THE DIRECT AND THE INVERSE LIMIT OF HYPERSTRUCTURES ASSOCIATED WITH FUZZY SETS OF TYPE 2

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Abstract. In this paper we study two important concepts, i.e. the direct and the inverse limit of hyperstructures associated with fuzzy sets of type 2, and show that the direct and the inverse limit of hyperstructures associated with fuzzy sets of type 2 are also hyperstructures associated with fuzzy sets of type 2.

1. **Introduction**

Fuzzy sets of type 2 have been introduced by Zadeh [11], to give a mathematical formalization of linguistic terms. They are a generalization of fuzzy sets and have been used as a tool in decision making [11] and neural networks [10].

Fuzzy sets of Type 2 can model uncertainty better than fuzzy sets, by Minimizing the total effect of all uncertainties.

Hypergroups , introduced by Marty [7] have been extensively studied in the last few decades. Today they have applications in many fields, including fuzzy set and rough set theory, automata, cryptography, codes, graphs and hypergraphs, probability, artificial intelligence, Euclidian and non Euclidean geometries[3].

We first recall some definitions which will be needed in the rest of the paper.

Let H be a nonempty set and $\mathcal{P}^*(H)$ be the set of all nonempty subsets of H.

A function $\circ : H \times H \longrightarrow \mathcal{P}^*(H)$ is called a *hyperoperation* on H and the hyperstructure $(H, \circ)$ is called a *hypergroupoid*.

We say that the hyperstructure $(H, \circ)$ is a *quasihypergroup* if, for any $x \in H$, we have $x \circ H = H \circ x = H$, where, for any nonempty subsets A, B of H, $A \circ B = \bigcup\limits_{a\in A,\, b\in B} a \circ b$. A quasihypergroup is called a *hypergroup* if the associativity law holds.

If $(H_1, \circ_1)$ and $(H_2, \circ_2)$ are hypergroupoids, then $f : H_1 \longrightarrow H_2$ is a *morphism* if, $\forall (x, y) \in H_1^2$, we have $f(x \circ_1 y) \subseteq f(x) \circ_2 f(y)$.

A *fuzzy set of type* 2 on a universal set X is a function $F : X \longrightarrow [0, 1]^J$, where J is a nonempty subset of $[0, 1]$.

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For any $x \in X$, denote $F(x)$ by F_x. We have $F_x : J \longrightarrow [0,1]$.

A connection between fuzzy sets of type 2 and hyperstructures has been established by Corsini [2], as follows:

If $F : X \longrightarrow [0,1]^J$ is a fuzzy set of type 2, then we define the following hyperoperation on J:

$$(*) \quad \forall (u,v) \in J^2,\ u \circ v = \{w \in J \mid \exists x \in X : F_x(u) \wedge F_x(v) \leq F_x(w) \leq F_x(u) \vee F_x(v)\}.$$

Then $(J, \circ)$ is a quasihypergroup called the *quasihypergroup associated with the fuzzy set of type* 2, F.

Direct limits of some particular classes of multialgebras have been discussed in [8]. Direct limits and inverse limits are particular cases of the categorical concepts of colimit and limit (see [1]). In the general case of universal hyperalgebras, the colimit and the limit are described and characterized in [4].

In this paper, we establish that the direct limit and the inverse limit of quasihypergroups associated with fuzzy sets of type 2 are also quasihypergroups associated with fuzzy sets of type 2 and we find such fuzzy sets of type 2.

2. Direct Limit of Quasihypergroups Associated with Fuzzy Sets of Type 2

The notion of a *direct limit* of a direct family of semihypergroups was introduced by Romeo [9].

The construction of a direct limit of hypergroupoidsis as follows.

Let $\{(J_i, \circ_i)\}_{i \in I}$ be a direct family of hypergroupoids and let $\{\varphi_{ik} : J_i \longrightarrow J_k\}_{i \leq k}$ be the corresponding family of hypergroupoid morphisms. In order to obtain its *direct limit*, we consider $J = \bigcup\limits_{i \in I} J_i$ and the following equivalence relation on J:

$u \sim v$ if and only if the following implication holds:

$(u,v) \in J_i \times J_k \Longrightarrow \exists \ell \in I,\ \ell \geq i,\ \ell \geq k$, such that $\varphi_{i\ell}(u) = \varphi_{k\ell}(v)$.

For $i \leq j$, denote $\varphi_{ij}(u_i)$ by u_j and let $\overline{J}$ be the set of equivalence classes.

We define the following hyperoperation on $\overline{J}$:

$\forall (\bar{u}, \bar{v}) \in \overline{J}^2,\ \bar{u} \bullet \bar{v} = \{\bar{w} \mid \exists i \in I,\ \exists u_i \in \bar{u} \cap J_i,\ \exists v_i \in \bar{v} \cap J_i,\ \exists w_i \in \bar{w} \cap J_i$ such that $w_i \in u_i \circ_i v_i\}$.

Then $(\overline{J}, \bullet)$ is the *direct limit* of the direct family $\{(J_i, \circ_i)\}_{i \in I}$ of hypergroupoids. Notice that the cardinality of $\overline{J}$ is at most c, since J is a subset of [0,1].

Now, we introduce the notion of 2 *f.s.morphism*:

Definition 2.1. Let $F_1 : X \longrightarrow [0,1]^{J_1}$ and $F_2 : X \longrightarrow [0,1]^{J_2}$ (where $J_1 \subset [0,1] \supset J_2$) be fuzzy sets of type 2 on X. The map $f : J_1 \longrightarrow J_2$ is called a 2 *f.s.morphism* if $\forall (u,v) \in J_1^2$, $\forall x \in X$, we have

$$F_{1x}(u) < F_{1x}(v) \Longleftrightarrow F_{2x}(f(u)) < F_{2x}(f(v)).$$

Using this notion, we can define the concept of a direct family of fuzzy sets of type 2, that we shall use in the following theorems.

Theorem 2.2. *Let $\{F_i \mid i \in I\}$ be a directed family of fuzzy sets of type 2 on X, $F_i : X \longrightarrow [0,1]^{J_i}$, where $\forall i \in I$, $J_i \subset [0,1]$ and $\{f_{ik} : J_i \longrightarrow J_k\}_{i \leq k}$ the corresponding family of 2 f.s.morphisms. If $\{(J_i, \circ_i)\}_{i \in I}$ is the associated family of hypergroupoids, defined by $(*)$, then $\{(J_i, \circ_i)\}_{i \in I}$ is a direct family.*

Theorem 2.3. *Let $\{F_i \mid i \in I\}$ be a direct family of fuzzy sets of type 2 and $\{f_{ik} : J_i \longrightarrow J_k\}_{i \leq k}$ be the corresponding family of 2 f.s.morphisms. If $\{(J_i, \circ_i)\}_{i \in I}$ is the associated family of quasihypergroups, then $(\overline{J}, \bullet)$ is a quasihypergroup associated with a fuzzy set of type 2, $\overline{F}$. Such a fuzzy set of type 2 is $\overline{F} : X \longrightarrow [0,1]^{\overline{J}}$ and satisfies the following conditions:*

$\forall (\bar{u}, \bar{v}) \in \overline{J}^2$ and $\forall x \in X$, we have $\overline{F}_x(\bar{u}) < \overline{F}_x(\bar{v})$ if and only if $\exists i_0 \in I$, $\exists u_{i_0} \in \bar{u} \cap J_{i_0}$, $\exists v_{i_0} \in \bar{v} \cap J_{i_0}$ such that $F_{i_0 x}(u_{i_0}) < F_{i_0 x}(v_{i_0})$.

Proof. First we verify that the following implication holds: $[\exists i_0 \in I, \exists u_{i_0} \in \bar{u} \cap J_{i_0}, \exists v_{i_0} \in \bar{v} \cap J_{i_0}$ such that $F_{i_0 x}(u_{i_0}) < F_{i_0 x}(v_{i_0})] \Longrightarrow [\forall k \in I, \forall u_k \in \bar{u} \cap J_k, \forall v_k \in \bar{v} \cap J_k,$ we have $F_{kx}(u_k) < F_{kx}(v_k)]$. Note that

- If $u_{i_0}, u'_{i_0} \in \bar{u} \cap J_{i_0}$, then there exists $k \in I$, $k \geq i_0$, such that $f_{i_0 k}(u_{i_0}) = f_{i_0 k}(u'_{i_0})$, that means $u_k = u'_k$. Suppose $F_{i_0 x}(u_{i_0}) < F_{i_0 x}(u'_{i_0})$, whence we get $F_{kx}(f_{i_0 k}(u_{i_0})) < F_{kx}(f_{i_0 k}(u'_{i_0}))$, which contradicts $u_k = u'_k$. Hence, for any $u_{i_0}, u'_{i_0} \in \bar{u} \cap J_{i_0}$, we have $F_{i_0 x}(u_{i_0}) = F_{i_0 x}(u'_{i_0})$.
- We now show that $\forall k \in I$, $F_{kx}(u_k) < F_{kx}(v_k)$. If $i_0 \leq k$, then from $F_{i_0 x}(u_{i_0}) < F_{i_0 x}(v_{i_0})$, we get $F_{kx}(f_{i_0 k}(u_{i_0})) < F_{kx}(f_{i_0 k}(v_{i_0}))$, that is $F_{kx}(u_k) < F_{kx}(v_k)$. Suppose that there exists $k \in I$, such that $F_{kx}(u_k) > F_{kx}(v_k)$. Set $t \in I$, $t \geq i_0$, $t \geq k$. We get $F_{tx}(f_{kt}(u_k)) > F_{tx}(f_{kt}(v_k))$, that is, $F_{tx}(u_t) > F_{tx}(v_t)$. On the other hand, from $F_{i_0 x}(u_{i_0}) < F_{i_0 x}(v_{i_0})$, we get $F_{tx}(u_t) < F_{tx}(v_t)$, a contradiction.

Finally, if the hyperoperation "$\circ$" associated with $\overline{F}$, is defined as follows:

$$\forall (\bar{u}, \bar{v}) \in \overline{J}^2, \quad \bar{u} \circ \bar{v} = \{\bar{w} \mid \exists x \in H : \overline{F}_x(\bar{u}) \wedge \overline{F}_x(\bar{v}) \leq \overline{F}_x(\bar{w}) \leq \overline{F}_x(\bar{u}) \vee \overline{F}_x(\bar{v})\},$$

then the hypergroupoids $(\overline{J}, \circ)$ and $(\overline{J}, \bullet)$ coincide.

Indeed, if we suppose $\overline{F}_x(\bar{u}) \leq \overline{F}_x(\bar{v})$, then $\bar{w} \in \bar{u} \circ \bar{v}$ means that $\exists x \in H$, $\exists (i,k) \in I^2$, $\exists u_i \in \bar{u} \cap J_i$, $\exists w_i \in \bar{w} \cap J_i$, $\exists w_k \in \bar{w} \cap J_k$, $\exists v_k \in \bar{v} \cap J_k$, such that:

$$F_{ix}(u_i) \leq F_{ix}(w_i) \ \text{ and } \ F_{kx}(w_k) \leq F_{kx}(v_k),$$

whence there exists $t \in I$, $t \geq i$, $t \geq k$, for which $F_{tx}(u_t) \leq F_{tx}(w_t) \leq F_{tx}(v_t)$, which implies that $w_t \in u_t \circ_t v_t$. Hence $\forall (\bar{u}, \bar{v}) \in \overline{J}^2$, we have $\bar{u} \circ \bar{v} = \bar{u} \bullet \bar{v}$. $\overline{J}$ is supposed to be a subset of $[0,1]$, but since it has the same cardinality as a subset $\overline{J}'$ of $[0,1]$, it is clear we can set $\overline{F}'_x(\bar{u}') = \overline{F}_x(\bar{u})$ and so, the hyperoperation is independent of the nature of the elements of $\overline{J}$. $\square$

Example 2.4. We can choose $\overline{F}$ in many ways. Two examples follow:

1) Set $i_0 \in I$ and for any $\bar{u} \in \overline{J}$, set $u_{i_0} \in \bar{u} \cap J_{i_0}$. For any $x \in X$, define $\overline{F}_x(\bar{u}) = F_{i_0 x}(u_{i_0})$.

2) If I' is a finite subset of I and $|I'|$ is the cardinal of I', then $\forall x \in X$, define $\overline{F}_x(\bar{u}) = \sum_{i \in I'} F_{ix}(u_i)/|I'|$, where $u_i \in \bar{u} \cap J_{i_0}$.

3. Inverse Limit of Quasihypergroups Associated with Fuzzy Sets of Type 2

First, we recall the construction of an *inverse family* of hypergroupoids ([3], [6]). Let $\{(J_i, \circ_i)\}_{i \in I}$ be an *inverse family* of hypergroupoids and denote the cor responding family of hypergroupoid morphisms by $\{\psi_{ik} : J_i \longrightarrow J_k\}_{i \geq k}$.

In order to obtain the *inverse limit* of the above family, we consider the direct product

$$\left(J = \prod_{i \in I} J_i, \odot \right)$$

and the subset $\widetilde{J}$ of J, defined as follows:

$$\widetilde{J} = \{u \in J \mid \psi_{ji}(u_j) = u_i, \ \forall i \leq j\}.$$

If $\widetilde{J} \neq \emptyset$, then we define the following hyperoperation on $\widetilde{J}$:

$$\forall (\widetilde{u}, \widetilde{v}) \in \widetilde{J}^2, \quad \widetilde{u} \,\square\, \widetilde{v} = \widetilde{u} \odot \widetilde{v} \cap \widetilde{J}.$$

Then $(\widetilde{J}, \square)$ is the *inverse limit* of the inverse family $\{(J_i, \circ_i)\}_{i \in I}$ of hypergroupoids.

The assumption $\widetilde{J} \neq \emptyset$ is necessary. In [5], Grätzer presents an example of an inverse family of nonempty sets, whose inverse limit is empty and he also proves that the inverse limit of a family of nonempty finite sets is always nonempty.

Another situation when the inverse limit of a family of nonempty sets $\{J_i\}_{i \in I}$ is nonempty is when

I has a maximum element.

In what follows, we consider $(I, \leq)$ a partially ordered set, such that $\widetilde{J} \neq \emptyset$.

Then we obtain the following theorem, which is the dual of Theorem 2.2:

Theorem 3.1. *If $\{F_i \mid i \in I\}$ is an inverse family of fuzzy sets of type 2 with the corresponding family $\{f_{ik} : J_i \longrightarrow J_k\}_{i \geq k}$ of 2 f.s morphisms, then the associated family of hypergroupoids $\{(J_i, \circ_i)\}_{i \in I}$ is also an inverse family.*

Theorem 3.2. *Let $\{F_i \mid i \in I\}$ be an inverse family of fuzzy sets of type 2, with the corresponding family $\{f_{ik} : J_i \longrightarrow J_k\}_{i \geq k}$ of 2 f.s morphisms. If $\{(J_i, \circ_i)\}_{i \in I}$ is the associated family of quasihypergroups and $\widetilde{J} \neq \emptyset$, and $\widetilde{J}$ has the cardinality at most c, then $(\widetilde{J}, \square)$ is a quasihypergroup associated with a fuzzy set of type 2, $\widetilde{F} : X \longrightarrow [0,1]^{\widetilde{J}}$, that satisfies the following condition:*

for all $(\tilde{u}, \tilde{v}) \in \widetilde{J}^2$, $\tilde{u} = (\tilde{u}_i)_{i \in I}$, $\tilde{v} = (\tilde{v}_i)_{i \in I}$, and all $x \in X$, we have $\widetilde{F}_x(\tilde{u}) < \widetilde{F}_x(\tilde{v})$ if and only if $\exists i_0 \in I$ such that $F_{i_0 x}(\tilde{u}_{i_0}) < F_{i_0 x}(\tilde{v}_{i_0})$.

Proof. **Step 1.** We show that the following implication holds:

$$[\exists i_0 \in I \text{ such that } F_{i_0 x}(\tilde{u}_{i_0}) < F_{i_0 x}(\tilde{v}_{i_0})] \Longrightarrow$$
$$\Longrightarrow [\forall j \in I, \text{ we have } F_{jx}(\tilde{u}_j) < F_{jx}(\tilde{v}_j)].$$

Indeed, if $j \leq i_0$, then $f_{i_0 j}$ is a 2 f.s. morphism so, from $F_{i_0 x}(\tilde{u}_{i_0}) < F_{i_0 x}(\tilde{v}_{i_0})$ it follows that $F_{jx}(f_{i_0 j}(\tilde{u}_{i_0})) < F_{jx}(f_{i_0 j}(\tilde{v}_{i_0}))$, i.e. $F_{jx}(\tilde{u}_j) < F_{jx}(\tilde{v}_j)$.

Suppose that there exists $p \in I$, such that $F_{px}(\tilde{u}_p) \geq F_{px}(\tilde{v}_p)$. Since I is a directed partially ordered set, it follows that $\exists t \in I$, $t \geq i_0$, $t \geq p$.

If $F_{tx}(\tilde{u}_t) < F_{tx}(\tilde{v}_t)$, then $F_{px}(f_{tp}(\tilde{u}_t)) < F_{px}(f_{tp}(\tilde{v}_t))$, i.e. $F_{px}(\tilde{u}_p) < F_{px}(\tilde{v}_p)$, which contradicts the original assumption.

If $F_{tx}(\tilde{u}_t) \geq F_{tx}(\tilde{v}_t)$, then $F_{i_0 x}(f_{ti_0}(\tilde{u}_t)) \geq F_{i_0 x}(f_{ti_0}(\tilde{v}_t))$, i.e. $F_{i_0 x}(\tilde{u}_{i_0}) \geq F_{i_0 x}(\tilde{v}_{i_0})$, in contradiction with the hypothesis.

Therefore, $\forall j \in I$, we have $F_{jx}(\tilde{u}_j) < F_{jx}(\tilde{v}_j)$.

Step 2. If we consider the hyperoperation "$\circ$" associated with $\widetilde{F}$, defined as follows:

$$\forall (\tilde{u}, \tilde{v}) \in \widetilde{J}^2,\ \tilde{u} \circ \tilde{v} = \{\tilde{w} \mid \exists x \in H \text{ such that}$$
$$\widetilde{F}_x(\tilde{u}) \wedge \widetilde{F}_x(\tilde{v}) \leq \widetilde{F}_x(\tilde{w}) \leq \widetilde{F}_x(\tilde{u}) \vee \widetilde{F}_x(\tilde{v})\}$$

then the hypergroupoids $(\widetilde{J}, \circ)$ and $(\widetilde{J}, \Box)$ coincide.

Indeed, if we suppose $\widetilde{F}_x(\tilde{u}) < \widetilde{F}_x(\tilde{v})$ then $\forall j \in I$, $F_{jx}(\tilde{u}_j) < F_{jx}(\tilde{v}_j)$. Hence

$$\begin{array}{rcl}
\tilde{u} \circ \tilde{v} & = & \{\tilde{w} \mid \exists x \in H,\ \widetilde{F}_x(\tilde{u}) \leq \widetilde{F}_x(\tilde{w}) \leq \widetilde{F}_x(\tilde{v})\} = \\
& = & \{\tilde{w} \mid \exists x \in H,\ \forall j \in I,\ F_{jx}(\tilde{u}_j) \leq F_{jx}(\tilde{w}_j) \leq\ F_{jx}(\tilde{v}_j)\} = \\
& \overset{\underline{step\ 1}}{=\!=} & \{\tilde{w} \mid \exists x \in H,\ \exists i \in I,\ F_{ix}(\tilde{u}_i) \leq F_{ix}(\tilde{w}_i) \leq\ F_{ix}(\tilde{v}_i)\} = \\
& = & \{\tilde{w} \mid \exists i \in I,\ \tilde{w}_i \in \tilde{u}_i \circ_i \tilde{v}_i\} = \tilde{u} \odot \tilde{v} \cap \widetilde{J} = \tilde{u} \Box \tilde{v},
\end{array}.$$

It follows that $(\widetilde{J}, \circ) and$ $(\widetilde{J}, \Box)$ coincide. $\widetilde{J}$ is supposed to be a subset of $[0,1]$, but, since $\widetilde{J}$ has the same cardinality with a subset $\widetilde{J}'$ of $[0,1]$, it is clear we can set $\widetilde{F}'_x(\widetilde{u}') = \widetilde{F}_x(\widetilde{u})$ and so, the hyperoperation is independent of the nature of the elements of $\widetilde{J}$. □

Example 3.3. There are many ways to choose $\widetilde{F}$. For instance, for all $x \in X$, we may consider

1. $\forall \tilde{u} \in \widetilde{J},\ \widetilde{F}_x(\tilde{u}) = F_{i_0 x}(\tilde{u}_{i_0})$ for some $i_0 \in I$.
2. $\forall \tilde{u} \in \widetilde{J},\ \widetilde{F}_x(\tilde{u}) = \sum_{i \in I'} F_{ix}(\tilde{u}_i)/|I'|$ where I' is a finite subset of I and $|I'|$ is the cardinal of I'.

Remark 3.4. The above results also hold if we consider an arbitrary real interval instead of $[0,1]$.

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