# ON TOPOLOGICAL EQ-ALGEBRAS

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Abstract. In this paper, by using a special family of filters $\mathcal{F}$ on an EQ-algebra $E$, we construct a topology $\mathcal{T}_{\mathcal{F}}$ on $E$ and show that $(E, \mathcal{T}_{\mathcal{F}})$ is a topological EQ-algebra. First of all, we give some properties of topological EQ-algebras and investigate the interaction of topological EQ-algebras and quotient topological EQ-algebras. Then we obtain the form of closure of each subset and show that $(E, \mathcal{T}_{\mathcal{F}})$ is a zero-dimensional space. Finally, we introduce the concept of convergence of sequences on topological EQ-algebras and give a condition under which the limit of a sequence is unique.

## 1. Introduction

Fuzzy logic is a logic to deal with uncertainty and approximate reasoning [22]. The well-known fuzzy logics include Łukasiewicz logic [3], BL-logic [12], MTL-logic [9] and $R_0$-logic [24], etc. The corresponding algebraic semantics are MV-algebras, BL-algebras, MTL-algebras and $R_0$-algebras, respectively. Note that the typical operations on these algebras are multiplication $\odot$ and implication $\rightarrow$ which are closely tied by adjointness property. EQ-algebras were proposed by Novák and De Baets [21] with the intent to develop an algebraic structure of truth values a higher-order fuzzy logic (a fuzzy type theory, FTT). EQ-algebras have three basic binary operations (meet, multiplication and fuzzy equality) and a top element. The motivation stems from the fact that until now, the truth values in FTT were assumed to form either an IMTL-algebra, a BL-algebra, or an MV-algebra, all of them are special kinds of residuated lattices in which the basic operations are the monoidal operation (multiplication) and its residuum. The latter is a natural interpretation of implication in fuzzy logic and the equivalence is then interpreted by the biresiduum, a derived operation. From the algebraic point of view, the class of EQ-algebras generalizes, in a certain sense, the class of residuated lattices and so they may become an interesting class of algebraic structures as such. There exists a list of references related to EQ-algebras, for instance [7, 14, 15, 16].

In the past ten years, many mathematicians have studied properties of algebras endowed with topologies. Di Nola and Leustean[6] defined BL-sheaf spaces and BL-algebras of global sections, and studied completely regular and compact BL-sheaf spaces. Ciungu [4] investigated some concepts of convergence in the class of perfect BL-algebras, Mi Ko and Kim [17] studied relationships between closure operators and BL-algebras and Haveshki et al. [11] applied filters to construct topologies on

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BL-algebras. They studied the properties of these topologies such as compactness regarding different filters. In [2] and [1], Borzooei et al. studied metrizability on (semi)topological BL-algebras and the relationship between separation axioms and (semi)topological quotient BL-algebras. In [13], Hoo introduced topological MV-algebras and studied some of its topological properties. Hoo's work reveals that the essential ingredients are the existence of an adjoint pair of operations and the fact that ideals of MV-algebras correspond to its congruences. Nganou [20] generalized Hoo's work in a more general context where similar ingredients are available, namely $\mathrm{FL}_{ew}$-algebras. Recently, Najafi et al. [19] studied (para, quasi)topological MV-algebras and discussed the relationship among semitopological MV-algebras, paratopological MV-algebras, quasitopological MV-algebras and topological MV-algebras. Zahiri and Borzooei [23] used a special family of filters on a BL-algebra $L$ to construct a topology $\tau$ such that all operations of $L$ are continuous with respect to $\tau$, that is, the pair $(L, \tau)$ is a topological BL-algebra. Also, some interesting results of $(L, \tau)$ were investigated, in particular, they introduced complete BL-algebra and constructed a complete BL-algebra from $L$.

It is well known that the logic background of EQ-algebras is different from other logic algebras mentioned above. We know that the current research of topological logic algebras was mainly based on filters. Given a prefilter $F$ on an EQ-algebra $E$, as usual, we define a relation $\approx_F$ on $E$ by $a \approx_F b$ if and only if $a \sim b \in F$, then $\approx_F$ is an equivalence relation, but it is not a congruence (see [7, Remark 1]). Based on the about reason Nov$\acute{a}$k[21] defined the filter on $E$ to add a condition, that is, if $a \to b \in F$, then $a \otimes c \to b \otimes c \in F$ for any $c \in E$, his main purpose is to make the equivalence relation becomes a congruence. As is well-known, there is a one-to-one correspondence between the set of all filters and the set of all congruences on the logic algebras based on residuated lattices, however, this property is not true in EQ-algebras. In fact, if $F$ is a filter of a residuated lattice $L$, we define a relation $\theta_F$ on $L$ by $(x, y) \in \theta_F$ if and only if $x \to y \in F$ and $y \to x \in F$, then $\theta_F$ is a congruence. Conversely, let $\theta$ be a congruence on $L$. Then the congruence class of an element 1 with respect to $\theta$, namely $[1]_\theta$ is a filter of $L$. However, we shall give an example to show that this property is generally incorrect in EQ-algebras (see Example 2.7.). For this reason, studying topological EQ-algebras is different from other topological logical algebras based on residuated lattices. From the algebraic point of view, EQ-algebras slightly generalize residuated lattices[21]. Indeed, in a residuated lattice $\mathcal{L} = (L, \wedge, \vee, \otimes, \to, 0, 1)$ when we define a biresiduation operation $\leftrightarrow$ by $x \leftrightarrow y := (x \to y) \wedge (y \to x)$, then $(L, \wedge, \otimes, \leftrightarrow, 1)$ is an EQ-algebra. Meanwhile, we observe that topological EQ-algebras (which are EQ-algebras with topologies such that all operations are continuous with respect to these topologies), in a certain sense, generalize topological residuated lattices. More explicitly, if $(\mathcal{L}, \tau)$ is a topological residuated lattice, then $(\mathcal{L}', \tau)$ is a topological EQ-algebra, where $\mathcal{L} = (L, \wedge, \vee, \otimes, \to, 0, 1)$ and $\mathcal{L}' = (L, \wedge, \otimes, \leftrightarrow, 1)$. It is sufficient to show that $\leftrightarrow$ is continuous for $\tau$. For this, we prove that the map $h : L \times L \to L$ defined by $h(x, y) = x \leftrightarrow y$ for any $x, y \in L$ is continuous. For $a \in L$, we define maps $f_a : L \to L$ and $k : L \times L \to L$ by $f_a(x) = a \to x$ and $k(x, y) = x \wedge y$ for any $x, y \in L$. It is easy to verify that maps $k$ and $f_a \times f_b$ are continuous where $f_a \times f_b$ is defined by $f_a \times f_b(x, y) = (a \to x, b \to y)$.

Since $h(x,y) = (k \circ (f_y \times f_x))(x,y)$ for any $x, y \in L$, it follows that $h$ is continuous. For detailed about topological residuated lattices we refer to[10]. Therefore, it is interesting to study topological EQ-algebras.

In this paper, we focus on a special type of topology induced by system of filters on an EQ-algebra. This paper is organized as follows: In Section 2, we review some facts about EQ-algebras and topologies, used in the sequel. In Section 3, we use a system of filters $\mathcal{F} = \{F_i \mid i \in \Lambda\}$ of an EQ-algebra $E$ to construct a topology $\mathcal{T}_{\mathcal{F}}$, prove that $(E, \mathcal{T}_{\mathcal{F}})$ is a topological EQ-algebra, and find the form of closure of each subset in this topological space. Then we study a special class of topological EQ-algebras, namely quotient topological EQ-algebras and show that the topological EQ-algebra $(E, \mathcal{T}_{\mathcal{F}})$ is a zero-dimensional space. Also, we show that in separated EQ-algebras the topology $\mathcal{T}_{\mathcal{F}}$ induced by $\mathcal{F}$ is Hausdorff if and only if $\bigcap \mathcal{F} = \{1\}$, and give an example to show that separability is necessary. Finally, the convergence of sequences on topological EQ-algebras is investigated.

## 2. **Preliminaries**

In this section, we summarize some definitions and results about EQ-algebras and topologies, which will be used in the following sections.

**Definition 2.1.** [15, 21] An EQ-algebra is an algebra $\mathcal{E} = (E, \wedge, \otimes, \sim, 1)$ of type (2,2,2,0) satisfying the following axioms: for any $x, y, z, t \in E$,

(E1) $(E, \wedge, 1)$ is a $\wedge$-semilattice with top element 1. We put $x \leq y$ if and only if $x \wedge y = x$;
(E2) $(E, \otimes, 1)$ is a commutative monoid and $\otimes$ is isotone with respect to $\leq$;
(E3) $x \sim x = 1$ ( reflexivity axiom );
(E4) $((x \wedge y) \sim z) \otimes (t \sim x) \leq z \sim (t \wedge y)$ (substitution axiom);
(E5) $(x \sim y) \otimes (z \sim t) \leq (x \sim z) \sim (y \sim t)$ (congruence axiom);
(E6) $(x \wedge y \wedge z) \sim x \leq (x \wedge y) \sim x$ (monotonicity axiom);
(E7) $x \otimes y \leq x \sim y$ (boundedness axiom).

We also put $x \to y = (x \wedge y) \sim x$ and $\widetilde{x} = x \sim 1$ for all $x, y \in E$.

In the following, we simply write an EQ-algebra $E$ instead of $(E, \wedge, \otimes, \sim, 1)$ when there is no chance for confusion.

**Proposition 2.2.** [8, 21] *Let $E$ be an EQ-algebra. Then the following properties hold: for any $x, y, z \in E$,*

(i) $x \otimes y \leq x \wedge y \leq x, y$;
(ii) $z \otimes (x \wedge y) \leq (z \otimes x) \wedge (z \otimes y)$;
(iii) $x \sim y \leq x \to y$;
(iv) $(x \sim y) \otimes (y \sim z) \leq x \sim z$;
(v) $(x \to y) \otimes (y \to z) \leq x \to z$;
(vi) $x \leq \widetilde{x}$, $\widetilde{1} = 1$;
(vii) $x \otimes (x \sim y) \leq \widetilde{y}$;
(xiii) $(x \to y) \otimes (y \to x) \leq x \sim y$.

**Definition 2.3.** [8, 21] Let $E$ be an EQ-algebra. We say that it is

(i) separated if for all $a, b \in E$, $a \sim b = 1$ implies $a = b$;
(ii) good if for all $a \in E$, $a \sim 1 = a$.

**Definition 2.4.** [8, 21] Let $E$ be an EQ-algebra. A subset $F$ of $E$ is called an EQ-filter (filter for short) of $E$ if for all $a, b, c \in E$ it holds that

(i) $1 \in F$;
(ii) if $a, a \to b \in F$, then $b \in F$;
(iii) if $a \to b \in F$, then $a \otimes c \to b \otimes c \in F$.

**Lemma 2.5.** [21] *Let $F$ be a filter of an EQ-algebra $E$. For all $a, b, a', b' \in F$ such that $a \sim b$ and $a' \sim b' \in F$ it holds that*

(a) $(a \wedge a') \sim (b \wedge b') \in F$;
(b) $(a \otimes c) \sim (b \otimes c) \in F$;
(c) $(a \sim a') \sim (b \sim b') \in F$.

Let $F$ be a filter of an EQ-algebra $E$. Define a binary relation $\approx_F$ on $E$ by

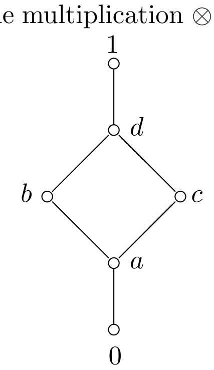
$$a \approx_F b \text{ if and only if } a \sim b \in F.$$

**Theorem 2.6.** [21] *Let $F$ be a filter of an EQ-algebra $E$. The relation $\approx_F$ is a congruence on $E$.*

If $F$ is a filter of an EQ-algebra $E$, the quotient EQ-algebra induced by the congruence $\approx_F$ will be denoted by $E/F$. In addition, the congruence class of an element $a \in E$ with respect to $\approx_F$ is often denoted by $a/F$, and $P_F$ denotes the natural projection $E \to E/F$.

In general, the congruence class of an element 1 in an EQ-algebra is not a filter.

**Example 2.7.** Consider $E = \{0, a, b, c, d, 1\}$ with a $\wedge$-semilattice structure defined by following Hasse diagram. The multiplication $\otimes$ and fuzzy equality $\sim$ are defined as follows:




Then $< E, \wedge, \otimes, \sim, 1 >$ is an EQ-algebra[21]. It is easily checked that $\theta = \{(0, 0), (a, a), (b, b), (c, c), (d, d), (1, 1)\}$ is a congruence on $E$. Clearly, $[1]_\theta = \{1\}$ is not a filter of $E$, because $1, 1 \to d = (1 \wedge d) \sim 1 = 1 \in [1]_\theta$ but $d \notin [1]_\theta$, which contradicts Definition 2.4 (ii).

**Definition 2.8.** [5] A poset $(D, \leq)$ is called an upward directed set if for any $x, y \in D$ there exists $z \in D$ such that $x \leq z$ and $y \leq z$. Dually, $(D, \leq)$ is called a down-directed set if for any $x, y \in D$ there exists $z \in D$ such that $z \leq x$ and $z \leq y$.

| $\otimes$ | 0 | $a$ | $b$ | $c$ | $d$ | 1 |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $a$ | 0 | 0 | 0 | 0 | 0 | $a$ |
| $b$ | 0 | 0 | $a$ | $a$ | $a$ | $b$ |
| $c$ | 0 | 0 | $a$ | 0 | $a$ | $c$ |
| $d$ | 0 | 0 | $a$ | $a$ | $a$ | $d$ |
| 1 | 0 | $a$ | $b$ | $c$ | $d$ | 1 |

| $\sim$ | 0 | $a$ | $b$ | $c$ | $d$ | 1 |
|---|---|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| $a$ | 0 | 1 | $d$ | $d$ | $d$ | $d$ |
| $b$ | 0 | $d$ | 1 | $d$ | $d$ | $d$ |
| $c$ | 0 | $d$ | $d$ | 1 | $d$ | $d$ |
| $d$ | 0 | $d$ | $d$ | $d$ | 1 | 1 |
| 1 | 0 | $d$ | $d$ | $d$ | 1 | 1 |

Let $\tau$ and $\tau'$ be two topologies on a set $X$. If $\tau \subseteq \tau'$, then we say that $\tau'$ is finer than $\tau$.

**Lemma 2.9.** [18] *Let $\beta$ and $\beta'$ be two bases for topologies $\tau$ and $\tau'$ on $X$, respectively. Then $\tau'$ finer than $\tau$ if and only if for any $x \in X$ and each basis element $B \in \beta$ containing $x$, there is a basis element $B' \in \beta'$ such that $x \in B' \subseteq B$.*

Let $(X, \tau)$ be a topological space. We have following separation axioms in $(X, \tau)$

$T_0$ : for any $x, y \in X$ and $x \neq y$, there is at least one in an open neighborhood excluding the other;

$T_1$ : for any $x, y \in X$ and $x \neq y$, each has an open neighborhood not containing the other;

$T_2$ : for any $x, y \in X$ and $x \neq y$, both have disjoint open neighborhoods $U, V \in \tau$ such that $x \in U$ and $y \in V$.

A topological space satisfying $T_i$ is called a $T_i$-space, for any $i = 0, 1, 2$. A $T_2$-space is also known as a Hausdorff space.

**Theorem 2.10.** [18] *Every finite point set in a Hausdorff space is closed.*

## 3. Topological EQ-algebras

The main purpose of this section is to endow a special topology $\mathcal{T}_{\mathcal{F}}$ on an EQ-algebra $E$ such that all operations of $E$ are continuous with respect to $\mathcal{T}_{\mathcal{F}}$, and some properties of $E$ with this topology are investigated.

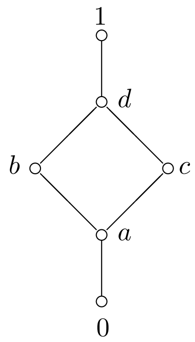
**Definition 3.1.** Let $E$ be an EQ-algebra and $\mathcal{T}$ be a topology on it. The pair $(E, \mathcal{T})$ is called a topological EQ-algebra (TEQ-algebra for short) if the operations $\wedge, \otimes$ and $\sim$ are continuous with respect to $\mathcal{T}$.

Note that operation $* \in \{\wedge, \otimes, \sim\}$ is continuous if and only if for any $x, y \in E$ and any open neighborhood $W$ of $x * y$, there exist two open neighborhoods $U$ and $V$ of $x$ and $y$, respectively, such that $U * V \subseteq W$.

Recall that a topological space $(X, \mathcal{U})$ is a discrete space if for any $x \in X$, $\{x\}$ is an open set (see [18]).

**Example 3.2.** Any EQ-algebra with a discrete topology is a TEQ-algebra. Let $\mathcal{T}$ be a discrete topology on an EQ-algebra $E$. Routine calculation shows that $(E, \mathcal{T})$ is a TEQ-algebra.

**Example 3.3.** Let $E = \{0, a, b, c, d, 1\}$ be a $\wedge$-semilattice which can be seen below Hasse diagram. Consider Cayley tables as follows:




| $\otimes$ | 0 | $a$ | $b$ | $c$ | $d$ | 1 |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $a$ | 0 | 0 | 0 | 0 | $a$ | $a$ |
| $b$ | 0 | 0 | 0 | 0 | $b$ | $b$ |
| $c$ | 0 | 0 | 0 | 0 | $c$ | $c$ |
| $d$ | 0 | $a$ | $b$ | $c$ | $d$ | $d$ |
| 1 | 0 | $a$ | $b$ | $c$ | $d$ | 1 |

| $\sim$ | 0 | $a$ | $b$ | $c$ | $d$ | 1 |
|---|---|---|---|---|---|---|
| 0 | 1 | 1 | $a$ | $a$ | $a$ | $a$ |
| $a$ | 1 | 1 | $a$ | $a$ | $a$ | $a$ |
| $b$ | $a$ | $a$ | 1 | $c$ | $b$ | $b$ |
| $c$ | $a$ | $a$ | $c$ | 1 | $c$ | $c$ |
| $d$ | $a$ | $a$ | $b$ | $c$ | 1 | $d$ |
| 1 | $a$ | $a$ | $b$ | $c$ | $d$ | 1 |

Routine calculation shows that $< E, \wedge, \otimes, \sim, 1 >$ is an EQ-algebra. We give a non-trivial topology $\mathcal{T} = \{\emptyset, \{0, a\}, \{b\}, \{c\}, \{d\}, \{1\}, \{b, c\}, \{b, d\}, \{b, 1\}, \{c, d\}, \{c, 1\}, \{d, 1\}, \{0, a, b\}, \{0, a, c\}, \{0, a, d\}, \{0, a, 1\}, \{b, c, d\}, \{b, c, 1\}, \{b, d, 1\}, \{c, d, 1\}, \{0, a, b, c\}, \{0, a, b, d\}, \{0, a, b, 1\}, \{0, a, c, d\}, \{0, a, c, 1\}, \{0, a, d, 1\}, \{b, c, d, 1\}, \{0, a, b, c, d, 1\}\}$ on $E$ induced by a base $\beta = \{\{0, a\}, \{b\}, \{c\}, \{d\}, \{1\}\}$. Then it is easy to check that $(E, \mathcal{T})$ is a TEQ-algebra.

**Definition 3.4.** Let $\mathcal{F}$ be a family of filters of an EQ-algebra $E$. We call $\mathcal{F}$ a system of filters of $E$ (system for short) if $(\mathcal{F}, \subseteq)$ is a down-directed set.

**Remark 3.5.** Note that in an EQ-algebra $E$ the definition of system denoted by $\mathcal{F} = \{F_i \mid i \in \Lambda\}$ is equivalent to the index set $\Lambda$ is an upward directed set and it satisfies if $i \leq j$ implies $F_j \subseteq F_i$ for any $i, j \in \Lambda$. In fact, suppose that poset $(\mathcal{F}, \subseteq)$ is a down-directed set, we define a relation on index set $\Lambda$ by $i \leq j$ if and only if $F_j \subseteq F_i$, then it is easy to verify that $(\Lambda, \leq)$ is an upward directed set. The converse is obviously.

**Proposition 3.6.** *Let $E$ be an EQ-algebra and $\mathcal{F}$ be a family of filters of $E$. If $\mathcal{F}$ is closed with respect to finite intersection, then $\mathcal{F}$ is a system of $E$.*

*Proof.* Let $\mathcal{F} = \{F_i \mid i \in I\}$ be a family of filters of $E$ which is closed with respect to finite intersection. It is sufficient to show that $(\mathcal{F}, \subseteq)$ is a down-directed set. Assume $F_i, F_j \in \mathcal{F}$, then $F_i \cap F_j \in \mathcal{F}$. Setting $F_k = F_i \cap F_j$, we conclude that $F_k \subseteq F_i, F_j$. Therefore, $(\mathcal{F}, \subseteq)$ is a down-directed set. $\square$

Recall that $\mathcal{T}_x$ denotes the set of all neighborhoods of $x$ in a topological space $(A, \mathcal{T})$. Then subfamily $\mathcal{V}_x$ of $\mathcal{T}_x$ is called a fundamental system of neighborhoods of $x$, if for each $U_x$ in $\mathcal{T}_x$, there exists a $V_x$ in $\mathcal{V}_x$ such that $V_x \subseteq U_x$ (see [18]).

**Proposition 3.7.** *Let $\mathcal{F} = \{F_i \mid i \in \Lambda\}$ be a system of an EQ-algebra $E$. Then there exists a topology $\mathcal{T}$ on $E$ with a base $\beta = \{x/F_i \mid x \in E, i \in \Lambda\}$. Moreover, $\mathcal{F}$ is a fundamental system of neighborhoods of 1.*

*Proof.* Let $\mathcal{F} = \{F_i \mid i \in \Lambda\}$ be a system of $E$ and $\beta = \{x/F_i \mid x \in E, i \in \Lambda\}$. Now we prove that $\mathcal{T} = \{U \subseteq E : \forall a \in U, \exists y/F_i \in \beta \ s.t. \ a \in y/F_i \subseteq U\}$ is a topology on $E$. Clearly, $\emptyset, A \in \mathcal{T}$. Let $\{U_\alpha\}$ be a subfamily of $\mathcal{T}$ and $a \in \bigcup U_\alpha$. Then $a \in U_\alpha$ for some $\alpha$. Hence there exists $y/F_i \in \beta$ such that $a \in y/F_i \subseteq U_\alpha$. It follows that $\bigcup U_\alpha \in \mathcal{T}$. Let $U_\alpha, U_\beta \in \mathcal{T}$ and $a \in U_\alpha \cap U_\beta$. Then there exist $y_1/F_i \in \beta$ and $y_2/F_j \in \beta$ such that $a \in y_1/F_i \subseteq U_\alpha$ and $a \in y_2/F_j \subseteq U_\beta$. Since $(\mathcal{F}, \subseteq)$ is a down-directed set and $F_i, F_j \in \mathcal{F}$, there exists $F_k \in \mathcal{F}$ such that $F_k \subseteq F_i, F_j$. Hence $F_k \subseteq F_i \cap F_j$. Now, we have

$$a \in a/F_k \subseteq (a/F_i) \cap (a/F_j) = (y_1/F_i) \cap (y_2/F_j) \subseteq U_\alpha \cap U_\beta.$$

It follows that $U_\alpha \cap U_\beta \in \mathcal{F}$. Clearly, $\beta$ is a base for $\mathcal{T}$. Now we prove that $\mathcal{F}$ is a fundamental system of neighborhoods of 1. Let $1 \in U \in \mathcal{T}$. Then there exists $y/F_i \in \beta$ such that $1 \in y/F_i \subseteq U$. Thus we can get $1 \in F_i = 1/F_i = y/F_i \subseteq U$. $\square$

From now on, if $\mathcal{F} = \{F_i \mid i \in \Lambda\}$ is a system of filters of an EQ-algebra $E$, then the topology in Proposition 3.7 will be denoted by $\mathcal{T}_\mathcal{F}$. We say that $\mathcal{T}_\mathcal{F}$ is a topology on $E$ induced by $\mathcal{F}$, unless otherwise stated.

**Example 3.8.** Let $E = \{0, a, b, 1\}$ be a chain $(0 < a < b < 1)$ with Cayley tables as follows:

| $\otimes$ | 0 | $a$ | $b$ | 1 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| $a$ | 0 | 0 | $a$ | $a$ |
| $b$ | 0 | $a$ | $b$ | $b$ |
| 1 | 0 | $a$ | $b$ | 1 |

| $\sim$ | 0 | $a$ | $b$ | 1 |
|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 0 |
| $a$ | 0 | 1 | $a$ | $a$ |
| $b$ | 0 | $a$ | 1 | 1 |
| 1 | 0 | $a$ | 1 | 1 |

Routine calculation shows that $< E, \wedge, \otimes, \sim >$ is an EQ-algebra. It is easy to check that $\mathcal{F} = \{F_i \mid i \in \Lambda\}$ is a system of $E$, where $\Lambda = \{1, 2\}$, $F_1 = \{0, a, b, 1\}$ and $F_2 = \{b, 1\}$. So we can easily calculate $\beta = \{\{0\}, \{a\}, \{b, 1\}, \{0, a, b, 1\}\}$ and $\mathcal{T}_\mathcal{F} = \{\emptyset, \{0\}, \{a\}, \{0, a\}, \{b, 1\}, \{0, b, 1\}, \{a, b, 1\}, \{0, a, b, 1\}\}$ is a topology on $E$ with a base $\beta$.

**Theorem 3.9.** *Let $\mathcal{F} = \{F_i \mid i \in \Lambda\}$ be a system of an EQ-algebra $E$ and $\mathcal{T}_\mathcal{F}$ be a topology on $E$ induced by $\mathcal{F}$. Then $(E, \mathcal{T}_\mathcal{F})$ is a TEQ-algebra.*

*Proof.* According to Proposition 3.7, $\mathcal{T}_\mathcal{F}$ is a topology on $E$ with a base $\beta = \{x/F_i \mid x \in E, i \in \Lambda\}$. Let $* \in \{\wedge, \otimes, \sim\}$ and $f_* : E \times E \to E$ be a map which is defined by $f_*(x, y) = x * y$ for any $x, y \in E$. Since $\beta$ is a base for $\mathcal{T}_\mathcal{F}$, it is sufficient to show that $f_*^{-1}(b/F_i)$ is an open subset of $E \times E$ for any $b/F_i \in \beta$. Let $(x, y) \in f_*^{-1}(b/F_i)$.

Then $x * y \in b/F_i$. Clearly, $x/F_i \times y/F_i$ is an open subset of $E \times E$ containing $(x, y)$. We shall show that $x/F_i \times y/F_i \subseteq f_*^{-1}(b/F_i)$. For any $(u, v) \in x/F_i \times y/F_i$, we have $u \in x/F_i,\ v \in y/F_i$, that is, $x \approx_{F_i} u, y \approx_{F_i} v$. Due to $\approx_{F_i}$ is a congruence of $E$, we get $x * y \approx_{F_i} u * v$. Thus $u * v \in b/F_i$. Hence $x/F_i \times y/F_i \subseteq f_*^{-1}(b/F_i)$. It follows that $f_*^{-1}(b/F_i)$ is open subset of $E \times E$ and so $f_*$ is a continuous map. Therefore, $(E, \mathcal{T}_\mathcal{F})$ is a TEQ-algebra. □

By Proposition 3.7 and Theorem 3.9, the pair $(E, \mathcal{T}_\mathcal{F})$ is a TEQ-algebra, where $\mathcal{T}_\mathcal{F}$ is a topology induced by a system $\mathcal{F} = \{F_i \mid i \in \Lambda\}$. For simply, we always use a sentence $(E, \mathcal{T}_\mathcal{F})$ is a TEQ-algebra induced by a system $\mathcal{F} = \{F_i \mid i \in \Lambda\}$ replace if there is no chance for confusion.

Let $(E,\mathcal{U})$ be a TEQ-algebra and $F$ be a filter of $E$. We define $\wedge, \otimes$ and $\sim$ on $E/F$ by

$$(a/F) \wedge (b/F) = (a \wedge b)/F,$$
$$(a/F) \otimes (b/F) = (a \otimes b)/F,$$
$$(a/F) \sim (b/F) = (a \sim b)/F.$$

It is easy to check that $< E/F, \wedge, \otimes, \sim, 1 >$ is an EQ-algebra. In the following we give conditions under which $E/F$ becomes a TEQ-algebra.

**Proposition 3.10.** *Let $(E,\mathcal{U})$ be a TEQ-algebra and $F$ be a filter of $E$. If natural projection $P_F$ is an open map, then the quotient EQ-algebra $E/F$ equipped with the quotient topology is a TEQ-algebra.*

*Proof.* It is sufficient to prove that $(x/F, y/F) \mapsto x/F * y/F = (x * y)/F$ is continuous with respect to quotient topology, where $* \in \{\wedge, \otimes, \sim\}$. Let $x, y \in E$ and $W$ be an open neighborhood of $(x * y)/F$. Then $P_F^{-1}(W)$ is an open subset in $E$ and $x * y \in P_F^{-1}(W)$. Since $(E,\mathcal{U})$ is a TEQ-algebra, there exist open neighborhoods $U_0$ of $x$ and $V_0$ of $y$, respectively, such that $U_0 * V_0 \subseteq P_F^{-1}(W)$. Taking $U = P_F(U_0)$ and $V = P_F(V_0)$. Then $U$ and $V$ are open subsets in $E/F$, since $P_F$ is open. Clearly, $x/F \in U$, $y/F \in V$ and then $U * V \subseteq P_F(U_0 * V_0) \subseteq W$. Therefore, $*$ is continuous. □

**Theorem 3.11.** *Let $(E, \mathcal{T}_\mathcal{F})$ be a TEQ-algebra induced by a system $\mathcal{F} = \{F_i \mid i \in \Lambda\}$ and $F$ be a filter of $E$. If $F \subseteq \bigcap\{F_i \mid i \in \Lambda\}$, then the natural projection $P_F : E \to E/F$ is an open map.*

*Proof.* Let $\mathcal{F} = \{F_i \mid i \in \Lambda\}$ and $\beta = \{x/F_i \mid x \in E, i \in \Lambda\}$. By Proposition 3.7, $\beta$ is a base for $\mathcal{T}_\mathcal{F}$, so that it is sufficient to show that $P_F(x/F_i)$ is an open subset in $E/F$ for any $x/F_i \in \beta$. Suppose $x/F_i \in \beta$. We show that $P_F^{-1}(P_F(x/F_i)) \in \mathcal{T}_\mathcal{T}$. Let $a$ be an arbitrary element of $P_F^{-1}(P_F(x/F_i))$. Then $P_F(a) \in P_F(x/F_i)$ and so $a/F \in (x/F_i)/F$. Hence there exists $b \in x/F_i$ such that $a/F = b/F$, that is, $a \approx_F b$. Since $b \approx_{F_i} x$ and $F \subseteq F_i$, we get $a \approx_{F_i} b$ and $b \approx_{F_i} x$. Thanks to $\approx_{F_i}$ is a congruence, it follows that $a \approx_{F_i} x$. Hence $a \in x/F_i$ and so $P_F^{-1}(P_F(x/F_i)) \subseteq x/F_i$. Clearly, $x/F_i \subseteq P^{-1}(P_F(x/F_i))$. Therefore, $P_F^{-1}(P_F(x/F_i)) = x/F_i \in \mathcal{T}_\mathcal{F}$. So $P_F$ is an open map. □

**Corollary 3.12.** *Let $(E, \mathcal{T}_{\mathcal{F}})$ be a TEQ-algebra induced by a system $\mathcal{F} = \{F_i \mid i \in \Lambda\}$ and $F$ be a filter of $E$. If $F \subseteq \bigcap\{F_i \mid i \in \Lambda\}$, then quotient EQ-algebra $E/F$ equipped with the quotient topology is a TEQ-algebra.*

*Proof.* It is directly follows from Proposition 3.10 and Theorem 3.11. □

**Proposition 3.13.** *Let $(E, \mathcal{T}_{\mathcal{F}})$ be a TEQ-algebra induced by a system $\mathcal{F} = \{F_i \mid i \in \Lambda\}$ and $F$ be a filter of $E$. If $F$ is open, then the quotient topology on $E/F$ is discrete.*

*Proof.* Note that for every $x \in E$, the equality $P_F^{-1}(\{x/F\}) = x/F$ holds. Therefore, it is enough to prove that $x/F$ is open for topology $\mathcal{T}_{\mathcal{F}}$ for any $x \in E$. Let $x \in E$. We define $f_x : E \to E$ by $f_x(y) = x \sim y$. Then $f_x$ is continuous. Since $F$ is open, we get $f_x^{-1}(F)$ is open. Clearly, $x/F = \{y \in E \mid x \sim y \in F\} = f_x^{-1}(F)$. Therefore, $x/F$ is an open set. □

**Proposition 3.14.** *Let $(E, \mathcal{T}_{\mathcal{F}})$ be a TEQ-algebra induced by a system $\mathcal{F} = \{F_i \mid i \in \Lambda\}$ and $F$ be a filter of $E$. If $F$ is open (closed), then for each $x \in E$, $x/F$ is open (closed).*

*Proof.* When $F$ is open, the proof has been given in the proof of Proposition 3.13. The case $F$ closed is handled similarly. □

**Proposition 3.15.** *Let $(E, \mathcal{T}_{\mathcal{F}})$ be a TEQ-algebra induced by a system $\mathcal{F} = \{F_i \mid i \in \Lambda\}$. Then every open filter is closed.*

*Proof.* Suppose that a filter $F$ is open. Then by Proposition 3.14, $x/F$ is open for any $x \in E$. It follows that $F = E \setminus \bigcup\{x/F \mid x \notin F\}$ is closed. □

Recall that a topological space $(X, \mathcal{U})$ is a zero-dimensional space if $\mathcal{U}$ has a clopen base (see [18]).

**Theorem 3.16.** *Let $(E, \mathcal{T}_{\mathcal{F}})$ be a TEQ-algebra induced by a system $\mathcal{F} = \{F_i \mid i \in \Lambda\}$. Then $(E, \mathcal{T}_{\mathcal{F}})$ is a zero-dimensional space.*

*Proof.* Let $\mathcal{F} = \{F_i \mid i \in \Lambda\}$ and $\beta = \{x/F_i \mid x \in E, i \in \Lambda\}$. According to Proposition 3.7, it is enough to show that $x/F_i$ is closed for any $x/F_i \in \beta$. Let $x/F_i \in \beta$. Clearly, $F_i = 1/F_i \in \beta$. By Proposition 3.15, it follows that $F_i$ is closed. According to Proposition 3.14, we conclude that $x/F_i$ is closed □

**Theorem 3.17.** *Let $(E, \mathcal{T}_{\mathcal{F}})$ be a TEQ-algebra induced by a system $\mathcal{F} = \{F_i \mid i \in \Lambda\}$, $S$ be a non-empty subset of $E$ and $S/F_i = \bigcup\{x/F_i \mid x \in S\}$. Then $\overline{S} = \bigcap\{S/F_i \mid i \in \Lambda\}$, where $\overline{S}$ is a topological closure of $S$.*

*Proof.* Let $x \in E$. Then $x \in \overline{S}$
$\Leftrightarrow x \in U$ implies $U \cap S \neq \emptyset$ for any $U \in \mathcal{T}_{\mathcal{F}}$
$\Leftrightarrow x \in a/F_i$ implies $(a/F_i) \cap S \neq \emptyset$ for any $a/F_i \in \beta$
$\Leftrightarrow (x/F_i) \cap S \neq \emptyset$ for any $i \in \Lambda$
$\Leftrightarrow x \in S/F_i$ for any $i \in \Lambda$
$\Leftrightarrow x \in \bigcap\{S/F_i \mid i \in \Lambda\}$. □

**Theorem 3.18.** *Let $(E, \mathcal{T}_{\mathcal{F}})$ be a TEQ-algebra induced by a system $\mathcal{F} = \{F_i \mid i \in \Lambda\}$. If $(E, \mathcal{T}_{\mathcal{F}})$ is a Hausdorff space, then $\bigcap\{F_i \mid i \in \Lambda\} = \{1\}$. The converse is true if $E$ is separated.*

*Proof.* Let $(E, \mathcal{T}_{\mathcal{F}})$ be a Hausdorff space. Then by Theorem 2.10, the set $\{1\}$ is closed. Hence according to Theorem 3.17, we have

$$\{1\} = \overline{\{1\}} = \bigcap\{1/F_i \mid i \in \Lambda\} = \bigcap\{F_i \mid i \in \Lambda\}.$$

Conversely, let $E$ be separated. Assume that $\bigcap\{F_i \mid i \in \Lambda\} = \{1\}$ and $x, y$ are distinct elements of $E$. It follows that $x \sim y \notin \bigcap\{F_i \mid i \in \Lambda\} = \{1\}$, and hence there is a $\lambda \in \Lambda$ such that $x \sim y \notin F_\lambda$. So $(x/F_\lambda) \bigcap (y/F_\lambda) = \emptyset$. Therefore, we conclude that $(E, \mathcal{T}_{\mathcal{F}})$ is a Hausdorff space. □

In the following we give an example to show that in Theorem 3.18 separability is necessary.

**Example 3.19.** Consider 6-element EQ-algebra $E$ given in Example 3.3. Note that this EQ-algebra is not separated because $0 \sim a = 1$ but $0 \neq a$. We give a system of $E$ by $\mathcal{F} = \{F_i \mid i \in \Lambda\}$, where $\Lambda = \{1, 2\}$, $F_1 = \{d, 1\}$ and $F_2 = \{1\}$. By Proposition 3.7, the topology $\mathcal{T}_{\mathcal{F}}$ with a base $\beta = \{\{0, a\}, \{b\}, \{c\}, \{d\}, \{1\}\}$. By Theorem 3.9, $(E, \mathcal{T}_{\mathcal{F}})$ is a TEQ-algebra. Clearly, $\mathcal{F}$ satisfies $\bigcap\{F_i \mid i \in \Lambda\} = \{1\}$. However, it is not a Hausdorff space because any open set of $\mathcal{T}_{\mathcal{F}}$ can not separate 0 and $a$.

**Proposition 3.20.** *Let $(E, \mathcal{T}_{\mathcal{F}})$ be a TEQ-algebra induced by a system $\mathcal{F} = \{F_i \mid i \in \Lambda\}$. The following statements are equivalent:*

(i) *$(E, \mathcal{T}_{\mathcal{F}})$ is a Hausdorff space;*
(ii) *$(E, \mathcal{T}_{\mathcal{F}})$ is a $T_1$-space;*
(iii) *$(E, \mathcal{T}_{\mathcal{F}})$ is a $T_0$-space.*

*Proof.* The proofs of (i)⇒(ii) and (ii)⇒(iii) are clear. Let $(E, \mathcal{T}_{\mathcal{F}})$ be a $T_0$-space and $x, y \in E$, $x \neq y$. By assumption, either there exists an open set $U$ containing $x$ but not containing $y$ or there exists open set $V$ containing $y$ but not containing $x$. If the former holds, then there exists $z/F_j$ such that $x \in z/F_j \subseteq U$. We can get $x/F_j = z/F_j$ and $y \notin x/F_j$, and so $(x/F_j) \bigcap (y/F_j) = \emptyset$. The rest situation is similar. Therefore, $(E, \mathcal{T}_{\mathcal{F}})$ is a Hausdorff space. □

**Proposition 3.21.** *Let $\mathcal{F} = \{F_i \mid i \in \Lambda\}$ and $\mathcal{G} = \{G_j \mid j \in \Gamma\}$ be two systems of an EQ-algebra $E$ and $\mathcal{T}_{\mathcal{F}}$, $\mathcal{T}_{\mathcal{G}}$ be the topologies induced by them, respectively. Then $\mathcal{T}_{\mathcal{F}}$ is finer than $\mathcal{T}_{\mathcal{G}}$ if and only if for any $j \in \Gamma$, there exists $i \in \Lambda$ such that $F_i \subseteq G_j$.*

*Proof.* Let $\beta = \{x/F_i \mid x \in E, i \in \Lambda\}$ and $\beta' = \{x/G_j \mid x \in E, j \in \Gamma\}$ be two bases for topologies $\mathcal{T}_{\mathcal{F}}$ and $\mathcal{T}_{\mathcal{G}}$ on $E$, respectively. We prove the sufficiently at first. Let $x \in E$ and $a/G_j$ be an arbitrary element of $\beta'$ containing $x$. Then by the assumption, there is $i \in \Lambda$ such that $F_i \subseteq G_j$. Clearly, $x \in x/F_i$. We claim that $x/F_i \subseteq a/G_j$. Let $u \in x/F_i$. Then $u \sim x \in F_i \subseteq G_j$. We can get $u \sim x \in G_j$ and so $u \approx_{G_j} x$. Since $x \in a/G_j$, we have $x \approx_{G_j} a$. It follows that $u \approx_{G_j} a$, that is, $u \in a/G_j$. Thus $x/F_\lambda \subseteq a/G_i$. Therefore, by Lemma 2.9, $\mathcal{T}_\mathcal{F}$ is finer than $\mathcal{T}_\mathcal{G}$. Now we prove the necessity. Suppose $\mathcal{T}_\mathcal{F}$ is finer than $\mathcal{T}_\mathcal{G}$ and $j \in \Gamma$. Clearly, $1 \in G_j = 1/G_j \in \beta'$. By Lemma 2.9, there exists $x/F_i \in \beta$ such that $1 \in x/F_i \subseteq 1/G_j$. It follows that $x/F_i = 1/F_i = F_i$. Hence $F_i \subseteq G_j$. □

**Theorem 3.22.** *Let $(E, \mathcal{T}_\mathcal{F})$ be a TEQ-algebra induced by a system $\mathcal{F} = \{F_i \mid i \in \Lambda\}$ and $F$ be a filter of $E$. If $F$ is closed, then quotient EQ-algebra $E/F$ equipped with the quotient topology is a TEQ-algebra.*

*Proof.* By Corollary 3.12, it is sufficient to show that $F \subseteq \bigcap\{F_i \mid i \in \Lambda\}$. According to Theorem 3.17, we have $F = \overline{F} = \bigcap\{F/F_i \mid i \in \Lambda\} \subseteq \bigcap\{1/F_i \mid i \in \Lambda\} = \bigcap\{F_i \mid i \in \Lambda\}$. □

**Proposition 3.23.** *Let $(E, \mathcal{T}_\mathcal{F})$ be a TEQ-algebra induced by a system $\mathcal{F} = \{F_i \mid i \in \Lambda\}$, $F$ be a filter of $E$ and $P_F : E \to E/F$ be the nature projection. If $E/F$ equipped with the quotient topology is a Hausdorff space, then $F$ is closed. Moreover, the converse is true if $P_F$ is open.*

*Proof.* Let quotient topology on $E/F$ be Hausdorff and $F$ be a filter of $E$. Suppose $x \in E \setminus F$, then $x/F \neq 1/F$. According to assumption, there exist two open subsets $V, W$ of $E/F$ such that $x/F \in W, 1/F \in V$ and $V \cap W = \emptyset$. Hence $x \in P_F^{-1}(W) \in \mathcal{U}_\mathcal{F}$, $F \subseteq P_F^{-1}(V) \in \mathcal{U}_\mathcal{F}$ and $F \cap P_F^{-1}(W) = \emptyset$. Since $P_F^{-1}(W)$ is an open subset of $E$ satisfies $x \in P_F^{-1}(W) \subseteq E \setminus F$, we deduce that $F$ is a closed subset of $E$. Conversely, let $P_F$ be open and $F$ be closed. Suppose $x/F \neq y/F$. Then we get that $x \sim y \notin F$ and so $x \sim y \in E \setminus F$. By assumption, $x \sim y \notin \overline{F}$. Thus there exists an open subset $V$ of $E$ such that $x \sim y \in V$ and $V \cap F = \emptyset$. Hence $1/F \notin V/F$. Considering a map $f : E \times E \to E$ which is defined by $f(a, b) = a \sim b$ for any $a, b \in E$. Since $(E, \mathcal{T}_\mathcal{F})$ is a TEQ-algebra, we have $f$ is a continuous map and so $P_F \circ f$ is continuous, too. As $P_F$ is an open map, we conclude that $P_F(V) = V/F$ is an open subset of $E/F$. Hence $(P_F \circ f)^{-1}(V/F)$ is an open subset of $E \times E$. By $x \sim y \in V$, we get $(P_F \circ f)(x, y) \in V/F$ and so $(x, y) \in (P_F \circ f)^{-1}(V/F)$. Then there exist $W, A \in \mathcal{T}_\mathcal{F}$ such that $(x, y) \in W \times A \subseteq (P_F \circ f)^{-1}(V/F)$. We get $x \in W, y \in A$ and so $x/F \in W/F, y/F \in A/F$. Clearly, $W/F, A/F$ are open subsets of $E/F$, since $P_F$ is an open map. Now, we show that $(W/F) \cap (A/F) = \emptyset$. Let $z/F \in (W/F) \cap (A/F)$. Then there exist $a \in W$ and $b \in A$ such that $a/F = z/F$ and $z/F = b/F$. It follows that $1/F = (z \sim z)/F = (z/F) \sim (z/F) = (a/F) \sim (b/F) = (a \sim b)/F = P_F(a \sim b) = P_F(f(a, b)) \in (P_F \circ f)(W \times A) \subseteq V/F$, which is a contradiction. Hence $(A/F) \bigcap (W/F) = \emptyset$ and so $E/F$ with quotient topology is a Hausdorff space. □

In the following we introduce notions of Cauchy sequences and convergent sequences in TEQ-algebras $(E, \mathcal{T}_\mathcal{F})$ induced by a system $\mathcal{F} = \{F_i \mid i \in \Lambda\}$.

**Definition 3.24.** Let $(E, \mathcal{T}_\mathcal{F})$ be a TEQ-algebra induced by a system $\mathcal{F} = \{F_i \mid i \in \Lambda\}$. A sequence $\{x_i\}_{i \in \Lambda}$ in $(E, \mathcal{T}_\mathcal{F})$ is called

(ii) convergent to the point $x$ if for any $\lambda \in \Lambda$, there exists $N_\lambda \in \Lambda$ such that $x_i \in x/F_\lambda$ for any $N_\lambda \leq i$ and $i \in \Lambda$;

(ii) Cauchy sequence if for any $\lambda \in \Lambda$, there exists $N_\lambda \in \Lambda$ such that $x_n/F_\lambda = x_m/F_\lambda$ for any $m, n \geq N_\lambda$ and $m, n \in \Lambda$.

If a sequence $\{x_i\}_{i\in\Lambda}$ converges to $x$, we denote $\lim x_i = x$ and say that $x$ is a limit of $\{x_i\}_{i\in\Lambda}$.

**Theorem 3.25.** *Let $(E, \mathcal{T}_\mathcal{F})$ be a TEQ-algebra induced by a system $\mathcal{F} = \{F_i \mid i \in \Lambda\}$ and let $\{x_i\}_{i\in\Lambda}$ and $\{y_i\}_{i\in\Lambda}$ be two sequences of $(E, \mathcal{T}_\mathcal{F})$. Then we have the following:*

(i) *if $(E, \mathcal{T}_\mathcal{F})$ is a Hausdorff space, $E$ is a separated EQ-algebra and $\{x_i\}_{i\in\Lambda}$ is convergent, then its limit is unique;*

(ii) *if $\lim y_i = y$ and $\lim x_i = x$ for some $x, y \in E$, then the sequence $\{x_i * y_i\}_{i\in\Lambda}$ is convergent and $\lim(x_i * y_i) = x * y$ for any operation $* \in \{\wedge, \otimes, \sim\}$;*

(iii) *any convergent sequence of $(E, \mathcal{T}_\mathcal{F})$ is a Cauchy sequence.*

*Proof.* (i) Let $\lim x_i = x$ and $\lim x_i = y$ for some $x, y \in E$. For $\lambda \in \Lambda$, there exist $N_\lambda \in \Lambda$ and $N_\lambda^{'} \in \Lambda$ such that $x_m \in x/F_\lambda$, $x_{m'} \in y/F_\lambda$ for any $m, m^{'} \in \Lambda$ and $N_\lambda \leq m, N_\lambda^{'} \leq m^{'}$. Since $\Lambda$ is an upward directed set, there exists $\mu \in \Lambda$ such that $N_\lambda \leq \mu$ and $N_\lambda^{'} \leq \mu$. If $n \in \Lambda$ and $\mu \leq n$, then $x_n \in x/F_\lambda$ and $x_n \in y/F_\lambda$ and so $x \approx_{F_\lambda} y$, that is, $x \sim y \in F_\lambda$. By Proposition 3.18 and $(E, \mathcal{T}_\mathcal{F})$ is a Hausdorff space, we conclude that $x \sim y \in \bigcap\{F_i \mid i \in \Lambda\} = \{1\}$, and so $x \sim y = 1$. As $E$ is separated, we have $x = y$.

(ii) Let $* \in \{\wedge, \otimes, \sim\}$, $\lim x_i = x$ and $\lim y_i = y$. Suppose that $\lambda \in \Lambda$. Then there exist $N_\lambda, N_\lambda^{'} \in \Lambda$ such that $x_m \in x/F_\lambda$ and $y_{m'} \in y/F_\lambda$ for any $m, m^{'} \in \Lambda$ and $N_\lambda \leq m, N_\lambda^{'} \leq m^{'}$. Since $\Lambda$ is an upward directed set, there exists $N \in \Lambda$ such that $N_\lambda \leq N$ and $N_\lambda^{'} \leq N$. If $n \in \Lambda$ and $N \leq n$, then $x_n \in x/F_\lambda$ and $y_n \in y/F_\lambda$. Hence $x_n \approx_{F_\lambda} x$, $y_n \approx_{F_\lambda} y$. It follows that $x_n * y_n \approx_{F_\lambda} x * y$ for any $n \in \Lambda, N \leq n$. Therefore, $\lim x_n * y_n = x * y$.

(iii) Let $\{x_i\}_{i\in\Lambda}$ converge to the point $x$ of $E$. For $\lambda \in \Lambda$, there is $N_\lambda \in \Lambda$ such that $x_i \in x/F_\lambda$ for any $N_\lambda \leq i$. Suppose that $m, n \in \Lambda$ and $m, n \geq N_\lambda$. Then $x_n \in x/F_\lambda$ and $x_m \in x/F_\lambda$, and so $x_n/F_\lambda = x/F_\lambda = x_m/F_\lambda$. Therefore, $\{x_i\}_{i\in\Lambda}$ is a Cauchy sequence. □

## 4. Conclusion

In this paper, we use a system of an EQ-algebra to construct a TEQ-algebra and investigate some topological properties of TEQ-algebras and quotient TEQ-algebras. Currently, studying topologies of logic algebras based on residuated lattices mainly through filters of corresponding algebras, which are prefilters corresponding to EQ-algebras. A prefilter is a subset $F$ of an EQ-algebra $E$ satisfies: (i) $1 \in F$, and (ii) for any $x, y \in E$, if $x \in F, x \to y \in F$, then $y \in F$. Unlike the method of recent papers [10, 19, 23], prefilters of $E$ can not induce a topology on $E$ such that every operation on $E$ is continuous with respect to this topology, since prefilters can not induce congruences on $E$. So we consider the system of filters on $E$, which can induce a topology such that all binary operations of $E$ are continuous. In our future work, we shall try to use such filters to study sheaf representations of TEQ-algebras and completeness of TEQ-algebras.

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