finite set of states,

(ii) $\Sigma = \{a_1, a_2, \ldots, a_m\}$ is a finite set of input symbols,

(iii) $\tilde{R}_\ell$ is an L-valued subset of Q, called an initial state,

(iv) $Z = \{b_1, b_2, \ldots, b_k\}$ is a finite set of output symbols,

(v) $\omega_\ell : Q \times Z \longrightarrow L$ is the output function,

(vi) $\tilde{\delta}_\ell : (Q \times L) \times \Sigma^* \times Q \longrightarrow L$ is the augmented transition function,

(vii) $F_1 : L \times L \longrightarrow L$ is called a membership assignment function.

Note that

$$\tilde{\delta}_\ell((p, \mu^t(p)), \Lambda, q) = \begin{cases} 1, & if \quad p = q \\ 0, & o.w \end{cases} \quad ; \quad \tilde{\delta}_\ell((p, \mu^t(p)), a, q) = \tilde{\delta}((p, \mu^t(p)), a, q);$$

$$\tilde{\delta}_\ell((p, \mu^t(p)), aob, q) = \sum_{r \in Q_{act}(t+1)} \tilde{\delta}_\ell((p, \mu^t(p)), a, r) \otimes \tilde{\delta}_\ell((r, \mu^{t+1}(r)), b, q);$$

$$\tilde{\delta}_\ell((p, \mu^{t_0}(p)), a_1 o a_2 o \ldots o a_n, q) = \sum \tilde{\delta}_\ell((p, \mu^{t_0}(p)), a_1, p_1)$$

$$\otimes \tilde{\delta}_\ell((p_1, \mu^{t_1}(p_1)), a_2, p_2) \otimes \cdots \otimes \tilde{\delta}_\ell((p_{n-1}, \mu^{t_{n-1}}(p_{n-1})), a_n, q);$$

$$p_1 \in Q_{act}(t_1), p_2 \in Q_{act}(t_2), \ldots, p_{n-1} \in Q_{act}(t_{n-1}).$$

$F_2 : L^* \longrightarrow L$ is called a multi-membership resolution function and it resolves the multi-membership active states and assigns a single membership value to each.

Example 3.2. Consider the GFA in Figure 1. It is specified as $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}, \omega, F_1, F_2)$ where $Q = \{q_0, q_1, q_2\}$ is the set of states, $\Sigma = \{a, b\}$ is the set of input symbols, $\tilde{R} = \{(q_0, 1)\}$, $Z = \{b_0, b_1, b_2\}$ s.t. $\omega(q_0) = b_0$, $\omega(q_1) = b_1$, $\omega(q_2) = b_2$, and $F_1(\mu, \delta) = \delta$, $F_2() = \mu^{t+1}(q_m) = \bigwedge_{i=1}^{n}(F_1(\mu^t(q_i), \delta(q_i, a_k, q_m)))$ where n is the number of simultaneous to the active state q_m at time $t + 1$.

Then we have:

$$\mu^{t_0}(q_0) = 1,$$
$$\mu^{t_1}(q_1) = \tilde{\delta}((q_0, \mu^{t_0}(q_0)), a, q_1) = F_1(\mu^{t_0}(q_0), \delta(q_0, a, q_1)) = F_1(1, 0.4) = 0.4,$$
$$\mu^{t_2}(q_2) = \tilde{\delta}((q_1, \mu^{t_1}(q_1)), a, q_2) = F_1(\mu^{t_1}(q_1), \delta(q_1, a, q_2)) = F_1(0.4, 0.8) = 0.8,$$
$$\mu^{t_3}(q_2) = \tilde{\delta}((q_2, \mu^{t_2}(q_2)), b, q_2) = F_1(\mu^{t_2}(q_2), \delta(q_2, b, q_2)) = F_1(0.8, 0.3) = 0.3,$$

Table 1 demonstrates the following results:

time	t_0	t_1	t_2	t_3
input	$\wedge$	a	a	b
$Q_{act}(t_i)$	q_0	q_1	q_2	q_2
mv	1	0.4	0.8	0.3

Table 1. Active States and Their Membership Values (mvs)

in Example 3.2

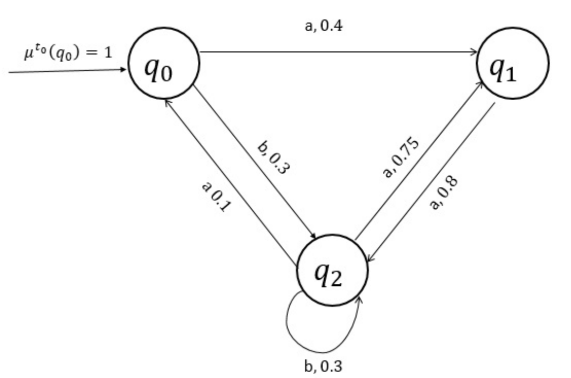


Figure 1. The GFA of Example 3.2

According to the definition of L-general fuzzy automata we have L-GFA $\tilde{F}_\ell = (Q, \Sigma, \tilde{R} = \tilde{R}_\ell, Z, \tilde{\delta}_\ell, \omega_\ell, F_1, F_2)$ where $\ell = \langle L, +, ., \otimes, \rho\rangle$ is a complete residuated lattice.

Let $L=[0,1]$ and $\otimes, \rho$ are two binary operations as follows:

$$\forall x,y\in L \qquad x\otimes y=(x\wedge y), \qquad x\rho y=\begin{cases}1 & x<y\\ y & \text{otherwise}\end{cases}.$$

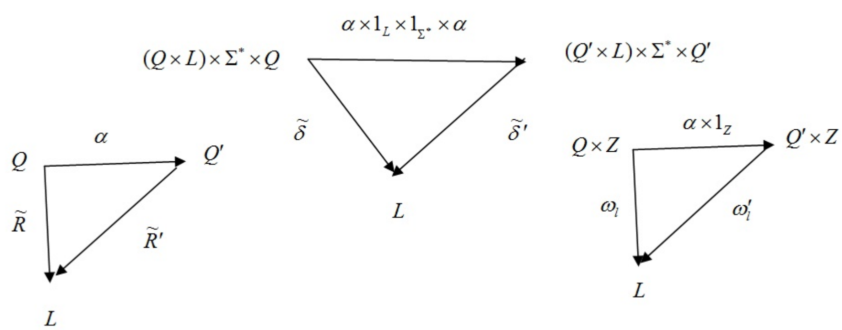
To show the definition of $\tilde{\delta}_\ell$ we have:

$$\begin{aligned}
\tilde{\delta}_\ell((q_0,\mu^{t_0}(q_0)),aob,q_2) &= \tilde{\delta}((q_0,\mu^{t_0}(q_0)),a,q_1)\otimes\tilde{\delta}((q_1,\mu^{t_1}(q_1)),b,q_2)\\
&= 0.4\otimes 0=(0.4\wedge 0)=0,\\
\tilde{\delta}_\ell((q_0,\mu^{t_0}(q_0)),aoaob,q_2) &= \tilde{\delta}((q_0,\mu^{t_0}(q_0)),a,q_1)\otimes\tilde{\delta}((q_1,\mu^{t_1}(q_1)),a,q_2)\\
\otimes\tilde{\delta}((q_2,\mu^{t_2}(q_2)),b,q_2) &= 0.4\otimes 0.8\otimes 0.3=0.3.
\end{aligned}$$

According to the above definition of ω, we will be able to define:

$$\omega_\ell(q_0,b_0)=0.5, \omega_\ell(q_1,b_1)=0.3, \omega_\ell(q_2,b_2)=0.4.$$

Definition 3.3. Given two L-GFAs $\tilde{F}_\ell=(Q,\Sigma,\tilde{R}_\ell,Z_\ell,\tilde{\delta}_\ell,\omega_\ell,F_1,F_2)$ and $\tilde{F}'_\ell=(Q',\Sigma,\tilde{R}'_\ell,Z_\ell,\tilde{\delta}'_\ell,\omega'_\ell,F_1,F_2)$ a morphism $f:\tilde{F}_\ell\to\tilde{F}'_\ell$ is a map $\alpha: Q\times Q'\longrightarrow L$ such that the following diagrams L-commutes:



where $\alpha\times\beta:(A\times A')\times(B\times B')\longrightarrow L$ is defined by

$$\alpha\times\beta((a,a'),(b,b'))=\alpha(a,a').\beta(b,b').$$

That is, $\forall m\in\Sigma^*$, $p',q'\in Q'$,

$$\begin{aligned}
\tilde{\delta}'_\ell((p',\mu^{t_0}(p')),m,q') &= \sum_{p,q\in Q}\tilde{\delta}_\ell((p,\mu^{t_0}(p)),m,q)\otimes(\alpha(p,p').\alpha(q,q'));\\
\tilde{R}'_\ell(p') &= \sum_{p\in Q}\tilde{R}_\ell(p)\otimes\alpha(p,p');\\
\omega'_\ell(q',b_i) &= \sum_{q\in Q}\omega_\ell(q,b_i)\otimes\alpha(q,q');
\end{aligned}$$

Example 3.4. Consider the L-general fuzzy automata $\tilde{F}_\ell$ in Example 3.2 and L-general fuzzy automata $\tilde{F}'_\ell$ as follows (Figure 2):

Let $\tilde{F}'=(Q',\Sigma,\tilde{R}',Z,\tilde{\delta}',\omega',F_1,F_2)$ be a general fuzzy automata where $Q'=\{q'_0,q'_1\}$ is the set of states, $\Sigma=\{a,b\}$ is the set of input symbols, $\tilde{R}'=\{(q'_0,1)\}$,

$Z = \{b_0, b_1, b_2\}$ s.t. $\omega'(q'_0) = b_0$, $\omega'(q'_1) = b_1$,

$$F_1(\mu, \delta) = \delta, F_2() = \mu^{t+1}(q_m) = \bigwedge_{i=1}^{n} (F_1(\mu^t(q_i), \delta(q_i, a_k, q_m)))$$

time	t_0	t_1	t_2	t_3
input	$\wedge$	a	a	b
$Q_{act}(t_i)$	q'_0	q'_1	q'_1	q'_0
mv	1	0.3	0.4	0.1

Table 2. Active States and Their Membership Values (mvs) in Example 3.4

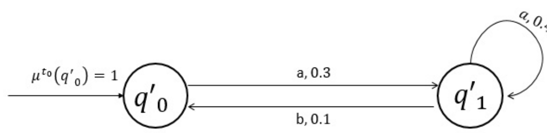


Figure 2. The GFA of Example 3.4

Then we have $\tilde{F}'_\ell = (Q', \Sigma, \tilde{R}'_\ell = \tilde{R}', Z, \tilde{\delta}'_\ell, \omega'_\ell, F_1, F_2)$ where $\omega'_\ell(q'_0, b_0) = 0.5$, $\omega'_\ell(q'_1, b_1) = 0.3$.

We define the morphism $f : \tilde{F}_\ell \longrightarrow \tilde{F}'_\ell$ as follows:

$\alpha : Q \longrightarrow Q'$ s.t. $\alpha(q_0, q'_0) = 1$, $\alpha(q_1, q'_0) = 0.2$, $\alpha(q_2, q'_1) = 0.3$, $\alpha(q_1, q'_1) = 0.3$, $\alpha(q_2, q'_0) = 0.1$.

In order to illustrate that α is morphism, the following instance are as:

$$\tilde{R}_\ell(q_0) \otimes \alpha(q_0, q'_0) = 1 \otimes 1 = 1 = \tilde{R}'_\ell(q'_0)$$
$$\omega_\ell(q_0, b_0) \otimes \alpha(q_0, q'_0) = 0.5 \otimes 1 = 0.5 = \omega'_\ell(q'_0, b_0)$$
$$\tilde{\delta}'_\ell = ((q'_0, \mu^{t_0}(q'_0)), aoa, q'_1) = \tilde{\delta}'((q'_0, \mu^{t_0}(q'_0)), a, q'_1)$$
$$\otimes \tilde{\delta}'((q'_1, \mu^{t_1}(q'_1)), a, q'_1) = 0.3 \otimes 0.4 = 0.3 \ (1)$$
$$\tilde{\delta}_\ell((q_0, \mu^{t_0}(q_0)), aoa, q_2) \otimes [\alpha(q_0, q'_0) \cdot \alpha(q_2, q'_1)] = 0.8 \otimes 0.3 = 0.3 \ (2)$$
$$(1), (2) \Longrightarrow \tilde{\delta}'_\ell((q'_0, \mu^{t_0}(q'_0)), aoa, q'_1) = \tilde{\delta}_\ell((q_0, \mu^{t_0}(q_0)), aoa, q_2)$$
$$\otimes [\alpha(q_0, q'_0), \alpha(q_2, q'_1)]$$

Proposition 3.5. *The class of L-GFAs and their morphisms forms a category unter component-wise composition of maps.*

Proof. With regard to the composition of maps, it is obvious. □

Definition 3.6. The behaviour of $\tilde{F}_\ell = (Q, \Sigma, \tilde{R}_\ell, Z, \tilde{\delta}_\ell, \omega_\ell, F_1, F_2)$ is given by an output L-valued set, $Out_L(\tilde{F}_\ell)$ as follows:

$m \in Out_L(\tilde{F}_\ell)$ whenever $\exists p, q \in Q, ((p \in \tilde{R}_\ell) \& ((p, \mu^{t_0}(p)), m, q) \in \tilde{\delta}_\ell \& (q \in \tilde{T}_\ell))$.

It is clear that

$$[Out_L(\tilde{F}_\ell)(m)] = Out_L(\tilde{F}_\ell)(m) = \sum_{p,q\in Q} \tilde{R}_\ell(p) \otimes \tilde{\delta}_\ell((p, \mu^{t_0}(p)), m, q) \otimes \tilde{T}_\ell(q),$$

where $\tilde{T}_\ell : Q \longrightarrow L$ is an L-valued subset of Q, called final L-valued states.

Intuitively, the behavior can be thought as an L-valued subset of all the elements of Σ^*, which can act on an initial state to produce a final state.

Definition 3.7. Let $\ell = \langle L, +, ., \otimes, \rho\rangle$ be a complete residuated lattice and (Σ^*, o) a monoid with unit element Λ. Suppose that Q is a nonempty set, and $\Re \subseteq L^{Q\times Q}$ is an L-valued subset of $Q \times Q$ and $\bullet$ a binary operation on $\Re$, such that $(\Re, \bullet)$ is a monoid. Then, a surjective homomorphism $F : (\Sigma^*, o) \longrightarrow (\Re, \bullet)$ is called a generalized L-GFA over Q.

Definition 3.8. Let $F_i : (\Sigma^*, o) \longrightarrow (\Re_i, \bullet)(i = 1, 2)$ be a generalized L-GFAs over $Q_i(i = 1, 2)$. Then, α is a morphism from F_1 to F_2 if

(1) $\alpha : Q_1 \longrightarrow Q_2$ is a map and $\alpha \oplus \alpha : Q_1 \times Q_1 \longrightarrow Q_2 \times Q_2$ is defined by

$$(\alpha \oplus \alpha)(p, q) = (\alpha(p), \alpha(q));$$

(2) $(\alpha \oplus \alpha, \omega) : (Q_1 \times Q_1, \Re_1) \longrightarrow (Q_2 \times Q_2, \Re_2)$ is a morphism in ψ_L, where

$$\omega = \{(F_1(m), F_2(m)) : m \in \Sigma^*\}$$

satisfying $\exists l \in L/0, F_1(m)(p, q) \longrightarrow F_2(m)(\alpha(p), \alpha(q)) \geq l$. In particular, if $L = \{0, 1\}$, then, $l = 1$ and $\omega = \{(F_1(m), F_2(m)) : m \in \Sigma^*\}$ is satisfying to $F_1(m)(p, q) \leq F_2(m))(\alpha(p), \alpha(q))$.

By ψ_G we denote the category of generalized L-GFAs with morphism defined as above.

Theorem 3.9. *There exists a functor $F : NA^0_M \longrightarrow \varsigma_G$ such that:*

$$=^\ell m \in Out(A) \longleftrightarrow Out_L(F(A))(m) = 1, \text{ for any } A \in NA^0_M.$$

Similarly, there exists a functor $K : NA_M \longrightarrow \varsigma_G$.

Proof. Let $A = (Q_1, \delta, I, T)$ be an object of NA^0_M, and let

$$F(A) = (Q_1, \tilde{\delta}_\ell, \tilde{R}_\ell, \Sigma, \omega_\ell, Z, F_1, F_2),$$

where $\tilde{\delta}_\ell : (Q_1 \times L) \times \Sigma^* \times \times Q_1 \longrightarrow L$ is defined by

$$\tilde{\delta}_\ell((p, \mu^{t_0}(p)), m, q) = \begin{cases} 1, & if \quad q \in \delta(p, m) \\ 0, & if \quad q \notin \delta(p, m), \end{cases}$$

and

$$\tilde{R}_\ell(q) = \begin{cases} 1, & if \quad q \in I \\ 0, & if \quad q \notin I; \end{cases} \qquad \tilde{T}_\ell(q) = \begin{cases} 1, & if \quad q \in T \\ 0, & if \quad q \notin T; \end{cases}$$

$$\omega_\ell(q, m) = \begin{cases} 1, & if \quad m \in Out(A) \\ 0, & if \quad m \notin Out(A); \end{cases}$$

$\forall p, q \in Q_1$, $\forall m \in \Sigma^*$.

Then, we will get $F(A) \in \varsigma_G$ and consequently

$$=^\ell m \in Out(A) \longleftrightarrow Out_L(F(A))(m) = 1.$$

Now, for objects $A = (Q_1, \delta, I, T)$, $B = (Q_2, \delta', J, S)$ of NA^0_M, and $\alpha \in Mor(A, B) \subseteq Mor(\mathrm{NA}^0_M)$, i.e., $\alpha \in Mor((Q_1, \delta), (Q_2, \delta'))$ we have

$$\alpha\left(\bigcup_{r\in Q_1, \alpha(r)=\alpha(p)} \delta(r, m)\right) = \delta'(\alpha(p), m) \cap \alpha(Q_1), \forall p \in Q_1, \forall m \in \Sigma^*$$

and $\alpha(I) = J \cap \alpha(Q_1), \alpha(T) = S \cap \alpha(Q_1)$.

Let $F(B) = (Q_2, \tilde{\delta}'_\ell, \tilde{R}'_\ell, \Sigma, \omega'_\ell, Z, F_1, F_2) \in object(\varsigma_G)$, i.e.,

$$\tilde{\delta}'_\ell((p, \mu^{t_0}(p)), m, q) = \begin{cases} 1, & if \quad q \in \delta'(p, m) \\ 0, & if \quad q \notin \delta'(p, m), \end{cases}$$

and

$$\tilde{R}'_\ell(q) = \begin{cases} 1, & if \quad q \in J \\ 0, & if \quad q \notin J; \end{cases} \qquad \tilde{T}'_\ell(q) = \begin{cases} 1, & if \quad q \in S \\ 0, & if \quad q \notin S; \end{cases}$$

$$\omega'_\ell(q, m) = \begin{cases} 1, & if \quad m \in Out(B) \\ 0, & if \quad m \notin Out(B); \end{cases}$$

$\forall p, q \in Q_2$, $\forall m \in \Sigma^*$.

We show that $F(\alpha) \in Mor_L(F(A), F(B))$, where

$$F(\alpha)(p, q) = \begin{cases} 1, & if \quad \alpha(p) = q \\ 0, & \quad otherwise \end{cases}.$$

It suffices to verify that, for any $q, q' \in \alpha(Q_1)$, and $m \in \Sigma^*$

$$\begin{aligned} \tilde{R}'_\ell(q) \otimes \tilde{\delta}'_\ell((q, \mu^{t_0}(q)), m, q') \quad &\otimes \quad \tilde{T}'_\ell(q') = \sum_{(p,p')\in Q_1\times Q_1} \tilde{R}_\ell(p) \\ &\otimes \quad \tilde{\delta}_\ell((p, \mu^{t_0}(p)), m, p') \\ &\otimes \quad \tilde{T}_\ell(p') \otimes (F(\alpha)(p, q).F(\alpha)(p', q')). \end{aligned}$$

To this end, from the Definition 3.3, it is enough to show that $\forall q, q' \in Q_2$, and $m \in \Sigma^*$,

(1) $\tilde{R}'_\ell(q) = \sum_{p\in Q_1} \tilde{R}(p) \otimes F(\alpha)(p, q)$;

(2) $\tilde{\delta}'_\ell((q, \mu^{t_0}(q)), m, q') = \sum_{p,p'\in Q_1} \tilde{\delta}_\ell((p, \mu^{t_0}(p)), m, p') \otimes (F(\alpha)(p, q).F(\alpha)(p', q'))$;

(3) $\omega'_\ell(q, m) = \sum_{p\in Q_1} \omega_\ell(p, m) \otimes F(\alpha)(p, q)$.

We only check (1), the others are similar to verify. Clearly, if $F(\alpha)(p, q) = 1$, $\tilde{R}_\ell(p) = 1$, then, $q \in J$, so $\tilde{R}'_\ell(q) = 1$, and so (1) holds. Otherwise, $\tilde{R}'_\ell(q) = 0$, i.e., $q \notin \alpha(I)$. Thus, for any $\tilde{R}_\ell(p) = 1$, $p \in I$, we have $\alpha(p) \neq q$, and hence $F(\alpha)(p, q) = 0$.

Therefore, $\sum_{p\in Q_1} \tilde{R}(p) \otimes F(\alpha)(p, q) \leq \tilde{R}(p) \wedge F(\alpha)(p, q) = 0 = \tilde{R}'_\ell(q)$. And again (1) holds. □

Theorem 3.10. *The category ς_G is isomorphic to a subcategory of ψ_G.*

Proof. We define D a subcategory of ψ_G as follows:
(1) Objects in D are monoid homomorphisms $B : (\Sigma^*, o) \longrightarrow (\Re, \bullet)$ such that $\Re$ is a monoid of L-subsets of $Q \times Q$ with monoid operation $\bullet$ satisfying

$$(fog)(p, q) = \sum_{r \in Q} f(p, r) \otimes g(r, q), \forall f, g \in \Re, \forall p, q \in Q.$$

(2) If $F_i : (\Sigma^*, o) \longrightarrow (\Re_i, \bullet), (i = 1, 2)$, are objects in D over Q_1, Q_2, respectively, then, $\alpha : F_1 \longrightarrow F_2$ is a morphism in D if it is a morphism in $\psi_{\{0,1\}}$ and we have

$$F_2(m)(\alpha \oplus \alpha)(p, q) = \sum_{p', q' \in Q_1} F_1(m)(p', q') \otimes (\alpha(p', \alpha(p)).\alpha(q', \alpha(q))),$$

$\forall m \in \Sigma^*, p, q \in Q_1$.

Now, we show that ς_G and D are isomorphic.

First, we consider the functor $\hbar : \varsigma_G \longrightarrow D$ such that

$$\forall \tilde{F}_\ell = (Q, \Sigma, \tilde{R}_\ell, Z_\ell, \tilde{\delta}_\ell, \omega_\ell, F_1, F_2) \in \varsigma_G, \hbar(\tilde{F}_\ell) : (\Sigma^*, o) \longrightarrow (\Re, \bullet),$$

satisfying

$$\hbar(\tilde{F}_\ell)(m)(p, q) = \tilde{\delta}_\ell((p, \mu^{t_0}(p)), m, q), \forall m \in \Sigma^*, p, q \in Q,$$

and $\Re = \{\hbar(\tilde{F}_\ell)(m) : m \in \Sigma^*\}$. Then, $(\Re, \bullet)$ is a monoid.

Let $\alpha : Q_1 \longrightarrow Q_2$ be a morphism in ς_G then, $h(\alpha) : B_1 \longrightarrow B_2$ is a morphism in D while

$$B_i = \hbar((Q_i, \Sigma, \tilde{R}_{\ell_i}, Z_{\ell_i}, \tilde{\delta}_{\ell_i}, \omega_{\ell_i}, F_1, F_2)) : (\Sigma^*, o) \longrightarrow (\Re_i, \bullet)(i = 1, 2),$$

because we have

$$\begin{aligned}
& B_2(m)(\alpha \oplus \alpha)(p, q) = \tilde{\delta}_{\ell_2}((\alpha(p), \mu^{t_0}(\alpha(p))), m, \alpha(q)) \\
= & \sum_{p', q' \in Q_1} \tilde{\delta}_{\ell_1}((\alpha(p'), \mu^{t_0}(\alpha(p'))), m, \alpha(q')) \otimes (\alpha(p', \alpha(p)).\alpha(q', \alpha(q))) \\
= & \sum_{p', q' \in Q_1} B_1(m)(p', q') \otimes (\alpha(p', \alpha(p)).\alpha(q', \alpha(q))) \\
\geq & \; B_1(m)(p, q)
\end{aligned}$$

$\forall m \in \Sigma^*,\ p, q \in Q_1$.

Conversely, we define a functor $\lambda : D \longrightarrow \varsigma_G$. Let $B : (\Sigma^*, o) \longrightarrow (\Re, \bullet)$ be an object in D over Q. We set $\lambda(B) = (Q, \Sigma, \tilde{R}_\ell, Z_\ell, \tilde{\delta}_\ell, \omega_\ell, F_1, F_2)$, where

$$\tilde{\delta}_\ell((p, \mu^t(p)), m, q) = B(m)(p, q), \forall m \in \Sigma^*.$$

From the definition of $\bullet$ in D, we get that $\lambda(B)$ is an L-GFA. Let $B_i : (\Sigma^*, o) \longrightarrow (\Re_i, \bullet)(i = 1, 2)$ be two objects in D over Q_1, Q_2, respectively, and let $\alpha : B_1 \longrightarrow B_2$ be a morphism in D. Then, $\alpha : Q_1 \longrightarrow Q_2$ is a morphism in ς_G from

$$\lambda(B_1) = (Q_1, \Sigma, \tilde{R}_{\ell_1}, Z_{\ell_1}, \tilde{\delta}_{\ell_1}, \omega_{\ell_1}, F_1, F_2)$$

to

$$\lambda(B_2) = (Q_2, \Sigma, \tilde{R}_{\ell_2}, Z_{\ell_2}, \tilde{\delta}_{\ell_2}, \omega_{\ell_2}, F_1, F_2).$$

In fact, $\forall m \in \Sigma^*$, $p, q \in Q_1$ we have

$$\tilde{\delta}_{\ell_2}((\alpha(p), \mu^{t_0}(\alpha(p))), m, \alpha(q)) = B_2(m)(\alpha(p), \alpha(q))$$

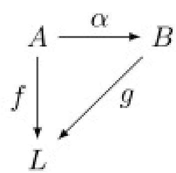
$$= \sum_{p', q' \in Q_1} B_1(m)(p', q') \otimes (\alpha(p', \alpha(p)).\alpha(q', \alpha(q)))$$

$$= \sum_{p', q' \in Q_1} \tilde{\delta}_{\ell_1}((p', \mu^{t_0}(p')), m, q') \otimes (\alpha(p', \alpha(p)).\alpha(q', \alpha(q))).$$

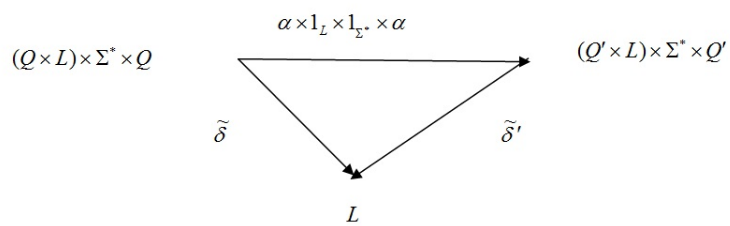
It follows that the functors $\hbar$ and λ are mutually inverse. □

4. A Categorical View of the Relation Between L-GFAs and NDAs

In this section, the concept of L-commute in section 3 will be changed, i.e., we begin with the supposition: for $\ell = \langle L, +, ., \otimes, \rho \rangle$, L-valued subsets f, g of A, B, respectively (i.e., $f \in L^A$, $g \in L^B$), and $\alpha \in Mor_L(A, B)$, we call a diagram



L-commute if $\forall b \in B$, $g(b) = \sum_{a \in A} f(a)$; and the Definition 3.3 is changed as: the objects of category ς_G are L-GFAs over Σ^*, defined in Definition 3.1, and the morphism between the objects $\tilde{F}_\ell = (Q, \Sigma, \tilde{R}_\ell, Z_\ell, \tilde{\delta}_\ell, \omega_\ell, F_1, F_2)$ and $\tilde{F}'_\ell = (Q', \Sigma, \tilde{R}'_\ell, Z_\ell, \tilde{\delta}'_\ell, \omega'_\ell, F_1, F_2)$ are maps $\alpha : Q \times Q' \longrightarrow L$ such that the following diagram L-commutes:



That is,

$$\tilde{\delta}'_\ell((p', \mu^{t_0}(p')), m, q') = \sum_{p, q \in Q, \alpha(p, p') = \alpha(q, q')} \tilde{\delta}_\ell((p, \mu^{t_0}(p)), m, q), \ \forall m \in \Sigma^*, p', q' \in Q'.$$

Considering the above definitions we have:

Theorem 4.1. *Let $l \in L/0$. Then, there exists a functor $\lambda_l : \varsigma_G \longrightarrow NA^0_M$ such that for any $\tilde{F}_\ell \in \varsigma_G$, $m \in \Sigma^*$, $=^\ell Out_L(\tilde{F}_\ell)(m) \geq l \longleftrightarrow m \in Out(\lambda_l(\tilde{F}_\ell))$.*

Proof. For $\tilde{F}_\ell = (Q, \Sigma, \tilde{R}_\ell, Z_\ell, \tilde{\delta}_\ell, \omega_\ell, F_1, F_2) \in \varsigma_G$ and $l \in L/0$, we set

$$\lambda_l(\tilde{F}_\ell) = (Q, \delta_l, J_l, S_l)$$

where $\delta_l(q,m)=\{p\in Q|\tilde{\delta}_\ell((q,\mu^{t_0}(q)),m,p)\geq l\}$. Now, let J_l and S_l are the l-level sets of the corresponding L-valued sets, i.e., $J_l=\{q\in Q|\tilde{R}_\ell(q)\geq l\}$, $S_l=\{q\in Q|\tilde{T}_\ell(q)\geq l\}$.

Then, $\lambda_l(\tilde{F}_\ell)\in obj(\mathrm{NA}^0_M)$. Since

$$=^\ell (\forall m\in\Sigma^*)\tilde{R}_\ell,\tilde{\delta}_\ell,\tilde{T}_\ell\geq l\longleftrightarrow(\exists q\in J_l)(\exists p\in S_l)p\in S_l\cap\delta_l(q,m),$$

we get the required relationship between $Out_L(\tilde{F}_\ell)$ and $Out(\lambda_l(\tilde{F}_\ell))$.

Finally, we show that λ_l is a functor. Let $\alpha\in Mor(\varsigma_G)$,

$$\alpha:\tilde{F}_{\ell_1}=(Q_1,\Sigma,\tilde{R}_{\ell_1},Z_{\ell_1},\tilde{\delta}_{\ell_1},\omega_{\ell_1},F_1,F_2)\longrightarrow\tilde{F}_{\ell_2}=(Q_2,\Sigma,\tilde{R}_{\ell_2},Z_{\ell_2},\tilde{\delta}_{\ell_2},\omega_{\ell_2},F_1,F_2)$$

and $\lambda_l(\tilde{F}_{\ell_i})=(Q_i,\delta^i_l,J^i_l,S^i_l)$, $i=1,2$. Then $\beta=\lambda_l(\alpha)$ is a morphism and we have

$$\tilde{T}_{\ell_2}(\beta(q))=\sum_{\beta(p)=\beta(q)}\tilde{T}_{\ell_1}(p),\tilde{R}_{\ell_2}(\beta(q))=\sum_{\beta(p)=\beta(q)}\tilde{R}_{\ell_1}(p).$$

Using this equalities and the fact that Q_1 is finite, we prove $\beta(J^1_l)=J^2_l\cap\beta(Q_1)$, for $p\in\beta(J^1_l)$, we only need to show $p\in J^2_l$, it is obvious that $p\in\beta(Q_1)$. Now, if $q_1\in J^1_l$ and $\beta(q_1)=p$, $\tilde{R}_{\ell_1}(q_1)\geq l$. Since

$$\tilde{R}_{\ell_2}(p)=\tilde{R}_{\ell_2}(\beta(q_1))=\sum_{\beta(q)=\beta(q_1)}\tilde{R}_{\ell_1}(q)\geq\tilde{R}_{\ell_1}(q_1)\geq l,$$

we have $p\in J^2_l$. Conversely, if $p\in J^2_l\cap\beta(Q_1)$, then, from $p\in J^2_l$, we get that $\tilde{R}_{\ell_2}(p)\geq l$. since $\tilde{R}_{\ell_2}(p)=\sum_{\beta(p)=p}\tilde{R}_{\ell_1}(q)$, so there exists $q_1\in Q_1$ such that $\beta(q_1)=p$ and $\tilde{R}_{\ell_1}(q_1)=\tilde{R}_{\ell_2}(p)\geq l$, i.e., $q_1\in J^1_l$, $\beta(q_1)=p\in\beta(J^1_l)$.

Similarly, $\beta(S^1_l)=S^2_l\cap\beta(Q_1)$ holds. Thus, we conclude that

$$\beta\in Mor(\lambda_l(\tilde{F}_{\ell_1}),\lambda_l(\tilde{F}_{\ell_2}))\subseteq Mor(NA^0_M).$$

□

Definition 4.2. Let CNA_M be the category with objects $B=\{(Q,\delta_l)|l\in L/0\}$, where

(1) $(Q,\delta_l)\in obj(NA_M)$ for any $l\in L/0$;

(2) $=^\ell (\forall l,\tau\in L)(\forall q\in Q)(\forall m\in\Sigma^*)\{l\leq\tau\longrightarrow\delta_\tau(q,m)\subseteq\delta_l(q,m)\}$;

(3) $=^\ell (\forall p,q\in Q)(\forall m\in\Sigma^*)\left\{\omega=\sum_{q\in\delta_l(p,m)}l\longrightarrow q\in\delta_\omega(p,m)\right\}$.

If $B_i=\{(Q_i,\delta^i_l)|l\in L/0\}\in obj(CNA_M), i=1,2$, then $\alpha:B_1\longrightarrow B_2$ is a morphism for any l and $\alpha\in Mor((Q_1,\delta^1_l),(Q_2,\delta^2_l))$.

Theorem 4.3. *There exist the functors $J:\varsigma_G\longrightarrow CNA_M$ and $K:CNA_M\longrightarrow\varsigma_G$ such that J and K are mutually inverse isomorphisms, that is, the categories ς_G and CNA_M are isomorphic.*

As the related proof is the same as the one with Theorem 3.4 in [15]*, it is not repeated in this part.*

Definition 4.4. By CNA_M^0, we denote the category, whose object is the set $\{(Q, \delta_l, I_l, T_l) | l \in L/0\}$, where $I_l = \{q \in Q, I \in L^Q | I(q) \geq l\}$, $T_l = \{q \in Q, T \in L^Q | T(q) \geq l\}$ such that:

(1) $\{(Q, \delta_l) | l \in L/0\} \in Obj(CNA_M)$ and for any $l \in L/0$, $(Q, \delta_l, I_l, T_l) \in Obj(CNA_M^0)$;

(2) for $R_i = \{(Q_i, \delta_l^i, I_l^i, T_l^i) | l \in L/0\} \in Obj(CNA_M^0), i = 1, 2$, then,

$$\alpha \in Mor(R_1, R_2) \subseteq Mor(CNA_M^0)$$

i.e., $\alpha \in Mor(CNA_M^0)$; for any $l \in L/0$, $\alpha \in Mor((Q_1, \delta_l^1), (Q_2, \delta_l^2)) \in Mor(CNA_M)$, and for all $l \in L/0$, $\alpha(I_l^1) = I_l^2 \cap \alpha(Q_1), \alpha(T_l^1) = T_l^2 \cap \alpha(Q_1)$.

Similar to what we did for NDAs with initial and final states, we may define the output L-valued sets for automata from the category CNA_M^0.

In fact, for $R = \{R_l = (Q, \delta_l, I_l, T_l) | l \in L/0\} \in Obj(CNA_M^0)$, then, R defines the output $Out_L(R) \in L^M$ such that

$$Out_L(R)(m) = [Out_L(R)(m)] = \sum_{m \in Out(R_l)} l$$

for any $m \in \Sigma^*$.

The following conclusions are analogous to Theorems 4.3 and 3.8 for L-GFAs.

Theorem 4.5. *There exist the functors $U : \varsigma_G \longrightarrow CNA_M^0$ and $V : CNA_M^0 \longrightarrow \varsigma_G$ such that U and V are mutually inverse isomorphisms, i.e., the categories ς_G and CNA_M^0 are isomorphic, and for any $R \in obj(CNA_M^0)$ we have*

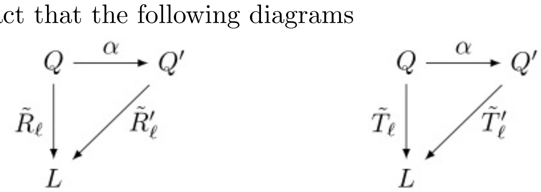
$$=^\ell m \in Out(R) \longleftrightarrow m \in Out_L(V(R))(m) = 1.$$

Proof. The functor U is defined as follows: Let $\tilde{F}_\ell = (Q, \Sigma, \tilde{R}_\ell, Z_\ell, \tilde{\delta}_\ell, \omega_\ell, F_1, F_2) \in \varsigma_G$. We set $U(\tilde{F}_\ell) = \{G_l(\tilde{F}_\ell) = (Q, \delta_l, I_l, T_l), l \in L/0\}$, where $G_l(F) = (Q, \delta_l, I_l, T_l)$ is the same as that in Theorem 4.1, i.e., $\delta_l(q, m) = \{p \in Q : \tilde{\delta}_l((q, \mu^{t_0}(q)), m, p) \geq l\}$, and I_l, T_l are corresponding 1-level set of $\tilde{R}_\ell$, $\tilde{T}_\ell$, respectively. Clearly, we have $U(\tilde{F}_\ell) \in CNA_M^0$.

Let $\alpha \in Mor(\tilde{F}_l, \tilde{F}_l')$ wehre

$$\tilde{F}_\ell = (Q, \Sigma, \tilde{R}_\ell, Z_\ell, \tilde{\delta}_\ell, \omega_\ell, F_1, F_2), \tilde{F}_\ell' = (Q', \Sigma, \tilde{R}_\ell', Z_\ell, \tilde{\delta}_\ell', \omega_\ell', F_1, F_2) \in Obj(\varsigma_G).$$

Then, from the fact that the following diagrams



L-commutes and that Q is a finite set, for any $\tau \in L/0$, we have $\alpha(I_\tau) = J_\tau \cap \alpha(Q)$, and $\alpha(T_\tau) = S_\tau \cap \alpha(Q)$. Therefore, according to Theorem 4.1, $\alpha : U(\tilde{F}_l) \longrightarrow U(\tilde{F}_l')$ is in $Mor(CNA_M^0)$, and U is a functor.

Furthermore, let $R = \{(Q, \delta_l, I_l, T_l) | l \in L/0\} \in Obj(CNA_M^0)$ and let

$$\tilde{F}_\ell = (Q, \Sigma, \tilde{R}_\ell, Z_\ell, \tilde{\delta}_\ell, \omega_\ell, F_1, F_2) = G(\{(Q, \delta_l) : l \in L/0\}),$$

where G is a functor from category CNA_M to ς_G as in Theorem 4.3. L-valued subsets $\tilde{R}_\ell$ and $\tilde{T}_\ell$ of Q are defined such that $\tilde{R}_\ell(q) = \sum\{\tau : q \in I_\tau\}$, $\tilde{T}_\ell(q) = \sum\{\tau : q \in T_\tau\}$ for all $q \in Q$. Then, we put $V(R) = \tilde{F}_\ell = (Q, \Sigma, \tilde{R}_\ell, Z_\ell, \tilde{\delta}_\ell, \omega_\ell, F_1, F_2)$. From Definition 4.4, it follows that $V(R) \in \varsigma_G$. Moreover, let $R_i = \{Q_i, \delta_l^i, I_l^i, T_l^i)|l \in L/0\} \in Obj(CNA_M^0)(i = 1, 2)$, and let $\alpha \in Mor(CNA_M^0) : R_1 \to R_2$. Then,

$$\alpha : \tilde{F}_{\ell_1} = (Q_1, \Sigma, \tilde{R}_{\ell_1}, Z_{\ell_1}, \tilde{\delta}_{\ell_1}, \omega_{\ell_1}, F_1, F_2) \longrightarrow \tilde{F}_{\ell_2} = (Q_2, \Sigma, \tilde{R}_{\ell_2}, Z_{\ell_2}, \tilde{\delta}_{\ell_2}, \omega_{\ell_2}, F_1, F_2)$$

and $\alpha \in Mor(\varsigma_G)$, where

$$(Q_i, \Sigma, \tilde{R}_{\ell_i}, Z_{\ell_i}, \tilde{\delta}_{\ell_i}, \omega_{\ell_i}, F_1, F_2) = G(\{(Q_i, \delta_l^i) : l \in L/0\}).$$

Moreover, if $V(R_i) = (Q_i, \Sigma, \tilde{R}_{\ell_i}, Z_{\ell_i}, \tilde{\delta}_{\ell_i}, \omega_{\ell_i}, F_1, F_2)$, then, for any

$$q \in Q_1, \tilde{R}_{\ell_2}(\alpha(q)) = \sum_{\alpha(p)=\alpha(q)} \tilde{R}_{\ell_1}(p), \tilde{T}_{\ell_2}(\alpha(q)) = \sum_{\alpha(p)=\alpha(q)} \tilde{T}_{\ell_1}(p),$$

as it follows from the fact that $\alpha \in Mor(CNA_M^0)$. Hence V is a functor. From Definition 4.4, and by a simple computation, we can also obtain that V and U are mutually inverse functors.

Let $R = \{(Q, \delta_l, I_l, T_l)|l \in L/0\} \in Obj(CNA_M^0)$, and let

$$V(R) = (Q, \Sigma, \tilde{R}_\ell, Z_\ell, \tilde{\delta}_\ell, \omega_\ell, F_1, F_2).$$

From the above, we have

$$\tilde{\delta}_l((q, \mu^{t_0}(q)), m, p) = \sum_{p \in \delta_l(q,m)} l, \tilde{R}_\ell(q) = \sum_{q \in I_\tau} \tau, \tilde{T}_\ell(q) = \sum_{q \in T_\tau} \tau,$$

for any $m \in \Sigma^*$, $p, q \in Q$. Now, let us denote, for any $m \in \Sigma^*$,

$$N_0(m) = \{\tau \geq 0 : \exists p, q \in Q, \tau = \tilde{R}_\ell(q) \otimes \tilde{\delta}_l((q, \mu^{t_0}(q)), m, p) \otimes \tilde{T}_\ell(p)\},$$
$$N_1(m) = \{\tau \geq 0 : \exists p, q \in Q, \tau \leq \tilde{R}_\ell(q) \otimes \tilde{\delta}_l((q, \mu^{t_0}(q)), m, p) \otimes \tilde{T}_\ell(p)\},$$
$$N_2(m) = \{l \geq 0 : \exists q \in I_l, \delta_l(q, m) \cap T_l \neq \emptyset\}.$$

Then we have

$$=^\ell m \in Out(R) \longleftrightarrow \sum_{l \in N_2(m)} l = 1, Out_L(V(R))(m) = \sum_{\tau \in N_0(m)} \tau = \sum_{\tau \in N_1(m)} \tau.$$

Let $l \in N_2(m)$, then, there exist $q \in I_l$ and $p \in Q$ such that $p \in \delta_l(q, m) \cap T_l$, which follows $\tilde{R}_\ell(q) \otimes \tilde{\delta}_\ell((q, \mu^{t_0}(q)), m, p) \otimes \tilde{T}_\ell(p) \geq l$. Hence $l \in N_1(m)$.

Conversely, let $\tau \in N_0(m)$, i.e., $\tau = \tilde{R}_\ell(q) \otimes \tilde{\delta}_l((q, \mu^{t_0}(q)), m, p) \otimes \tilde{T}_\ell(p)$ for some $q, p \in Q$. Then, $\tau \leq \tilde{R}_\ell(q) = \sum_{q \in I_l} l$. If $\tau = \sum_{q \in I_l} l$, then, according to Definition 4.4, we have $q \in I_\tau$; if $\tau < \sum_{q \in I_l} l$, then, there exists $l > \tau$ such that $q \in I_l \subseteq I_\tau$. Hence $\tau \in N_2(m)$, and we complete the proof. □

Example 4.6. Let $\tilde{F}_\ell = (Q, \Sigma, \tilde{R}_\ell, Z_\ell, \tilde{\delta}_\ell, \omega_\ell, F_1, F_2)$, $\tilde{F}'_\ell = (Q', \Sigma, \tilde{R}'_\ell, Z_\ell, \tilde{\delta}'_\ell, \omega'_\ell, F_1, F_2) \in \varsigma_G$, $Q = \{p, q\}$, $Q' = \{p', q', q_1\}$. For some $m \in \Sigma^*$,

$$\tilde{\delta}((p, \mu^{t_0}(p)), m, q) = \frac{1}{2}, \tilde{\delta}'((p', \mu^{t_0}(p')), m, q') = \frac{1}{4}, \tilde{\delta}'((p', \mu^{t_0}(p')), m, q_1) = \frac{1}{2},$$

and $\alpha : Q \longrightarrow Q'$ is defined by $\alpha(p, p') = \frac{1}{3}$, $\alpha(q, q') = \frac{1}{3}$, $\alpha(q, q_1) = 1$. Suppose that λ_l is a functor from ς_G to CNA_M, where $l = \frac{1}{2}$, $\lambda_l(\tilde{F}) = (Q, \delta_l^1)$, $\lambda_l(\tilde{F}'_l) = (Q', \delta_l^2)$, and

$$\begin{aligned} \delta_l^1(p, m) &= \{r : \tilde{\delta}((p, \mu^{t_0}(p)), m, r) \geq l\} = \{q\}, \\ \delta_l^2(p', m) &= \{r : \tilde{\delta}'((p', \mu^{t_0}(p')), m, r) \geq l\} = \{q_1\}. \end{aligned}$$

Since

$$\tilde{\delta}'((p', \mu^{t_0}(p')), m, q') \otimes (\alpha(p, p').\alpha(q, q')) = \frac{1}{4} \otimes \frac{1}{3} < \frac{1}{3} < \tilde{\delta}((p, \mu^{t_0}(p)), m, q),$$

we have $\alpha \in Mor_L(\tilde{F}_l, \tilde{F}'_l)$. Naturally, let $\beta = \lambda_l(\alpha)$ be defined as: $\beta(p) = p'$, $\beta(q) = q'$.

We have

$$\beta\left(\bigcup_{p \in Q, \beta(p) = \beta(q)} \delta_l^1(p, m)\right) \neq \delta_l^2(\beta(p), m) \bigcap \beta(Q)$$

since $q \in \bigcup_{p \in Q, \beta(p) = \beta(q)} \delta_l^1(p, m)$,i.e., $\tilde{\delta}_l((p, \mu^{t_0}(p)), m, q) = \frac{1}{2} \geq l$, but

$$\tilde{\delta}'((\beta(p), \mu^{t_0}(\beta(p))), m, q) = \tilde{\delta}'((p', \mu^{t_0}(p')), m, q') = \frac{1}{4} < l.$$

That is, $\beta(q) \notin \delta_l^2(\beta(p), m) \cap \beta(Q)$, and it indicates that $\beta \notin Mor((Q, \delta_l^1), (Q, \delta_l^2))$.

5. Concluding Remarks

As it has been mentioned in Section 1, category theory has important applications in studying dynamic systems. In addition, fuzzy systems have been used to solve meaningful problems such as intelligent interface design, and neural networks. In this study, the researchers have proved that the general fuzzy automata can be represented by a chain of non-deterministic automata with initial and final states. In addition, the existence of isomorphism between the category of L-GFAs and the subcategory of generalized L-GFAs and between the category of L-GFAs and the category of sets of NDAs have been demonstrated in details. Moreover, there are very close relationship between the output functions of both these systems. As to extend some studies in this respect, the researchers suggest further investigation of these systems and their relationship to be done for BL-general fuzzy automata, where instead of fuzzy sets an L-valued set for a general BL-algebra will be taken into account.

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