Iranian Journal of Fuzzy Systems **Vol. 14, No. 6,** (2017) pp. 135-148

ROBUSTNESS OF THE TRIPLE IMPLICATION INFERENCE METHOD BASED ON THE WEIGHTED LOGIC METRIC

J. LI AND C. FU

Abstract. This paper focuses on the robustness problem of full implication triple implication inference method for fuzzy reasoning. First of all, based on strong regular implication, the weighted logic metric for measuring distance between two fuzzy sets is proposed. Besides, under this metric, some robustness results of the triple implication method are obtained, which demonstrates that the triple implication method possesses a good behavior of robustness.

1. Introduction

It is well known that fuzzy reasoning is an important component of fuzzy control. The basic reasoning forms of fuzzy reasoning can be expressed as follows:

FMP: for given $A \to B$ (rule) and A^* (input), calculate B^* (output).

FMT: for given $A \to B$ (rule) and B^* (input), calculate A^* (output).

where FMP and FMT denote fuzzy modus ponens and fuzzy modus tollens respectively, $\to$ denotes fuzzy implication R. $A, A^* \in \mathcal{F}(X)$ and $B, B^* \in \mathcal{F}(Y)$ ($\mathcal{F}(X)$ and $\mathcal{F}(Y)$ denote the set of all fuzzy subsets of universe X and Y respectively).Zadeh [31] proposed the compositional rule of inference (CRI method for short) for solving the fundamental forms of fuzzy reasoning mentioned above. Subsequently, many other methods of fuzzy reasoning have been known (see [7, 8, 25, 29]). In particular, Wang [25] proposed a novel method, called the full implication triple implication method (triple I method for short), which has been widely investigated by some scholars (see [13, 14, 15, 16, 17, 19, 20, 21, 22, 27, 28]).

When fuzzy reasoning method is applied, the robustness of the fuzzy reasoning method becomes one of the important problems. Recently, many scholars discussed the robustness of CRI method for fuzzy reasoning. The goal of the robustness of fuzzy reasoning method is to discuss how errors in inputs affect outputs in fuzzy reasoning. In other words, if a small perturbation of input always causes small changes of the output, then this method has a good behavior of robustness. Based on this fact, we may ask two questions: One is what is the meaning of "small perturbation". The other is how to measure the perturbation of input. Researchers in different disciplines introduced different concepts to measure the perturbation between two fuzzy sets. Pappis [18] proposed the definition of approximately equal of two fuzzy sets. Hong and Hwang [9] defined the α-similarity of two fuzzy sets.

Received: January 2016; Revised: February 2017; Accepted: June 2017

Key words and phrases: Robustness, Triple implication method, Weighted logic metric, Weighted logic similarity degree, Fuzzy reasoning.

Cai [2, 3] introduced the δ-equalities of fuzzy sets and discussed robustness of CRI method based on δ-equalities of fuzzy sets. Ying [30] introduced the concepts of maximum and average perturbations of fuzzy sets. Dai et al. [5] considered the concept of perturbation based on the normalized Minkowski distances. Cheng and Fu [4] discussed simple perturbation and interval perturbation of fuzzy sets. Li et al. [12] presented the concept of sensitivity of connectives for measuring the robustness of fuzzy reasoning. In previous work, the perturbation of fuzzy sets are based on the Chebyshev distance or Minkowski distances. However, the robustness of fuzzy reasoning mainly depends on fuzzy connectives and fuzzy implications. For this reason, Dai et al. [6] introduced the logic similarity degree between fuzzy sets and analyzed the robustness of triple I method. Wang and Duan [23] investigated robustness of triple I method based on average logic similarity degree. Both the logic similarity degree and the average logic similarity degree are a new perturbation parameter, which can be expressed as follows respectively:

$$S(A,B)=\bigwedge_{i=1}^{n}[R(A(x_i),B(x_i))\wedge R(B(x_i),A(x_i))],\ \text{for } A,B\in\mathcal{F}(X).$$

$$S^*(A,B)=\frac{1}{n}\sum_{i=1}^{n}[R(A(x_i),B(x_i))\wedge R(B(x_i),A(x_i))],\ \text{for } A,B\in\mathcal{F}(X).$$

where R is a fuzzy implication. However, by comparing and analyzing these concepts, we discover that the difference between them is largely due to the underlying measurements adopted, and some shortages of them are pointed out (please see Remark 3.2 and Remark 3.4). Thus, a new measurement induced by implication operators will be discussed in this paper, and a revised concept of the logic similarity degree and the average logic similarity degree, called the weighted logic similarity degree, is proposed under the unified framework of strong regular implication, which can naturally induce a fuzzy metric. Based on this new concept, robustness of triple I inference method will be investigated.

The rest of this paper is organized as follows. In section 2, we recall some necessary concepts and propositions. In section 3, some definitions of the logic similarity degree and the average logic similarity degree are analyzed and the weighted logic metric are proposed. In section 4, we discuss the robustness of the triple I method by utilizing the weighted logic metric. We conclude with a short summary in section 5.

2. **Preliminaries**

In this section, we firstly review some basic concepts and results.

Definition 2.1. [8, 10] A mapping T:$[0,1]^2\to[0,1]$ is called t-norm if and only if

(i) T is nondecreasing in each argument;
(ii) T is commutative;
(iii) T is associative;
(iv) $T(a,1)=a$ for all $a\in[0,1]$.

Definition 2.2. [10, 21] A t-norm T is a left-continuous t-norm if for every non-empty index set I, the following equality holds:

$$T(\bigvee_{i\in I} a_i, b) = \bigvee_{i\in I} T(a_i, b). \tag{1}$$

Proposition 2.3. [10, 20] *Let T be a left-continuous t-norm, then there exists a binary operation R on $[0,1]$ such that (T,R) satisfies the residuated principle, i.e., $T(a,b) \leq c$ iff $a \leq R(b,c)$, then (T,R) is called a residuated pair on $[0,1]$, and $([0,1],R,T)$ is called a residuated lattice, where R is given by the following equality:*

$$R(a,b) = \vee\{c \in [0,1] | T(a,c) \leq b\}.$$

and R is called a residuated implication(or regular implication)induced by T.

In the following we say that R is a regular implication induced by T.

Proposition 2.4. [21] *Suppose that R is a regular implication induced by a left-continuous t-norm, then the following properties hold for any $a,b,c \in [0,1]$:*

(i) $R(b,c) = 1$ *iff* $b \leq c$.
(ii) $a \leq R(b,c)$ *iff* $b \leq R(a,c)$.
(iii) $R(a,R(b,c)) = R(b,R(a,c))$.
(iv) $R(1,c) = c$.
(v) $R(b, \bigwedge_{i\in I} c_i) = \bigwedge_{i\in I} R(b,c_i)$, $R(\bigvee_{i\in I} b_i, c) = \bigwedge_{i\in I} R(b_i,c)$.
(vi) $R(a,b)$ *is non-increasing on* a*, and non-decreasing on* b.
(vii) $R(b,c) \leq R(R(a,b),R(a,c))$.
(viii) $R(a,b) \leq R(T(a,c),T(b,c))$.

Example 2.5. [21] The following are four important left-continuous t-norms and the corresponding residuated implications.

(i) Łukasiewicz t-norm and Łukasiewicz implication

$$T_L(a,b) = (a+b-1) \vee 0, \quad R_L(a,b) = (1-a+b) \wedge 1.$$

(ii) Minimum t-norm and Gödel implication

$$T_G(a,b) = a \wedge b, \quad R_G(a,b) = \begin{cases} 1, & a \leq b, \\ b, & a > b. \end{cases}$$

(iii) Product t-norm and Goguen implication

$$T_p(a,b) = ab, \quad R_p(a,b) = \begin{cases} 1, & a \leq b, \\ \dfrac{b}{a}, & a > b. \end{cases}$$

(iv) Nilpotent minimum t-norm and R_0 implication

$$T_0(a,b) = \begin{cases} a \wedge b, & a+b > 1, \\ 0, & a+b \leq 1. \end{cases} \quad R_0(a,b) = \begin{cases} 1, & a \leq b, \\ (1-a) \vee b, & a > b. \end{cases}$$

Definition 2.6. [11] (i) A left-continuous t-norm T is said to be strong left-continuous t-norm for any $a, b \in [0,1]$, if T satisfies the following inequality:

$$T_L(a,b) \leq T(a,b).$$

Where T_L is the Łukasiewicz t-norm.
(ii) A regular implication R is said to be strong regular implication for any $a, b \in [0,1]$, if R satisfies the following inequality:

$$R(a,b) \leq R_L(a,b),$$

where R_L is the Łukasiewicz implication.

It's easy to verify that Łukasiewicz implication, Goguen implication, Gödel implication and R_0 implication are all strong regular implication. Furthermore, Łukasiewicz implication is the greatest strong regular implication, Gödel implication is the smallest strong regular implication, Łukasiewicz t-norm is the smallest strong left continuous t-norm, while Gödel t-norm is the greatest one. Finally, A regular implication R induced by strong left continuous t-norm T is strong regular implication.

Lemma 2.7. *Assume that* R *is a regular implication on* $[0,1]$*, then for any* $a, b, c \in [0,1]$*,* R *is a strong regular implication on* $[0,1]$ *if and only if* $R(a,c)+1 \geq R(a,b)+R(b,c)$.

Proof. Firstly, if R is a strong regular implication on $[0,1]$, then it follows from Proposition 2.4(vii) and Definition 2.6(ii) that

$$R(b,c) \leq R(R(a,b), R(a,c)) \leq R_L(R(a,b), R(a,c)).$$

If $R(a,b) \leq R(a,c)$, then $R(b,c) \leq R_L(R(a,b), R(a,c)) = 1$. Thus we have $R(a,c) + 1 \geq R(a,b) + R(b,c)$.
If $R(a,b) > R(a,c)$, then $R(b,c) \leq R_L(R(a,b), R(a,c)) = 1 - R(a,b) + R(a,c)$. Then we have $R(a,c) + 1 \geq R(a,b) + R(b,c)$.

Secondly, suppose that $R(a,c) + 1 \geq R(a,b) + R(b,c)$. Let $a = 1$, by using Proposition 2.4(iv), we obtain $1 - b + c \geq R(b,c)$.
Moreover, since $1 \geq R(b,c)$, then we can get $(1 - b + c) \wedge 1 \geq R(b,c)$.

i.e.,

$$R_L(b,c) \geq R(b,c).$$

This completes the proof of the Lemma 2.7. □

Definition 2.8. [10] Let $([0,1], T, R)$ be a residuated lattice. Define

$$\rho_R(a,b) = a \leftrightarrow b = R(a,b) \wedge R(b,a) \quad a, b \in [0,1].$$

Definition 2.9. [24] Let X be a nonempty set, if a mapping $d : X^2 \to [0,1]$ satisfies the following conditions:

(i) $d(a,b) \geq 0$, *and* $d(a,b) = 0$ if and only if $a = b$.
(ii) $d(a,b) = d(b,a)$.
(iii) $d(a,c) \leq d(a,b) + d(b,c)$.

then d is called a metric on the set X.

Lemma 2.10. *Assume that R is a strong regular implication on $[0,1]$. Define*

$$h(a,b) = 1 - \rho_R(a,b) = 1 - R(a,b) \wedge R(b,a) \quad a,b \in [0,1]. \tag{2}$$

Then h is a metric on $[0,1]$.

Proof. Firstly, it is easy to verify that $h(a,b) = 0$ iff $a = b$, and $h(a,b) = h(b,a)$. Secondly, in order to prove that $h(a,c) \leq h(a,b) + h(b,c)$, we need only to prove that

$$\rho_R(a,b) + \rho_R(b,c) \leq 1 + \rho_R(a,c) \quad \forall a,b,c \in [0,1]. \tag{3}$$

Since R is a strong regular implication, according to Lemma 2.7, we have

$$R(a,c) \geq R(a,b) + R(b,c) - 1.$$

In addition, since $R(a,b) \geq \rho_R(a,b)$, then we can obtain

$$R(a,c) \geq R(a,b) + R(b,c) - 1 \geq \rho_R(a,b) + \rho_R(b,c) - 1.$$

Similarly, we can obtain

$$R(c,a) \geq R(c,b) + R(b,a) - 1 \geq \rho_R(a,b) + \rho_R(b,c) - 1.$$

Hence

$$\rho_R(a,c) = R(a,c) \wedge R(c,a) \geq \rho_R(a,b) + \rho_R(b,c) - 1.$$

Therefore, we can conclude that h is a metric on $[0,1]$. □

In the following, we present the triple I principles and triple I solutions based on the strong regular implication.

The triple I principle for FMP (See[20, 25]): The solution B^* of FMP should be the smallest fuzzy set on Y such that the following formula attains the greatest value for all $x_i \in X, y_j \in Y$:

$$R(R(A(x_i), B(y_j)), R(A^*(x_i), B^*(y_j))).$$

The triple I principle for FMT (See[20, 25]): The solution A^* of FMT should be the greatest fuzzy set on X such that the following formula attains the greatest value for all $x_i \in X, y_j \in Y$:

$$R(R(A(x_i), B(y_j)), R(A^*(x_i), B^*(y_j))).$$

Proposition 2.11. [19, 27] *Suppose that R is a strong regular implication, (R,T) is a residuated pair, then*

(i) the triple I solution B^* of FMP can be expressed as follows:

$$B^*(y_j) = \bigvee_{i=1}^{n} T(A^*(x_i), R(A(x_i), B(y_j))), \ y_j \in Y; \tag{4}$$

(ii) the α-triple I solution B^*_α of FMP can be expressed as follows:

$$B^*_\alpha(y) = T(\alpha, \bigvee_{i=1}^{n} T(A^*(x_i), R(A(x_i), B(y_j)))) = T(\alpha, B^*(y_j)), \ y_j \in Y; \tag{5}$$

Proposition 2.12. [19, 27] *Suppose that R is a strong regular implication, then* *(i) the triple I solution A^* of FMT can be expressed as follows:*

$$A^*(x_i) = \bigwedge_{j=1}^{m} R(R(A(x_i), B(y_j)), B^*(y_j)),\ x_i \in X; \tag{6}$$

(ii) the α-triple I solution A^*_α of FMT can be expressed as follows:

$$A^*_\alpha(x_i) = \bigwedge_{j=1}^{m} R(\alpha, R(R(A(x_i), B(y_j)), B^*(y_j))) = R(\alpha, A^*(x_i)),\ x_i \in X; \tag{7}$$

3. **Weighted Logic Metric Between Two Fuzzy Sets**

In this section, we will compare different similarity degree between two fuzzy sets and point out some blemishes of them. Besides, the weighted logic metric for measuring distance of two fuzzy sets is proposed.

Because in most of practical problems, the universe of fuzzy sets usually involves finite elements, and the computer can only store finite elements, thus throughout this paper we always suppose that universes X and Y are finite and $X = \{x_1, x_2, \ldots, x_n\}$, $Y = \{y_1, y_2, \ldots, y_m\}$.

Definition 3.1. [1, 6] Let R be a fuzzy implication on $[0, 1]$. For $A, B \in \mathcal{F}(X)$, define

$$S(A, B) = \bigwedge_{i=1}^{n} \rho_R(A(x_i), B(x_i)) = \bigwedge_{i=1}^{n} [R(A(x_i), B(x_i)) \wedge R(B(x_i), A(x_i))]. \tag{8}$$

$S(A, B)$ is called the logic similarity degree of A and B.

Remark 3.2. For each object $x_i \in X$, if we think of $\rho_R(A(x_i), B(x_i))$ as the local similarity degree of A and B in object x_i, then the logic similarity degree $S(A, B)$ given by (8) is the infimum of all local similarity degrees. Especially, when the universe X is finite, $S(A, B)$ is essentially the minimum value of all local similarity degrees. In this way, information aggregation is only concerned with the object x_i whose local similarity degree is the minimum value of all local similarity degrees, while the information provided by the rest objects will be completely abandoned, this will inevitably result in the loss of information.

Definition 3.3. [23, 26] For $A, B \in \mathcal{F}(X)$, define

$$\begin{aligned} S^*(A, B) &= \frac{1}{n} \sum_{i=1}^{n} \rho_R(A(x_i), B(x_i)) \\ &= \frac{1}{n} \sum_{i=1}^{n} [R(A(x_i), B(x_i)) \wedge R(B(x_i), A(x_i))]. \end{aligned} \tag{9}$$

$S^*(A, B)$ is called the average logic similarity degree of A and B. Where R are only the four important implications from example 2.5.

Remark 3.4. To remedy the shortage of logic similarity degree, Wang and Duan (see[23]) introduced the average logic similarity degree $S^*(A, B)$ of fuzzy sets A and B given by (9). Essentially, $S^*(A, B)$ is the arithmetic mean of all local similarity degrees, in other words, the weights of each object x_i are same. But in many practical applications, the importance of each object x_i may be different. Hence, in order to make up this defect, we will propose an improved similarity degree, named the weighted logic similarity degree, by giving different weights for each object x_i, the average logic similarity degree is just a special case of the weighted logic similarity degree.

Definition 3.5. Let R be a strong regular implication on $[0,1]$. For $A, B \in \mathcal{F}(X)$, define

$$\begin{aligned} S_R(A,B) &= \sum_{i=1}^{n} \alpha_i \rho_R(A(x_i), B(x_i)) \\ &= \sum_{i=1}^{n} \alpha_i [R(A(x_i), B(x_i)) \wedge R(B(x_i), A(x_i))]. \end{aligned} \tag{10}$$

Where $\sum\limits_{i=1}^{n} \alpha_i = 1$ and $0 < \alpha_i < 1$. $S_R(A,B)$ is called the weighted logic similarity degree of A and B.

Obviously, Definition 3.3 is a special case($\alpha_i = \frac{1}{n}$) of Definition 3.5. The weighted logic similarity degree S_R will be written as S_{R_L}, S_{R_G}, S_{R_P} and S_{R_0}, whenever R is R_L, R_G, R_P and R_0, respectively.

Proposition 3.6. *Let X be a universe of discourse. For $A, B \in \mathcal{F}(X)$.*

(i) If R is the Łukasiewicz implication R_L, then

$$S_{R_L}(A,B) = 1 - \sum_{i=1}^{n} \alpha_i |A(x_i) - B(x_i)|.$$

(ii) If R is the Gödel implication R_G, then

$$S_{R_G}(A,B) = \left[\sum_{A(x_i) \le B(x_i)} \alpha_i A(x_i) \right] + \left[\sum_{A(x_i) > B(x_i)} \alpha_i B(x_i) \right].$$

(iii) If R is the Goguen implication R_P, then

$$S_{R_P}(A,B) = \left[\sum_{A(x_i) \le B(x_i)} \alpha_i \frac{A(x_i)}{B(x_i)} \right] + \left[\sum_{A(x_i) > B(x_i)} \alpha_i \frac{B(x_i)}{A(x_i)} \right].$$

(iv) If R is the R_0 implication, then

$$S_{R_0}(A,B) = \left[\sum_{A(x_i) \le B(x_i)} \alpha_i (1 - B(x_i)) \vee A(x_i) \right] + \left[\sum_{A(x_i) > B(x_i)} \alpha_i (1 - A(x_i)) \vee B(x_i) \right].$$

Proposition 3.7. *Suppose that R is a strong regular implication on $[0,1]$. For $A, B \in \mathcal{F}(X)$, define*

$$d_R(A,B) = 1 - S_R(A,B) = 1 - \sum_{i=1}^{n} \alpha_i [R(A(x_i), B(x_i)) \wedge R(B(x_i), A(x_i))]. \tag{11}$$

Where $\sum\limits_{i=1}^{n} \alpha_i = 1$ and $0 < \alpha_i < 1$. Then d_R is a metric on $\mathcal{F}(X)$. d_R is called the weighted logic metric of A and B. If $d_R(A,B) \le \varepsilon$, then B is called a ε perturbation of A, where $\varepsilon \in [0,1]$.

Proof. (i) It is easy to prove that $d_R(A,B)=0$ if and only if $A=B$.

(ii) $d_R(A,B)=d_R(B,A)$ is obviously true.

(iii) we need only to prove that $S_R(A,B)+S_R(B,C)\le 1+S_R(A,C)$, for any $A,B,C\in\mathcal{F}(X)$. In fact, it follows from (3) that

$$\rho_R(A(x_i),B(x_i))+\rho_R(B(x_i),C(x_i))\le 1+\rho_R(A(x_i),C(x_i))\quad \forall x_i\in X.$$

Thus

$$\sum_{i=1}^{n}\alpha_i\rho_R(A(x_i),B(x_i))+\sum_{i=1}^{n}\alpha_i\rho_R(B(x_i),C(x_i))\le 1+\sum_{i=1}^{n}\alpha_i\rho_R(A(x_i),C(x_i)).$$

i.e.,

$$S_R(A,B)+S_R(B,C)\le 1+S_R(A,C).$$

□

Remark 3.8. The weighted logic metric d_R will be written as d_{R_L}, d_{R_G}, d_{R_P} and d_{R_0}, whenever R is R_L, R_G, R_P and R_0, respectively. In particular, if R is R_L, then

$$\begin{aligned}d_{R_L}(A,B)&=1-\sum_{i=1}^{n}\alpha_i(R_L(A(x_i),B(x_i))\wedge R_L(B(x_i),A(x_i)))\\&=1-\sum_{i=1}^{n}\alpha_i(((1-A(x_i)+B(x_i))\wedge 1)\wedge((1-B(x_i)+A(x_i))\wedge 1))\\&=1-\sum_{i=1}^{n}\alpha_i((1-A(x_i)+B(x_i))\wedge(1-B(x_i)+A(x_i))).\end{aligned}$$

Since $\forall i(1\le i\le n)$, we have

$$\begin{aligned}(1-A(x_i)+B(x_i))\wedge(1-B(x_i)+A(x_i))&=\begin{cases}1-A(x_i)+B(x_i), & A(x_i)>B(x_i),\\ 1-B(x_i)+A(x_i), & A(x_i)\le B(x_i).\end{cases}\\&=1-|A(x_i)-B(x_i)|.\end{aligned}$$

therefore

$$\begin{aligned}d_{R_L}(A,B)&=1-\sum_{i=1}^{n}\alpha_i(1-|A(x_i)-B(x_i)|)\\&=1-\sum_{i=1}^{n}(\alpha_i-\alpha_i|A(x_i)-B(x_i)|)\\&=1-\sum_{i=1}^{n}\alpha_i+\sum_{i=1}^{n}\alpha_i|A(x_i)-B(x_i)|.\end{aligned}$$

it follows from $\sum_{i=1}^{n}\alpha_i=1$ that

$$d_{R_L}(A,B)=\sum_{i=1}^{n}\alpha_i|A(x_i)-B(x_i)|.$$

$d_{R_L}(A,B)$ is just right the weighted Hamming distance on $\mathcal{F}(X)$.

Proposition 3.9. *Suppose that R is a strong regular implication on $[0,1]$. For $a,b\in[0,1]$, then*

$$\rho_{R_G}(a,b)\le\rho_R(a,b)\le\rho_{R_L}(a,b).$$

Proof. Firstly, by (ii) of Definition 2.6, we obtain

$$R(a,b) \leq R_L(a,b) \quad \text{and} \quad R(b,a) \leq R_L(b,a).$$

Thus

$$\rho_R(a,b) = R(a,b) \wedge R(b,a) \leq R_L(a,b) \wedge R_L(b,a) = \rho_{R_L}(a,b). \tag{12}$$

Secondly, since $R_G(a,b) = \begin{cases} 1, & a \leq b, \\ b, & a > b. \end{cases}$

If $a \leq b$, then we obtain $R_G(a,b) = 1$ and $R(a,b) = 1$, thus $R_G(a,b) = R(a,b)$.
If $a > b$, then we obtain $R_G(a,b) = b$, again because $b = R(1,b) \leq R(a,b)$, thus

$$R_G(a,b) \leq R(a,b).$$

Hence, for any $a, b \in [0,1]$, we have

$$\rho_{R_G}(a,b) \leq \rho_R(a,b). \tag{13}$$

Consequently, it follows from (12) and (13) that

$$\rho_{R_G}(a,b) \leq \rho_R(a,b) \leq \rho_{R_L}(a,b).$$

This completes the proof of the proposition 3.9. □

From Proposition 3.9, we can get the following proposition.

Proposition 3.10. *Suppose that R is a strong regular implication, then for any $A, B \in \mathcal{F}(X)$, we have*

$$d_{R_L}(A,B) \leq d_R(A,B) \leq d_{R_G}(A,B).$$

4. The Robustness of Triple I Method Based on Weighted Logic Metric

In this section, we discuss the robustness of triple I method and α-triple I method for fuzzy reasoning. In order to verify the robustness of triple I method, some lemmas will be given in the following.

Lemma 4.1. [20] *Suppose that $X = \{x_1, x_2, \ldots, x_n\}$, (R,T) is a residuated pair on[0,1], then we have the following inequality for any $A, B, C, D \in \mathcal{F}(X), x_i \in X$.*

$$\rho_R(A(x_i) \circ B(x_i), C(x_i) \circ D(x_i)) \geq \rho_R(A(x_i), C(x_i)) + \rho_R(B(x_i), D(x_i)) - 1.$$

Where $\circ \in \{T, R, \vee, \wedge\}$.

Proof. We only prove the case of $\circ = R$. The proofs of $\circ = T$, $\circ = \vee$, and $\circ = \wedge$ can similarly be attained. It follows from (3) that

$$\begin{aligned} &\rho_R(R(A(x_i), B(x_i)), R(C(x_i), D(x_i))) \\ \geq\ &\rho_R(R(A(x_i), B(x_i)), R(A(x_i), D(x_i))) + \\ &\rho_R(R(A(x_i), D(x_i)), R(C(x_i), D(x_i))) - 1. \end{aligned}$$

In the literature[20], we can obtain the following inequality for any $x_i \in X$:

$$T(\rho_R(A(x_i), B(x_i)), \rho_R(C(x_i), D(x_i))) \leq \rho_R(R(A(x_i), C(x_i)), R(x(B_i), D(x_i))).$$

Thus

$$\begin{aligned}
&\rho_R(R(A(x_i),B(x_i)),R(C(x_i),D(x_i)))\\
\geq\ &\rho_R(R(A(x_i),B(x_i)),R(A(x_i),D(x_i)))+\\
&\rho_R(R(A(x_i),D(x_i)),R(C(x_i),D(x_i)))-1\\
\geq\ &T(\rho_R(R(A(x_i),A(x_i))),\rho_R(R(B(x_i),D(x_i))))+\\
&T(\rho_R(R(A(x_i),C(x_i))),\rho_R(R(D(x_i),D(x_i))))-1\\
\geq\ &\rho_R(R(A(x_i),C(x_i)))+\rho_R(R(B(x_i),D(x_i)))-1.
\end{aligned}$$

This completes the proof. □

Lemma 4.2. *Suppose that* $X=\{x_1,x_2,\ldots,x_n\}$, $A,A^*,B,B^*,C,C^*\in\mathcal{F}(X),x_i\in X$, *and* (T,R) *is a residuated pair on[0,1]. Then we obtain*

$$\begin{aligned}
&\rho_R(C(x_i)\circ_1(A(x_i)\circ_2 B(x_i)),C^*(x_i)\circ_1(A^*(x_i)\circ_2 B^*(x_i)))\\
\geq\ &\rho_R(A(x_i),A^*(x_i))+\rho_R(B(x_i),B^*(x_i))+\rho_R(C(x_i),C^*(x_i))-2.
\end{aligned}$$

Where $\circ_1\in\{T,R,\vee,\wedge\},\circ_2\in\{T,R,\vee,\wedge\}$.

Proof. It follows from Lemma 4.1 that

$$\begin{aligned}
&\rho_R(C(x_i)\circ_1(A(x_i)\circ_2 B(x_i)),C^*(x_i)\circ_1(A^*(x_i)\circ_2 B^*(x_i)))\\
\geq\ &\rho_R(C(x_i),C^*(x_i))+\rho_R(A(x_i)\circ_2 B(x_i),A^*(x_i)\circ_2 B^*(x_i))-1\\
\geq\ &\rho_R(A(x_i),A^*(x_i))+\rho_R(B(x_i),B^*(x_i))+\rho_R(C(x_i),C^*(x_i))-2.
\end{aligned}$$

This completes the proof. □

Remark 4.3. Lemma 4.1 and Lemma 4.2 still hold when R is a strong regular implication.

Lemma 4.4. [23] *Assume that* $a_i,b_i\in[0,1]$ $(i=1,2,\cdots,n)$, *then the following inequalities hold:*

(i) $\rho_R(\bigvee_{i=1}^{n}a_i,\bigvee_{i=1}^{n}b_i)\geq\sum_{i=1}^{n}\rho_R(a_i,b_i)-(n-1)$.

(ii) $\rho_R(\bigwedge_{i=1}^{n}a_i,\bigwedge_{i=1}^{n}b_i)\geq\sum_{i=1}^{n}\rho_R(a_i,b_i)-(n-1)$.

Proof. We prove (i) by induction. The proof of (ii) can similarly be attained.
If $n=1$, then (i) is obviously true. Suppose that (i) is true for $n=k$, then it follows from Lemma 4.1 and the induction hypothesis that

$$\begin{aligned}
\rho_R(\bigvee_{i=1}^{k+1}a_i,\bigvee_{i=1}^{k+1}b_i)&\geq\rho_R(\bigvee_{i=1}^{k}a_i,\bigvee_{i=1}^{k}b_i)+\rho_R(a_{k+1},b_{k+1})-1\\
&\geq\sum_{i=1}^{k}\rho_R(a_i,b_i)-(k-1)+\rho_R(a_{k+1},b_{k+1})-1\\
&\geq\sum_{i=1}^{k+1}\rho_R(a_i,b_i)-((k+1)-1).
\end{aligned}$$

Thus (i) is valid for $n=k+1$. This completes the proof. □

Proposition 4.5. *Suppose that* $X=\{x_1,x_2,\ldots,x_n\}, Y=\{y_1,y_2,\ldots,y_m\}$, $A, A^{'}$, $A^{*}, A^{'*}\in\mathcal{F}(X)$, $B, B^{'}, B^{*}, B^{'*}\in\mathcal{F}(Y), x_i\in X$. *$(T,R)$ and R are the residuated pair on [0,1] and the strong regular implication, respectively. B^{*} and $B^{'*}$ are the triple I solutions of the forms FMP(A,B,A^{*}) and FMP$(A^{'},B^{'},A^{'*})$, respectively. Then we have*

$$d_R(B^{*},B^{'*})\leq 2n-\sum_{i=1}^{n}\rho_R(A^{*}(x_i),A^{'*}(x_i))-\sum_{i=1}^{n}\rho_R(A(x_i),A^{'}(x_i))+nd_R(B,B^{'}). \quad (14)$$

Where $\sum_{j=1}^{m}\beta_j=1$ *and* $0<\beta_j<1$.

Proof. From Lemma 4.1, 4.2, 4.4 and equation(4), we have

$$\begin{aligned}
&d_R(B^{*},B^{'*})=1-S_R(B^{*},B^{'*})\\
&=1-\sum_{j=1}^{m}\beta_j\rho_R(B^{*}(y_j),B^{'*}(y_j))\\
&=1-\sum_{j=1}^{m}\beta_j\rho_R(\bigvee_{i=1}^{n}T(A^{*}(x_i),R(A(x_i),B(y_j))),\bigvee_{i=1}^{n}T(A^{'*}(x_i),R(A^{'}(x_i),B^{'}(y_j))))\\
&\leq 1-\sum_{j=1}^{m}\beta_j\left[\sum_{i=1}^{n}\rho_R(T(A^{*}(x_i),R(A(x_i),B(y_j))),T(A^{'*}(x_i),R(A^{'}(x_i),B^{'}(y_j))))-(n-1)\right]\\
&\leq 1-\sum_{j=1}^{m}\beta_j\left[\sum_{i=1}^{n}(\rho_R(A^{*}(x_i),A^{'*}(x_i))+\rho_R(A(x_i),A^{'}(x_i))+\rho_R(B(y_j),B^{'}(x_i))-2)-(n-1)\right]\\
&=1-\sum_{j=1}^{m}\beta_j\left[\sum_{i=1}^{n}\rho_R(A^{*}(x_i),A^{'*}(x_i))+\sum_{i=1}^{n}\rho_R(A(x_i),A^{'}(x_i))+n\rho_R(B(y_j),B^{'}(y_j))-2n-(n-1)\right]\\
&=1-\sum_{i=1}^{n}\rho_R(A^{*}(x_i),A^{'*}(x_i))-\sum_{i=1}^{n}\rho_R(A(x_i),A^{'}(x_i))-n\sum_{j=1}^{m}\beta_j\rho_R(B(y_j),B^{'}(y_j))+3n-1\\
&=3n-\sum_{i=1}^{n}\rho_R(A^{*}(x_i),A^{'*}(x_i))-\sum_{i=1}^{n}\rho_R(A(x_i),A^{'}(x_i))-n\sum_{j=1}^{m}\beta_j\rho_R(B(y_j),B^{'}(y_j))\\
&=3n-\sum_{i=1}^{n}\rho_R(A^{*}(x_i),A^{'*}(x_i))-\sum_{i=1}^{n}\rho_R(A(x_i),A^{'}(x_i))-nS_R(B,B^{'})\\
&=3n-\sum_{i=1}^{n}\rho_R(A^{*}(x_i),A^{'*}(x_i))-\sum_{i=1}^{n}\rho_R(A(x_i),A^{'}(x_i))-n(1-d_R(B,B^{'}))\\
&=3n-\sum_{i=1}^{n}\rho_R(A^{*}(x_i),A^{'*}(x_i))-\sum_{i=1}^{n}\rho_R(A(x_i),A^{'}(x_i))-n+nd_R(B,B^{'})\\
&=2n-\sum_{i=1}^{n}\rho_R(A^{*}(x_i),A^{'*}(x_i))-\sum_{i=1}^{n}\rho_R(A(x_i),A^{'}(x_i))+nd_R(B,B^{'}).
\end{aligned}$$

This completes the proof. □

We can obtain the following three propositions and they can be proved in the way similar to Proposition 4.5.

Proposition 4.6. *Suppose that* $X=\{x_1,x_2,\ldots,x_n\}, Y=\{y_1,y_2,\ldots,y_m\}$, $A, A^{'}$, $A^{*}, A^{'*}\in\mathcal{F}(X)$, $B, B^{'}, B^{*}, B^{'*}\in\mathcal{F}(Y), x_i\in X$. *$(T,R)$ and R are the residuated pair on [0,1] and the strong regular implication, respectively. A^{*} and $A^{'*}$ are the triple I solutions of the forms FMT(A,B,B^{*}) and FMT$(A^{'},B^{'},B^{'*})$, respectively. Then we have*

$$d_R(A^{*},A^{'*})\leq 2m-\sum_{j=1}^{m}\rho_R(B^{*}(y_j),B^{'*}(y_j))-\sum_{j=1}^{m}\rho_R(B(y_j),B^{'}(y_j))+md_R(A,A^{'}).$$

Where $\sum_{i=1}^{n}\alpha_i=1$ *and* $0<\alpha_i<1$.

Proposition 4.7. *Suppose that* $X = \{x_1, x_2, \ldots, x_n\}, Y = \{y_1, y_2, \ldots, y_m\}$, $A, A^{'}$, $A^{*}, A^{'*} \in \mathcal{F}(X)$, $B, B^{'}, B^{*}, B^{'*} \in \mathcal{F}(Y), x_i \in X$. (T, R) *and* R *are the residuated pair on [0,1] and the strong regular implication, respectively.* B^{*}_{α} *and* $B^{'*}_{\alpha}$ *are the* α*-triple I solutions of the forms FMP*(A, B, A^{*}) *and FMP*$(A^{'}, B^{'}, A^{'*})$*, respectively. Then we have*

$$d_R(B^{*}_{\alpha}, B^{'*}_{\alpha}) \leq 2n - \sum_{i=1}^{n} \rho_R(A^{*}(x_i), A^{'*}(x_i)) - \sum_{i=1}^{n} \rho_R(A(x_i), A^{'}(x_i)) + n d_R(B, B^{'}).$$

Where $\sum_{j=1}^{m} \beta_j = 1$ *and* $0 < \beta_j < 1$.

Proposition 4.8. *Suppose that* $X = \{x_1, x_2, \ldots, x_n\}, Y = \{y_1, y_2, \ldots, y_m\}$, $A, A^{'}$, $A^{*}, A^{'*} \in \mathcal{F}(X)$, $B, B^{'}, B^{*}, B^{'*} \in \mathcal{F}(Y), x_i \in X$. (T, R) *and* R *are the residuated pair on [0,1] and the strong regular implication, respectively.* A^{*}_{α} *and* $A^{'*}_{\alpha}$ *are the* α*-triple I solutions of the forms FMT*(A, B, B^{*}) *and FMT*$(A^{'}, B^{'}, B^{'*})$*, respectively. Then we have*

$$d_R(A^{*}_{\alpha}, A^{'*}_{\alpha}) \leq 2m - \sum_{j=1}^{m} \rho_R(B^{*}(y_j), B^{'*}(y_j)) - \sum_{j=1}^{m} \rho_R(B(y_j), B^{'}(y_j)) + m d_R(A, A^{'}).$$

Where $\sum_{i=1}^{n} \alpha_i = 1$ *and* $0 < \alpha_i < 1$.

Remark 4.9. Assume that $d_R(A^{*}, A^{'*}) \to 0$, $d_R(A, A^{'}) \to 0$ and $d_R(B, B^{'}) \to 0$, i.e., $\rho_R(A^{*}(x_i), A^{'*}(x_i)) \to 1$, $\rho_R(A(x_i), A^{'}(x_i)) \to 1$ and $\rho_R(B(y_j), B^{'}(y_j)) \to 1$. Then it follows from Proposition 4.5 and Proposition 4.7 that $d_R(B^{*}, B^{'*}) \to 0$ and $d_R(B^{*}_{\alpha}, B^{'*}_{\alpha}) \to 0$. In particular, if $d_R(A^{*}, A^{'*}) = 0$, $d_R(A, A^{'}) = 0$ and $d_R(B, B^{'}) = 0$, then $d_R(B^{*}, B^{'*}) = 0$. Under this condition, we can know that small perturbations of input always cause small changes of output, thus we say that the triple (α-triple) I method of FMP has a good behavior of robustness. According to the similar analysis, we find that the triple (α-triple) I method of FMT also has a good behavior of robustness.

We can obtain the following two corollaries from Proposition 4.5 and 4.6 when $A = A^{'}$ and $B = B^{'}$.

Corollary 4.10. *Suppose that* $X = \{x_1, x_2, \ldots, x_n\}, Y = \{y_1, y_2, \ldots, y_m\}$, A, A^{*}, $A^{'*} \in \mathcal{F}(X)$, $B, B^{*}, B^{'*} \in \mathcal{F}(Y), x_i \in X$. (T, R) *and* R *are the residuated pair on [0,1] and the strong regular implication, respectively.* B^{*} *and* $B^{'*}$ *are the triple (*α*-triple) I solutions of the forms FMP*(A, B, A^{*}) *and FMP*$(A, B, A^{'*})$*, respectively. Thus we have*

$$d_R(B^{*}, B^{'*}) \leq n - \sum_{i=1}^{n} \rho_R(A^{*}(x_i), A^{'*}(x_i)).$$

Corollary 4.11. *Suppose that* $X = \{x_1, x_2, \ldots, x_n\}, Y = \{y_1, y_2, \ldots, y_m\}$, A, A^{*}, $A^{'*} \in \mathcal{F}(X)$, $B, B^{*}, B^{'*} \in \mathcal{F}(Y), x_i \in X$. (T, R) *and* R *are the residuated pair on [0,1] and the strong regular implication, respectively.* A^{*} *and* $A^{'*}$ *are the triple (*α*-triple) I solutions of the forms FMT*(A, B, B^{*}) *and FMT*$(A, B, B^{'*})$*, respectively.*

Then we have

$$d_R(A^*, A^{'*}) \leq m - \sum_{j=1}^{m} \rho_R(B^*(y_j), B^{'*}(y_j)).$$

5. **Conclusions**

In this paper, we analyzed some shortages of previous fuzzy metric and presented a novel weighted logic metric for measuring distance between two fuzzy sets, which was defined by means of weighted average of all local measurements. In addition, we investigated the robustness of triple I method by using the weighted logic metric and proved that the triple I method possesses a good behavior of robustness under the weighted logic metric. In future, we will investigate the robustness of other fuzzy reasoning methods based on the weighted logic metric.

Acknowledgements. Authors would like to express their sincere thanks to the valuable comments and suggestions from editors and reviewers. This work was supported by the National Natural Science Foundation of China (No. 11261032).

References

[1] R. Bělohlávek, *Fuzzy Relational Systems, Foundations and Principles*, Kluwer Academic Publishers, Dordrecht, (2002), 40-116.

[2] K. Y. Cai, *δ-equalities of fuzzy sets*, Fuzzy Sets and Systems, **76(1)** (1995), 97-112.

[3] K. Y. Cai, *Robustness of fuzzy reasoning and δ-equalities of fuzzy sets*, IEEE Transactions on Fuzzy Systems, **9(5)** (2001), 738-750.

[4] G. S. Cheng and Y. X. Fu, *Error estimation of perturbations under CRI*, IEEE Transactions on Fuzzy Systems, **14(6)** (2006), 709-715.

[5] S. S. Dai, D. W. Pei and S. M. Wang, *Perturbation of fuzzy sets and fuzzy reasoning based on normalized Minkowski distances*, Fuzzy Sets and Systems, **189** (2012), 63-73.

[6] S. S. Dai, D. W. Pei and D. H. Guo, *Robustness analysis of full implication inference method*, International Journal of Approximate Reasoning, **54(5)** (2013), 653-666.

[7] D. Dubois, J. Lang and H. Prade, *Fuzzy sets in approximate reasoning, parts 1 and 2*, Fuzzy Sets and Systems, **40(1)** (1991), 143-244.

[8] P. Hájek, *Metamathematics of Fuzzy Logic*, Kluwer Academic Publishers, Dordrecht, (1998), 27-29.

[9] D. H. Hong and S. Y. Hwang, *A note on the value similarity of fuzzy systems variables*, Fuzzy Sets and Systems, **66(3)** (1994), 383-386.

[10] E. P. Klement, R. Mesiar and E. Pap, *Triangular Norms*, Kluwer Academic Publishers, Dordrecht, (2000), 4-6.

[11] J. Li and J. T. Yao, *Theory of integral truth degrees of formula in SMTL propositional logic*, Acta Electronica Sinica, **41(5)** (2013), 878-883.

[12] Y. M. Li, D. C. Li, W. Pedrycz and J. J. Wu, *An approach to measure the robustness of fuzzy reasoning*, International Journal of Intelligent Systems, **20(4)** (2005), 393-413.

[13] H. W. Liu and G. J. Wang, *Continuity of triple I methods based on several implications*, Computers and Mathematics with Applications, **56(8)** (2008), 2079-2087.

[14] H. W. Liu and G. J. Wang, *A note on the unified forms of triple I method*, Computers and Mathematics with Applications, **52(10)** (2006), 1609-1613.

[15] H. W. Liu and G. J. Wang, *Unified forms of fully implicational restriction methods for fuzzy reasoning*, Information Sciences, **177(3)** (2007), 956-966.

[16] H. W. Liu and G. J. Wang, *Triple I method based on pointwise sustaining degrees*, Computers and Mathematics with Applications, **55(11)** (2008), 2680-2688.
[17] M. X. Luo and N. Yao, *Triple I algorithms based on Schweizer- Sklar operators in fuzzy reasoning*, International Journal of Approximate Reasoning, **54(5)** (2013), 640-652.
[18] C. P. Pappis, *Value approximation of fuzzy systems variables*, Fuzzy Sets and Systems, **39(1)** (1991), 111-115.
[19] D. W. Pei, *Unified full implication algorithms of fuzzy reasoning*, Information Sciences, **178(2)** (2008), 520-530.
[20] G. J. Wang, *Non-classical Mathematical Logic and Approximate Reasoning, second ed.*, Science Press, Beijing, China, (2008), 155-165.
[21] G. J. Wang, *Introduction to Mathematical Logic and Resolution Principle, second ed.*, Science Press, Beijing, China, (2006), 160-162.
[22] G. J. Wang and H. Wang, *Non-fuzzy versions of fuzzy reasoning in classical logics*, Information Sciences, **138** (2001), 211-236.
[23] G. J. Wang and J. Y. Duan, *On robustness of the full implication triple I inference method with respect to finer measurements*, International Journal of Approximate Reasoning, **55(3)** (2014),787-796.
[24] S. W. Wang and W. X. Zheng, *Real Variable Function and Functional Analysis*, Higher Education Press, Beijing, China, (2005), 4-5.
[25] G. J. Wang, *The full implication triple I method of fuzzy reasoning*, SCIENCE CHINA Ser. E **29** (1999), 43-53.
[26] G. J. Wang and J. Y. Duan, *Two types of fuzzy metric spaces suitable for fuzzy reasoning*, Science China Information Sciences, **44(5)** (2014), 623-632.
[27] G. J. Wang and L. Fu, *Unified forms of triple I method*, Computers and Mathematics with Applications, **49(5)** (2005), 923-932.
[28] G. J. Wang, *Formalized theory of general fuzzy reasoning*, Information Sciences, **160(1)** (2004), 251-266.
[29] R. R. Yager, *On some new classes of implication operators and their role in approximate reasoning*, Information Sciences, **167(1-4)** (2004), 193-216.
[30] M. S. Ying, *Perturbation of fuzzy reasoning*, IEEE Transactions on Fuzzy Systems, **7(5)** (1999), 625-629.
[31] L. A. Zadeh, *Outline of a new approach to the analysis of complex systems and decision processes*, IEEE Transactions on Systems Man and Cybernetics, **3(1)** (1973), 28-33.

Jun Li*, School of Science, Lanzhou University of Technology, Lanzhou 730050, Gansu, China
E-mail address: lijun@lut.cn

Chao Fu, School of Science, Lanzhou University of Technology, Lanzhou 730050, Gansu, China
E-mail address: fuchao45612@sina.com

*Corresponding author