BEST APPROXIMATION SETS IN α-n-NORMED SPACE CORRESPONDING TO INTUITIONISTIC FUZZY n-NORMED LINEAR SPACE

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Abstract. The aim of this paper is to present the new and interesting notion of ascending family of $\alpha - n$-norms corresponding to an intuitionistic fuzzy n-normed linear space. The notion of best aproximation sets in an $\alpha - n$-normed space corresponding to an intuitionistic fuzzy n-normed linear space is also defined and several related results are obtained.

1. Introduction

In [7, 8], S.Gähler introduced the theory of 2-norms and n-norms on a linear space. A systematic development of n-normed linear spaces is due to S.S. Kim and Y.J. Cho [10], R. Malceski [12], A. Misiak [13] and Hendra Gunawan [9]. In [9], Hendra Gunawan and Mashadi have given a simple method of deriving an $(n-1)$-norm from the n-norm and shown that any n-normed space is also an $(n-1)$-normed space. A detailed theory of fuzzy normed linear spaces can be found in [3, 4, 5, 6, 11]. In [14], we have extended n-normed linear spaces to fuzzy n-normed linear spaces. Intuitionistic fuzzy sets have been studied in detail in [1, 2] and recently we have introduced the notion of intuitionistic fuzzy n-normed linear spaces [15].

The purpose of this paper is to generalize the notion of intuitionistic fuzzy n-normed linear spaces [15]. In this regard, the notion of ascending families of $\alpha - n$-norms corresponding to an intuitionistic fuzzy n-normed linear space, which is a generalization of ascending families of $\alpha - n$-norms corresponding to fuzzy n-norms [14], is introduced. Furthermore, the concept of best approximation sets in an $\alpha - n$-normed space corresponding to an intuitionistic fuzzy n-normed linear space is considered as a generalization of best approximation sets in α-n-normed linear space [14].

2. Preliminaries

Definition 2.1. [9] Let $n \in N$(natural numbers) and X be a real linear space of dimension $d \geq n$. A real valued function $||\bullet, ..., \bullet||$ on $\underbrace{X \times ... \times X}_{n} = X^n$, satisfying the following conditions is called an n-norm on X.

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(1) $||x_1, x_2, ..., x_n|| = 0$ if any only if $x_1, x_2, ..., x_n$ are linearly dependent
(2) $||x_1, x_2, ..., x_n||$ is invariant under any permutation
(3) $||x_1, x_2, ..., kx_n|| = |k| \; ||x_1, x_2, ..., x_n||$, for any $k \in R$(set of real numbers)
(4) $||x_1, x_2, ..., x_{n-1}, y + z|| \leq ||x_1, x_2, ..., x_{n-1}, y|| + ||x_1, x_2, ..., x_{n-1}, z||$

The pair $(X, ||\bullet, ..., \bullet||)$ is called an n-normed linear space.

Definition 2.2. [14] Let X be a linear space over a field F. A fuzzy subset N of $X^n \times R$ (set of real numbers) is called a fuzzy n-norm on X if and only if:

$(N1)$ For all $t \in R$ with $t \leq 0$, $N(x_1, x_2, ..., x_n, t) = 0$
$(N2)$ For all $t \in R$ with $t > 0$, $N(x_1, x_2, ..., x_n, t) = 1$ if and only if $x_1, x_2, ..., x_n$ are linearly dependent
$(N3)$ $N(x_1, x_2, ..., x_n, t)$ is invariant under any permutation of $x_1, x_2, ..., x_n$
$(N4)$ For all $t \in R$ with $t > 0$,
$N(x_1, x_2, ..., cx_n, t) = N(x_1, x_2, ..., x_n, \frac{t}{|c|})$, if $c \neq 0, c \in F$(field).
$(N5)$ For all $s, t \in R$
$N(x_1, x_2, ..., x_n + x_n^{'}, s + t) \geq \min\Big\{N(x_1, x_2, ..., x_n, s), N(x_1, x_2, ..., x_n^{'}, t)\Big\}$
$(N6)$ $N(x_1, x_2, ..., x_n, t)$ is a non-decreasing function of $t \in R$
and $\lim_{t \to \infty} N(x_1, x_2, ..., x_n, t) = 1$

Then (X, N) is called a fuzzy n-normed linear space (in short f-n-NLS).

Theorem 2.3. [14] Let (X, N) be a f-n-NLS such that that

$(N7)$ $N(x_1, x_2, ..., x_n, t) > 0$ for all $t > 0$ implies $x_1, x_2, ..., x_n$ are linearly dependent.
Define $||x_1, x_2, ..., x_n||_\alpha$=inf$\{t :$ N$(x_1, x_2, ..., x_n, t) \geq \alpha\}$,$\alpha \in$(0,1).

Then $\{||\bullet, \bullet, ..., \bullet||_\alpha : \alpha \in (0, 1)\}$ is an ascending family of n-norms on X. These n-norms will be called the $\alpha - n$-norms on X corresponding to the fuzzy n-norm on X.

Definition 2.4. [15] An intuitionistic fuzzy n-normed linear space (in short i-f-n-NLS) is an object of the form

$A = \{(X, N(x_1, x_2, ..., x_n, t), M(x_1, x_2, ..., x_n, t))|(x_1, x_2, ..., x_n) \in X^n\}$,
where X is a linear space over a field F, $*$ is a continuous t-norm, $\diamond$ is a continuous t-co-norm and N, M are fuzzy sets on $X^n \times (0, \infty)$. N and M respectively denote the degree of membership and the degree of non-membership of $(x_1, x_2, ..., x_n, t) \in X^n \times (0, \infty)$, and satisfy the following conditions:

(1) $N(x_1, x_2, ..., x_n, t) + M(x_1, x_2, ..., x_n, t) \leq 1$
(2) $N(x_1, x_2, ..., x_n, t) > 0$

(3) $N(x_1, x_2, ..., x_n, t) = 1$ if and only if $x_1, x_2, ..., x_n$ are linearly dependent
(4) $N(x_1, x_2, ..., x_n, t)$ is invariant under any permutation of $x_1, x_2, ..., x_n$
(5) $N(x_1, x_2, ..., cx_n, t) = N(x_1, x_2, ..., x_n, \frac{t}{|c|})$, if $c \neq 0, c \in F$(field)
(6) $N(x_1, x_2, ..., x_n, s) * N(x_1, x_2, ..., x_n^{'}, t) \leq N(x_1, x_2, ..., x_n + x_n^{'}, s + t)$
(7) $N(x_1, x_2, ..., x_n, t) : (0, \infty) \to [0, 1]$ is continuous in t
(8) $M(x_1, x_2, ..., x_n, t) > 0$
(9) $M(x_1, x_2, ..., x_n, t) = 0$ if and only if $x_1, x_2, ..., x_n$ are linearly dependent
(10) $M(x_1, x_2, ..., x_n, t)$ is invariant under any permutation of $x_1, x_2, ..., x_n$
(11) $M(x_1, x_2, ..., cx_n, t) = M(x_1, x_2, ..., x_n, \frac{t}{|c|})$, if $c \neq 0, c \in F$(field)
(12) $M(x_1, x_2, ..., x_n, s) \diamond M(x_1, x_2, ..., x_n^{'}, t) \geq M(x_1, x_2, ..., x_n + x_n^{'}, s + t)$
(13) $M(x_1, x_2, ..., x_n, t) : (0, \infty) \to [0, 1]$ is continuous in t.

3. Intuitionistic Fuzzy n-normed Linear Space

The following example illustrates the concept in Definition 2.4 .

Example 3.1. Let $(X, ||\bullet, \bullet, ..., \bullet||)$ be an n-normed linear space.
Define

$$\mathrm{N}(x_1, x_2, ..., x_n, t) = \begin{cases} \dfrac{t}{t + ||x_1, x_2, ..., x_n||}, & \text{when } t > 0, t \in R, \\ & (x_1, x_2, ..., x_n) \in X^n \\ 0, & \text{when } t \leq 0. \end{cases}$$

and

$$\mathrm{M}(x_1, x_2, ..., x_n, t) = \begin{cases} \dfrac{||x_1, x_2, ..., x_n||}{t + ||x_1, x_2, ..., x_n||}, & \text{when } t > 0, t \in R, \\ & (x_1, x_2, ..., x_n) \in X^n \\ 1, & \text{when } t \leq 0. \end{cases}$$

Then
$A = \{(X, N(x_1, x_2, ..., x_n, t), M(x_1, x_2, ..., x_n, t)) | (x_1, x_2, ..., x_n) \in X^n\}$
is an i-f-n-NLS.

Remark 3.2. If we define

$$\mathrm{N}(x_1, x_2, ..., x_n, t) = \begin{cases} \dfrac{t}{t + k||x_1, x_2, ..., x_n||}, & \text{when } t > 0, t \in R, \\ & (x_1, x_2, ..., x_n) \in X^n, \\ & \text{for any real constant } k > 1 \\ 0, & \text{when } t \leq 0. \end{cases}$$

and

$$\mathrm{M}(x_1, x_2, ..., x_n, t) = \begin{cases} \dfrac{||x_1, x_2, ..., x_n||}{t + ||x_1, x_2, ..., x_n||}, & \text{when } t > 0, t \in R, \\ & (x_1, x_2, ..., x_n) \in X^n \\ 1, & \text{when } t \leq 0. \end{cases}$$

in Example 3.2, then A is an i-f-n-NLS with $N + M < 1$ for $t > 0$.

Definition 3.3. Let A be an i-f-n-NLS. Suppose that
(14) for all $t > 0$ $N(x_1, x_2, ..., x_n, t) > 0$ implies $x_1, x_2, ..., x_n$ are linearly dependent.
then the n-norms
$||x_1, x_2, ..., x_n||_\alpha = \inf\{t : \mathrm{N}(x_1, x_2, ..., x_n, t) \geq \alpha\}$, $\alpha \in (0,1)$.
are called α-n-norms on X corresponding to the i-f-n-NLS A.

Theorem 3.4. $\{||\bullet, \bullet, ..., \bullet||_\alpha : \alpha \in (0, 1)\}$ in the above definition is an ascending family of n-norms on X.

Proof.
(1)$||x_1, x_2, ..., x_n||_\alpha = 0$
$\Rightarrow \inf\{t : N(x_1, x_2, ..., x_n, t) \geq \alpha,\ M(x_1, x_2, ..., x_n, t) \leq 1 - \alpha\} = 0$.
$\Rightarrow$ For all $t \in$R, $t > 0, N(x_1, x_2, ..., x_n, t) \geq \alpha > 0$ and
$M(x_1, x_2, ..., x_n, t) \leq 1 - \alpha < 1,\ \alpha \in (0, 1)$.
$\Rightarrow$ By (13) $x_1, x_2, ..., x_n$ are linearly dependent.
Conversely, assume that $x_1, x_2, ..., x_n$ are linearly dependent.
By (3) and (4), $N(x_1, x_2, ..., x_n, t) = 1$ and $M(x_1, x_2, ..., x_n, t) = 0$,
for all $t > 0$.
$\Rightarrow$ For all $\alpha \in (0, 1)$,
$\inf\{t : N(x_1, x_2, ..., x_n, t) \geq \alpha,\ M(x_1, x_2, ..., x_n, t) \leq 1 - \alpha\} = 0$.
$\Rightarrow ||x_1, x_2, ..., x_n||_\alpha = 0$.
(2) As $N(x_1, x_2, ..., x_n, t)$ and $M(x_1, x_2, ..., x_n, t)$ are invariant under any permutation of $x_1, x_2, ..., x_n$, it follows that $||x_1, x_2, ..., x_n||_\alpha$ is invariant under any permutation.

(3) If $c \neq 0$ then,
$||x_1, x_2, ..., cx_n||_\alpha$
$= \inf\{s : N(x_1, x_2, ..., cx_n, s) \geq \alpha,\ M(x_1, x_2, ..., cx_n, s) \leq 1 - \alpha\}$
$= \inf\left\{s : N(x_1, x_2, ..., x_n, \frac{s}{|c|}) \geq \alpha,\ M(x_1, x_2, ..., x_n, \frac{s}{|c|}) \leq 1 - \alpha\right\}$.
Let $t = \dfrac{s}{|c|}$ then,

$||x_1, x_2, ..., cx_n||_\alpha$
$= \inf\{\,|c|\,t : N(x_1, x_2, ..., x_n, t) \geq \alpha\ , M(x_1, x_2, ..., x_n, t) \leq 1 - \alpha\}$
$= |c|\ \inf\{t : N(x_1, x_2, ..., x_n, t) \geq \alpha\ , M(x_1, x_2, ..., x_n, t) \leq 1 - \alpha\}$
$= |c|\ ||x_1, x_2, ..., x_n||_\alpha$.

If $c = 0$ then,

$$\begin{aligned}||x_1, x_2, ..., cx_n||_\alpha \\ &= ||x_1, x_2, ..., 0||_\alpha \\ &= 0 \\ &= 0||x_1, x_2, ..., x_n||_\alpha \\ &= |c|\ ||x_1, x_2, ..., x_n||_\alpha, \text{ for all } c \in F(\text{field}).\end{aligned}$$

$$\begin{aligned}(4)&||x_1, x_2, ..., x_n||_\alpha + ||x_1, x_2, ..., x_n^{'}||_\alpha \\ &= \inf\{t : N(x_1, x_2, ..., x_n, t) \geq \alpha,\ M(x_1, x_2, ..., x_n, t) \leq 1 - \alpha\} \\ &\quad + \inf\Big\{s : N(x_1, x_2, ..., x_n^{'}, s) \geq \alpha\ , M(x_1, x_2, ..., x_n^{'}, t) \leq 1 - \alpha\Big\} \\ &= \inf\{t + s : N(x_1, x_2, ..., x_n, t) \geq \alpha, N(x_1, x_2, ..., x_n^{'}, s) \geq \alpha \text{ and} \\ &\qquad M(x_1, x_2, ..., x_n, t) \leq 1 - \alpha, M(x_1, x_2, ..., x_n^{'}, t) \leq 1 - \alpha\} \\ &\geq \inf\{t + s : N(x_1, x_2, ..., x_n + x_n^{'}, t + s) \geq \alpha \\ &\qquad M(x_1, x_2, ..., x_n + x_n^{'}, t + s) \leq 1 - \alpha\} \\ &\geq \inf\Big\{r : N(x_1, x_2, ..., x_n + x_n^{'}, r) \geq \alpha\ M(x_1, x_2, ..., x_n + x_n^{'}, r) \leq 1 - \alpha\Big\}, \\ &\qquad\qquad r = t + s \\ &= ||x_1, x_2, ..., x_n + x_n^{'}||_\alpha.\end{aligned}$$

Therefore,

$$||x_1, x_2, ..., x_n + x_n^{'}||_\alpha \leq ||x_1, x_2, ..., x_n||_\alpha + ||x_1, x_2, ..., x_n^{'}||_\alpha.$$

Thus $\{||\bullet, \bullet, ..., \bullet||_\alpha : \alpha \in (0, 1)\}$ is an α-n-norm on X.
Let $0 < \alpha_1 < \alpha_2$. Then

$$\begin{aligned}||x_1, x_2, ..., x_n||_{\alpha_1} &= \inf\{t : N(x_1, x_2, ..., x_n, t) \geq \alpha_1, \\ &\qquad M(x_1, x_2, ..., x_n, t) \leq 1 - \alpha_1\} \\ ||x_1, x_2, ..., x_n||_{\alpha_2} &= \inf\{t : N(x_1, x_2, ..., x_n, t) \geq \alpha_2, \\ &\qquad M(x_1, x_2, ..., x_n, t) \leq 1 - \alpha_2\}.\end{aligned}$$

As $\alpha_1 < \alpha_2$,

$$\begin{aligned}&\{t : N(x_1, x_2, ..., x_n, t) \geq \alpha_2, M(x_1, x_2, ..., x_n, t) \leq 1 - \alpha_2\} \\ &\qquad \subset \{t : N(x_1, x_2, ..., x_n, t) \geq \alpha_1, M(x_1, x_2, ..., x_n, t) \leq 1 - \alpha_1\} \\ &\Rightarrow \inf\{t : N(x_1, x_2, ..., x_n, t) \geq \alpha_2, M(x_1, x_2, ..., x_n, t) \leq 1 - \alpha_2\} \\ &\qquad \geq \inf\{t : N(x_1, x_2, ..., x_n, t) \geq \alpha_1, M(x_1, x_2, ..., x_n, t) \leq 1 - \alpha_1\} \\ &\Rightarrow ||x_1, x_2, ..., x_n||_{\alpha_2} \geq ||x_1, x_2, ..., x_n||_{\alpha_1}.\end{aligned}$$

Hence $\{||\bullet, \bullet, ..., \bullet||_\alpha : \alpha \in (0, 1)\}$ is an ascending family of $\alpha - n$-norms on X corresponding to the i-f-n-NLS A. □

4. Best Approximation Sets in α-n-normed Space

Remark 4.1. Let $(X, ||\bullet, \bullet, ..., \bullet||_\alpha)$ be an $\alpha - n$-normed space corresponding to the i-f-n-NLS A. Let G be an arbitrary non-empty subset of X and $x_0 \in X$. Then for every $x \in X$ and for every set $x_2, x_3, ..., x_n \subset X \backslash G$ independent of x and x_0,we have

$$d_{x_2,x_3,...,x_n}(x, G) \leq ||x - x_0, x_2, x_3, ..., x_n||_\alpha + d_{x_2,x_3,...,x_n}(x_0, G),$$

where

$$d_{x_2,x_3,\ldots,x_n}(x,G) = \inf_{g\in G} ||x-g,x_2,x_3,...,x_n||_\alpha. \quad (4.1)$$

For each $G \subset$X and $x_0 \in X$.

Definition 4.2. For $x_0 \in G$ and $x_2, x_3, ..., x_n \subset X \backslash G$ which is independent of x and x_0, we define

$$D_{x_2,x_3,\ldots,x_n}(x_0,G) = \{x \in X : d_{x_2,x_3,\ldots,x_n}(x,G) = ||x-x_0,x_2,x_3,...,x_n||_\alpha + d_{x_2,x_3,\ldots,x_n}(x_0,G)\} \quad (4.2)$$

for any $x_2, x_3, ..., x_n \in X\backslash G$ which are independent of x and x_0.

We write: $P_{G,x_2,x_3,\ldots,x_n}(x) = \{g_0 \in G : ||x-g_0,x_2,x_3,...,x_n||_\alpha = d_{x_2,x_3,\ldots,x_n}(x,G)\}$ and

$P^{-1}_{G,x_2,x_3,\ldots,x_n}(x_0) = \{x \in \mathrm{X} : ||x-x_0,x_2,x_3,...,x_n||_\alpha = d_{x_2,x_3,\ldots,x_n}(x,G)\}$.

Example 4.2. Let $X = R^3$ be a linear space over R.

Define $||\bullet,\bullet||$: $X \times X \to R$ by

$||x_1,x_2||_1 = \max\{|a_1b_2 - a_2b_1|, |b_1c_2 - b_2c_1|, |a_1c_2 - a_2c_1|\}$ and

$||x_1,x_2||_2 = \frac{1}{2}\{\max\{|a_1b_2 - a_2b_1|, |b_1c_2 - b_2c_1|, |a_1c_2 - a_2c_1|\}\}$,

where $x_i = (a_i, b_i, c_i) \in \mathrm{R}^3$; i=1,2.

Then $(X, ||\bullet,\bullet||_1)$ and $(X, ||\bullet,\bullet||_2)$ are 2-normed linear spaces.

Define $N : X \times X \times R \to [0,1]$ by

$$N(x_1,x_2,t) = \begin{cases} 1, & \text{if } t > ||x_1,x_2||_1 \\ 0.5, & \text{if } ||x_1,x_2||_2 < t \le ||x_1,x_2||_1 \\ 0, & \text{if } t \le ||x_1,x_2||_2. \end{cases}$$

and $M : X \times X \times R \to [0,1]$ by

$$M(x_1,x_2,t) = \begin{cases} 0, & \text{if } t > ||x_1,x_2||_1 \\ 0.5, & \text{if } ||x_1,x_2||_2 < t \le ||x_1,x_2||_1 \\ 1, & \text{if } t \le ||x_1,x_2||_2. \end{cases}$$

Then A is an i-f-2-NLS.

Define $||x_1,x_2||_\alpha = \inf\{t : N(x_1,x_2,t) \ge \alpha, M(x_1,x_2,t) \le 1-\alpha\}$, $\alpha \in$(0,1).

The α-2-norms are given by

$$\begin{aligned} ||x_1,x_2||_\alpha &= ||x_1,x_2||_1, \text{ when } 1 > \alpha > 0.5, \\ &= ||x_1,x_2||_2, \text{ when } 0 < \alpha \le 0.5. \end{aligned}$$

Let G=$\{(a,0,0) : a \in R\}$ be a subset of X.

Choose $x_0 = (0,1,1)$ and $x_2 \in$K=$\{(0,0,k) : k \in$ R$\setminus \{0\}\}$.

Then
$D_{x_2}(x_0, G) = \{x = (0,b,0), b \in R^+ \setminus \{0\} : d_{x_2}(x,G) = ||x - x_0, x_2||_\alpha + d_{x_2}(x_0,G)\}$

and

$$P_{G,x_2}(x) = \{g = (a,0,0) : -1 \leq a \leq 1\}.$$

Example 4.3. Let X =R^{n+1} be a linear space over R.

Define $||\bullet, \bullet, ..., \bullet||$: $X^n \to R$ by
$||x_1, x_2, ..., x_n||_1 = \max\{\Delta_1, \Delta_2, ..., \Delta_n\}$ and
$||x_1, x_2, ..., x_n||_2 = \frac{1}{2}\{\max\{\Delta_1, \Delta_2, ..., \Delta_n\}\}$,

where $\Delta_1 = \begin{vmatrix} a_{12} & a_{13} & \cdots & a_{1(n+1)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n2} & a_{n3} & \cdots & a_{n(n+1)} \end{vmatrix}$, $\Delta_2 = \begin{vmatrix} a_{13} & \cdots & a_{1(n+1)} & a_{11} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n3} & \cdots & a_{n(n+1)} & a_{n1} \end{vmatrix}$,
..., $\Delta_n = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$ and

$x_i = (a_{i1}, a_{i2}, ..., a_{i(n+1)}) \in$ R^{n+1}, $i = 1, 2, ..., n$.

Then $(X, ||\bullet, \bullet, ..., \bullet||_1)$ and $(X, ||\bullet, ..., \bullet, \bullet||_2)$ are n-normed linear spaces.

Define $N : X^n \times$ R $\to$[0,1] and $M : X^n \times$ R $\to$[0,1] as follows:

$$N(x_1, x_2, ..., x_n, t) = \begin{cases} 1, & \text{if } t > ||x_1, x_2, ..., x_n||_1 \\ 0.5, & \text{if } ||x_1, x_2, ..., x_n||_2 < t \leq ||x_1, x_2, ..., x_n||_1 \\ 0, & \text{if } t \leq ||x_1, x_2, ..., x_n||_2. \end{cases}$$

and

$$M(x_1, x_2, ..., x_n, t) = \begin{cases} 0, & \text{if } t > ||x_1, x_2, ..., x_n||_1 \\ 0.5, & \text{if } ||x_1, x_2, ..., x_n||_2 < t \leq ||x_1, x_2, ..., x_n||_1 \\ 1, & \text{if } t \leq ||x_1, x_2, ..., x_n||_2. \end{cases}$$

Then A is an i-f-n-NLS.

Define $||x_1, x_2, ..., x_n||_\alpha$=inf$\{t : N(x_1, x_2, ..., x_n, t) \geq \alpha, M(x_1, x_2, ..., x_n, t) \leq 1 - \alpha\}$,
$\alpha \in$(0,1).

The $\alpha - n$-norms are given by
$||x_1, x_2, ..., x_n||_\alpha$
$= ||x_1, x_2, ..., x_n||_1$,when $1 > \alpha > 0.5$,

$= ||x_1, x_2, ..., x_n||_2$,when $0 < \alpha \leq 0.5$

Let G=$\{(a, 0, 0, ...$n times$) : a \in$ R$\}$ be a subset of X.

Choose $x_0 = (0, 1, 1, ...,$ n times) and

$$x_2, x_3, ..., x_n \in \text{K}= \left\{(0, 0, k_3^{(i)}, .., k_{n+1}^{(i)}) : k_3^{(i)}, ..., k_{n+1}^{(i)} \in \text{R}\setminus \{0\}\right\}.$$

In other words, $x_2 = (0, 0, k_3^{(2)}, ..., k_{n+1}^{(2)})$,

$$x_3 = (0, 0, k_3^{(3)}, ..., k_{n+1}^{(3)}), ..., x_n = (0, 0, k_3^{(n)}, ..., k_{n+1}^{(n)})$$

Then,

$D_{x_2,x_3,...,x_n}(x_0, G) = \{x = (0, b, 0, ..., \ (n\text{-}1) \text{ times}), b \in \ \text{R}^+\setminus \{0\} :$
$d_{x_2,x_3,...,x_n}(x, G) = ||x - x_0, x_2, x_3, ..., x_n||_\alpha + d_{x_2,x_3,...,x_n}(x_0, G)\}$

where $d_{x_2,x_3,...,x_n}(x, G) = \max\{|b|\Delta, |a|\Delta\}$,

$$\Delta = \begin{vmatrix} k_3^{(2)} & k_4^{(2)} & \cdots & k_{n+1}^{(2)} \\ k_3^{(3)} & k_4^{(3)} & \cdots & k_{n+1}^{(3)} \\ \vdots & \vdots & \ddots & \vdots \\ k_3^{(n)} & k_4^{(n)} & \cdots & k_{n+1}^{(n)} \end{vmatrix},$$

$$||x - x_0, x_2, ..., x_n||_\alpha = |b - 1|\Delta,$$

$$d_{x_2,x_3,...,x_n}(x_0, G) = \max\{\Delta, |a|\Delta\}$$

and also

$$P_{G,x_2,x_3,...,x_n}(x) = \{g = (a, 0, ..., \text{n times}) : -1 \leq a \leq 1\}.$$

Theorem 4.4. For $x \in D_{x_2,x_3,...,x_n}(x_0, G)$ and $y \in D_{x_2,x_3,...,x_n}(x, G)$ we have:
(i) $||y - x_0, x_2, x_3, ..., x_n||_\alpha = ||y - x, x_2, x_3, ..., x_n||_\alpha + ||x - x_0, x_2, x_3, ..., x_n||_\alpha$
(ii) $y - x + x_0 \in D_{x_2,x_3,...,x_n}(x_0, G)$.

Proof. (i) Let $x \in D_{x_2,x_3,...,x_n}(x_0, G)$ and $y \in D_{x_2,x_3,...,x_n}(x, G)$.
Then by (4.2) we have
$d_{x_2,x_3,...,x_n}(x, G) = ||x - x_0, x_2, x_3, ..., x_n||_\alpha + d_{x_2,x_3,...,x_n}(x_0, G)$,
$d_{x_2,x_3,...,x_n}(y, G) = ||y - x, x_2, x_3, ..., x_n||_\alpha + d_{x_2,x_3,...,x_n}(x, G)$.
Now
$||y - x_0, x_2, x_3, ..., x_n||_\alpha$
$= ||y - x_0 - x + x, x_2, x_3, ..., x_n||_\alpha$
$= ||(y - x) + (x - x_0), x_2, x_3, ..., x_n||_\alpha$

$$\begin{aligned}
&\le ||y-x,x_2,x_3,...,x_n||_\alpha + ||x-x_0,x_2,x_3,...,x_n||_\alpha \\
&= (d_{x_2,x_3,...,x_n}(y,G) - d_{x_2,x_3,...,x_n}(x,G)) + (d_{x_2,x_3,...,x_n}(x,G) \\
&\qquad -d_{x_2,x_3,...,x_n}(x_0,G)) \\
&= d_{x_2,x_3,...,x_n}(y,G) - d_{x_2,x_3,...,x_n}(x_0,G) \\
&\le ||y-x_0,x_2,x_3,...,x_n||_\alpha .
\end{aligned}$$

Therefore,

$$||y-x_0,x_2,x_3,...,x_n||_\alpha = ||y-x,x_2,x_3,...,x_n||_\alpha + ||x-x_0,x_2,x_3,...,x_n||_\alpha .$$

(ii) By (4.1), we have

$$\begin{aligned}
&d_{x_2,x_3,...,x_n}(y-x+x_0,G) \\
&\ge d_{x_2,x_3,...,x_n}(y,G) - ||y-(y-x+x_0),x_2,x_3,...,x_n||_\alpha \\
&= d_{x_2,x_3,...,x_n}(y,G) - ||x-x_0,x_2,x_3,...,x_n||_\alpha \\
&= (||y-x,x_2,x_3,...,x_n||_\alpha + d_{x_2,x_3,...,x_n}(x,G)) - ||x-x_0,x_2,x_3,...,x_n||_\alpha \\
&= ||y-x,x_2,x_3,...,x_n||_\alpha + (||x-x_0,x_2,x_3,...,x_n||_\alpha + d_{x_2,x_3,...,x_n}(x_0,G)) \\
&\qquad -||x-x_0,x_2,x_3,...,x_n||_\alpha \\
&= ||y-x,x_2,x_3,...,x_n||_\alpha + d_{x_2,x_3,...,x_n}(x_0,G) \\
&= ||(y-x+x_0)-x_0,x_2,x_3,...,x_n||_\alpha + d_{x_2,x_3,...,x_n}(x_0,G).
\end{aligned}$$

Again it follows from (4.1) that

$$\begin{aligned}
&d_{x_2,x_3,...,x_n}(y-x+x_0,G) = ||(y-x+x_0)-x_0,x_2,x_3,...,x_n||_\alpha + d_{x_2,x_3,...,x_n}(x_0,G) \\
&\qquad \Rightarrow y-x+x_0 \in D_{x_2,x_3,...,x_n}(x_0,G).
\end{aligned}$$

□

Theorem 4.5. Let $x \in D_{x_2,x_3,...,x_n}(x_0,G)$. Then
(i) $[x_0,x] = \{\lambda x_0 + (1-\lambda)x : 0 \le \lambda \le 1\} \subset D_{x_2,x_3,...,x_n}(x_0,G)$
(ii) $D_{x_2,x_3,...,x_n}(x,G) \subset D_{x_2,x_3,...,x_n}(x_0,G)$.

Proof. (i) Let $y = \lambda x_0 + (1-\lambda)x$ such that $0 \le \lambda \le 1$.

Then

$$\begin{aligned}
&d_{x_2,x_3,...,x_n}(y,G) \\
&\ge d_{x_2,x_3,...,x_n}(x,G) - ||x-y,x_2,x_3,...,x_n||_\alpha \\
&= ||x-x_0,x_2,x_3,...,x_n||_\alpha + d_{x_2,x_3,...,x_n}(x_0,G) - ||x-y,x_2,x_3,...,x_n||_\alpha \\
&= ||y-x_0,x_2,x_3,...,x_n||_\alpha + d_{x_2,x_3,...,x_n}(x_0,G).
\end{aligned}$$

By (4.1) we have

$$\begin{aligned}
&d_{x_2,x_3,...,x_n}(y,G) = ||y-x_0,x_2,x_3,...,x_n||_\alpha + d_{x_2,x_3,...,x_n}(x_0,G) \\
&\qquad \Rightarrow y \in D_{x_2,x_3,...,x_n}(x_0,G).
\end{aligned}$$

(ii) Let $y \in D_{x_2,x_3,...,x_n}(x,G)$.

Thus by (4.2) and Theorem 4.4(i),

$$\begin{aligned}
&d_{x_2,x_3,...,x_n}(y,G) \\
&= ||y-x,x_2,x_3,...,x_n||_\alpha + d_{x_2,x_3,...,x_n}(x,G) \\
&= ||y-x,x_2,x_3,...,x_n||_\alpha + (||x-x_0,x_2,x_3,...,x_n||_\alpha + d_{x_2,x_3,...,x_n}(x_0,G))
\end{aligned}$$

$= ||y - x_0, x_2, x_3, ..., x_n||_\alpha + d_{x_2,x_3,...,x_n}(x_0, G)$
$\Rightarrow y \in D_{x_2,x_3,...,x_n}(x_0, G)$.

Therefore $D_{x_2,x_3,...,x_n}(x, G) \subset D_{x_2,x_3,...,x_n}(x_0, G)$. □

Theorem 4.6. Let $x_0, y_0 \in$ X and $\lambda \neq 0$. Then

(i) $D_{x_2,x_3,...,x_n}(x_0, G) + y_0 = D_{x_2,x_3,...,x_n}(x_0 + y_0, G + y_0)$
(ii) $D_{x_2,x_3,...,x_n}(x_0, \lambda G) = \lambda D_{x_2,x_3,...,x_n}(\frac{x_0}{\lambda}, G)$.

Proof. (i) Let $x \in D_{x_2,x_3,...,x_n}(x_0, G)$. Then

$$\begin{aligned}
&d_{x_2,x_3,...,x_n}(x + y_0, G + y_0)\\
&= d_{x_2,x_3,...,x_n}(x, G)\\
&= ||x - x_0, x_2, x_3, ..., x_n||_\alpha + d_{x_2,x_3,...,x_n}(x_0, G)\\
&= ||x + y_0 - (x_0 + y_0), x_2, x_3, ..., x_n||_\alpha + d_{x_2,x_3,...,x_n}(x_0, G).
\end{aligned}$$

Therefore $x + y_0 \in D_{x_2,x_3,...,x_n}(x_0 + y_0, G + y_0)$.

Conversely, let $y \in D_{x_2,x_3,...,x_n}(x_0 + y_0, G + y_0)$.

Then,

$$\begin{aligned}
&d_{x_2,x_3,...,x_n}(y - y_0, G)\\
&= d_{x_2,x_3,...,x_n}(y, G + y_0)\\
&= ||y - y_0 - x_0, x_2, x_3, ..., x_n||_\alpha + d_{x_2,x_3,...,x_n}(x_0 + y_0, G + y_0)\\
&= ||(y - y_0) - x_0, x_2, x_3, ..., x_n||_\alpha + d_{x_2,x_3,...,x_n}(x_0, G).
\end{aligned}$$

Therefore $y - y_0 \in D_{x_2,x_3,...,x_n}(x_0, G)$ and so

$D_{x_2,x_3,...,x_n}(x_0, G) + y_0 = D_{x_2,x_3,...,x_n}(x_0 + y_0, G + y_0)$.

(ii) Let $x \in D_{x_2,x_3,...,x_n}(x_0, \lambda G)$. Then

$$\begin{aligned}
&d_{x_2,x_3,...,x_n}(\tfrac{x}{\lambda}, G)\\
&= \tfrac{1}{|\lambda|} d_{x_2,x_3,...,x_n}(x, \lambda G)\\
&= \tfrac{1}{|\lambda|}(||x - x_0, x_2, x_3, ..., x_n||_\alpha + d_{x_2,x_3,...,x_n}(x_0, \lambda G))\\
&= ||\tfrac{x}{\lambda} - \tfrac{x_0}{\lambda}, x_2, x_3, ..., x_n||_\alpha + d_{x_2,x_3,...,x_n}(\tfrac{x_0}{\lambda}, G).
\end{aligned}$$

Therefore $\frac{x}{\lambda} \in D_{x_2,x_3,...,x_n}(\frac{x_0}{\lambda}, G)$.

Conversely, let $x \in D_{x_2,x_3,...,x_n}(\frac{x_0}{\lambda}, G)$.

Then

$$\begin{aligned}
&d_{x_2,x_3,...,x_n}(\lambda x, \lambda G)\\
&= |\lambda| d_{x_2,x_3,...,x_n}(x, G)\\
&= |\lambda|(||x - \tfrac{x_0}{\lambda}, x_2, x_3, ..., x_n||_\alpha + d_{x_2,x_3,...,x_n}(\tfrac{x_0}{\lambda}, G))\\
&= ||\lambda x - x_0, x_2, x_3, ..., x_n||_\alpha + d_{x_2,x_3,...,x_n}(x_0, \lambda G).
\end{aligned}$$

Therefore $\lambda x \in D_{x_2,x_3,...,x_n}(x_0, \lambda G)$.

Thus $D_{x_2,x_3,...,x_n}(x_0, \lambda G) = \lambda D_{x_2,x_3,...,x_n}(\frac{x_0}{\lambda}, G)$. □

Theorem 4.7. Let $G \subset G_1$ and $x_0 \in$ X,where G_1 is a subset of X such that

$$d_{x_2,x_3,...,x_n}(x_0, G) = d_{x_2,x_3,...,x_n}(x_0, G_1). \tag{4.3}$$

Then $D_{x_2,x_3,...,x_n}(x_0, G_1) \subset D_{x_2,x_3,...,x_n}(x_0, G)$.

Proof. Let $x \in D_{x_2,x_3,...,x_n}(x_0, G_1)$.

By (4.3) we have,

$$\begin{aligned}
&d_{x_2,x_3,...,x_n}(x, G)\\
&\quad \geq d_{x_2,x_3,...,x_n}(x, G_1)\\
&\quad = ||x - x_0, x_2, x_3, ..., x_n||_\alpha + d_{x_2,x_3,...,x_n}(x_0, G_1)\\
&\quad = ||x - x_0, x_2, x_3, ..., x_n||_\alpha + d_{x_2,x_3,...,x_n}(x_0, G).
\end{aligned}$$

From (4.1) it follows that

$$\begin{aligned}
d_{x_2,x_3,...,x_n}(x, G) = ||x - x_0, x_2, x_3, ..., x_n||_\alpha + d_{x_2,x_3,...,x_n}(x_0, G)\\
\Rightarrow x \in D_{x_2,x_3,...,x_n}(x_0, G).
\end{aligned}$$

Hence $D_{x_2,x_3,...,x_n}(x_0, G_1) \subset D_{x_2,x_3,...,x_n}(x_0, G)$. □

Theorem 4.8. (i) $P_{G,x_2,x_3,...,x_n}(x_0) \subset P_{G,x_2,x_3,...,x_n}(x)$, for every $x \in D_{x_2,x_3,...,x_n}(x_0, G)$

(ii) $D_{x_2,x_3,...,x_n}(x_0, G) = P^{-1}_{G,x_2,x_3,...,x_n}(x_0)$ for every $x_0 \in \overline{G}$.

Proof. (i) Let $x \in D_{x_2,x_3,...,x_n}(x_0, G)$ and $g \in P_{G,x_2,x_3,...,x_n}(x_0)$.

Now,

$$\begin{aligned}
&d_{x_2,x_3,...,x_n}(x, G)\\
&\quad = ||x - x_0, x_2, x_3, ..., x_n||_\alpha + d_{x_2,x_3,...,x_n}(x_0, G)\\
&\quad = ||x - x_0, x_2, x_3, ..., x_n||_\alpha + ||x_0 - g_0, x_2, x_3, ..., x_n||_\alpha.
\end{aligned}$$

By Theorem 4.4(i) we have,

$$\begin{aligned}
d_{x_2,x_3,...,x_n}(x, G) = ||x - g_0, x_2, x_3, ..., x_n||_\alpha\\
\Rightarrow g_0 \in P_{G,x_2,x_3,...,x_n}(x)\\
\Rightarrow P_{G,x_2,x_3,...,x_n}(x_0) \subset P_{G,x_2,x_3,...,x_n}(x).
\end{aligned}$$

(ii)Let $x_0 \in \overline{G}$ and $x \in D_{x_2,x_3,...,x_n}(x_0, G)$.

Then,

$$\begin{aligned}
&d_{x_2,x_3,...,x_n}(x, G)\\
&\quad = ||x - x_0, x_2, x_3, ..., x_n||_\alpha + d_{x_2,x_3,...,x_n}(x_0, G)
\end{aligned}$$

$= ||x - x_0, x_2, x_3, ..., x_n||_\alpha$, where $x_0 \in \overline{G}$.
$\Rightarrow x \in P^{-1}_{G,x_2,x_3,...,x_n}(x_0)$.

So, $D_{x_2,x_3,...,x_n}(x_0, G) \subset P^{-1}_{G,x_2,x_3,...,x_n}(x_0)$. (4.4)

Conversely, let $x \in P^{-1}_{G,x_2,x_3,...,x_n}(x_0)$.

Then $x_o \in P_{G,x_2,x_3,...,x_n}(x)$.

Since $x_0 \in \overline{G}$,$d_{x_2,x_3,...,x_n}(x_0, G) = 0$.

Hence we have,
$d_{x_2,x_3,...,x_n}(x, G)$
$= ||x - x_0, x_2, x_3, ..., x_n||_\alpha + d_{x_2,x_3,...,x_n}(x_0, G)$
$\Rightarrow x \in D_{x_2,x_3,...,x_n}(x_0, G)$
$\Rightarrow P^{-1}_{G,x_2,x_3,...,x_n}(x_0) \subset D_{x_2,x_3,...,x_n}(x_0, G)$. (4.5)

From (4.4) and (4.5) it follows that

$D_{x_2,x_3,...,x_n}(x_0, G) = P^{-1}_{G,x_2,x_3,...,x_n}(x_0)$. □

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