A NOVEL TRIANGULAR INTERVAL TYPE-2 INTUITIONISTIC FUZZY SETS AND THEIR AGGREGATION OPERATORS

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ABSTRACT. The objective of this work is to present a triangular interval type-2 (TIT2) intuitionistic fuzzy sets and their corresponding aggregation operators, namely, TIT2 intuitionistic fuzzy weighted averaging, TIT2 intuitionistic fuzzy ordered weighted averaging and TIT2 intuitionistic fuzzy hybrid averaging based on Frank norm operation laws. Furthermore, based on these operators, an approach to multi-criteria decision-making, in which assessments are in the form of TIT2 intuitionistic fuzzy numbers has been developed. A practical example to illustrate the decision-making process has been presented and compared their results with the existing operator results.

1. Introduction

In a real-life, uncertainties play a dominant role during the analysis and without handling of its, the decision maker(s) can't give their preferences to an accurate level. The main objective of an analysis is to handle the data so as to minimize their uncertainties level. For handling this, fuzzy set (FS) [38] and its extensions such as intuitionistic fuzzy set (IFSs) [1,3], interval-valued intuitionistic fuzzy set (IVIFSs) [2] etc., have been proposed by the researchers during the last decades. Under these environments, the various researchers pay more attention to aggregate the different alternatives using weighted and ordered weighted aggregation operators during the information fusion process. For instance, Xu [37], Xu and Yager [36] presented geometric, as well as, averaging aggregation operators for aggregating the different intuitionistic fuzzy numbers (IFNs). Later on, Wang and Liu [33] extended these operators by using Einstein norm operations. Garg [11] had presented generalized intuitionistic fuzzy interactive geometric interaction operators using Einstein norm operations for aggregating the different intuitionistic fuzzy information. Garg [17], further, proposed some series of interactive aggregations operators for IFNs. Liu [24] presented weighted averaging aggregation operators under the Hamacher-norm operations. Apart from that, recently, many authors [7,12–16,18–20,22,29,35] have shown the great interest in the study of the decision-making problems under the IFS or IVIFSs environments.

Since all the above works have been investigated under the ordinary fuzzy set (henceforth called as a type-1 fuzzy set) environment, in which they have been assumed that the membership function corresponding to their element is exact. But

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however, in some circumstances, it is difficult for the decision-maker to determine an exact membership function for a fuzzy set. In order to overcome it, the concept of the type-2 fuzzy sets (T2FSs) was introduced by Zadeh [39], as an extension of the type-1 fuzzy set. Such sets are fuzzy sets whose membership values themselves are type-1 fuzzy set and is characterized by primary function, secondary function and a footprint of uncertainty (FOU). It is the new third dimension of T2FSs and the footprint of uncertainty can reflect more additional degrees of freedom that make it more capable of handling imprecision and imperfect information in the real-world application. Hence, T2FSs are receiving more and more attentions from researchers and have been successfully developed both in theoretical and in practical aspects. But due to the high complexities of T2FSs, it is difficult to apply in the real situation. For this, an interval type-2 fuzzy sets (IT2FSs) has been considered [25] which contain membership values from zero to one. Castillo et al. [4] discussed the short remarks on FS, IT2FS, general T2FS and IFS. Chen and Lee [5], Chen et al. [6] presented a fuzzy decision -making method based on the ranking values as well as the arithmetic operations of IT2FSs. As all the existing works have been examined under the T2FSs environment by considering only the degree of the membership during an analysis. But, in the real-life situation, it is not possible to make a decision without considering the degree of non-membership (also called as a dissatisfactory degree), as it is difficult for the person to give their preferences towards an object in terms of single or exact number. Thus, for handling it, there is a need for the degree of non-membership into the analysis. Therefore, to overcome it, a degree of membership, non-membership and their corresponding FOU have been considered during the present analysis and called a theory as an interval type-2 intuitionistic fuzzy sets (IT2IFSs).

For a decision-making problem, an essential important step is how to aggregate the decision-making information in different formats with aggregation operators. In that direction, various authors have investigated the problems of the DM under the T2FSs environment by using different aggregation operators as well as information measures [8, 21, 23, 26–28, 30, 32, 34, 40]. All aforementioned aggregation operators are mainly based on the algebraic operational laws of general t-norms and their dual t-conorms. The most widely used norm operations are product and probabilistic sum [9], because these pairs of the triangular norm are useful to carry out computing. However, the main drawbacks of the probability triangular norms are that lack of flexibility and robustness. Apart from them, some of the existing aggregation operators are derived by either the fuzzy extension principle or based on the triangular norms. As the fundamental of information fusion, triangle norms play an important role and have been successfully used, to overcome the drawbacks of the probability triangular norms, to derive various fuzzy aggregation operators [11, 17, 33, 36]. However, until now, there is no research about interval type-2 intuitionistic fuzzy aggregation operator based on triangular norm operations. In order to fill this gap, we will utilize the Frank triangle norms to develop some desirable interval type-2 intuitionistic fuzzy aggregation operators in this paper.

Frank triangle norms [10], is one of the most important norm operations and the generalizations of probabilistic and product t-norm and t-conorm. Further, frank triangle norms are the only triangle norm which satisfying the compatibility property. Since the Frank triangle norms involve the parameter, this can provide more flexibility and robustness in the process of information fusion and make it more adequate to model practical decision-making problems than other triangle norms. To the best of our knowledge, very fewer investigations on aggregation operators based on Frank t-norms for the decision-making problems have been done by the authors. For instance, Qin and Liu [30] have presented an aggregation operator for triangular interval type-2 fuzzy set during the decision-making process. Nancy and Garg [29] proposed the weighted averaging and geometric aggregation operator based on the Frank norm operations and called as single-valued neutrosophic Frank weighted averaging and geometric operators respectively. Qin et al. [31] presented a hesitant fuzzy aggregation operator based on the Frank t-norm operations. Therefore, it is meaningful to study aggregation operators based on Frank triangle norms operations and their application in decision-making under triangle interval type-2 intuitionistic fuzzy sets environment.

Thus, keeping inspiration from the fact that IT2IFSs have great powerful ability to model the imprecise and ambiguous information in real-world applications, the present paper have developed the various aggregation operators for the triangular IT2IFSs based on the Frank t-norms for decision-making problems. For it, firstly the basic operational laws based on the Frank t-norms have been defined on the triangular interval type-2 intuitionistic fuzzy numbers (TIT2IFNs). Based on these laws, some aggregation operators, namely triangular interval type-2 intuitionistic fuzzy (TIT2IF) weighted averaging, ordered weighted averaging and hybrid averaging have been proposed. Finally, a decision-making method approach has been presented for solving the decision-making problem under the triangular IT2IFS environment.

The remainder of this paper is shown as follows. In section 2, we briefly introduce some basic concepts of T2FSs, IT2FSs, and Frank t-norms. In section 3, we establish the T2IFSs, TIT2IFSs, and their corresponding Frank t-norms operation laws. In Section 4, we present some newly weighted aggregation operators and discuss some desirable properties in detail. In Section 5, we develop an approach to decision-making process under the TIT2IF environment and then illustrate with a practical example. Section 6 gives the concluding remarks.

2. Preliminaries

2.1. **Type 2 fuzzy set [26].** In this section, we will briefly review some of the basic concepts of type-2 fuzzy sets and the Frank t-norms.

Definition 2.1. A T2FS α in the universe of discourse X, is characterized by a type-2 membership function $\mu_\alpha(x,u)$ and is defined as follows:

$$\alpha = \left\{((x,u),\mu_\alpha(x,u)) \mid x \in X, u \in j_x \subseteq [0,1]\right\}$$

in which $0 \leq \mu_\alpha(x,u) \leq 1$. Moreover, α can also be expressed as

$$\alpha = \int_{x\in X} \mu_\alpha(x)/x = \int_{x\in X} \left[\int_{u\in j_x} f_x(u)/u\right]/x,$$

where $\mu_\alpha(x) = \int_{x\in j_x} f_x(u)/u$ is the grade of the membership, f_x is named as a secondary membership function and the value of $f_x(u)$ is named as secondary grade or secondary membership. In addition, u is an argument of the secondary membership function and j_x is named as the primary membership function of x.

Definition 2.2. (Interval type-2) [7] An interval type-2 fuzzy set is one in which the membership grade of every domain point is a crisp set whose domain is some interval contained in [0,1].

Definition 2.3. (Footprint of uncertainty) [7] Uncertainty in the primary memberships of a type-2 fuzzy set consists of boundary region that we call the "footprint of the uncertainty" (FOU). Mathematically, it is the union of all primary membership functions, i.e. $FOU(\alpha) = \cup_{x\in X} j_x$.

2.2. **Triangular interval type-2 fuzzy set [8].** The operations on the T2FSs are very complex and hence cannot be applied in a real-life situations. For it, the interval type-2 fuzzy sets [8] are usually taken in some simplified formations in applications in which the upper membership function (UMF) and the lower membership function (LMF) are represented by the triangular fuzzy numbers.

Definition 2.4. Let $\alpha = \langle [a,b], c, [d,e]\rangle$ be a triangular interval type-2 fuzzy set (TIT2FS) defined on X shown in Fig. 1, where a, b, c, d, e are reference points of the set, satisfying $0 \le a \le b \le c \le d \le e \le 1$. The footprint of uncertainty (FOU) of the membership function of α is depicted as a shaded portion in Fig. 1. If X is a set consists of all real numbers, then a TIT2FS in X is called triangular interval type-2 fuzzy number (TIT2FN).

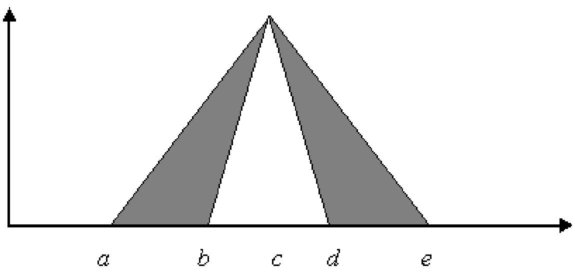

FIGURE 1. Representation of TIT2FS

2.3. **Frank t-norms.** The t-norms (T) and t-conorms (S) defined by $T, S : [0,1]^2 \to [0,1]$, related by $S(x,y) = 1 - T(1-x, 1-y), \forall x, y \in [0,1]$. Among the various existing t-norm and t-conorms, Frank norm operations [10] which are an interesting generalizations of probabilistic and Lukasiewicz t-norm and t-conorm, is a general and flexible family of continuous triangular norms. Frank operations include the Frank product($\otimes_F$) and Frank sum ($\oplus_F$) which are defined in the following ways:

$$\begin{aligned} x \oplus_F y &= 1 - \log_\lambda \left(1 + \frac{(\lambda^{1-x}-1)(\lambda^{1-y}-1)}{\lambda - 1}\right), \lambda > 1 \quad \forall (x,y) \in [0,1]^2 \\ x \otimes_F y &= \log_\lambda \left(1 + \frac{(\lambda^{x}-1)(\lambda^{y}-1)}{\lambda - 1}\right), \lambda > 1 \quad \forall (x,y) \in [0,1]^2 \end{aligned}$$

Further, it can be easily verified that the Frank sum and product satisfy the following properties, i.e.,

- $\big(x \oplus_F y\big) + \big(x \otimes_F y\big) = x + y$
- $\frac{\partial(x \oplus_F y)}{\partial x} + \frac{\partial(x \otimes_F y)}{\partial x} = \frac{\partial(x \oplus_F y)}{\partial y} + \frac{\partial(x \otimes_F y)}{\partial y} = 1$

In addition, Frank t-norm and t-conorm have also two special cases given as follows.

(i) When $\lambda \to 1$, we have $x \oplus_F y = x + y - xy$ and $x \otimes_F y = xy$ which are the algebraic t-norm and the algebraic t-conorm, respectively.

(ii) When $\lambda \to +\infty$, we have $x \oplus_F y = \min(x + y, 1)$, $x \otimes_F y = \max(0, x + y - 1)$, which are the Lukasiewicz t-norm and Lukasiewicz t-conorm, respectively.

3. Type-2 intuitionistic fuzzy set and its operational laws

3.1. **Type-2 intuitionistic fuzzy set.** In this section, we have extended the T2FSs to the T2IFSs and hence presented some basic concepts of it.

Definition 3.1. A type-2 intuitionistic fuzzy set (T2IFS) α is characterized by a type-2 membership function $f_x(u_1)$ and type-2 non-membership function $g_x(u_2)$ and is represented as [32, 40]

$$\alpha = \big\{\langle (x, u_1, u_2), \mu_\alpha(x, u_1), \nu_\alpha(x, u_2)\rangle \mid x \in X, u_1 \in j_x^1, u_2 \in j_x^2\big\}$$

where the element of the domain (x, u_1, u_2) called as primary membership functions (u_1) and non-membership functions (u_2) of $x \in X$ while $\mu_\alpha(x, u_1)$ and $\nu_\alpha(x, u_2)$ be the membership of the primary membership functions called as the secondary membership and non-membership functions respectively where $u_1 \in j_x^1 \subseteq [0, 1]$, $u_2 \in j_x^2 \subseteq [0, 1]$. In addition, u_1 and u_2 are arguments of the secondary membership and non-membership functions and j_x^1, j_x^2 are named as the primary membership and the non-membership functions of x.

Definition 3.2. Let α be a T2IFS in X represented by $\mu_\alpha(x, u_1)$ and $\nu_\alpha(x, u_2)$. If all $\mu_\alpha(x, u_1) = 1$ and $\nu_\alpha(x, u_1) = 0$ then α is called an interval type-2 intuitionistic fuzzy set (IT2IFS). An IT2IFS can be regarded as a special case of the T2IFS.

3.2. **Triangular interval type-2 Intuitionistic fuzzy sets.** Since due to the high complexities of T2IFS, it is difficult to apply in the real situation. For this, an interval type-2 intuitionistic fuzzy set (IT2IFS) has been considered in which the upper and lower membership and non-membership functions are represented by the triangular fuzzy numbers.

Definition 3.3. Let $\alpha = \langle [a, b], c, [d, e];\ [A, B], C, [D, E]\rangle$ be a triangular IT2IFS (TIT2IFS) defined on X, shown in Fig. 2, where $0 \le a \le b \le c \le d \le e \le 1$ and $0 \le E \le D \le C \le B \le A \le 1$ such that $e + E \le 1$, $a + A \le 1$ is characterized by a linear upper and lower membership and non-membership functions, which are defined as follows:

$$\mathrm{UMF}_\mu(x) = \begin{cases} \frac{x-a}{c-a} & ; \quad a \le x < c \\ 1 & ; \quad x = c \\ \frac{e-x}{e-c} & ; \quad c \le x < e \end{cases} \quad ; \quad \mathrm{LMF}_\mu(x) = \begin{cases} \frac{x-b}{c-b} & ; \quad b \le x < c \\ 1 & ; \quad x = c \\ \frac{d-x}{d-c} & ; \quad c \le x < d \end{cases}$$

and

$$\mathrm{UMF}_\nu(x)=\begin{cases}\frac{C-x}{C-E} & ; \quad E\le x<C\\ 0 & ; \quad x=C\\ \frac{x-C}{A-C} & ; \quad C\le x<A\end{cases} \quad ; \quad \mathrm{LMF}_\nu(x)=\begin{cases}\frac{C-x}{C-D} & ; \quad D\le x<C\\ 0 & ; \quad x=C\\ \frac{x-C}{B-C} & ; \quad C\le x<B\end{cases}$$

The FOU of the membership and non-membership functions are depicted as a shaded portion in Fig. 2. If X is a set consists of all real numbers, then a TIT2IFS in X is called triangular interval type-2 intuitionistic fuzzy number (TIT2IFN).

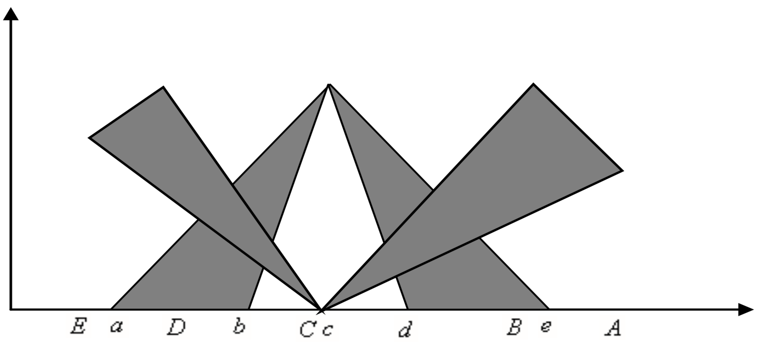

FIGURE 2. Representation of TIT2IFS

Definition 3.4. A compliment of TIT2IFN $\alpha=\langle[a,b],c,[d,e];[A,B],\ C,\ [D,E]\rangle$ is given by $\alpha^c=\langle[A,B],C,[D,E];[a,b],c,[d,e]\rangle$.

Definition 3.5. In the process of applying these sets to practical problems, it is necessary to rank these numbers. For this, we have defined the ranking value for a TIT2IFN $\alpha=\langle[a,b],c,[d,e];[A,B],\ C,\ [D,E]\rangle$ as follows:

$$\mathrm{Rank}(\alpha)=\left(\frac{(a-A)+(e-E)}{2}+1\right)\left(\frac{(a-A)+(b-B)+4(c-C)+(d-D)+(e-E)}{8}\right) \quad (1)$$

The larger the Rank(α), the greater the TIT2IFN.

Definition 3.6. For any two TIT2IFNs α and β, the ordering can be defined as $\alpha\prec\beta$ if $\mathrm{Rank}(\alpha)<\mathrm{Rank}(\beta)$ and $\alpha=\beta$ if $\mathrm{Rank}(\alpha)=\mathrm{Rank}(\beta)$.

Example 3.7. Let $\alpha=\langle[0.3,0.4],0.5,[0.6,0.7];[0.6,0.5],0.3,[0.2,0.1]\rangle$ and $\beta=\langle[0.4,0.45],\ 0.50,\ [0.60,0.70];\ [0.6,0.4],\ 0.35,[0.25,0.15]\rangle$ be two TIT2IFNs. Then, by using Eq. (1), we get $\mathrm{Rank}(\alpha)=0.1725$ and $\mathrm{Rank}(\beta)=0.1982$. Since $\mathrm{Rank}(\alpha)<\mathrm{Rank}(\beta)$ and thus, we have $\beta\succ\alpha$.

Definition 3.8. Let $\alpha_1=\langle[a_1,b_1],c_1,[d_1,e_1];\ [A_1,B_1],C_1,[D_1,E_1]\rangle$ and $\alpha_2=\langle[a_2,b_2],\ c_2,[d_2,e_2];\ [A_2,B_2],\ C_2,[D_2,E_2]\rangle$ be two TIT2IFNs, we denote the partial order as $\alpha_1\succeq_P\alpha_2$ if and only if $a_1\ge a_2,b_1\ge b_2,\ldots,e_1\ge e_2$ and $A_1\le A_2,B_1\le B_2,\ldots,E_1\le E_2$. Especially, $\alpha_1=\alpha_2$ if and only if $a_1=a_2,b_2=b_2,\ldots,e_1=e_2$ and $A_1=A_2,\ldots,E_1=E_2$.

From the above definition, it has been observed that if $\alpha_1 \succeq_P \alpha_2$ which indicate that Rank$(\alpha_1) \geq$ Rank(α_2). If Rank$(\alpha_1) >$ Rank(α_2) then $\alpha_1 \succ \alpha_2$; if Rank$(\alpha_1) =$ Rank(α_2) and since $a_1 \geq a_2, \ldots, e_1 \geq e_2$ and $A_1 \leq A_2, \ldots, E_1 \leq E_2$ then $a_1 = a_2, \ldots, e_1 = e_2, A_1 = A_2, \ldots, E_1 = E_2$, which indicates that $\alpha_1 = \alpha_2$. Thus, we can say if $\alpha_1 \succeq_P \alpha_2$ then, we have $\alpha_1 \succeq \alpha_2$.

3.3. **Frank Operational laws of TIT2IFNs.** Let $\alpha = \langle [a,b], c, [d,e]; [A,B], C, [D,E] \rangle$, $\alpha_1 = \langle [a_1,b_1], c_1, [d_1,e_1]; [A_1,B_1], C_1, [D_1,E_1] \rangle$ and $\alpha_2 = \langle [a_2,b_2], c_2, [d_2, e_2]; [A_2,B_2], C_2, [D_2,E_2] \rangle$ be three TIT2IFNs and $k > 0$, $\lambda > 1$ be two real numbers. Then, the operational laws on these numbers based on the Frank t-norm and t-conorm are defined as below:

(i) Addition operations: $\alpha_1 \oplus_F \alpha_2$

$$
\begin{aligned}
= \ & \Bigg\langle \left[1-\log_\lambda\left(1+\frac{(\lambda^{1-a_1}-1)(\lambda^{1-a_2}-1)}{\lambda-1}\right), 1-\log_\lambda\left(1+\frac{(\lambda^{1-b_1}-1)(\lambda^{1-b_2}-1)}{\lambda-1}\right)\right], \\
& 1-\log_\lambda\left(1+\frac{(\lambda^{1-c_1}-1)(\lambda^{1-c_2}-1)}{\lambda-1}\right), \left[1-\log_\lambda\left(1+\frac{(\lambda^{1-d_1}-1)(\lambda^{1-d_2}-1)}{\lambda-1}\right),\right. \\
& \left. 1-\log_\lambda\left(1+\frac{(\lambda^{1-e_1}-1)(\lambda^{1-e_2}-1)}{\lambda-1}\right)\right]; \left[\log_\lambda\left(1+\frac{(\lambda^{A_1}-1)(\lambda^{A_2}-1)}{\lambda-1}\right),\right. \\
& \left.\log_\lambda\left(1+\frac{(\lambda^{B_1}-1)(\lambda^{B_2}-1)}{\lambda-1}\right)\right], \log_\lambda\left(1+\frac{(\lambda^{C_1}-1)(\lambda^{C_2}-1)}{\lambda-1}\right), \\
& \left[\log_\lambda\left(1+\frac{(\lambda^{D_1}-1)(\lambda^{D_2}-1)}{\lambda-1}\right), \log_\lambda\left(1+\frac{(\lambda^{E_1}-1)(\lambda^{E_2}-1)}{\lambda-1}\right)\right] \Bigg\rangle
\end{aligned}
$$

(ii) Multiplication operations: $\alpha_1 \otimes_F \alpha_2$

$$
\begin{aligned}
= \ & \Bigg\langle \left[\log_\lambda\left(1+\frac{(\lambda^{a_1}-1)(\lambda^{a_2}-1)}{\lambda-1}\right), \log_\lambda\left(1+\frac{(\lambda^{b_1}-1)(\lambda^{b_2}-1)}{\lambda-1}\right)\right], \\
& \log_\lambda\left(1+\frac{(\lambda^{c_1}-1)(\lambda^{c_2}-1)}{\lambda-1}\right), \left[\log_\lambda\left(1+\frac{(\lambda^{d_1}-1)(\lambda^{d_2}-1)}{\lambda-1}\right),\right. \\
& \left.\log_\lambda\left(1+\frac{(\lambda^{e_1}-1)(\lambda^{e_2}-1)}{\lambda-1}\right)\right]; \left[1-\log_\lambda\left(1+\frac{(\lambda^{1-A_1}-1)(\lambda^{1-A_2}-1)}{\lambda-1}\right),\right. \\
& \left. 1-\log_\lambda\left(1+\frac{(\lambda^{1-B_1}-1)(\lambda^{1-B_2}-1)}{\lambda-1}\right)\right], 1-\log_\lambda\left(1+\frac{(\lambda^{1-C_1}-1)(\lambda^{1-C_2}-1)}{\lambda-1}\right), \\
& \left[1-\log_\lambda\left(1+\frac{(\lambda^{1-D_1}-1)(\lambda^{1-D_2}-1)}{\lambda-1}\right), 1-\log_\lambda\left(1+\frac{(\lambda^{1-E_1}-1)(\lambda^{1-E_2}-1)}{\lambda-1}\right)\right] \Bigg\rangle
\end{aligned}
$$

(iii) Multiplication by ordinary number: $k \cdot_F \alpha$

$$
\begin{aligned}
= \ & \Bigg\langle \left[1-\log_\lambda\left(1+\frac{(\lambda^{1-a}-1)^k}{(\lambda-1)^{k-1}}\right), 1-\log_\lambda\left(1+\frac{(\lambda^{1-b}-1)^k}{(\lambda-1)^{k-1}}\right)\right], \\
& 1-\log_\lambda\left(1+\frac{(\lambda^{1-c}-1)^k}{(\lambda-1)^{k-1}}\right), \left[1-\log_\lambda\left(1+\frac{(\lambda^{1-d}-1)^k}{(\lambda-1)^{k-1}}\right), 1-\log_\lambda\left(1+\frac{(\lambda^{1-e}-1)^k}{(\lambda-1)^{k-1}}\right)\right]; \\
& \left[\log_\lambda\left(1+\frac{(\lambda^{A}-1)^k}{(\lambda-1)^{k-1}}\right), \log_\lambda\left(1+\frac{(\lambda^{B}-1)^k}{(\lambda-1)^{k-1}}\right)\right], \log_\lambda\left(1+\frac{(\lambda^{C}-1)^k}{(\lambda-1)^{k-1}}\right), \\
& \left[\log_\lambda\left(1+\frac{(\lambda^{D}-1)^k}{(\lambda-1)^{k-1}}\right), \log_\lambda\left(1+\frac{(\lambda^{E}-1)^k}{(\lambda-1)^{k-1}}\right)\right] \Bigg\rangle
\end{aligned}
$$

(iv) Power operation: α^k

$$\begin{aligned}
= \; & \Bigg\langle \Bigg[\log_\lambda\left(1+\frac{(\lambda^a-1)^k}{(\lambda-1)^{k-1}}\right), \log_\lambda\left(1+\frac{(\lambda^b-1)^k}{(\lambda-1)^{k-1}}\right)\Bigg], \log_\lambda\left(1+\frac{(\lambda^c-1)^k}{(\lambda-1)^{k-1}}\right),\\
& \Bigg[\log_\lambda\left(1+\frac{(\lambda^d-1)^k}{(\lambda-1)^{k-1}}\right), \log_\lambda\left(1+\frac{(\lambda^e-1)^k}{(\lambda-1)^{k-1}}\right)\Bigg]; \Bigg[1-\log_\lambda\left(1+\frac{(\lambda^{1-A}-1)^k}{(\lambda-1)^{k-1}}\right),\\
& 1-\log_\lambda\left(1+\frac{(\lambda^{1-B}-1)^k}{(\lambda-1)^{k-1}}\right)\Bigg], 1-\log_\lambda\left(1+\frac{(\lambda^{1-C}-1)^k}{(\lambda-1)^{k-1}}\right),\\
& \Bigg[1-\log_\lambda\left(1+\frac{(\lambda^{1-D}-1)^k}{(\lambda-1)^{k-1}}\right), 1-\log_\lambda\left(1+\frac{(\lambda^{1-E}-1)^k}{(\lambda-1)^{k-1}}\right)\Bigg]\Bigg\rangle
\end{aligned}$$

Theorem 3.9. *If α, α_1 and α_2 be three TIT2IFNs; then, $\alpha_3 = \alpha_1 \oplus_F \alpha_2$, $\alpha_4 = \alpha_1 \otimes_F \alpha_2$, $\alpha_5 = k \cdot_F \alpha$ and $\alpha_6 = \alpha^k$, $k > 0$ are also TIT2IFNs.*

Proof. Consider $\alpha_3 = \alpha_1 \oplus \alpha_2 = \langle [a_3, b_3], c_3, [d_3, e_3]; [A_3, B_3], C_3, [D_3, E_3]\rangle$ where

$$\begin{aligned}
a_3 &= 1-\log_\lambda\left(1+\frac{(\lambda^{1-a_1}-1)(\lambda^{1-a_2}-1)}{\lambda-1}\right) \quad ; \quad b_3 = 1-\log_\lambda\left(1+\frac{(\lambda^{1-b_1}-1)(\lambda^{1-b_2}-1)}{\lambda-1}\right)\\
c_3 &= 1-\log_\lambda\left(1+\frac{(\lambda^{1-c_1}-1)(\lambda^{1-c_2}-1)}{\lambda-1}\right) \quad ; \quad d_3 = 1-\log_\lambda\left(1+\frac{(\lambda^{1-d_1}-1)(\lambda^{1-d_2}-1)}{\lambda-1}\right)\\
e_3 &= 1-\log_\lambda\left(1+\frac{(\lambda^{1-e_1}-1)(\lambda^{1-e_2}-1)}{\lambda-1}\right) \quad ; \quad A_3 = \log_\lambda\left(1+\frac{(\lambda^{A_1}-1)(\lambda^{A_2}-1)}{\lambda-1}\right)\\
B_3 &= \log_\lambda\left(1+\frac{(\lambda^{B_1}-1)(\lambda^{B_2}-1)}{\lambda-1}\right) \quad ; \quad C_3 = \log_\lambda\left(1+\frac{(\lambda^{C_1}-1)(\lambda^{C_2}-1)}{\lambda-1}\right)\\
D_3 &= \log_\lambda\left(1+\frac{(\lambda^{D_1}-1)(\lambda^{D_2}-1)}{\lambda-1}\right) \quad ; \quad E_3 = \log_\lambda\left(1+\frac{(\lambda^{E_1}-1)(\lambda^{E_2}-1)}{\lambda-1}\right)
\end{aligned}$$

Then, in order to show α_3 be TIT2IFN we have to show that $0 \le a_3 \le b_3 \le c_3 \le d_3 \le e_3 \le 1$, $1 \ge A_3 \ge B_3 \ge C_3 \ge D_3 \ge E_3 \ge 0$ and $e_3 + E_3 \le 1$, $a_3 + A_3 \le 1$. Since α_j, $(j = 1, 2)$ be TIT2IFNs which implies that $0 \le a_j \le b_j \le c_j \le d_j \le e_j \le 1$; $1 \ge A_j \ge B_j \ge C_j \ge D_j \ge E_j \ge 0$, $a_j + A_j \le 1$ and $e_j + E_j \le 1$ for $j = 1, 2$, then, for $\lambda > 1$ be a real number, we have

$$\begin{aligned}
& \frac{(\lambda^{1-e_1}-1)(\lambda^{1-e_2}-1)}{\lambda-1} \le \frac{(\lambda^{1-d_1}-1)(\lambda^{1-d_2}-1)}{\lambda-1} \le \frac{(\lambda^{1-c_1}-1)(\lambda^{1-c_2}-1)}{\lambda-1}\\
& \qquad \le \frac{(\lambda^{1-b_1}-1)(\lambda^{1-b_2}-1)}{\lambda-1} \le \frac{(\lambda^{1-a_1}-1)(\lambda^{1-a_2}-1)}{\lambda-1}\\
\Leftrightarrow \quad & 0 \le \log_\lambda\left(1+\frac{(\lambda^{1-e_1}-1)(\lambda^{1-e_2}-1)}{\lambda-1}\right) \le \log_\lambda\left(1+\frac{(\lambda^{1-d_1}-1)(\lambda^{1-d_2}-1)}{\lambda-1}\right)\\
& \quad \le \log_\lambda\left(1+\frac{(\lambda^{1-c_1}-1)(\lambda^{1-c_2}-1)}{\lambda-1}\right) \le \log_\lambda\left(1+\frac{(\lambda^{1-b_1}-1)(\lambda^{1-b_2}-1)}{\lambda-1}\right)\\
& \quad \le \log_\lambda\left(1+\frac{(\lambda^{1-a_1}-1)(\lambda^{1-a_2}-1)}{\lambda-1}\right) \le 1\\
\Leftrightarrow \quad & 0 \le a_3 \le b_3 \le c_3 \le d_3 \le e_3 \le 1
\end{aligned}$$

and

$$\begin{aligned}
&\frac{(\lambda^{A_1}-1)(\lambda^{A_2}-1)}{\lambda-1} \geq \frac{(\lambda^{B_1}-1)(\lambda^{B_2}-1)}{\lambda-1} \geq \frac{(\lambda^{C_1}-1)(\lambda^{C_2}-1)}{\lambda-1} \\
&\quad \geq \frac{(\lambda^{D_1}-1)(\lambda^{D_2}-1)}{\lambda-1} \geq \frac{(\lambda^{E_1}-1)(\lambda^{E_2}-1)}{\lambda-1} \\
\Leftrightarrow \quad & 1 \geq \log_\lambda\left(1+\frac{(\lambda^{A_1}-1)(\lambda^{A_2}-1)}{\lambda-1}\right) \geq \log_\lambda\left(1+\frac{(\lambda^{B_1}-1)(\lambda^{B_2}-1)}{\lambda-1}\right) \\
&\quad \geq \log_\lambda\left(1+\frac{(\lambda^{C_1}-1)(\lambda^{C_2}-1)}{\lambda-1}\right) \geq \log_\lambda\left(1+\frac{(\lambda^{D_1}-1)(\lambda^{D_2}-1)}{\lambda-1}\right) \\
&\quad \geq \log_\lambda\left(1+\frac{(\lambda^{E_1}-1)(\lambda^{E_2}-1)}{\lambda-1}\right) \geq 0 \\
\Leftrightarrow \quad & 1 \geq A_3 \geq B_3 \geq C_3 \geq D_3 \geq E_3 \geq 0
\end{aligned}$$

Finally, $e_3+E_3 = 1-\log_\lambda\left(1+\frac{(\lambda^{1-e_1}-1)(\lambda^{1-e_2}-1)}{\lambda-1}\right)+\log_\lambda\left(1+\frac{(\lambda^{E_1}-1)(\lambda^{E_2}-1)}{\lambda-1}\right) \leq 1-\log_\lambda\left(1+\frac{(\lambda^{E_1}-1)(\lambda^{E_2}-1)}{\lambda-1}\right)+\log_\lambda\left(1+\frac{(\lambda^{E_1}-1)(\lambda^{E_2}-1)}{\lambda-1}\right) \leq 1$. Similarly, $a_3+A_3 \leq 1$ which indicates that $\alpha_3 = \alpha_1 \oplus_F \alpha_2$ is TIT2IFN. Similarly, for others. □

Theorem 3.10. *Let α, α_1 and α_2 be three TIT2IFNs and k, k_1, k_2 be three positive real numbers then, we have*

(i) $\alpha_1 \oplus_F \alpha_2 = \alpha_2 \oplus_F \alpha_1$
(ii) $\alpha_1 \otimes_F \alpha_2 = \alpha_2 \otimes_F \alpha_1$
(iii) $k \cdot_F (\alpha_1 \oplus_F \alpha_2) = k \cdot_F \alpha_1 \oplus_F k \cdot_F \alpha_2$
(iv) $(\alpha_1 \otimes_F \alpha_2)^k = \alpha_1^k \otimes_F \alpha_2^k$
(v) $(k_1 \cdot_F \alpha) \oplus_F (k_2 \cdot_F \alpha) = (k_1+k_2) \cdot_F \alpha$
(vi) $\alpha^{k_1} \otimes_F \alpha^{k_2} = \alpha^{k_1+k_2}$

Proof. We prove the parts (i), (iii) and (v) and hence similar for others.

(i) It is trivial.
(iii) Since α_1 and α_2 be two TIT2IFNs and then by addition operations, we get

$$\begin{aligned}
&\alpha_1 \oplus_F \alpha_2 \\
= \quad & \Bigg\langle \Bigg[1-\log_\lambda\left(1+\frac{(\lambda^{1-a_1}-1)(\lambda^{1-a_2}-1)}{\lambda-1}\right), 1-\log_\lambda\left(1+\frac{(\lambda^{1-b_1}-1)(\lambda^{1-b_2}-1)}{\lambda-1}\right)\Bigg], \\
& 1-\log_\lambda\left(1+\frac{(\lambda^{1-c_1}-1)(\lambda^{1-c_2}-1)}{\lambda-1}\right), \Bigg[1-\log_\lambda\left(1+\frac{(\lambda^{1-d_1}-1)(\lambda^{1-d_2}-1)}{\lambda-1}\right), \\
& 1-\log_\lambda\left(1+\frac{(\lambda^{1-e_1}-1)(\lambda^{1-e_2}-1)}{\lambda-1}\right)\Bigg]; \Bigg[\log_\lambda\left(1+\frac{(\lambda^{A_1}-1)(\lambda^{A_2}-1)}{\lambda-1}\right), \\
& \log_\lambda\left(1+\frac{(\lambda^{B_1}-1)(\lambda^{B_2}-1)}{\lambda-1}\right)\Bigg], \log_\lambda\left(1+\frac{(\lambda^{C_1}-1)(\lambda^{C_2}-1)}{\lambda-1}\right), \\
& \Bigg[\log_\lambda\left(1+\frac{(\lambda^{D_1}-1)(\lambda^{D_2}-1)}{\lambda-1}\right), \log_\lambda\left(1+\frac{(\lambda^{E_1}-1)(\lambda^{E_2}-1)}{\lambda-1}\right)\Bigg]\Bigg\rangle
\end{aligned}$$

Thus,

$$
\begin{aligned}
& k \cdot_F (\alpha_1 \oplus_F \alpha_2) \\
= & \Bigg\langle \Bigg[1-\log_\lambda\left(1+\frac{(\lambda^{1-a_1}-1)^k(\lambda^{1-a_2}-1)^k}{(\lambda-1)^{2k-1}}\right), 1-\log_\lambda\left(1+\frac{(\lambda^{1-b_1}-1)^k(\lambda^{1-b_2}-1)^k}{(\lambda-1)^{2k-1}}\right)\Bigg], \\
& 1-\log_\lambda\left(1+\frac{(\lambda^{1-c_1}-1)^k(\lambda^{1-c_2}-1)^k}{(\lambda-1)^{2k-1}}\right), \Bigg[1-\log_\lambda\left(1+\frac{(\lambda^{1-d_1}-1)^k(\lambda^{1-d_2}-1)^k}{(\lambda-1)^{2k-1}}\right), \\
& 1-\log_\lambda\left(1+\frac{(\lambda^{1-e_1}-1)^k(\lambda^{1-e_2}-1)^k}{(\lambda-1)^{2k-1}}\right)\Bigg]; \Bigg[\log_\lambda\left(1+\frac{(\lambda^{A_1}-1)^k(\lambda^{A_2}-1)^k}{(\lambda-1)^{2k-1}}\right), \\
& \log_\lambda\left(1+\frac{(\lambda^{B_1}-1)^k(\lambda^{B_2}-1)^k}{(\lambda-1)^{2k-1}}\right)\Bigg], \log_\lambda\left(1+\frac{(\lambda^{C_1}-1)^k(\lambda^{C_2}-1)^k}{(\lambda-1)^{2k-1}}\right), \\
& \Bigg[\log_\lambda\left(1+\frac{(\lambda^{D_1}-1)^k(\lambda^{D_2}-1)^k}{(\lambda-1)^{2k-1}}\right), \log_\lambda\left(1+\frac{(\lambda^{E_1}-1)^k(\lambda^{E_2}-1)^k}{(\lambda-1)^{2k-1}}\right)\Bigg]\Bigg\rangle \\
= & \Bigg\langle \Bigg[1-\log_\lambda\left(1+\frac{(\lambda^{1-a_1}-1)^k}{(\lambda-1)^{k-1}}\right), 1-\log_\lambda\left(1+\frac{(\lambda^{1-b_1}-1)^k}{(\lambda-1)^{k-1}}\right)\Bigg], \\
& 1-\log_\lambda\left(1+\frac{(\lambda^{1-c_1}-1)^k}{(\lambda-1)^{k-1}}\right), \Bigg[1-\log_\lambda\left(1+\frac{(\lambda^{1-d_1}-1)^k}{(\lambda-1)^{k-1}}\right), \\
& 1-\log_\lambda\left(1+\frac{(\lambda^{1-e_1}-1)^k}{(\lambda-1)^{k-1}}\right)\Bigg]; \Bigg[\log_\lambda\left(1+\frac{(\lambda^{A_1}-1)^k}{(\lambda-1)^{k-1}}\right), \log_\lambda\left(1+\frac{(\lambda^{B_1}-1)^k}{(\lambda-1)^{k-1}}\right)\Bigg], \\
& \log_\lambda\left(1+\frac{(\lambda^{C_1}-1)^k}{(\lambda-1)^{k-1}}\right), \Bigg[\log_\lambda\left(1+\frac{(\lambda^{D_1}-1)^k}{(\lambda-1)^{k-1}}\right), \log_\lambda\left(1+\frac{(\lambda^{E_1}-1)^k}{(\lambda-1)^{k-1}}\right)\Bigg]\Bigg\rangle \\
\oplus_F & \Bigg\langle \Bigg[1-\log_\lambda\left(1+\frac{(\lambda^{1-a_2}-1)^k}{(\lambda-1)^{k-1}}\right), 1-\log_\lambda\left(1+\frac{(\lambda^{1-b_2}-1)^k}{(\lambda-1)^{k-1}}\right)\Bigg], \\
& 1-\log_\lambda\left(1+\frac{(\lambda^{1-c_2}-1)^k}{(\lambda-1)^{k-1}}\right), \Bigg[1-\log_\lambda\left(1+\frac{(\lambda^{1-d_2}-1)^k}{(\lambda-1)^{k-1}}\right), \\
& 1-\log_\lambda\left(1+\frac{(\lambda^{1-e_2}-1)^k}{(\lambda-1)^{k-1}}\right)\Bigg]; \Bigg[\log_\lambda\left(1+\frac{(\lambda^{A_2}-1)^k}{(\lambda-1)^{k-1}}\right), \log_\lambda\left(1+\frac{(\lambda^{B_2}-1)^k}{(\lambda-1)^{k-1}}\right)\Bigg], \\
& \log_\lambda\left(1+\frac{(\lambda^{C_2}-1)^k}{(\lambda-1)^{k-1}}\right), \Bigg[\log_\lambda\left(1+\frac{(\lambda^{D_2}-1)^k}{(\lambda-1)^{k-1}}\right), \log_\lambda\left(1+\frac{(\lambda^{E_2}-1)^k}{(\lambda-1)^{k-1}}\right)\Bigg]\Bigg\rangle \\
= & k \cdot_F \alpha_1 \oplus_F k \cdot_F \alpha_2
\end{aligned}
$$

Hence, $k \cdot_F (\alpha_1 \oplus_F \alpha_2) = k \cdot_F \alpha_1 \oplus_F k \cdot_F \alpha_2$.

(v) For $k_i > 0$, for $i = 1, 2$, we have

$$
\begin{aligned}
k_i \cdot_F \alpha = & \Bigg\langle \Bigg[1-\log_\lambda\left(1+\frac{(\lambda^{1-a}-1)^{k_i}}{(\lambda-1)^{k_i-1}}\right), 1-\log_\lambda\left(1+\frac{(\lambda^{1-b}-1)^{k_i}}{(\lambda-1)^{k_i-1}}\right)\Bigg], \\
& 1-\log_\lambda\left(1+\frac{(\lambda^{1-c}-1)^{k_i}}{(\lambda-1)^{k_i-1}}\right), \Bigg[1-\log_\lambda\left(1+\frac{(\lambda^{1-d}-1)^{k_i}}{(\lambda-1)^{k_i-1}}\right), \\
& 1-\log_\lambda\left(1+\frac{(\lambda^{1-e}-1)^{k_i}}{(\lambda-1)^{k_i-1}}\right)\Bigg]; \Bigg[\log_\lambda\left(1+\frac{(\lambda^{A}-1)^{k_i}}{(\lambda-1)^{k_i-1}}\right), \\
& \log_\lambda\left(1+\frac{(\lambda^{B}-1)^{k_i}}{(\lambda-1)^{k_i-1}}\right)\Bigg], \log_\lambda\left(1+\frac{(\lambda^{C}-1)^{k_i}}{(\lambda-1)^{k_i-1}}\right), \\
& \Bigg[\log_\lambda\left(1+\frac{(\lambda^{D}-1)^{k_i}}{(\lambda-1)^{k_i-1}}\right), \log_\lambda\left(1+\frac{(\lambda^{E}-1)^{k_i}}{(\lambda-1)^{k_i-1}}\right)\Bigg]\Bigg\rangle
\end{aligned}
$$

Thus, $k_1 \cdot_F \alpha \oplus_F k_2 \cdot_F \alpha$

$$
\begin{aligned}
= & \Bigg\langle \Bigg[1-\log_\lambda \left(1+\frac{\frac{(\lambda^{1-a}-1)^{k_1}}{(\lambda-1)^{k_1-1}}\frac{(\lambda^{1-a}-1)^{k_2}}{(\lambda-1)^{k_2-1}}}{(\lambda-1)}\right), 1-\log_\lambda \left(1+\frac{\frac{(\lambda^{1-b}-1)^{k_1}}{(\lambda-1)^{k_1-1}}\frac{(\lambda^{1-b}-1)^{k_2}}{(\lambda-1)^{k_2-1}}}{(\lambda-1)}\right)\Bigg], \\
& 1-\log_\lambda \left(1+\frac{\frac{(\lambda^{1-c}-1)^{k_1}}{(\lambda-1)^{k_1-1}}\frac{(\lambda^{1-c}-1)^{k_2}}{(\lambda-1)^{k_2-1}}}{(\lambda-1)}\right), \Bigg[1-\log_\lambda \left(1+\frac{\frac{(\lambda^{1-d}-1)^{k_1}}{(\lambda-1)^{k_1-1}}\frac{(\lambda^{1-d}-1)^{k_2}}{(\lambda-1)^{k_2-1}}}{(\lambda-1)}\right), \\
& 1-\log_\lambda \left(1+\frac{\frac{(\lambda^{1-e}-1)^{k_1}}{(\lambda-1)^{k_1-1}}\frac{(\lambda^{1-e}-1)^{k_2}}{(\lambda-1)^{k_2-1}}}{(\lambda-1)}\right)\Bigg]; \Bigg[\log_\lambda \left(1+\frac{\frac{(\lambda^{A}-1)^{k_1}}{(\lambda-1)^{k_1-1}}\frac{(\lambda^{A}-1)^{k_2}}{(\lambda-1)^{k_2-1}}}{\lambda-1}\right), \\
& \log_\lambda \left(1+\frac{\frac{(\lambda^{B}-1)^{k_1}}{(\lambda-1)^{k_1-1}}\frac{(\lambda^{B}-1)^{k_2}}{(\lambda-1)^{k_2-1}}}{\lambda-1}\right)\Bigg], \log_\lambda \left(1+\frac{\frac{(\lambda^{C}-1)^{k_1}}{(\lambda-1)^{k_1-1}}\frac{(\lambda^{C}-1)^{k_2}}{(\lambda-1)^{k_2-1}}}{\lambda-1}\right), \\
& \Bigg[\log_\lambda \left(1+\frac{\frac{(\lambda^{D}-1)^{k_1}}{(\lambda-1)^{k_1-1}}\frac{(\lambda^{D}-1)^{k_2}}{(\lambda-1)^{k_2-1}}}{\lambda-1}\right), \log_\lambda \left(1+\frac{\frac{(\lambda^{E}-1)^{k_1}}{(\lambda-1)^{k_1-1}}\frac{(\lambda^{E}-1)^{k_2}}{(\lambda-1)^{k_2-1}}}{\lambda-1}\right)\Bigg]\Bigg\rangle \\
= & \Bigg\langle \Bigg[1-\log_\lambda \left(1+\frac{(\lambda^{1-a}-1)^{k_1+k_2}}{(\lambda-1)^{k_1+k_2-1}}\right), 1-\log_\lambda \left(1+\frac{(\lambda^{1-b}-1)^{k_1+k_2}}{(\lambda-1)^{k_1+k_2-1}}\right)\Bigg], \\
& 1-\log_\lambda \left(1+\frac{(\lambda^{1-c}-1)^{k_1+k_2}}{(\lambda-1)^{k_1+k_2-1}}\right), \Bigg[1-\log_\lambda \left(1+\frac{(\lambda^{1-d}-1)^{k_1+k_2}}{(\lambda-1)^{k_1+k_2-1}}\right), \\
& 1-\log_\lambda \left(1+\frac{(\lambda^{1-e}-1)^{k_1+k_2}}{(\lambda-1)^{k_1+k_2-1}}\right)\Bigg]; \Bigg[\log_\lambda \left(1+\frac{(\lambda^{A}-1)^{k_1+k_2}}{(\lambda-1)^{k_1+k_2-1}}\right), \\
& \log_\lambda \left(1+\frac{(\lambda^{B}-1)^{k_1+k_2}}{(\lambda-1)^{k_1+k_2-1}}\right)\Bigg], \log_\lambda \left(1+\frac{(\lambda^{C}-1)^{k_1+k_2}}{(\lambda-1)^{k_1+k_2-1}}\right), \\
& \Bigg[\log_\lambda \left(1+\frac{(\lambda^{D}-1)^{k_1+k_2}}{(\lambda-1)^{k_1+k_2-1}}\right), \log_\lambda \left(1+\frac{(\lambda^{E}-1)^{k_1+k_2}}{(\lambda-1)^{k_1+k_2-1}}\right)\Bigg]\Bigg\rangle \\
= & (k_1+k_2)\cdot_F \alpha
\end{aligned}
$$

Hence $k_1 \cdot_F \alpha \oplus_F k_2 \cdot_F \alpha = (k_1+k_2)\cdot_F \alpha$. □

Theorem 3.11. *Let α_1 and α_2 be two TIT2IFNs then,*

(i) $\alpha_1^c \oplus_F \alpha_2^c = (\alpha_1 \otimes_F \alpha_2)^c$
(ii) $\alpha_1^c \otimes_F \alpha_2^c = (\alpha_1 \oplus_F \alpha_2)^c$

Proof. The proof is trivial, so it is omitted here. □

4. Aggregation Operators for TIT2IFNs

In this section, some series of weighted aggregation operators for TIT2IFNs have been proposed based on above defined Frank t-norm operations.

4.1. Weighted averaging Operator.

Definition 4.1. Let $\alpha_j, j = 1, 2, \ldots, n$ be a collection of TIT2IFNs and if

$$\text{TIT2IFWA}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \omega_1 \cdot_F \alpha_1 \oplus_F \omega_2 \cdot_F \alpha_2 \oplus_F \ldots \oplus_F \omega_n \cdot_F \alpha_n$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the weight vector of α_j such that $\omega_j > 0$, $\sum_{j=1}^{n} \omega_j = 1$ then, TIT2IFWA is called a TIT2IF weighted averaging operator.

Theorem 4.2. *The aggregated value by using TIT2IFWA operator for a collection of TIT2IFNs* $\alpha_j = \langle [a_j, b_j], c_j, [d_j, e_j]; [A_j, B_j], C_j, [D_j, E_j] \rangle, j = 1, 2, \ldots, n$ *is still TIT2IFN and is given by*

$$\begin{aligned} &TIT2IFWA(\alpha_1, \alpha_2, \ldots, \alpha_n) = \Bigg\langle \Bigg[1 - \log_\lambda \left(1 + \prod_{j=1}^{n} (\lambda^{1-a_j} - 1)^{\omega_j} \right), \\ &1 - \log_\lambda \left(1 + \prod_{j=1}^{n} (\lambda^{1-b_j} - 1)^{\omega_j} \right) \Bigg], 1 - \log_\lambda \left(1 + \prod_{j=1}^{n} (\lambda^{1-c_j} - 1)^{\omega_j} \right), \\ &\Bigg[1 - \log_\lambda \left(1 + \prod_{j=1}^{n} (\lambda^{1-d_j} - 1)^{\omega_j} \right), 1 - \log_\lambda \left(1 + \prod_{j=1}^{n} (\lambda^{1-e_j} - 1)^{\omega_j} \right) \Bigg]; \\ &\Bigg[\log_\lambda \left(1 + \prod_{j=1}^{n} (\lambda^{A_j} - 1)^{\omega_j} \right), \log_\lambda \left(1 + \prod_{j=1}^{n} (\lambda^{B_j} - 1)^{\omega_j} \right) \Bigg], \log_\lambda \left(1 + \prod_{j=1}^{n} (\lambda^{C_j} - 1)^{\omega_j} \right), \\ &\Bigg[\log_\lambda \left(1 + \prod_{j=1}^{n} (\lambda^{D_j} - 1)^{\omega_j} \right), \log_\lambda \left(1 + \prod_{j=1}^{n} (\lambda^{E_j} - 1)^{\omega_j} \right) \Bigg] \Bigg\rangle \end{aligned} \tag{2}$$

Proof. We will prove the Eq. (2) by mathematical induction on n. Since for each j, α_j is a TIT2IFN. Then, the following steps of the mathematical induction have been followed:

Step 1: For $n = 2$, we have $\text{TIT2IFWA}(\alpha_1, \alpha_2) = \omega_1 \cdot_F \alpha_1 \oplus_F \omega_2 \cdot_F \alpha_2$. Thus, by the operation of TIT2IFNs for $i = 1, 2$, we get

$$\begin{aligned} \omega_i \cdot_F \alpha_i \;=\; &\Bigg\langle \Bigg[1 - \log_\lambda \left(1 + \frac{(\lambda^{1-a_i} - 1)^{\omega_i}}{(\lambda - 1)^{\omega_i - 1}} \right), 1 - \log_\lambda \left(1 + \frac{(\lambda^{1-b_i} - 1)^{\omega_i}}{(\lambda - 1)^{\omega_i - 1}} \right) \Bigg], \\ &1 - \log_\lambda \left(1 + \frac{(\lambda^{1-c_i} - 1)^{\omega_i}}{(\lambda - 1)^{\omega_i - 1}} \right), \Bigg[1 - \log_\lambda \left(1 + \frac{(\lambda^{1-d_i} - 1)^{\omega_i}}{(\lambda - 1)^{\omega_i - 1}} \right), \\ &1 - \log_\lambda \left(1 + \frac{(\lambda^{1-e_i} - 1)^{\omega_i}}{(\lambda - 1)^{\omega_i - 1}} \right) \Bigg]; \Bigg[\log_\lambda \left(1 + \frac{(\lambda^{A_i} - 1)^{\omega_i}}{(\lambda - 1)^{\omega_i - 1}} \right), \\ &\log_\lambda \left(1 + \frac{(\lambda^{B_i} - 1)^{\omega_i}}{(\lambda - 1)^{\omega_i - 1}} \right) \Bigg], \log_\lambda \left(1 + \frac{(\lambda^{C_i} - 1)^{\omega_i}}{(\lambda - 1)^{\omega_i - 1}} \right), \\ &\Bigg[\log_\lambda \left(1 + \frac{(\lambda^{D_i} - 1)^{\omega_i}}{(\lambda - 1)^{\omega_i - 1}} \right), \log_\lambda \left(1 + \frac{(\lambda^{E_i} - 1)^{\omega_i}}{(\lambda - 1)^{\omega_i - 1}} \right) \Bigg] \Bigg\rangle \end{aligned}$$

Thus, by addition law of TIT2IFNs, we get

$$\begin{aligned} &\text{TIT2IFWA}(\alpha_1, \alpha_2) \\ = \;&\Bigg\langle \Big[1 - \log_\lambda \left((\lambda^{1-a_1} - 1)^{\omega_1} (\lambda^{1-a_2} - 1)^{\omega_2} \right), 1 - \log_\lambda \Big((\lambda^{1-b_1} - 1)^{\omega_1} (\lambda^{1-b_2} - 1)^{\omega_2} \Big) \Big], \\ &1 - \log_\lambda \left((\lambda^{1-c_1} - 1)^{\omega_1} (\lambda^{1-c_2} - 1)^{\omega_2} \right), \Big[1 - \log_\lambda \Big((\lambda^{1-d_1} - 1)^{\omega_1} (\lambda^{1-d_2} - 1)^{\omega_2} \Big), \\ &1 - \log_\lambda \left((\lambda^{1-e_1} - 1)^{\omega_1} (\lambda^{1-e_2} - 1)^{\omega_2} \right) \Big]; \Big[\log_\lambda \Big((\lambda^{A_1} - 1)^{\omega_1} (\lambda^{A_2} - 1)^{\omega_2} \Big), \\ &\log_\lambda \Big((\lambda^{B_1} - 1)^{\omega_1} (\lambda^{B_2} - 1)^{\omega_2} \Big) \Big], \log_\lambda \Big((\lambda^{C_1} - 1)^{\omega_1} (\lambda^{C_2} - 1)^{\omega_2} \Big), \\ &\Big[\log_\lambda \Big((\lambda^{D_1} - 1)^{\omega_1} (\lambda^{D_2} - 1)^{\omega_2} \Big), \log_\lambda \Big((\lambda^{E_1} - 1)^{\omega_1} (\lambda^{E_2} - 1)^{\omega_2} \Big) \Big] \Bigg\rangle \end{aligned}$$

Thus, result hold for $n = 2$.

Step 2: If Eq. (2) holds for $n = k$, then for $n = k + 1$, we have

$$\begin{aligned}
&\text{TIT2IFWA}(\alpha_1, \alpha_2, \ldots, \alpha_{k+1}) = \text{TIT2IFWA}(\alpha_1, \alpha_2, \ldots, \alpha_k) \oplus_F \omega_{k+1} \cdot_F \alpha_{k+1} \\
= \ & \Bigg\langle \left[1 - \log_\lambda\left(1 + \prod_{j=1}^{k}(\lambda^{1-a_j} - 1)^{\omega_j}\right), 1 - \log_\lambda\left(1 + \prod_{j=1}^{k}(\lambda^{1-b_j} - 1)^{\omega_j}\right)\right], \\
& 1 - \log_\lambda\left(1 + \prod_{j=1}^{k}(\lambda^{1-c_j} - 1)^{\omega_j}\right), \left[1 - \log_\lambda\left(1 + \prod_{j=1}^{k}(\lambda^{1-d_j} - 1)^{\omega_j}\right),\right. \\
& \left. 1 - \log_\lambda\left(1 + \prod_{j=1}^{k}(\lambda^{1-e_j} - 1)^{\omega_j}\right)\right]; \left[\log_\lambda\left(1 + \prod_{j=1}^{k}(\lambda^{A_j} - 1)^{\omega_j}\right),\right. \\
& \left. \log_\lambda\left(1 + \prod_{j=1}^{k}(\lambda^{B_j} - 1)^{\omega_j}\right)\right], \log_\lambda\left(1 + \prod_{j=1}^{k}(\lambda^{C_j} - 1)^{\omega_j}\right), \\
& \left[\log_\lambda\left(1 + \prod_{j=1}^{k}(\lambda^{D_j} - 1)^{\omega_j}\right), \log_\lambda\left(1 + \prod_{j=1}^{k}(\lambda^{E_j} - 1)^{\omega_j}\right)\right] \Bigg\rangle \\
\oplus_F \ & \Bigg\langle \left[1 - \log_\lambda\left(1 + \frac{(\lambda^{1-a_{k+1}} - 1)^{\omega_{k+1}}}{(\lambda - 1)^{\omega_{k+1}-1}}\right), 1 - \log_\lambda\left(1 + \frac{(\lambda^{1-b_{k+1}} - 1)^{\omega_{k+1}}}{(\lambda - 1)^{\omega_{k+1}-1}}\right)\right], \\
& 1 - \log_\lambda\left(1 + \frac{(\lambda^{1-c_{k+1}} - 1)^{\omega_{k+1}}}{(\lambda - 1)^{\omega_{k+1}-1}}\right), \left[1 - \log_\lambda\left(1 + \frac{(\lambda^{1-d_{k+1}} - 1)^{\omega_{k+1}}}{(\lambda - 1)^{\omega_{k+1}-1}}\right),\right. \\
& \left. 1 - \log_\lambda\left(1 + \frac{(\lambda^{1-e_{k+1}} - 1)^{\omega_{k+1}}}{(\lambda - 1)^{\omega_{k+1}-1}}\right)\right]; \left[\log_\lambda\left(1 + \frac{(\lambda^{A_{k+1}} - 1)^{\omega_{k+1}}}{(\lambda - 1)^{\omega_{k+1}-1}}\right),\right. \\
& \left. \log_\lambda\left(1 + \frac{(\lambda^{B_{k+1}} - 1)^{\omega_{k+1}}}{(\lambda - 1)^{\omega_{k+1}-1}}\right)\right], \log_\lambda\left(1 + \frac{(\lambda^{C_{k+1}} - 1)^{\omega_{k+1}}}{(\lambda - 1)^{\omega_{k+1}-1}}\right), \\
& \left[\log_\lambda\left(1 + \frac{(\lambda^{D_{k+1}} - 1)^{\omega_{k+1}}}{(\lambda - 1)^{\omega_{k+1}-1}}\right), \log_\lambda\left(1 + \frac{(\lambda^{E_{k+1}} - 1)^{\omega_{k+1}}}{(\lambda - 1)^{\omega_{k+1}-1}}\right)\right] \Bigg\rangle \\
= \ & \Bigg\langle \left[1 - \log_\lambda\left(1 + \frac{\prod_{j=1}^{k+1}(\lambda^{1-a_j} - 1)^{\omega_j}}{(\lambda - 1)^{\sum_{j=1}^{k+1}\omega_j - 1}}\right), 1 - \log_\lambda\left(1 + \frac{\prod_{j=1}^{k+1}(\lambda^{1-b_j} - 1)^{\omega_j}}{(\lambda - 1)^{\sum_{j=1}^{k+1}\omega_j - 1}}\right)\right], \\
& 1 - \log_\lambda\left(1 + \frac{\prod_{j=1}^{k+1}(\lambda^{1-c_j} - 1)^{\omega_j}}{(\lambda - 1)^{\sum_{j=1}^{k+1}\omega_j - 1}}\right), \left[1 - \log_\lambda\left(1 + \frac{\prod_{j=1}^{k+1}(\lambda^{1-d_j} - 1)^{\omega_j}}{(\lambda - 1)^{\sum_{j=1}^{k+1}\omega_j - 1}}\right),\right. \\
& \left. 1 - \log_\lambda\left(1 + \frac{\prod_{j=1}^{k+1}(\lambda^{1-e_j} - 1)^{\omega_j}}{(\lambda - 1)^{\sum_{j=1}^{k+1}\omega_j - 1}}\right)\right]; \left[\log_\lambda\left(1 + \frac{\prod_{j=1}^{k+1}(\lambda^{A_j} - 1)^{\omega_j}}{(\lambda - 1)^{\sum_{j=1}^{k+1}\omega_j - 1}}\right),\right. \\
& \left. \log_\lambda\left(1 + \frac{\prod_{j=1}^{k+1}(\lambda^{B_j} - 1)^{\omega_j}}{(\lambda - 1)^{\sum_{j=1}^{k+1}\omega_j - 1}}\right)\right], \log_\lambda\left(1 + \frac{\prod_{j=1}^{k+1}(\lambda^{C_j} - 1)^{\omega_j}}{(\lambda - 1)^{\sum_{j=1}^{k+1}\omega_j - 1}}\right), \\
& \left[\log_\lambda\left(1 + \frac{\prod_{j=1}^{k+1}(\lambda^{D_j} - 1)^{\omega_j}}{(\lambda - 1)^{\sum_{j=1}^{k+1}\omega_j - 1}}\right), \log_\lambda\left(1 + \frac{\prod_{j=1}^{k+1}(\lambda^{E_j} - 1)^{\omega_j}}{(\lambda - 1)^{\sum_{j=1}^{k+1}\omega_j - 1}}\right)\right] \Bigg\rangle
\end{aligned}$$

$$= \Bigg\langle \Bigg[1-\log_\lambda\Bigg(1+\prod_{j=1}^{k+1}(\lambda^{1-a_j}-1)^{\omega_j}\Bigg), 1-\log_\lambda\Bigg(1+\prod_{j=1}^{k+1}(\lambda^{1-b_j}-1)^{\omega_j}\Bigg)\Bigg],$$

$$1-\log_\lambda\Bigg(1+\prod_{j=1}^{k+1}(\lambda^{1-c_j}-1)^{\omega_j}\Bigg), \Bigg[1-\log_\lambda\Bigg(1+\prod_{j=1}^{k+1}(\lambda^{1-d_j}-1)^{\omega_j}\Bigg),$$

$$1-\log_\lambda\Bigg(1+\prod_{j=1}^{k+1}(\lambda^{1-e_j}-1)^{\omega_j}\Bigg)\Bigg]; \Bigg[\log_\lambda\Bigg(1+\prod_{j=1}^{k+1}(\lambda^{A_j}-1)^{\omega_j}\Bigg),$$

$$\log_\lambda\Bigg(1+\prod_{j=1}^{k+1}(\lambda^{B_j}-1)^{\omega_j}\Bigg)\Bigg], \log_\lambda\Bigg(1+\prod_{j=1}^{k+1}(\lambda^{C_j}-1)^{\omega_j}\Bigg),$$

$$\Bigg[\log_\lambda\Bigg(1+\prod_{j=1}^{k+1}(\lambda^{D_j}-1)^{\omega_j}\Bigg), \log_\lambda\Bigg(1+\prod_{j=1}^{k+1}(\lambda^{E_j}-1)^{\omega_j}\Bigg)\Bigg]\Bigg\rangle$$

Thus, results holds for $n = k+1$ and hence, by the principle of mathematical induction, result given in Eq. (2) holds for all positive integer n. □

It has been observed from the TIT2IFWA operator that it satisfies the certain properties such as boundedness, idempotent and monotonicity, invariance etc., which can be demonstrated as follows:

Theorem 4.3. *(Idempotency:) If $\alpha_j = \alpha_0$ for all j, then*

$$\mathit{TIT2IFWA}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \alpha_0$$

The proof of Theorem 4.3 is provided in Appendix.

Theorem 4.4. *(Boundedness:) Let $\alpha^- = \big\langle [\min_j\{a_j\}, \min_j\{b_j\}], \min_j\{c_j\}, [\min_j\{d_j\}, \min_j\{e_j\}]; [\max_j\{A_j\}, \max_j\{B_j\}], \max_j\{C_j\}, [\max_j\{D_j\}, \max_j\{E_j\}]\big\rangle$ and $\alpha^+ = \big\langle [\max_j\{a_j\}, \max_j\{b_j\}], \max_j\{c_j\}, [\max_j\{d_j\}, \max_j\{e_j\}]; [\min_j\{A_j\}, \min_j\{B_j\}], \min_j\{C_j\}, [\min_j\{D_j\}, \min_j\{E_j\}]\big\rangle$ then*

$$\alpha^- \leq \mathit{TIT2IFWA}(\alpha_1, \alpha_2, \ldots, \alpha_n) \leq \alpha^+$$

The proof of Theorem 4.4 is provided in Appendix.

Theorem 4.5. *(Monotonicity:) If α_j and β_j be collections of two TIT2IFNs such that $\alpha_j \leq \beta_j$ then*

$$\mathit{TIT2IFWA}(\alpha_1, \alpha_2, \ldots, \alpha_n) \leq \mathit{TIT2IFWA}(\beta_1, \beta_2, \ldots, \beta_n)$$

Theorem 4.6. *(Shift-invariance:) If β be another TIT2IFN, then*

$$\mathit{TIT2IFWA}(\alpha_1 \oplus_F \beta, \alpha_2 \oplus_F \beta, \ldots, \alpha_n \oplus_F \beta) = \mathit{TIT2IFWA}(\alpha_1, \alpha_2 \ldots, \alpha_n) \oplus_F \beta$$

Theorem 4.7. *(Homogeneity:) If $\beta > 0$ be a real number, then*

$$\mathit{TIT2IFWA}(\beta \cdot_F \alpha_1, \beta \cdot_F \alpha_2, \ldots, \beta \cdot_F \alpha_n) = \beta \cdot_F \mathit{TIT2IFWA}(\alpha_1, \alpha_2 \ldots, \alpha_n)$$

Proof. The proof of these Theorems can be easily derived from the Frank operational laws of TIT2IFNs; thus, it is omitted here due to the space limitations. □

4.2. **Ordered weighted averaging Operator.**

Definition 4.8. Suppose Ω be a family of TIT2IFNs α_j for $j = 1, 2, \ldots, n$ and TIT2IFOWA : $\Omega^n \longrightarrow \Omega$, if

$$\text{TIT2IFOWA}(\alpha_1, \ldots, \alpha_n) = \omega_1 \cdot_F \alpha_{\delta(1)} \oplus_F \omega_2 \cdot_F \alpha_{\delta(2)} \oplus_F \ldots \oplus_F \omega_n \cdot_F \alpha_{\delta(n)}$$

where $\omega = (\omega_1, \omega_2 \ldots, \omega_n)^T$ is the weight vector of α_j, $(\delta(1), \delta(2), \ldots, \delta(n))$ is a permutation of $(1, 2, 3, \ldots, n)$ such that $\alpha_{\delta(j-1)} \geq \alpha_{\delta(j)}$ for $j = 2, 3, \ldots, n$ then, TIT2IFOWA is called triangular interval type-2 intuitionistic fuzzy ordered weighted averaging operator

Theorem 4.9. *The aggregated value by using TIT2IFOWA operator for a collection of TIT2IFNs α_j, $(j = 1, 2, \ldots n)$ is again TIT2IFN, and is given by*

$$\begin{aligned}
&\mathit{TIT2IFOWA}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \Bigg\langle \Bigg[1 - \log_\lambda \Bigg(1 + \prod_{j=1}^{n} (\lambda^{1-a_{\delta(j)}} - 1)^{\omega_j}\Bigg), \\
&1 - \log_\lambda \Bigg(1 + \prod_{j=1}^{n} (\lambda^{1-b_{\delta(j)}} - 1)^{\omega_j}\Bigg)\Bigg], 1 - \log_\lambda \Bigg(1 + \prod_{j=1}^{n} (\lambda^{1-c_{\delta(j)}} - 1)^{\omega_j}\Bigg), \\
&\Bigg[1 - \log_\lambda \Bigg(1 + \prod_{j=1}^{n} (\lambda^{1-d_{\delta(j)}} - 1)^{\omega_j}\Bigg), 1 - \log_\lambda \Bigg(1 + \prod_{j=1}^{n} (\lambda^{1-e_{\delta(j)}} - 1)^{\omega_j}\Bigg)\Bigg]; \\
&\Bigg[\log_\lambda \Bigg(1 + \prod_{j=1}^{n} (\lambda^{A_{\delta(j)}} - 1)^{\omega_j}\Bigg), \log_\lambda \Bigg(1 + \prod_{j=1}^{n} (\lambda^{B_{\delta(j)}} - 1)^{\omega_j}\Bigg)\Bigg], \log_\lambda \Bigg(1 + \prod_{j=1}^{n} (\lambda^{C_{\delta(j)}} - 1)^{\omega_j}\Bigg), \\
&\Bigg[\log_\lambda \Bigg(1 + \prod_{j=1}^{n} (\lambda^{D_{\delta(j)}} - 1)^{\omega_j}\Bigg), \log_\lambda \Bigg(1 + \prod_{j=1}^{n} (\lambda^{E_{\delta(j)}} - 1)^{\omega_j}\Bigg)\Bigg] \Bigg\rangle
\end{aligned}$$

Proof. Proof of this result is similar to Theorem 4.2, so we omit here. □

Example 4.10. Let $\alpha_1 = \langle [0.4, 0.5], 0.5, [0.6, 0.7];\ [0.5, 0.4], 0.3, [0.2, 0.1] \rangle$, $\alpha_2 = \langle [0.2, 0.3], 0.4,\ [0.4,\ 0.5];\ [0.7, 0.6],\ 0.5, [0.3, 0.2] \rangle$, $\alpha_3 = \langle [0.3, 0.5], 0.5, [0.6, 0.7];\ [0.6, 0.4],\ 0.4,\ [0.3, 0.2] \rangle$ be three TIT2IFNs and $\omega = (0.3, 0.4, 0.3)^T$ be their corresponding weight vector. Now, based on the ranking formula, we have $\alpha_1 \geq \alpha_3 \geq \alpha_2$ thus, we have $\alpha_{\delta(1)} = \alpha_1$, $\alpha_{\delta(2)} = \alpha_3$ and $\alpha_{\delta(3)} = \alpha_2$. If we take $\lambda = 2$ for simplification, then we have, $1 - \log_\lambda \big(1 + \prod_{j=1}^{3} (\lambda^{1-a_{\delta(j)}} - 1)^{\omega_j}\big) = 0.3034$, $\log_\lambda \big(1 + \prod_{j=1}^{3} (\lambda^{A_{\delta(j)}} - 1)^{\omega_j}\big) = 0.5959$, $1 - \log_\lambda \big(1 + \prod_{j=1}^{3} (\lambda^{1-b_{\delta(j)}} - 1)^{\omega_j}\big) = 0.4456$, $1 - \log_\lambda \big(1 + \prod_{j=1}^{3} (\lambda^{1-c_{\delta(j)}} - 1)^{\omega_j}\big) = 0.4716$, $1 - \log_\lambda \big(1 + \prod_{j=1}^{3} (\lambda^{1-d_{\delta(j)}} - 1)^{\omega_j}\big) = 0.5470$, $1 - \log_\lambda \big(1 + \prod_{j=1}^{3} (\lambda^{1-e_{\delta(j)}} - 1)^{\omega_j}\big) = 0.6491$, $\log_\lambda \big(1 + \prod_{j=1}^{3} (\lambda^{B_{\delta(j)}} - 1)^{\omega_j}\big) = 0.4530$, $\log_\lambda \big(1 + \prod_{j=1}^{3} (\lambda^{C_{\delta(j)}} - 1)^{\omega_j}\big) = 0.3933$, $\log_\lambda \big(1 + \prod_{j=1}^{3} (\lambda^{D_{\delta(j)}} - 1)^{\omega_j}\big) = 0.2660$, $\log_\lambda \big(1 + \prod_{j=1}^{3} (\lambda^{E_{\delta(j)}} - 1)^{\omega_j}\big) = 0.1629$. Thus, $\text{TIT2IFOWA}(\alpha_1, \alpha_2, \alpha_3) = \big\langle [0.3034, 0.4456], 0.4716, [0.5470, 0.6491]; [0.5959, 0.4530], 0.3933, [0.2660, 0.1629] \big\rangle$.

As similar to those of TIT2IFWA operator, the TIT2IFOWA operator also follows the boundedness, idempotency and monotonicity properties. Besides the aforementioned properties, the TIT2IFOWA operator has the following desirable results.

Theorem 4.11. *For a collection of TIT2IFNs $\alpha_j (j = 1, 2, \ldots, n)$, we have the following:*

(i) If $\omega = (1, 0, \ldots, 0)^T$ then TIT2IFOWA$(\alpha_1, \alpha_2, \ldots, \alpha_n) = \max\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$
(ii) If $\omega = (0, 0, \ldots, 1)^T$ then TIT2IFOWA$(\alpha_1, \alpha_2, \ldots, \alpha_n) = \min\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$
(iii) If $\omega_j = 1$ and $\omega_i = 0 (i \neq j)$, then TIT2IFOWA$(\alpha_1, \alpha_2, \ldots, \alpha_n) = \alpha_{\delta(j)}$ where $\alpha_{\delta(j)}$ is the j^{th} largest of α_j, $(j = 1, 2, \ldots, n)$.

4.3. **Hybrid Averaging Operator.** Since TIT2IFWA operator weights only the TIT2IFNs while TIT2IFOWA operator weights the ordered positions of TIT2IFNs. However, in order to combine these two aspects in one, we now introduce hybrid aggregation operator, which weight both the given TIT2IFNs and their ordered positions.

Definition 4.12. A triangular interval type-2 intuitionistic fuzzy hybrid averaging (TIT2IFHA) operator is a mapping TIT2IFHA : $\Omega^n \longrightarrow \Omega$, such that

$$\text{TIT2IFHA}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \phi_1 \cdot_F \dot{\alpha}_{\delta(1)} \oplus_F \phi_2 \cdot_F \dot{\alpha}_{\delta(2)} \oplus_F \ldots \oplus_F \phi_n \cdot_F \dot{\alpha}_{\delta(n)}$$

where Ω is the set of all TIT2IFNs, and $\phi = (\phi_1, \phi_2, \ldots, \phi_n)^T$ is the weighted vector associated with TIT2IFHA, such that $\phi_j > 0$ and $\sum\limits_{j=1}^{n} \phi_j = 1$; $\dot{\alpha}_j = (n\omega_j) \cdot_F \alpha_j, j = 1, 2, \ldots, n$, $\dot{\alpha}_{\delta(j)}$ is the j^{th} largest of the weighted TIT2IFNs $\dot{\alpha}_j$ and $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector of α_j with $\omega_j > 0$, $\sum\limits_{j=1}^{n} \omega_j = 1$.

Theorem 4.13. *For a collection of TIT2IFNs, $\alpha_j (j = 1, 2, \ldots, n)$, the aggregated value based on the TIT2IFHA operator is also TIT2IFN and can be expressed as*

$$\begin{aligned}
&TIT2IFHA(\alpha_1, \alpha_2, \ldots, \alpha_n) = \Bigg\langle \Bigg[1 - \log_\lambda \left(1 + \prod_{j=1}^{n} (\lambda^{1-a_{\delta(j)}} - 1)^{\phi_j}\right), \\
&1 - \log_\lambda \left(1 + \prod_{j=1}^{n} (\lambda^{1-b_{\delta(j)}} - 1)^{\phi_j}\right)\Bigg], 1 - \log_\lambda \left(1 + \prod_{j=1}^{n} (\lambda^{1-c_{\delta(j)}} - 1)^{\phi_j}\right), \\
&\Bigg[1 - \log_\lambda \left(1 + \prod_{j=1}^{n} (\lambda^{1-d_{\delta(j)}} - 1)^{\phi_j}\right), 1 - \log_\lambda \left(1 + \prod_{j=1}^{n} (\lambda^{1-e_{\delta(j)}} - 1)^{\phi_j}\right)\Bigg]; \\
&\Bigg[\log_\lambda \left(1 + \prod_{j=1}^{n} (\lambda^{A_{\delta(j)}} - 1)^{\phi_j}\right), \log_\lambda \left(1 + \prod_{j=1}^{n} (\lambda^{B_{\delta(j)}} - 1)^{\phi_j}\right)\Bigg], \log_\lambda \left(1 + \prod_{j=1}^{n} (\lambda^{C_{\delta(j)}} - 1)^{\phi_j}\right), \\
&\Bigg[\log_\lambda \left(1 + \prod_{j=1}^{n} (\lambda^{D_{\delta(j)}} - 1)^{\phi_j}\right), \log_\lambda \left(1 + \prod_{j=1}^{n} (\lambda^{E_{\delta(j)}} - 1)^{\phi_j}\right)\Bigg]\Bigg\rangle
\end{aligned}$$

Proof. The proof is similar to Theorem 4.2, so it is omitted here. □

Similar to those of TIT2IFWA and TIT2IFOWA operators, the TIT2IFHA operator has also follows the same properties.

Theorem 4.14. *The TIT2IFWA operator as defined in Theorem 4.2 is a special case of the TIT2IFHA operator.*

Proof. Let $\phi = (1/n, 1/n, \ldots, 1/n)^T$ then, $\text{TIT2IFHA}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \frac{1}{n}(\dot{\alpha}_{\delta(1)} \oplus_F \dot{\alpha}_{\delta(2)} \oplus_F \ldots \oplus_F \dot{\alpha}_{\delta(n)}) = \omega_1 \cdot_F \alpha_1 \oplus_F \ldots \oplus_F \omega_n \cdot_F \alpha_n = \text{TIT2IFWA}(\alpha_1, \alpha_2, \ldots, \alpha_n)$. □

Theorem 4.15. *The TIT2IFOWA operator as defined in Theorem 4.9 is a special case of the TIT2IFHA operator.*

Proof. Let $\omega = (1/n, 1/n, \ldots, 1/n)^T$ then, $\dot{\alpha}_j = (n\omega_j) \cdot_F \alpha_j = \alpha_j$, for all j. Thus, $\text{TIT2IFHA}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \phi_1 \cdot_F \dot{\alpha}_{\delta(1)} \oplus_F \phi_2 \cdot_F \dot{\alpha}_{\delta(2)} \oplus_F \ldots \oplus_F \phi_n \cdot_F \dot{\alpha}_{\delta(n)} = \phi_1 \cdot_F \alpha_{\delta(1)} \oplus_F \phi_2 \cdot_F \alpha_{\delta(2)} \oplus_F \ldots \oplus_F \phi_n \cdot_F \alpha_{\delta(n)} = \text{TIT2IFOWA}(\alpha_1, \alpha_2, \ldots, \alpha_n)$. □

5. Decision-making approach based on proposed operators

In this section, a decision-making method by using above defined aggregation operators has been presented followed by an illustrative example for demonstrating the approach.

5.1. **Decision-making approach.** Let $A = \{A_1, A_2, \ldots, A_m\}$ be the set of alternatives and $G = \{G_1, G_2, \ldots, G_n\}$ be a set of criteria with the associated weight $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ satisfying $\omega_j > 0$ and $\sum_{j=1}^{n} \omega_j = 1$. The characteristics of each alternative $A_i (i = 1, 2, \ldots, m)$ with respect to each criteria $G_j (j = 1, 2, \ldots, n)$ is characterized in terms of TIT2IFNs $\alpha_{ij} = \langle [a_{ij}, b_{ij}], c_{ij}, [d_{ij}, e_{ij}]; [A_{ij}, B_{ij}], C_{ij}, [D_{ij}, E_{ij}] \rangle$ where $a_{ij} \leq b_{ij} \leq c_{ij} \leq d_{ij} \leq e_{ij}$ and $A_{ij} \geq B_{ij} \geq C_{ij} \geq D_{ij} \geq E_{ij}$ such that $e_{ij} + E_{ij} \leq 1$ and $a_{ij} + A_{ij} \leq 1$. Then, in the following, we develop an approach based on the proposed operator to solve the decision-making problems with TIT2IF information, which involves the following steps.

Step 1: Collect the information as decision matrix $D = (\alpha_{ij})_{m \times n}$.

Step 2: If all the attributes are of same type, then there is no need of normalization. But, if there are different types of criterion say profit (B) and cost (C), then we convert profit into cost by the following normalized formula:

$$r_{ij} = \begin{cases} \alpha_{ij} & ; \quad j \in B \\ \alpha_{ij}^c & ; \quad j \in C \end{cases}$$

where α_{ij}^c is the complement of α_{ij}. Hence, we obtain the normalized TIT2IFN decision matrix $R = (r_{ij})_{m \times n}$.

Step 3: Aggregate the TIT2IFNs $r_{ij} (j = 1, 2, \ldots, n)$ for each alternative $A_i (i = 1, 2, \ldots, m)$ into the overall preference value r_i either by using the proposed TIT2IFWA, TIT2IFOWA or TIT2IFHA operators.

Step 4: Determine the ranking value of each aggregated value $r_i, (i = 1, 2, \ldots, m)$ by using Eq. (1) and hence select the best one(s).

Step 5: Do the sensitivity analysis on the parameter λ according to decision makers' preferences.

5.2. **Numerical Example.** The above decision-making procedure has been illustrated with the case study that a person wants to invest a money in the market. For this, they have chosen the four multinational companies namely A_1: Infosys, A_2: Wipro, A_3: Dell and A_4: Apple. In order to assess these alternatives, the investors have brought a panel with three experts e_1, e_2 and e_3 whose weight vector is 0.35, 0.35 and 0.30. These three experts have evaluated the each alternative A_i, $i = 1, 2, 3, 4$ with respect to the four attributes namely G_1 : growth analysis, G_2 : development of society, G_3 : technical support and G_4 : quality whose weight vector is $\omega = (0.3, 0.2, 0.1, 0.4)^T$. Then, the following steps have been performed based on the proposed decision-making approach to find the most desirable alternative(s).

Step 1: The three experts have evaluated the given alternatives A_i, $i = 1, 2, 3, 4$ with respect to attributes G_j, $j = 1, 2, 3, 4$ in the form of TIT2IFNs which are represented in Tables 1-3 respectively.

	G_1	G_2	G_3	G_4
A_1	$\langle[0.4, 0.5], 0.6, [0.7, 0.8]; [0.5, 0.4], 0.3, [0.2, 0.1]\rangle$	$\langle[0.5, 0.6], 0.7, [0.8, 0.9]; [0.4, 0.3], 0.3, [0.2, 0.1]\rangle$	$\langle[0.6, 0.7], 0.8, [0.9, 0.95]; [0.3, 0.25], 0.2, [0.1, 0.05]\rangle$	$\langle[0.6, 0.7], 0.8, [0.8, 0.9]; [0.3, 0.25], 0.2, [0.2, 0.1]\rangle$
A_2	$\langle[0.5, 0.6], 0.7, [0.8, 0.9]; [0.4, 0.3], 0.3, [0.2, 0.1]\rangle$	$\langle[0.5, 0.6], 0.7, [0.8, 0.9]; [0.4, 0.3], 0.3, [0.2, 0.1]\rangle$	$\langle[0.4, 0.5], 0.6, [0.7, 0.8]; [0.5, 0.4], 0.4, [0.3, 0.2]\rangle$	$\langle[0.6, 0.7], 0.8, [0.8, 0.9]; [0.4, 0.2], 0.2, [0.2, 0.1]\rangle$
A_3	$\langle[0.3, 0.4], 0.5, [0.6, 0.7]; [0.6, 0.5], 0.4, [0.3, 0.2]\rangle$	$\langle[0.4, 0.5], 0.6, [0.7, 0.8]; [0.45, 0.4], 0.3, [0.2, 0.1]\rangle$	$\langle[0.2, 0.3], 0.4, [0.5, 0.6]; [0.6, 0.5], 0.4.[0.3, 0.2]\rangle$	$\langle[0.3, 0.4], 0.5, [0.6, 0.7]; [0.7, 0.5], 0.4, [0.3, 0.2]\rangle$
A_4	$\langle[0.2, 0.3], 0.4, [0.5, 0.6]; [0.7, 0.6], 0.5, [0.4, 0.3]\rangle$	$\langle[0.6, 0.7], 0.8, [0.85, 0.9]; [0.4, 0.2], 0.2, [0.15, 0.05]\rangle$	$\langle[0.3, 0.4], 0.5, [0.6, 0.7]; [0.6, 0.5], 0.4, [0.3, 0.2]\rangle$	$\langle[0.4, 0.5], 0.6, [0.7, 0.8]; [0.5, 0.4], 0.3, [0.2, 0.1]\rangle$

TABLE 1. TIT2 Intuitionistic fuzzy decision matrix given by Expert $e_1, (D^1)$

	G_1	G_2	G_3	G_4
A_1	$\langle[0.2, 0.3], 0.35, [0.4, 0.5]; [0.7, 0.6], 0.50, [0.4, 0.3]\rangle$	$\langle[0.3, 0.35], 0.4, [0.4, 0.5]; [0.6, 0.5], 0.5, [0.4, 0.3]\rangle$	$\langle[0.3, 0.5], 0.6, [0.6, 0.7]; [0.6, 0.4], 0.4, [0.3, 0.2]\rangle$	$\langle[0.3, 0.4], 0.5, [0.5, 0.6]; \langle[0.6, 0.3], 0.3, [0.2, 0.1]\rangle$
A_2	$\langle[0.25, 0.30], 0.45, [0.35, 0.40]; [0.65, 0.55], 0.45, [0.35, 0.25]\rangle$	$\langle[0.3, 0.4], 0.4, [0.5, 0.6]; [0.6, 0.5], 0.4, [0.4, 0.3]\rangle$	$\langle[0.3, 0.5], 0.55, [0.6, 0.7]; [0.7, 0.6], 0.4, [0.4, 0.2]\rangle$	$\langle[0.2, 0.4], 0.45, [0.5, 0.6]; [0.7, 0.5], 0.45, [0.4, 0.3]\rangle$
A_3	$\langle[0.45, 0.50], 0.50, [0.6, 0.7]; [0.4, 0.3], 0.3, [0.2, 0.1]\rangle$	$\langle[0.5, 0.6], 0.6, [0.7, 0.8]; [0.5, 0.4], 0.3, [0.2, 0.1]\rangle$	$\langle[0.4, 0.6], 0.7, [0.8, 0.9]; [0.5, 0.4], 0.35, [0.2, 0.1]\rangle$	$\langle[0.3, 0.4], 0.55, [0.6, 0.7]; [0.6, 0.5], 0.4, [0.3, 0.2]\rangle$
A_4	$\langle[0.5, 0.55], 0.55, [0.6, 0.75]; [0.4, 0.3], 0.2, [0.2, 0.1]\rangle$	$\langle[0.6, 0.7], 0.8, [0.85, 0.9]; [0.4, 0.35], 0.2, [0.1, 0.05]\rangle$	$\langle[0.4, 0.5], 0.6, [0.7, 0.8]; [0.5, 0.4], 0.3, [0.2, 0.1]\rangle$	$\langle[0.5, 0.6], 0.65, [0.7, 0.8]; [0.4, 0.3], 0.35, [0.3, 0.2]\rangle$

TABLE 2. TIT2 Intuitionistic fuzzy decision matrix given by Expert $e_2, (D^2)$

	G_1	G_2	G_3	G_4
A_1	$\langle[0.4, 0.5], 0.6, [0.7, 0.8]; [0.5, 0.4], 0.3, [0.2, 0.1]\rangle$	$\langle[0.5, 0.6], 0.7, [0.8, 0.9]; [0.45, 0.4], 0.25, [0.15, 0.1]\rangle$	$\langle[0.4, 0.5], 0.6, [0.7, 0.8]; [0.45, 0.35], 0.3, [0.3, 0.25]\rangle$	$\langle[0.5, 0.6], 0.7, [0.8, 0.9]; [0.4, 0.3], 0.3, [0.2, 0.1]\rangle$
A_2	$\langle[0.3, 0.4], 0.5, [0.6, 0.7]; [0.6, 0.5], 0.4, [0.3, 0.2]\rangle$	$\langle[0.6, 0.7], 0.8, [0.8, 0.9]; [0.35, 0.3], 0.2, [0.15, 0.1]\rangle$	$\langle[0.4, 0.5], 0.6, [0.7, 0.8]; [0.45, 0.40], 0.3, [0.2, 0.1]\rangle$	$\langle[0.6, 0.7], 0.8, [0.8, 0.9]; [0.3, 0.2], 0.2, [0.15, 0.05]\rangle$
A_3	$\langle[0.1, 0.2], 0.3, [0.4, 0.5]; [0.6, 0.5], 0.4, [0.3, 0.2]\rangle$	$\langle[0.6, 0.7], 0.8, [0.9, 0.9]; [0.3, 0.2], 0.2, [0.2, 0.1]\rangle$	$\langle[0.3, 0.4], 0.5, [0.6, 0.7]; [0.55, 0.45], 0.4, [0.3, 0.2]\rangle$	$\langle[0.4, 0.5], 0.6, [0.7, 0.8]; [0.5, 0.4], 0.35, [0.3, 0.2]\rangle$
A_4	$\langle[0.2, 0.3], 0.4, [0.5, 0.6]; [0.7, 0.6], 0.5, [0.4, 0.3]\rangle$	$\langle[0.2, 0.3], 0.4, [0.5, 0.6]; [0.6, 0.5], 0.4, [0.3, 0.2]\rangle$	$\langle[0.1, 0.2], 0.3, [0.4, 0.5]; [0.5, 0.4], 0.3, [0.2, 0.1]\rangle$	$\langle[0.3, 0.4], 0.5, [0.6, 0.7]; [0.6, 0.5], 0.4, [0.4, 0.3]\rangle$

TABLE 3. TIT2 Intuitionistic fuzzy decision matrix given by Expert $e_3, (D^3)$

Step 2: Since all the attributes are of same types, so there is no need of the normalization.

Step 3: Let $\phi = (0.35, 0.25, 0.25, 0.15)^T$ be the weight vector associated with TIT2IFHA operator. Without loss of generality, here we utilize TIT2IFHA and TIT2IFWA operators to aggregate the given data by taking $\lambda = 2$. For this, firstly we compute $\dot{\alpha}_{ij}^k = 4\omega_j \alpha_{ij}^k$; $k = 1, 2, 3$ and hence the corresponding values for each decision makers e_k, $k = 1, 2, 3$ are summarized in Tables 4-6 respectively.

	G_1	G_2	G_3	G_4
A_1	⟨[0.7845, 0.8678], 0.9332, [0.9332, 0.9788]; [0.1322, 0.0972], 0.0668, [0.0668, 0.0212]⟩	⟨[0.4622, 0.5700], 0.6731, [0.7705, 0.8604]; [0.4300, 0.3269], 0.2295, [0.1396, 0.0599]⟩	⟨[0.4208, 0.5131], 0.6106, [0.7158, 0.8345]; [0.4869, 0.3894], 0.3894, [0.2842, 0.1655]⟩	⟨[0.2920, 0.3616], 0.4475, [0.5685, 0.6638]; [0.6384, 0.5982], 0.5525, [0.4315, 0.3362]⟩
A_2	⟨[0.7845, 0.8678], 0.9332, [0.9332, 0.9788]; [0.2155, 0.0668], 0.0668, [0.0668, 0.0212]⟩	⟨[0.5700, 0.6731], 0.7705, [0.8604, 0.9401]; [0.3269, 0.2295], 0.2295, [0.1396, 0.0599]⟩	⟨[0.4208, 0.5131], 0.6106, [0.7158, 0.8345]; [0.4869, 0.3894], 0.3894, [0.2842, 0.1655]⟩	⟨[0.1785, 0.2320], 0.2920, [0.3616, 0.4475]; [0.7680, 0.7080], 0.7080, [0.6384, 0.5525]⟩
A_3	⟨[0.4434, 0.5707], 0.6849, [0.7845, 0.8678]; [0.5566, 0.3151], 0.2155, [0.1322, 0.0668]⟩	⟨[0.3507, 0.4622], 0.5700, [0.6731, 0.7705]; [0.5378, 0.4300], 0.3269, [0.2295, 0.1396]⟩	⟨[0.3321, 0.4208], 0.5131, [0.6106, 0.7158]; [0.5336, 0.4869], 0.3894, [0.2842, 0.1655]⟩	⟨[0.0839, 0.1295], 0.1785, [0.2320, 0.2920]; [0.8215, 0.7680], 0.7080, [0.6384, 0.5525]⟩
A_4	⟨[0.5707, 0.6849], 0.7845, [0.8678, 0.9332]; [0.3151, 0.2155], 0.1322, [0.0668, 0.0212]⟩	⟨[0.5131, 0.6106], 0.7158, [0.7729, 0.8345]; [0.4869, 0.2842], 0.2842, [0.2271, 0.0961]⟩	⟨[0.2362, 0.3507], 0.4622, [0.5700, 0.6731]; [0.6493, 0.5378], 0.4300, [0.3269, 0.2295]⟩	⟨[0.1295, 0.1785], 0.2320, [0.2920, 0.3616]; [0.8215, 0.7680], 0.7080, [0.6384, 0.5525]⟩

TABLE 4. Decision maker preferences corresponding to Expert $e_1(D^1)$

	G_1	G_2	G_3	G_4
A_1	⟨[0.4434, 0.5707], 0.6849, [0.6849, 0.7845]; [0.4293, 0.1322], 0.1322, [0.0668, 0.0212]⟩	⟨[0.2362, 0.3507], 0.4069, [0.4622, 0.5700]; [0.6493, 0.5378], 0.4300, [0.3269, 0.2295]⟩	⟨[0.2463, 0.2889], 0.3321, [0.3321, 0.4208]; [0.6679, 0.5792], 0.5792, [0.4869, 0.3894]⟩	⟨[0.1295, 0.2320], 0.2920, [0.2920, 0.3616]; [0.8215, 0.7080], 0.7080, [0.6384, 0.5525]⟩
A_2	⟨[0.3048, 0.5707], 0.6295, [0.6849, 0.7845]; [0.5566, 0.3151], 0.2634, [0.2155, 0.1322]⟩	⟨[0.2938, 0.3507], 0.5166, [0.4069, 0.4622]; [0.5931, 0.4834], 0.3778, [0.2774, 0.1835]⟩	⟨[0.2463, 0.3321], 0.3321, [0.4208, 0.5131]; [0.6679, 0.5792], 0.4869, [0.4869, 0.3894]⟩	⟨[0.1295, 0.2320], 0.2610, [0.2920, 0.3616]; [0.8705, 0.8215], 0.7080, [0.7080, 0.5525]⟩
A_3	⟨[0.4434, 0.5707], 0.7366, [0.7845, 0.8678]; [0.4293, 0.3151], 0.2155, [0.1322, 0.0668]⟩	⟨[0.5166, 0.5700], 0.5700, [0.6731, 0.7705]; [0.3269, 0.2295], 0.2295, [0.1396, 0.0599]⟩	[⟨0.4208, 0.5131], 0.5131, [0.6106, 0.7158]; [0.5792, 0.4869], 0.3894, [0.2842, 0.1655]⟩	⟨[0.1785, 0.2920], 0.3616, [0.4475, 0.5685] [0.7680, 0.7080], 0.6747, [0.5525, 0.4315]⟩
A_4	⟨[0.6849, 0.7845], 0.8283, [0.8678, 0.9332]; [0.2155, 0.1322], 0.1717, [0.1322, 0.0668]⟩	[0.5700, 0.6222], 0.6222, [0.6731, 0.8165]; [0.3269, 0.2295], 0.1396, [0.1396, 0.0599]⟩	[0.5131, 0.6106], 0.7158, [0.7729, 0.8345]; [0.4869, 0.4389], 0.2842, [0.1655, 0.0961]⟩	[0.1785, 0.2320], 0.2920, [0.3616, 0.4475]; [0.7680, 0.7080], 0.6384, [0.5525, 0.4315]⟩

TABLE 5. Decision maker preferences corresponding to Expert $e_2(D^2)$

	G_1	G_2	G_3	G_4
A_1	⟨[0.6849, 0.7845], 0.8678, [0.9332, 0.9788]; [0.2155, 0.1322], 0.1322, [0.0668, 0.0212]⟩	⟨[0.4622, 0.5700], 0.6731, [0.7705, 0.8604]; [0.4300, 0.3269], 0.2295, [0.1396, 0.0599]⟩	⟨[0.4208, 0.5131], 0.6106, [0.7158, 0.8345]; [0.5336, 0.4869], 0.3380, [0.2271, 0.1655]⟩	⟨[0.1785, 0.2320], 0.2920, [0.3616, 0.4475]; [0.7390, 0.6747], 0.6384, [0.6384, 0.5982]⟩
A_2	⟨[0.7845, 0.8678], 0.9332, [0.9332, 0.9788]; [0.1322, 0.0668], 0.0668, [0.0413, 0.0068]⟩	⟨[0.5131, 0.6106], 0.7158, [0.7158, 0.8345]; [0.4389, 0.3894], 0.2842, [0.2271, 0.1655]⟩	⟨[0.3507, 0.4622], 0.5700, [0.6731, 0.7705]; [0.5378, 0.4300], 0.3269, [0.2295, 0.1396]⟩	⟨[0.1785, 0.2320], 0.2920, [0.3616, 0.4475]; [0.7390, 0.7080], 0.6384, [0.5525, 0.4315]⟩
A_3	⟨[0.5707, 0.6849], 0.7845, [0.8678, 0.9332]; [0.3151, 0.2155], 0.1717, [0.1322, 0.0668]⟩	⟨[0.5131, 0.6106], 0.7158, [0.8345, 0.8345]; [0.3894, 0.2842], 0.2842, [0.2842, 0.1655]⟩	⟨[0.1191, 0.2362], 0.3507, [0.4622, 0.5700]; [0.5378, 0.4300], 0.3269, [0.2295, 0.1396]⟩	⟨[0.1295, 0.1785], 0.2320, [0.2920, 0.3616]; [0.7954, 0.7390], 0.7080, [0.6384, 0.5525]⟩
A_4	⟨[0.4434, 0.5707], 0.6849, [0.7845, 0.8678]; [0.4293, 0.3151], 0.2155, [0.2155, 0.1322]⟩	⟨[0.2362, 0.3507], 0.4622, [0.5700, 0.6731]; [0.6493, 0.5378], 0.4300, [0.3269, 0.2295]⟩	⟨[0.1626, 0.2463], 0.3321, [0.4208, 0.5131]; [0.6679, 0.5792], 0.4869, [0.3894, 0.2842]⟩	⟨[0.0409, 0.0839], 0.1295, [0.1785, 0.2320]; [0.7680, 0.7080], 0.6384, [0.5525, 0.4315]⟩

TABLE 6. Decision maker preferences corresponding to Expert $e_3(D^3)$

Now, aggregate these three expert matrices corresponding to the expert weight vector $0.35, 0.35$ and 0.30 into the collective TIT2IF matrix $R = (r_{ij})_{4\times4}$ by using TIT2IFWA operator. The result corresponding to these are summarized in Table 7. Utilize the TIT2IFWA operator, corresponding to the weight vector $\phi = (0.35,$

	G_1	G_2	G_3	G_4
A_1	⟨[0.6601, 0.7664], 0.8573, [0.8835, 0.9516]; [0.2336, 0.1188], 0.1043, [0.0668, 0.0212]⟩	⟨[0.3902, 0.5016], 0.5950, [0.6878, 0.7906]; [0.4984, 0.3906], 0.2873, [0.1891, 0.0966]⟩	⟨[0.3638, 0.4424], 0.5271, [0.6122, 0.7387]; [0.5599, 0.4795], 0.4305, [0.3227, 0.2248]⟩	⟨[0.2034, 0.2795], 0.3499, [0.4204, 0.5087]; [0.7295, 0.6582], 0.6300, [0.5583, 0.4778]⟩
A_2	⟨[0.6688, 0.7978], 0.8762, [0.8835, 0.9516]; [0.2639, 0.1165], 0.1090, [0.0879, 0.0289]⟩	⟨[0.4668, 0.5590], 0.6806, [0.7080, 0.8197]; [0.4419, 0.3511], 0.2920, [0.2060, 0.1208]⟩	⟨[0.3419, 0.4387], 0.5131, [0.6172, 0.7309]; [0.5613, 0.4620], 0.4002, [0.3236, 0.2139]⟩	⟨[0.1615, 0.2320], 0.2813, [0.3379, 0.4186]; [0.7937, 0.7463], 0.6865, [0.6345, 0.5135]⟩
A_3	⟨[0.4845, 0.6082], 0.7357, [0.8136, 0.8921]; [0.4301, 0.2815], 0.2014, [0.1322, 0.0668]⟩	⟨[0.4619, 0.5480], 0.6194, [0.7324, 0.7918]; [0.4114, 0.3060], 0.2772, [0.2062, 0.1096]⟩	⟨[0.3073, 0.4058], 0.4684, [0.5703, 0.6776]; [0.5505, 0.4692], 0.3696, [0.2666, 0.1573]⟩	⟨[0.1313, 0.2034], 0.2619, [0.3307, 0.4204]; [0.7947, 0.7380], 0.6962, [0.6072, 0.5073]⟩
A_4	⟨[0.5820, 0.6961], 0.7764, [0.8467, 0.9179]; [0.3039, 0.2045], 0.1680, [0.1211, 0.0551]⟩	⟨[0.4635, 0.5487], 0.6182, [0.6864, 0.7888]; [0.4644, 0.3217], 0.2531, [0.2146, 0.1064]⟩	⟨[0.3254, 0.4282], 0.5366, [0.6204, 0.7069]; [0.5932, 0.5127], 0.3874, [0.2731, 0.1816]⟩	⟨[0.1213, 0.1704], 0.2243, [0.2852, 0.3573]; [0.7864, 0.7286], 0.6621, [0.5814, 0.4710]⟩

TABLE 7. Collective information by the Decision maker (R)

$0.25,\ 0.25,\ 0.15)^T$, to aggregate all the performance values $r_{ij}, (j = 1, 2, 3, 4)$ of the i^{th} alternative and get the overall performance value r_i corresponding to alternative $A_i (i = 1, 2, 3, 4)$ are $r_1 = \langle[0.4726,\ 0.5792],\ 0.6807,\ [0.7398,\ 0.8464];\ [0.4221,\ 0.2996],$

0.2559, [0.1802, 0.0913]⟩, r_2 = ⟨[0.4857, 0.6062], 0.7059, [0.7393, 0.8466]; [0.4326, 0.2930], 0.2596, [0.2066, 0.1074]⟩, r_3 = ⟨[0.3910, 0.4960], 0.5939, [0.6918, 0.7817]; [0.4984, 0.3807], 0.3090, [0.2239, 0.1287]⟩ and r_4 = ⟨[0.4357, 0.5387], 0.6259, [0.7059, 0.8016]; [0.4649, 0.3535], 0.2852, [0.2193, 0.1225]⟩.

Step 4: The ranking value corresponding to these aggregated numbers are obtained by using Eq. (1) as $\text{Rank}(r_1) = 0.5863$, $\text{Rank}(r_2) = 0.5975$, $\text{Rank}(r_3) = 0.3608$ and $\text{Rank}(r_4) = 0.4446$.

Step 5: Therefore, the ranking order of the four alternatives is $A_2 \succ A_1 \succ A_4 \succ A_3$, and found that A_2 is the most desirable one while A_3 is the least one.

5.3. **Effect of λ on the ranking.** To analyze the effect of the parameters λ on the most desirable alternatives of the given attributes, we use the different values of λ in the proposed approach for describing the changing trend of ranking order as well as the measuring value corresponding to each alternative. The complete variation of the ranking value of each alternative with respect to parameter λ is summarized in Fig. 3, while the ranking value for some parametric values of $\lambda \to 1, 2, 2.5, 5, 7.5, 10$ are summarized in Table 8. From this table and figure, it has been seen that with the increase in the value of λ, the ranking value corresponding to each alternative is also increases but the ranking order of these alternative remain same i.e., $A_2 \succ A_1 \succ A_4 \succ A_3$ and hence the best alternative is Wipro (A_2) for investing a money in the market.

Alternative	$\lambda \to 1$	$\lambda = 2$	$\lambda = 2.5$	$\lambda = 5$	$\lambda = 7.5$	$\lambda = 10$
A_1	0.5814	0.5863	0.5880	0.5934	0.5966	0.5989
A_2	0.5952	0.5975	0.5984	0.6015	0.6034	0.6048
A_3	0.3548	0.3608	0.3629	0.3692	0.3728	0.3753
A_4	0.4404	0.4446	0.4460	0.4507	0.4535	0.4554

Table 8. Ranking values for the different values of λ for each alternative

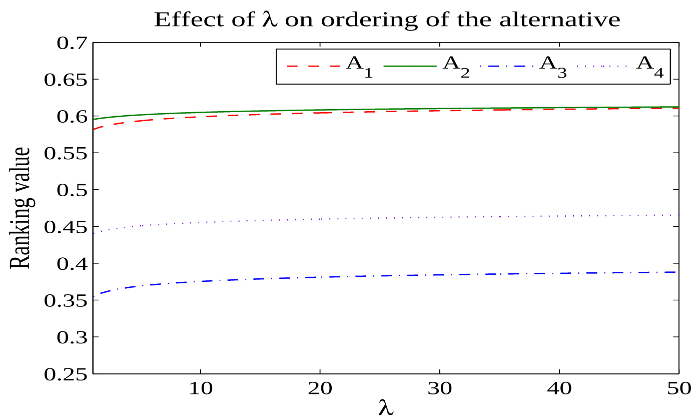

Figure 3. Effect of λ on the ranking order of the alternative

5.4. **Comparative studies.** In order to compare the performance of the proposed methods with some existing methods, a comparative studies has been taken in which the existing operators based on algebraic averaging operator [36], geometric operator [37], Einstein based t-norm [33], Hamacher based t-norm operator [24] etc have been considered. The results corresponding to it has been has been shown in Table 9. From this table, it has been analyzed that the best company for investing the money in the market is Wipro (A_2) than others and this result has been overlapped with the existing studies results which validates the stability of the approach. Furthermore, the proposed operator involve a certain parameter λ, which makes them more flexible in the process of information fusion and is more adequate to model practical decision making problems. Thus, the proposed technique can be suitably utilized to solve the problem of decision-making problem than the other existing measures.

Method	Parameter	Order of alternatives
Xu and Yager [37]	None	$A_1 \succ A_2 \succ A_4 \succ A_3$
Xu [36]	None	$A_2 \succ A_1 \succ A_4 \succ A_3$
Wang and Liu [33]	None	$A_2 \succ A_1 \succ A_4 \succ A_3$
Liu [24]	$\gamma = 2$	$A_2 \succ A_1 \succ A_4 \succ A_3$
Proposed operator	$\lambda = 2$	$A_2 \succ A_1 \succ A_4 \succ A_3$

TABLE 9. Comparative analysis

According to the above comparison analysis, the proposed method for addressing the decision-making problems has the following merits with respect to the existing ones.

(i) Compared with IFWA operator (or IFWG operator) proposed by Xu [36] (or Xu and Yager [37]), the IFWA operator (or IFWG operator) is only a special case of our proposed operators when parameter $\lambda \to 1$. So, our methods are more general. Furthermore, the proposed operators based on Frank t-norms, are more robust and can capture the relationship between the arguments. Moreover, when $\lambda \to \infty$, the proposed operators are reduced to the operators based on Lukasiewicz product and Lukasiewicz sum. Therefore, the Frank aggregation operators can contain almost all of the arithmetic aggregation operators and geometric aggregation operators for TIT2IFNs according to different values of parameter λ.

(ii) The proposed methods include a parameter, which can adjust the aggregate value based on the real decision needs, and capture many existing hesitant fuzzy aggregation operators. Therefore, the benefit is that the proposed operators come with their higher generality and flexibility. In other words, the decision maker can use the appropriate parameter value based on their risk preference and actual needs.

6. **Conclusion**

The objective of the present manuscript is to present triangular interval type 2 intuitionistic fuzzy aggregation operators by considering the Frank operational laws. For this, some operational laws based on Frank t-norm and conorm has been presented under the triangular type 2 intuitionistic fuzzy environment and then based on its some series of weighted averaging operators such as TIT2IFWA, TIT2IFOWA and TIT2IFHA have been proposed. Various desirable properties of its have also been stated and discussed in details. An illustrative example related to decision making process has been taken for demonstrating the approach. A comparative study with some existing operators shows that the proposed operators and their corresponding techniques provides an alternative ways to solve MCDM problem in a more effective manner. A sensitivity analysis has also been conducted for showing the impact of the decision parameters on to the ranking of the alternatives. In addition, the proposed results corresponding to different values of λ will offer the various choices for the decision maker for assessing the decisions.

7. Appendix

Proof of Theorem 4.3: Since for all j, $\alpha_j = \alpha_0 = \langle [a_0, b_0], c_0, [d_0, e_0];\ [A_0, B_0], C_0,\ [D_0, E_0]\rangle$, and $\sum_{j=1}^{n} \omega_j = 1$ so by Theorem 4.2, we have

$$\begin{aligned}
\text{TIT2IFWA}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \Bigg\langle & \Bigg[1 - \log_\lambda \Bigg(1 + \prod_{j=1}^{n} (\lambda^{1-a_0} - 1)^{\omega_j}\Bigg), 1 - \log_\lambda \Bigg(1 + \prod_{j=1}^{n} (\lambda^{1-b_0} - 1)^{\omega_j}\Bigg)\Bigg], \\
& 1 - \log_\lambda \Bigg(1 + \prod_{j=1}^{n} (\lambda^{1-c_0} - 1)^{\omega_j}\Bigg), \Bigg[1 - \log_\lambda \Bigg(1 + \prod_{j=1}^{n} (\lambda^{1-d_0} - 1)^{\omega_j}\Bigg), \\
& 1 - \log_\lambda \Bigg(1 + \prod_{j=1}^{n} (\lambda^{1-e_0} - 1)^{\omega_j}\Bigg)\Bigg]; \Bigg[\log_\lambda \Bigg(1 + \prod_{j=1}^{n} (\lambda^{A_0} - 1)^{\omega_j}\Bigg), \log_\lambda \Bigg(1 + \prod_{j=1}^{n} (\lambda^{B_0} - 1)^{\omega_j}\Bigg)\Bigg], \\
& \log_\lambda \Bigg(1 + \prod_{j=1}^{n} (\lambda^{C_0} - 1)^{\omega_j}\Bigg), \Bigg[\log_\lambda \Bigg(1 + \prod_{j=1}^{n} (\lambda^{D_0} - 1)^{\omega_j}\Bigg), \log_\lambda \Bigg(1 + \prod_{j=1}^{n} (\lambda^{E_0} - 1)^{\omega_j}\Bigg)\Bigg]\Bigg\rangle \\
= & \Bigg\langle \Bigg[1 - \log_\lambda \lambda^{1-a_0}, 1 - \log_\lambda \lambda^{1-b_0}\Bigg], 1 - \log_\lambda \lambda^{1-c_0}, \Bigg[1 - \log_\lambda \lambda^{1-d_0}, 1 - \log_\lambda \lambda^{1-e_0}\Bigg]; \\
& \Bigg[\log_\lambda \lambda^{A_0}, \log_\lambda \lambda^{B_0}\Bigg], \log_\lambda \lambda^{C_0}, \Bigg[\log_\lambda \lambda^{D_0}, \log_\lambda \lambda^{E_0}\Bigg]\Bigg\rangle \\
= & \Bigg\langle [a_0, b_0], c_0, [d_0, e_0]; [A_0, B_0], C_0, [D_0, E_0]\Bigg\rangle \\
= & \alpha_0
\end{aligned}$$

Thus, proof is completed.

Proof of Theorem 4.4: As for all j, we have $\min_j\{a_j\} \le a_j \le \max_j\{a_j\}$, this implies that $1 - \max_j\{a_j\} \le 1 - a_j \le 1 - \min_j\{a_j\}$. Hence, for $\lambda > 1$, we have $\lambda^{1-\max_j\{a_j\}} - 1 \le \lambda^{1-a_j} - 1 \le \lambda^{1-\min_j\{a_j\}} - 1 \Leftrightarrow \lambda^{1-\max_j\{a_j\}} - 1 \le \prod_{j=1}^{n} (\lambda^{1-a_j} - 1)^{\omega_j} \le \lambda^{1-\min_j\{a_j\}} - 1 \Leftrightarrow 1 - \max_j\{a_j\} \le \log_\lambda \big(1 + \prod_{j=1}^{n} (\lambda^{1-a_j} - 1)^{\omega_j}\big) \le 1 - \min_j\{a_j\}$.

Therefore,

$$\min_j\{a_j\} \leq 1 - \log_\lambda \left(1 + \prod_{j=1}^{n}(\lambda^{1-a_j} - 1)^{\omega_j}\right) \leq \max_j\{a_j\} \tag{3}$$

Furthermore, for all j, we have $\min_j\{A_j\} \leq A_j \leq \max_j\{A_j\}$, this implies $\lambda^{\min_j\{A_j\}} - 1 \leq \lambda^{A_j} - 1 \leq \lambda^{\max_j\{A_j\}} - 1 \Leftrightarrow \lambda^{\min_j\{A_j\}} - 1 \leq \prod_{j=1}^{n}(\lambda^{A_j} - 1)^{\omega_j} \leq \lambda^{\max_j\{A_j\}} - 1$. Thus,

$$\min_j\{A_j\} \leq \log_\lambda \left(1 + \prod_{j=1}^{n}(\lambda^{A_j} - 1)^{\omega_j}\right) \leq \max_j\{A_j\} \tag{4}$$

Let $\text{TIT2IFWA}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \langle [a, b], c, [d, e]; [A, B], C, [D, E]\rangle$. Then from Eq. (3) and (4) we have, $\min_j\{a_j\} \leq a \leq \max_j\{a_j\}$ and $\min_j\{A_j\} \leq A \leq \max_j\{A_j\}$. Similarly, we have $\min_j\{b_j\} \leq b \leq \max_j\{b_j\}, \min_j\{c_j\} \leq c \leq \max_j\{c_j\}, \min_j\{d_j\} \leq d \leq \max_j\{d_j\}$, $\min_j\{e_j\} \leq e \leq \max_j\{e_j\}, \min_j\{B_j\} \leq B \leq \max_j\{B_j\}, \min_j\{C_j\} \leq C \leq \max_j\{C_j\}$, $\min_j\{D_j\} \leq D \leq \max_j\{D_j\}, \min_j\{E_j\} \leq E \leq \max_j\{E_j\}$. Hence, by using Eq. (1), we have

$$\begin{aligned}
\text{Rank }(\alpha) &= \left(\frac{(a-A)+(e-E)}{2}+1\right)\left(\frac{a-A+b-B+4(c-C)+d-D+e-E}{8}\right)\\
&\leq \left(\frac{\max_j\{a_j\} - \min_j\{A_j\} + \max_j\{e_j\} - \min_j\{E_j\}}{2}+1\right)\times\\
&\times \left(\frac{\begin{array}{c}\max_j\{a_j\} - \min_j\{A_j\} + \max_j\{b_j\} - \min_j\{B_j\} + 4\max_j\{c_j\}\\ - 4\min_j\{C_j\} + \max_j\{d_j\} - \min_j\{D_j\} + \max_j\{e_j\} - \min_j\{E_j\}\end{array}}{8}\right)\\
&= \text{Rank }(\alpha^+)
\end{aligned}$$

and

$$\begin{aligned}
\text{Rank }(\alpha) &= \left(\frac{(a-A)+(e-E)}{2}+1\right)\left(\frac{a-A+b-B+4(c-C)+d-D+e-E}{8}\right)\\
&\geq \left(\frac{\min_j\{a_j\} - \max_j\{A_j\} + \min_j\{e_j\} - \max_j\{E_j\}}{2}+1\right)\times\\
&\times \left(\frac{\begin{array}{c}\min_j\{a_j\} - \max_j\{A_j\} + \min_j\{b_j\} - \max_j\{B_j\} + 4\min_j\{c_j\}\\ - 4\max_j\{C_j\} + \min_j\{d_j\} - \max_j\{D_j\} + \min_j\{e_j\} - \max_j\{E_j\}\end{array}}{8}\right)\\
&= \text{Rank }(\alpha^-)
\end{aligned}$$

Therefore, $\alpha^- \leq \text{TIT2IFWA}(\alpha_1, \alpha_2, \ldots, \alpha_n) \leq \alpha^+$.

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