FUZZY ROUGH N-ARY SUBHYPERGROUPS

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Abstract. Fuzzy rough n-ary subhypergroups are introduced and characterized.

1. Introduction

n-ary structures, particularly n-ary groups, were introduced by Dörnte in [10] and since then, n-ary systems have been studied in depth in different contexts. These systems have many applications in various fields of science, e.g. physics, quantum group theory, codes and topology. In [13] Davvaz and Vougiouklis defined n-ary hyperstructures, which are a generalization of the hyperstructures introduced by Marty [20], and constitute a field of algebra with applications in geometry, graphs and hypergraphs, binary relations, lattices, fuzzy and rough sets, automata, cryptography, codes, artificial intelligence, probability (see [4]). Some applications of n-ary hypergroups to binary relations and lattices were studied by Leoreanu Fotea and Davvaz in [18] and [19].

On the other hand, Rosenfeld [30] introduced fuzzy sets in the context of group theory and formulated the concept of a fuzzy subgroup of a group. Since then, many researchers have extended the concepts of abstract algebra to a fuzzy framework. Several basic definitions and results about fuzzy algebra can be found in [21]. Fuzzy hypergroups have been considered by Zahedi at al. [31] and studied by Davvaz [6] and Davvaz and Corsini [7], [8]. Rough set theory, introduced by Pawlak (see [23]), represents a mathematical tool for dealing with vagueness or uncertainty. Rough sets are especially useful in data analysis, artificial intelligence and the cognitive sciences (see [24]–[29]). Several basic aspects of the research of rough sets and applications are presented in [24], [25] by Z. Pawlak and A. Skowron.

Dubois and Prade [11] introduced fuzzy rough sets as a fuzzy generalization of rough sets. Nanda and Majumdar [22] analyzed the concept of a fuzzy rough set. J.N. Mordeson [14] studied rough groups and B. Davvaz studied roughness based on fuzzy ideals [3]. Notes on the lower and upper approximations in a fuzzy group and rough ideals in semigroups were presented by W. Cheng, Z.W. Mo and J. Wang in [1]. Roughness in H_v–structures was considered by Davvaz [5]. The upper and lower approximations of a subset were studied by Leoreanu-Fotea (see [15]) in the context of hypergroups.

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In this paper we introduce and characterize fuzzy rough n-ary subhypergroups. In other words, we consider the problem of fuzzification of rough n-ary subhypergroups. Fuzzy algebraic structures - in particular, fuzzy lattices and join hyperoperations - have been used in several engineering and computer science applications [12], [26], [27]. Hence we expect that the results of this work may in the future prove useful in similar applications. The present research can be exyended to the analysis some particular classes (e.g. closed, invertible, ultraclosed) of fuzzy rough n-ary subhypegroups, starting from the corresponding classes of n-ary subhypegroups.

2. n-hypergroups and Approximations in n-ary Hypergroups

We first present some basic notions and results about n-hypergroups (see [9]), which are needed in this paper.

Let H be a nonempty set and $n \in \mathbb{N}$, $n \geq 2$.

Consider $f : H^n \to P^*(H)$, where $P^*(H)$ is the set of all nonempty subsets of H. Then f is called an *n-ary hyperoperation* on H and the pair (H, f) is called an *n-hypergroupoid*. If $A_1, A_2, ..., A_n$ are subsets of H, then we define $f(A_1, A_2, ..., A_n) = \cup\{f(a_1, .., a_n) \mid a_i \in A_i,\ i \in \{1, 2, ..., n\}\}$.

We shall denote the sequence $a_i, a_{i+1}, ..., a_j$ by a_i^j. For $j < i$, the symbol a_i^j is the empty set.

Definition 2.1.

1°. The n-hypergroupoid (H, f) is called an *n-ary semihypergroup* if for $i, j \in \{1, 2, ..., n\}$ and a_1^{2n-1} of H, we have

$$f(a_1^{i-1}, f(a_i^{n+i-1}), a_{n+i}^{2n-1}) = f(a_1^{j-1}, f(a_j^{n+j-1}), a_{n+j}^{2n-1}).$$

2°. We say that (H, f) is an *n-ary quasihypergroup* if for all $a_0, a_1, ..., a_n \in H$ and fixed $i \in \{1, ..., n\}$ there exists $x \in H$ such that

$$a_0 \in f(a_1^{i-1}, x, a_{i+1}^n).$$

3°. An *n-ary hypergroup* is both an n-ary semihypergroup and an n-ary quasihypergroup.

4°. An n-ary hypergroup (H, f) is *commutative* if for all a_1^n of H, and any permutation σ of $\{1, 2, ..., n\}$, we have $f(a_1^n) = f(a_{\sigma(1)}, ..., a_{\sigma(n)})$.

Definition 2.2. Let (H, f) be an n-ary hypergroup and K a nonempty subset of H. If K is closed under the n-ary hyperoperation f then we say that K is an *n-ary subsemihypergroup*. An n-ary subsemihypergroup K is called an *n-ary subhypergroup* of H if for all $k_0, k_1, ..., k_n \in K$ and fixed $i \in \{1, 2, ..., n\}$, there exists $x \in K$ such that $k_0 \in f(k_1^{i-1}, x, k_{i+1}^n)$.

The intersection of n-ary subhypergroups, as well as subhypergroups, can be empty (see [2]). This can happen since the element x in the above definition is not always unique.

In what follows, we shall assume that (H, f) is a commutative n-ary hypergroup.

We recall several types of n-ary subhypergroups together with the connections among them, in the context of a commutative n-ary hypergroup ([16]).

Let (K, f) be an n-ary subhypergroup of (H, f).

Definition 2.3. We say that

1°. K is *closed* if for all k_1^n of K and x of H, it follows from $k_1 \in f(x, k_2^n)$ that $x \in K$.

2°. K is *invertible* if it follows from $x \in f\big(y, \underbrace{K, ..., K}_{n-1}\big)$ that $y \in f\big(x, \underbrace{K, ..., K}_{n-1}\big)$.

3°. K is *ultraclosed* if for all $x \in H$, we have

$$f\big(x, \underbrace{K, ..., K}_{n-1}\big) \cap f\big(x, H - K, \underbrace{K, ..., K}_{n-2}\big) = \emptyset.$$

4°. K is *conjugable* if it is closed and for all $x \in H$, there exists $x' \in H$, such that $f\big(x', x, \underbrace{K, ..., K}_{n-2}\big) \subseteq K$.

Several examples of these types of n-ary subhypergroups are given in [16]. We mention just two:

Example 2.4.

1°. Let K be an ideal in a modular lattice $(L, \vee, \wedge)$. For all a_1^n of L and $i \in \{1, 2, ..., n\}$, denote $A_n^{(i)} = a_1 \vee ... \vee a_{i-1} \vee a_{i+1} \vee ... \vee a_n$ and $A_n = a_1 \vee ... \vee a_n$. We set $A_n^{(1)} = a_2 \vee ... \vee a_n$ and $A_n^{(n)} = a_1 \vee ... \vee a_{n-1}$.

We define the following n-ary hyperoperation on L:

$$f(a_1^n) = \{x \in L \mid x \vee A_n^{(i)} = A_n, \text{ for all } i \in \{1, 2, ..., n\}\}.$$

Then (L, f) is a commutative n-ary hypergroup and (K, f) is an n-ary subhypergroup of it.

Moreover, K is invertible. If $n \geq 4$, then K is not ultraclosed. Indeed, if $h \in L - K$ and $k \in K$, then

$$h \vee k \in f\big(h \vee k, \underbrace{K, ..., K}_{n-1}\big) \cap f\big(h \vee k, L - K, \underbrace{K, ..., K}_{n-2}\big).$$

2°. Let us consider the distributive lattice $(P(M), \cup, \cap)$ of the parts of a set M, which contains at least three elements. Define the following n-ary hyperoperation on $P(M)$: for all $X_1, ..., X_n \in P(M)$,

$$f(X_1, ..., X_n) = \{Z \in P(M) \mid X_1 \cap ... \cap X_n \subseteq Z \subseteq X_1 \cup ... \cup X_n\}.$$

Then $(P(M), f)$ is a commutative n-ary hypergroup. Let $a, b \in M$, $a \neq b$ and $K = \{M - \{a\}, M - \{a, b\}\}$. Then (K, f) is an n-ary subhypergroup of $(P(M), f)$ which is not closed.

Indeed, we have $M - \{a\} \in f\big(\{b\}, \underbrace{M - \{a, b\}, ..., M - \{a, b\}}_{n-1}\big\}$.

The following theorem establishes several connections between types of n-ary subhypergroups defined above ([16]).

Theorem 2.5. *Let (H, f) be an n-ary hypergroup.*

(i) *Any conjugable n-ary subhypergroup (K, f) is ultraclosed;*
(ii) *Any ultraclosed n-ary subhypergroup (K, f) is invertible;*
(iii) *Any invertible n-ary subhypergroup (K, f) is closed.*

Notice that the intersection of n-ary closed subhypergroups is a closed n-ary subhypergroup.

Let us now recall the notion of rough sets .

Let H be a nonempty set and $A \subset H$.

If R is an equivalence relation on H, then the pair $(\underline{R(A)}, \overline{R(A)})$ is called the *rough set* of A, with respect to R, where

$$\underline{R(A)} = \{x \in H \mid \bar{x} \subseteq A\} \quad \text{and} \quad \overline{R(A)} = \{x \in H \mid \bar{x} \cap A \neq \emptyset\}.$$

We have denoted the equivalence class of $x \in H$ by $\bar{x}$.

$\underline{R(A)}$ is called the *lower approximation* of A, while $\overline{R(A)}$ is called the *upper approximation* of A.

Let us consider now (H, f) a commutative n-ary hypergroup, $A \subseteq H$ and K an invertible n-ary hypergroup. Denote

$$\underline{Apr}_K(A) = \big\{x \in H \mid f\big(x, \underbrace{K, ..., K}_{n-1}\big) \subseteq A\big\} \qquad \text{and}$$
$$\overline{Apr}_K(A) = \big\{x \in H \mid f\big(x, \underbrace{K, ..., K}_{n-1}\big) \cap A \neq \emptyset\big\}.$$

The set $\underline{Apr}_K(A)$ is called the *lower approximation* of A with respect to K, while the set $\overline{Apr}_K(A)$ is called the *upper approximation* of A with respect to K. The pair $\big(\underline{Apr}_K(A), \overline{Apr}_K(A)\big)$ is called the *rough set* of A with respect to K.

Consider the equivalence relation R_K, defined as follows:

$$x\, R_k\, y \text{ if and only if } x \in f\big(y, \underbrace{K, ..., K}_{n-1}\big)$$

. Then, for all $x \in H$, we have $\bar{x} = f\big(x, \underbrace{K, ..., K}_{n-1}\big)$ and so

$$\underline{Apr}_K(A) = \underline{R}\,_K(A) \quad \text{and} \quad \overline{Apr}_K(A) = \overline{R}_K(A).$$

Roughness of n-ary hypergroups was introduced and analyzed in [17].

A subset A of an n-ary hypergroup (H, f) is called *definable* with respect to K if $\underline{Apr}_K(A) = A = \overline{Apr}_K(A)$. If $\underline{Apr}_K(A)$ and $\overline{Apr}_K(A)$ are n-ary subhypergroups of (H, f), then $Apr_K(A)$ is called a *rough n-ary subhypergroup* of (H, f).

In [17] it is proved that

Theorem 2.6. *If $K_1 \subseteq K_2$, K_1 is an invertible n-ary subhypergroup and K_2 is a closed n-ary subhypergroup of (H, f), then $Apr_{K_1}(K_2)$ is a rough n-ary subhypergroup of (H, f).*

3. **Fuzzy Rough n-ary Subhypergroups**

In order to introduce the notion of a fuzzy rough n-ary subhypergroup, we first recall the notion of an fuzzy n-ary subgroup.

Let (H, f) be an n-ary hypergroup. For all x, y_2^n of H, we denote $x/y_2^n = \{u \mid x \in f(u, y_2^n)\}$.

A fuzzy subset A is called a *fuzzy n-ary subhypergroup* if, for all x_1^n of H, the following conditions hold:

(1) for all $z \in f(x_1^n)$, we have $\mu_A(z) \geq \min\{\mu_A(x_1), ..., \mu_A(x_n)\}$;
(2) there exists $u \in x_1/x_2^n$ such that $\mu_A(u) \geq \min\{\mu_A(x_1), ..., \mu_A(x_n)\}$.

Fuzzy n-ary subhypergroups were introduced and analized by Davvaz and Corsini [8].

Now let $H \neq \emptyset$, $A \subseteq X \subseteq H$ and R be an equivalence relation on H.

A *fuzzy rough set* $(\underline{R}(A), \overline{R}(A))$ is characterized by a pair of maps

$$\mu_{\underline{R(A)}} : \underline{R(X)} \longrightarrow [0,1] \text{ and } \mu_{\overline{R(A)}} : \overline{R(X)} \longrightarrow [0,1]$$

such that $\mu_{\underline{R(A)}}(x) \leq \mu_{\overline{R(A)}}(x),$ for all $x \in \underline{R(X)}$.

Let S be an n-ary invertible subhypergroup of an n-ary hypergroup (H, f). If $Apr_S(X) = \big(\underline{Apr}_S(X), \overline{Apr}_S(X)\big)$ is a rough n-ary subhypergroup of H, then the set

$$\widehat{Apr_S}(X) = \overline{Apr}_S(X) - \underline{Apr}_S(X)$$

is called the *boundary region* of X.

Let $A \subseteq X$ and $Apr_S(A) = \big(\underline{Apr}_S(A), \overline{Apr}_S(A)\big)$ be a fuzzy rough set.

We define $\overline{\mu}_{\underline{Apr}_S(A)} : \overline{Apr}_S(X) \longrightarrow [0,1]$ as follows:

$$\overline{\mu}_{\underline{Apr}_S(A)}(x) = \begin{cases} \mu_{\underline{Apr}_S(A)}(x), & \text{if } x \in \underline{Apr}_S(X) \\ 0, & \text{if } x \in \widehat{Apr}_S(X). \end{cases}$$

For all $x \in \overline{Apr}_S(X)$, denote $\widetilde{\mu}_A(x) = \big[\overline{\mu}_{\underline{Apr}_S(A)}(x), \mu_{\overline{Apr}_S(A)}(x)\big]$.

An *interval–valued fuzzy subset* A is given by

$$A = \{(x, \widetilde{\mu}_A(x)) \mid x \in \overline{Apr}_S(X)\}.$$

Let $I_1 = [a_1, b_1]$, $I_2 = [a_2, b_2]$ be closed subintervals of $[0,1]$. We define

$$r\max(I_1, I_2) = [a_1 \vee a_2, b_1 \vee b_2]$$
$$r\min(I_1, I_2) = [a_1 \wedge a_2, b_1 \wedge b_2]$$

We say that $I_2 \leq I_1$ if $a_2 \leq a_1$ and $b_2 \leq b_1$.

We shall say that $\widetilde{\mu}_A(x)$ is a closed subinterval of $[0,1]$ even if it contains only one element, which means that

$$\overline{\mu}_{\underline{Apr}_S(A)}(x) = \mu_{\overline{Apr}_S(A)}(x).$$

Now we introduce the notion of a fuzzy rough n-ary subhypergroup, which generalizes the fuzzy rough subhypergroup notion, studied in [15].

Definition 3.1. If $Apr_S(X)$ is a rough n-ary subhypergroup of an n-ary hypergroup (H, f) and $A \subseteq X$, then the fuzzy rough set

$$Apr_S(A) = \big(\underline{Apr}_S(A), \overline{Apr}_S(A)\big)$$

is called a *fuzzy rough n-ary subhypergroup* if for all x_1^n of $\overline{Apr}_S(X)$, the following conditions hold:

(1) for all $z \in f(x_1^n)$, we have $\widetilde{\mu}_A(z) \geq r \min\big\{\widetilde{\mu}_A(x_1), ..., \widetilde{\mu}_A(x_n)\big\}$.
(2) there exists $u \in x_1/x_2^n$ such that $\widetilde{\mu}_A(u) \geq r \min\big\{\widetilde{\mu}_A(x_1), ..., \widetilde{\mu}_A(x_n)\big\}$.

In what follows, we present particular cases and examples of fuzzy rough n-ary subhypergroups.

Example 3.2.

1°. Any rough n-ary subhypergroup is a fuzzy rough n-ary subhypergroup.

Indeed, let $Apr_S(X)$ be a rough n-ary subhypergroup of an n-ary hypergroup (H, f) and $A \subseteq X$, such that $Apr_S(A) = \big(\underline{Apr}_S(A), \overline{Apr}_S(A)\big)$ is a rough n-ary subhypergroup. Define $\overline{\mu}_{\underline{Apr}_S(A)} = \chi_{\underline{Apr}_S(A)}$ and $\mu_{\overline{Apr}_S(A)} = \chi_{\overline{Apr}_S(A)}$. Clearly, we have $\chi_{\underline{Apr}_S(A)} \leq \chi_{\overline{Apr}_S(A)}$.

Since $\underline{Apr}_S(A)$ is a n-ary subhypergroup, it follows that for all x_1^n of $\overline{Apr}_S(X)$ and $z \in f(x_1^n)$, we have

$$\chi_{\underline{Apr}_S(A)}(z) \geq min\big\{\chi_{\underline{Apr}_S(A)}(x_1), ..., \chi_{\underline{Apr}_S(A)}(x_n)\big\}$$

and there exists $u \in x_1/x_2^n$ such that

$$\chi_{\underline{Apr}_S(A)}(u) \geq min\big\{\chi_{\underline{Apr}_S(A)}(x_1), ..., \chi_{\underline{Apr}_S(A)}(x_n)\big\}.$$

In other words, for all $z \in f(x_1^n)$, we have

$$\overline{\mu}_{\underline{Apr}_S(A)}(z) \geq min\big\{\overline{\mu}_{\underline{Apr}_S(A)}(x_1), ..., \overline{\mu}_{\underline{Apr}_S(A)}(x_n)\big\}$$

and there exists $u \in x_1/x_2^n$ such that

$$\overline{\mu}_{\underline{Apr}_S(A)}(u) \geq min\big\{\overline{\mu}_{\underline{Apr}_S(A)}(x_1), ..., \overline{\mu}_{\underline{Apr}_S(A)}(x_n)\big\}.$$

Similarly, since $\overline{Apr}_S(A)$ is a n-ary subhypergroup, it follows that for all $z \in f(x_1^n)$, we have

$$\mu_{\overline{Apr}_S(A)}(z) \geq min\big\{\mu_{\overline{Apr}_S(A)}(x_1), ..., \mu_{\overline{Apr}_S(A)}(x_n)\big\}$$

and there exists $v \in x_1/x_2^n$ such that

$$\mu_{\overline{Apr}_S(A)}(v) \geq min\big\{\mu_{\overline{Apr}_S(A)}(x_1), ..., \mu_{\overline{Apr}_S(A)}(x_n)\big\}.$$

If for all $i \in \{1, 2, ..., n\}$, we have $x_i \in \underline{Apr}_S(A)$, then $u \in \underline{Apr}_S(A)$ and we consider $v = u$. If there exists $i \in \{1, 2, ..., n\}$, such that $x_i \notin \underline{Apr}_S(A)$, then u can be any element of x_1/x_2^n and so, we can take $u = v$.

From $\widetilde{\mu}_A(x) = \left[\overline{\mu}_{\underline{Apr}_S(A)}(x), \mu_{\overline{Apr}_S(A)}(x)\right]$, it follows that for all $z \in f(x_1^n)$, we have

$$\widetilde{\mu}_A(z) \geq r \min\left\{\widetilde{\mu}_A(x_1), ..., \widetilde{\mu}_A(x_n)\right\}$$

and there exists $u \in x_1/x_2^n$ such that

$$\widetilde{\mu}_A(u) \geq r \min\left\{\widetilde{\mu}_A(x_1), ..., \widetilde{\mu}_A(x_n)\right\},$$

which means that $Apr_S(A) = \left(\underline{Apr}_S(A), \overline{Apr}_S(A)\right)$ is a fuzzy rough n-ary subhypergroup of (H, f).

2°. Let (L, f) be the commutative n-ary hypergroup defined in Example 2.4.,1°. If K is an ideal of $(L, \vee, \wedge)$, then (K, f) is an invertible n-ary subhypergroup of (L, f).

Let $K_1 \subseteq K_2$, where K_1 is an invertible n-ary subhypergroup and K_2 is a closed n-ary subhypergroup of (L, f). According to Theorem 2.6., $Apr_{K_1}(K_2)$ is a rough n-ary subhypergroup of (L, f).

If $A \subseteq K_2$, such that $\underline{Apr}_{K_1}(A)$ is a n-ary subhypergroup, then define $\overline{\mu}_{\underline{Apr}_{K_1}(A)} = \chi_{\underline{Apr}_{K_1}(A)}$ and let $\mu_{\overline{Apr}_{K_1}(A)}$ that satisfies the following conditions:

(α). for all $x \in \underline{Apr}_{K_1}(A)$, we have $\mu_{\overline{Apr}_{K_1}(A)}(x) = 1$;

(β). for all $x, y \in \overline{Apr}_{K_1}(K_2)$, we have

$$\mu_{\overline{Apr}_{K_1}(A)}(x \vee y) = min\{\mu_{\overline{Apr}_{K_1}(A)}(x), \mu_{\overline{Apr}_{K_1}(A)}(y)\}.$$

From the condition (α), we obtain $\chi_{\underline{Apr}_{K_1}(A)} \leq \mu_{\overline{Apr}_{K_1}(A)}$. On the other hand, for all $z \in f(x_1^n)$, we have $z \leq x_1 \vee x_2 \vee ... \vee x_n$, and so $z \vee x_1 \vee x_2 \vee ... \vee x_n = x_1 \vee x_2 \vee ... \vee x_n$. From ($\beta$), we obtain

$$min\{\mu_{\overline{Apr}_{K_1}(A)}(z), \mu_{\overline{Apr}_{K_1}(A)}(x_1), \mu_{\overline{Apr}_{K_1}(A)}(x_2), ..., \mu_{\overline{Apr}_{K_1}(A)}(x_n)\} =$$

$$min\{\mu_{\overline{Apr}_{K_1}(A)}(x_1), \mu_{\overline{Apr}_{K_1}(A)}(x_2), ..., \mu_{\overline{Apr}_{K_1}(A)}(x_n)\},$$

whence

$$\mu_{\overline{Apr}_{K_1}(A)}(z) \geq min\{\mu_{\overline{Apr}_{K_1}(A)}(x_1), \mu_{\overline{Apr}_{K_1}(A)}(x_2), ..., \mu_{\overline{Apr}_{K_1}(A)}(x_n)\}.$$

On the other hand, there exists $v = x_1 \vee x_2 \vee ... \vee x_n \in x_1/x_2^n$. For the n-ary hyperoperation f defined in Example 2.4.,1°, we have $x_1 \in f(v, x_n^2)$ if and only if $v \in f(x_1, x_n^2)$ and so, similarly, we obtain

$$\mu_{\overline{Apr}_{K_1}(A)}(v) \geq min\left\{\mu_{\overline{Apr}_{K_1}(A)}(x_1), ..., \mu_{\overline{Apr}_{K_1}(A)}(x_n)\right\}.$$

If for all $i \in \{1, 2, ..., n\}$, we have $x_i \in \underline{Apr}_{K_1}(A)$, then $v = x_1 \vee x_2 \vee ... \vee x_n \in f(x_1, ..., x_n) \subseteq \underline{Apr}_{K_1}(A)$ and we have

$$\overline{\mu}_{\underline{Apr}_{K_1}(A)}(v) \geq min\left\{\overline{\mu}_{\underline{Apr}_{K_1}(A)}(x_1), ..., \overline{\mu}_{\underline{Apr}_{K_n}(A)}(x_n)\right\}.$$

If there exists $i \in \{1, 2, ..., n\}$, such that $x_i \notin \underline{Apr}_{K_1}(A)$, then any element u of x_1/x_2^n satisfies

$$\overline{\mu}_{\underline{Apr}_{K_1}(A)}(u) \geq min\{\overline{\mu}_{\underline{Apr}_{K_1}(A)}(x_1), ..., \overline{\mu}_{\underline{Apr}_{K_n}(A)}(x_n)\}$$

and so, we can take $u = v$.

Since $\widetilde{\mu}_A(x) = \big[\overline{\mu}_{\underline{Apr}_{K_1}(A)}(x), \mu_{\overline{Apr}_{K_1}(A)}(x)\big]$, it follows that $\widetilde{\mu}_A$ satisfies the conditions (1) and (2) of Definition 3.1. and $Apr_{K_1}(A)$ is a fuzzy rough n-ary subhypergroup of (L, f).

3°. Let $(H, \cdot)$ be a hypergroup. If, for all $x_1, ..., x_n$ of H, we define

$$f(x_1, ..., x_n) = \prod_{i=1}^{n} x_i,$$

then (H, f) is an n-ary hypergroup. If $(S, \cdot)$ is an invertible subhypergroup of $(H, \cdot)$ and $apr_S(A) = \big(\underline{apr}_S(A), \overline{apr}_S(A)\big)$ is a fuzzy rough subhypergroup of $(H, \cdot)$, where

$$\underline{apr}_S(A) = \{x \mid xS \subseteq A\} \text{and} \overline{apr}_S(A) = \{x \mid xS \cap A \neq \emptyset\}$$

, then (S, f) is an invertible n-ary subhypergroup of (H, f) and $Apr_S(A) = \big(\underline{Apr}_S(A), \overline{Apr}_S(A)\big)$ is a fuzzy rough n-ary subhypergroup of (H, f), where

$$\underline{Apr}_S(A) = \{x \mid f(x, \underbrace{S, ..., S}_{n-1}) \subseteq A\},$$

$$\overline{Apr}_S(A) = \{x \mid f(x, \underbrace{S, ..., S}_{n-1}) \cap A \neq \emptyset\}.$$

Now we define the level rough set notion in the context of rough n-ary hypergroups.

Definition 3.3. Let $Apr_S(X)$ be a rough n-ary subhypergroup of an n-ary hypergroup (H, f), $A \subseteq X$ and $\big(\underline{Apr}_S(A), \overline{Apr}_S(A)\big)$ be a fuzzy rough set. For each $t \in [0, 1]$, we define

$$\underline{A}_t = \big\{x \in \underline{Apr}_S(X) \mid \mu_{\underline{Apr}_S(A)}(x) \geq t\big\},$$
$$\overline{A}_t = \big\{x \in \overline{Apr}_S(X) \mid \mu_{\overline{Apr}_S(A)}(x) \geq t\big\}.$$

The pair $\big(\underline{A}_t, \overline{A}_t\big)$ is called a *level rough set*. If both $\underline{A}_t$ and $\overline{A}_t$ are n-ary subhypergroups of H, then $\big(\underline{A}_t, \overline{A}_t\big)$ is called a *level rough n-ary subhypergroup* of H.

In what follows, we shall characterize fuzzy rough subhypergroups of a hypergroup, by using level rough sets. As a consequence, we can characterize fuzzy n-ary subhypergroups, using level subsets.

Let $Apr_S(X)$ be a rough closed n-ary subhypergroup of an n-ary hypergroup (H, f) and $A \subseteq X$, be such that $Apr_S(A) = \big(\underline{Apr}_S(A), \overline{Apr}_S(A)\big)$ is a fuzzy rough set.

Lemma 3.4. *If $Apr_S(A) = \big(\underline{Apr}_S(A), \overline{Apr}_S(A)\big)$ is a fuzzy rough n-ary subhypergroup, then for all $t \in [0,1]$, the pair $\big(\underline{A}_t, \overline{A}_t\big)$ is a level rough n-ary subhypergroup of (H, f).*

Proof. Set $t \in [0,1]$ and let x_1^n be arbitrary in $\underline{A}_t$.

Then for all i we have $\mu_{\underline{Apr}_S(A)}(x_i) \geq t$. Also,

$$r\min\left\{\widetilde{\mu}_A(x_1^n)\right\} \geq \left[t, \min\left\{\mu_{\overline{Apr}_S(A)}(x_1^n)\right\}\right] \geq \left[t, \min\left\{\mu_{\underline{Apr}_S(A)}(x_1^n)\right\}\right].$$

It follows by hypothesis that

$$\widetilde{\mu}_A(z) \geq \left[t, \min\left\{\mu_{\underline{Apr}_S(A)}(x_1^n))\right\}\right], \text{ for all } z \in f(x_1^n).$$

Since $x_1^n \in \underline{Apr}_S(X)$, we have $z \in \underline{Apr}_S(X)$ for all $z \in f(x_1^n)$. Hence $\mu_{\underline{Apr}_S(A)}(z) \geq t$, which means that $z \in \underline{A}_t$.

We now show that there exists $u \in \underline{A}_t$, such that $x_1 \in f(u, x_2^n)$. By hypothesis, there exists $u \in x_1/x_2^n$, such that $\widetilde{\mu}_A(u) \geq r\min\left\{\widetilde{\mu}_A(x_1^n)\right\}$. We obtain $\widetilde{\mu}_A(u) \geq \left[t, \min\left\{\mu_{\underline{Apr}_S(A)}(x_1^n)\right\}\right]$. Since $\underline{Apr}_S(X)$ is a closed n-ary subhypergroup and $x_1^n \in \underline{Apr}_S(X)$, it follows that $x_1/x_2^n \subseteq \underline{Apr}_S(X)$. Hence, $u \in \underline{Apr}_S(X)$ and $\mu_{\underline{Apr}_S(A)}(u) \geq t$, which means that $u \in \underline{A}_t$. Therefore, $\underline{A}_t$ is an n-ary subhypergroup of (H, f).

Now, let us consider $x_1^n \in \overline{A}_t$. For all i, we have $\mu_{\overline{Apr}_S(A)}(x_i) \geq t$, whence

$$r\min\left\{\widetilde{\mu}_A(x_1^n)\right\} \geq [0, t].$$

It follows that for all $z \in f(x_1^n)$, we have $\widetilde{\mu}_A(z) \geq [0, t]$ and there exists $u \in x_1/x_2^n$ such that $\widetilde{\mu}_A(u) \geq [0, t]$. Hence $\mu_{\overline{Apr}_S(A)}(z) \geq t$ and $\mu_{\overline{Apr}_S(A)}(u) \geq t$. Since $\overline{Apr}_S(X)$ is a closed n-ary subhypergroup we conclude that $z, u \in \overline{Apr}_S(X)$. Hence $z, u \in \overline{A}_t$. In other words, $\overline{A}_t$ is an n-ary subhypergroup of (H, f). □

Lemma 3.5. *If $Apr_S(A) = \big(\underline{Apr}_S(A), \overline{Apr}_S(A)\big)$ is a fuzzy rough n-ary subhypergroup, then for all $s, t \in [0,1]$, such that $\underline{A}_s \cap \overline{A}_t \neq \emptyset$, the intersection $\underline{A}_s \cap \overline{A}_t$ is an n-ary subhypergroup of (H, f).*

Proof. Let $s, t \in [0,1]$, such that $\underline{A}_s \cap \overline{A}_t \neq \emptyset$,. Let $x_1^n \in \underline{A}_s \cap \overline{A}_t$. It follows that $x_1^n \in \underline{Apr}_S(X)$. By hypothesis, there exists $u \in x_1/x_2^n$, such that $\widetilde{\mu}_A(u) \geq r\min\left\{\widetilde{\mu}_A(x_1^n)\right\}$. As above, we obtain that $u \in \underline{A}_s$. On the other hand, $\widetilde{\mu}_A(u) \geq \left[s, \min\left\{\mu_{\overline{Apr}_S(A)}(x_1^n)\right\}\right]$, whence $\mu_{\overline{Apr}_S(A)}(u) \geq \min\left\{\mu_{\overline{Apr}_S(A)}(x_1^n)\right\} \geq t$. Hence $u \in \overline{A}_t$. Similarly, for all $z \in f(x_1^n)$ we obtain $z \in \underline{A}_s \cap \overline{A}_t$. Therefore $\underline{A}_s \cap \overline{A}_t$ is an n-ary subhypergroup of (H, f). □

Lemma 3.6. *If for all $t \in [0,1]$, the pair $\big(\underline{A}_t, \overline{A}_t\big)$ is a level rough n-ary subhypergroup of (H, f) and for all $s, t \in [0,1]$, such that $\underline{A}_s \cap \overline{A}_t \neq \emptyset$ and the intersection $\underline{A}_s \cap \overline{A}_t$ is an n-ary subhypergroup of (H, f), then $Apr_S(A)$ is a fuzzy rough n-ary subhypergroup.*

Proof. For all $t \in [0,1]$, $\underline{A}_t$ and $\overline{A}_t$ are n-ary subhypergroups of (H,f). Let x_1^n be arbitrary in $\overline{Apr}_S(X)$. Denote $r\min\left\{\widetilde{\mu}_A(x_1),...,\widetilde{\mu}_A(x_n)\right\}$ by $[t_0,t_1]$. We have

$$\min\left\{\overline{\mu}_{\underline{Apr}_S(A)}(x_1^n)\right\} = t_0 \text{ and}$$
$$\min\left\{\mu_{\overline{Apr}_S(A)}(x_1^n)\right\} = t_1.$$

Since $x_1^n \in \overline{A}_{t_1}$, it follows that for all $z \in f(x_1^n)$, we have $z \in \overline{A}_{t_1}$. Now, we have the following cases:

1°. If there is i such that $x_i \in \widehat{Apr}_S(X)$, then $t_0 = 0$. For all $z \in f(x_1^n)$, we have $\overline{\mu}_{\underline{Apr}_S(A)}(z) \geq 0 = t_0$. On the other hand, from $x_1^n \in \overline{A}_{t_1}$, it follows that there exists $u \in x_1/x_2^n$, such that $u \in \overline{A}_{t_1}$. Moreover, $\overline{\mu}_{\underline{Apr}_S(A)}(u) \geq 0 = t_0$.

2°. If for all i, we have $x_i \in \underline{Apr}_S(X)$, then $\overline{\mu}_{\underline{Apr}_S(A)}(x_i) = \mu_{\underline{Apr}_S(A)}(x_i)$. We have $x_1^n \in \underline{A}_{t_0}$, whence $z \in \underline{A}_{t_0}$ for all $z \in f(x_1^n)$. Hence,

$$\widetilde{\mu}_A(z) = \left[\overline{\mu}_{\underline{Apr}_S(A)}(z), \mu_{\overline{Apr}_S(A)}(z)\right] \geq [t_0,t_1] = r\min\left\{\widetilde{\mu}_A(x_1^n)\right\}.$$

Now we show that there exists $u \in x_1/x_2^n$, such that $\widetilde{\mu}_A(u) \geq [t_0,t_1]$. Since $\overline{A}_{t_1} \cap \underline{A}_{t_0}$ is an n-ary subhypergroup of (H,f) and $x_1^n \in \overline{A}_{t_1} \cap \underline{A}_{t_0}$, it follows that there exists $u \in x_1/x_2^n$ such that $u \in \overline{A}_{t_1} \cap \underline{A}_{t_0}$. We obtain $\widetilde{\mu}_A(u) = \left[\mu_{\underline{Apr}_S(A)}(u), \mu_{\overline{Apr}_S(A)}(u)\right] \geq [t_0,t_1]$.

Therefore $Apr_S(A) = \left(\underline{Apr}_S(A), \overline{Apr}_S(A)\right)$ is a fuzzy rough n-ary subhypergroup. □

Theorem 3.7. *Let $Apr_S(X)$ be a rough closed n-ary subhypergroup of an n-ary hypergroup (H,f) and, for $A \subseteq X$, suppose that $Apr_S(A) = \left(\underline{Apr}_S(A), \overline{Apr}_S(A)\right)$ is a fuzzy rough set. Then the following conditions are equivalent:*

(1) *$Apr_S(A)$ is a fuzzy rough n-ary subhypergroup;*

(2) *for all $t \in [0,1]$, the pair $\left(\underline{A}_t, \overline{A}_t\right)$ is a level rough n-ary subhypergroup of (H,f) and for all $s,t \in [0,1]$, such that $\underline{A}_s \cap \overline{A}_t \neq \emptyset$, the intersection $\underline{A}_s \cap \overline{A}_t$ is an n-ary subhypergroup of (H,f).*

Proof. The proof follows immediately from the above three lemmas. □

Example 3.8. In Example 3.2.,2°, we have $\underline{A}_0 = \underline{Apr}_{K_1}(K_2)$ and for all $t > 0$, $\underline{A}_t = \underline{Apr}_{K_1}(A)$. Both $\underline{Apr}_{K_1}(K_2)$ and $\underline{Apr}_{K_1}(A)$ are n-ary subhypergroups of (L,f). By condition (β), we obtain that for all $s \in [0,1]$, $\overline{A}_s$ is an n-ary subhypergroup.

By condition (α), for all $s \in [0,1]$, $\underline{Apr}_{K_1}(A) \subseteq \overline{A}_s$, whence it follows that for all $s,t \in [0,1]$, $\underline{A}_t \cap \overline{A}_s \neq \emptyset$. We have $\underline{A}_0 \cap \overline{A}_s = \{x \in \underline{Apr}_{K_1}(K_2) \mid \mu_{\overline{Apr}_{K_1}(A)}(x) \geq s\}$ and for all $t,s \in [0,1]$, $\underline{A}_t \cap \overline{A}_s = \underline{Apr}_{K_1}(A) \cap \overline{A}_s = \{x \in \underline{Apr}_{K_1}(A) \mid \mu_{\overline{Apr}_{K_1}(A)}(x) \geq s\}$, which are also n-ary subhypergroups.

Therefore, it follows from the theorem that $Apr_{K_1}(A)$ is a fuzzy rough n-ary subhypergroup of (L,f).

Corollary 3.9. *A fuzzy subset A on an n-ary hypergroup (H, f) is a fuzzy n-ary subhypergroup if and only if each its non-empty level subset is an n-ary subhypergroup of (H, f).*

Proof. We apply the above theorem for a definable set A and $X = H$. then $\underline{Apr}_S(A) = \overline{Apr}_S(A) = A$ and $\underline{A}_t = \overline{A}_t = \{x \in X : \mu_A(x) \geq t\}$. □

4. Conclusion

The study of properties of fuzzy rough sets in the context of n ary-hypergroups is an new research topic of fuzzy set theory. The existing research on this topic deals only with fuzzy rough hyperstructures [15]and for this study, the approximations in n-ary hyperstructures are important. In this paper, we introduce and characterize fuzzy rough n-ary subhypergroups and give some examples. Our future work on this topic will be focused on the study of some particular classes of fuzzy rough n-ary subhypegroups, starting from the corresponding classes of n-ary subhypergroups.

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